On the construction of valuations and generating sequences on hypersurface singularities

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ABSTRACT

Suppose that (K, ν) is a valued field, $f(z) \in K[z]$ is a unitary and irreducible polynomial and (L, ω) is an extension of valued fields, where $L = K[z]/(f(z))$. Let A be a local domain with quotient field K dominated by the valuation ring of ν , and assume that $f(z)$ is in $A[z]$. We describe the structure of the associated graded ring $\text{gr}_\omega A[z]/(f(z))$ of $A[z]/(f(z))$ for the filtration defined by ω as an extension of the associated graded ring of A for the filtration defined by ν . This gives an algorithm which in many cases produces a finite set of elements of $A[z]/(f(z))$ whose images in $\text{gr}_\omega A[z]/(f(z))$ generate it as a $\text{gr}_\nu A$ -algebra as well as the relations between these images. We also work out the interactions of our method with phenomena such as the nontameness and the defect of an extension. For a valuation ν of rank 1 and a separable extension of valued fields $(K, \nu) \subset (L, \omega)$ as above, our algorithm produces a generating sequence in a local birational extension A_1 of A dominated by ν if and only if there is no defect. In this case, $\text{gr}_\omega A_1[z]/(f(z))$ is a finitely presented $\text{gr}_\nu A_1$ -module.

1. Introduction

Given a field K and an extension L of K , the study of the problem of extending a valuation from K to L has a long history motivated in part by its close relation with ramification theory, whether in number theory or in algebraic geometry. It has an incarnation in logic, the model theory of valued fields which provides another viewpoint on ramification theory. After fundamental work by E. Artin, H. Hasse, A. Ostrowski and others, S. Mac Lane created a method for describing all extensions of a discrete rank 1 valuation on a field K to a primitive extension $K(z)$, be it algebraic or transcendental. The method is based on the existence of *key polynomials* in $K[z]$ which provide successive approximations of a given extension of the valuation and, by the behavior of their degrees, a measure of its complexity.

On the side of algebraic geometry, O. Zariski's approach to resolution of singularities of algebraic varieties using local uniformization of valuations provides a strong motivation for the study of valuations on local domains essentially of finite type over a field, which waned after

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H. Hironaka's proof of resolution in characteristic zero but later revived as an approach to resolution in positive characteristic.

Here the emphasis shifts from valuations on fields to valuations on (quasi-excellent) local subrings of valuation rings, for which the extension problem is more involved, as we shall see below.

In the 1970s and 1980s appeared (see [Zar06, Spi90, GT00]) the idea that the associated graded ring $\mathrm{gr}_\nu A$ of a local domain A with respect to the filtration of A associated with a valuation ν of its field of fractions centered in A (nonnegative on A and positive on its maximal ideal) encoded in a geometric way essential characters of the valuation and could be used, at least in special cases, to obtain local uniformization. For example, representatives in A of the generators of the graded algebra associated with the unique valuation of a 1-dimensional integral complex analytic algebra can be used to embed the corresponding curve in an affine space where a single birational toric modification provides an embedded resolution of singularities (see [GT00]). It also became apparent that some of Mac Lane's essential definitions are better understood using associated graded rings.

Somewhat later, Mac Lane's theory was generalized by M. Vaquié, who extended to all Krull valuations the construction of sequences of key polynomials, now indexed by totally ordered sets (see [Vaq07a, Vaq07b, Vaq08] and Section 7 below). He also described the extension $\mathrm{gr}_\nu K \subset \mathrm{gr}_\omega K[z]$ of graded rings corresponding to an extension of valuations from ν on K to ω on $K[z]$, for z algebraic or transcendental over K . It appeared that the images of Mac Lane's and Vaquié's key polynomials in the graded algebra $\mathrm{gr}_\omega K[z]$ were related to its generation as a $\mathrm{gr}_\nu K$ -algebra.

In the last three decades or so, the problem of describing a *generating sequence* for a valuation, which is a set of elements of a ring A whose images in $\mathrm{gr}_\nu A$ provide a presentation by generators and relations, has become of major interest for the ramification theory of extensions of valued fields as well as for local uniformization in positive characteristic, which is still an open problem.

In fact, it has become apparent that given an extension $(A, \nu) \subset (B, \omega)$ of valued *rings*, the extension $\mathrm{gr}_\nu A \subset \mathrm{gr}_\omega B$ of the associated graded algebras, as well as the similar extensions obtained after birational extensions of A and B , encodes essential information about the ramification of the original extension in a comparatively simple language, such as the condition of being finitely generated. This concerns especially the defect and the possibility to uniformize ω on B if we can uniformize ν on A . But we can access this information only if we have descriptions of $\mathrm{gr}_\nu A$ and $\mathrm{gr}_\omega B$ by generators and relations, or of $\mathrm{gr}_\omega B$ as a $\mathrm{gr}_\nu A$ -algebra. This is the main motivation for this work.¹

Here we consider the case when the essence of the difficulty resides: suppose that (K, ν) is a valued field, $f(z) \in K[z]$ is a unitary and irreducible polynomial and (L, ω) is a finite field extension, where $L = K[z]/(f(z))$. Further suppose that A is a local domain with quotient field K such that ν dominates A and that $f(z)$ is in $A[z]$. We provide an algorithm producing the first significant part of a generating sequence for extensions of a valuation ν to $A[z]/(f(z))$.

¹We think that the problem of constructing generating sequences in a Noetherian local domain A which is dominated by a valuation ν is very difficult, and little is known about it in general. The difficulty reflects the fact that the structure of the semigroup of values $S^A(\nu) = \nu(A \setminus \{0\})$ is closely related to some of the birational maps providing embedded local uniformizations of ν and can be extremely complicated. It is well understood in the case when A has dimension 1 (see [Zar06, GT00]) and for regular local rings of dimension 2 [Spi90, CV14, Mou17]. It is known for certain valuations dominating 2-dimensional quotient singularities [Dut20] and for certain valuations dominating 3-dimensional regular local rings [Kas16].

The valuations ν and ω also induce filtrations of K and $K[z]/(f(z))$, respectively. The associated graded ring of $K[z]/(f(z))$ along ω has been constructed implicitly as an extension of the associated graded ring of K along ν in the papers [Mac36a, ML36b] of Mac Lane for discrete rank 1 valuations and for general valuations by Vaquié in [Vaq07a, Vaq07b, Vaq08]. Further papers on this topic, and comparison with the method of pseudoconvergent sequences (introduced by Ostrowski in [Ost35, Teil III, Section 11] and developed by I. Kaplansky in [Kap42]) are [Bro72, NS18, San17, HOMS14] and [DMS18]. Finding generating sequences for $A[z]/(f(z))$ in the case when A is no longer a field but an arbitrary Noetherian subring dominated by R_ν and with the same field of fractions is much more closely related to resolution of singularities via local uniformization and correspondingly more difficult.

This paper is devoted to this problem. We describe the relationship of our method with the key polynomials of Mac Lane and Vaquié. We also work out the interactions of our method of computation with phenomena which complicate the study of ramification in positive characteristic, such as the lack of tameness and the defect of an extension.

We now give more details about the content of this paper.

Let G_ν be the value group of ν and R_ν be the valuation ring of ν , with maximal ideal m_ν . Given a subring A of the field of fractions of R_ν , the associated graded ring of A along ν is defined as

$$\mathrm{gr}_\nu(A) = \bigoplus_{\gamma \in G_\nu} \mathcal{P}_\gamma(A) / \mathcal{P}_\gamma^+(A),$$

where

$$\mathcal{P}_\gamma(A) = \{g \in A \setminus \{0\} \mid \nu(g) \geq \gamma\} \quad \text{and} \quad \mathcal{P}_\gamma^+(A) = \{g \in A \setminus \{0\} \mid \nu(g) > \gamma\}.$$

The ring $\mathrm{gr}_\nu(A)$ is an algebra over its degree zero subring. It is a domain which is generally not Noetherian. In this text, we shall consider subrings of R_ν such that the semigroup $S^A(\nu)$ of values of elements of $A \setminus \{0\}$ which indexes the homogeneous components of $\mathrm{gr}_\nu(A)$ is contained in the positive part of G_ν . We shall see more about this semigroup below.

Important invariants of a finite extension $(K, \nu) \subset (L, \omega)$ of valued fields are the reduced ramification index and residue degree of ω over ν , which are

$$e(\omega/\nu) = [G_\omega : G_\nu] \quad \text{and} \quad f(\omega/\nu) = [R_\omega/m_\omega : R_\nu/m_\nu].$$

Another, very subtle, invariant is the defect $\delta(\omega/\nu)$ of the extension, which is a power of the characteristic p of the residue field R_ν/m_ν . The defect and its role in local uniformization are explained in [Kuh00]. We give the definition of the defect in (44) below. In the case when ω is the unique extension of ν to L , we have that

$$[L : K] = e(\omega/\nu) f(\omega/\nu) \delta(\omega/\nu). \tag{1}$$

If A and B are local domains with quotient fields K and L such that ω dominates B and B dominates A , we have a graded inclusion of graded domains

$$\mathrm{gr}_\nu(A) \rightarrow \mathrm{gr}_\omega(B).$$

The index of quotient fields is

$$[\mathrm{QF}(\mathrm{gr}_\omega(B)) : \mathrm{QF}(\mathrm{gr}_\nu(A))] = e(\omega/\nu) f(\omega/\nu)$$

by Proposition 3.3 of [Cut16a]. The defect seems to disappear, but it manifests itself in mysterious behavior in the extensions of associated graded rings of injections $A' \rightarrow B'$ of birational extensions of Noetherian local domains A and B . For instance, if ν has rational rank 1 but is not discrete,

the defect $\delta(\omega/\nu)$ is larger than 1 and A and B are 2-dimensional excellent local domains, then $\text{gr}_\omega(B')$ is not a finitely generated $\text{gr}_\nu(A')$ -algebra for any regular local rings $A' \rightarrow B'$ which are dominated by ω and dominate A and B , as shown in [Cut17].

The construction of generating sequences is closely related to the problem of local uniformization. In [CM19, Theorem 7.1], it is shown how reduction of multiplicity along a rank 1 valuation can be achieved in a defectless extension $A \rightarrow A[z]/(f(z))$. A similar statement is proven by J.-C. San Saturnino in [San17].

The statement “defectless” means that the rank 1 valuations ν and ω satisfy $\delta(\omega/\nu) = 1$. From this assumption, it follows that either $\omega(z - K)$ has a largest element, or the limsup of this set is ∞ . If the limsup of this set is ∞ , then in an appropriate extension, the valuation ω corresponds to a linear factor of $f(z)$, and it is not difficult to realize a reduction of multiplicity by blowing up. So assume that $\omega(z - K)$ has a largest element $\gamma \in G_\omega$. We then have $\gamma \notin G_\nu$. After a birational extension A_1 of A and a change of variables of z in $A_1[z]$, we obtain that $\omega(z) = \gamma$ and then after a Cremona transformation involving z , we obtain a reduction of the multiplicity of the strict transform of f .

In [Tei03] and [Tei14], it is shown how associated graded rings along a valuation can be used to prove local uniformization, at least when the associated graded rings are finitely generated algebras over A/m_A . A suitable toric resolution of singularities of the associated graded ring induces a local uniformization of the given valuation.

The subring of degree zero elements of the graded ring $\text{gr}_\nu(A)$ is $(\text{gr}_\nu(A))_0 = A/Q$, where Q is the prime ideal in A of elements of positive value. A generating sequence for ν on A is an ordered set of elements of A whose classes in $\text{gr}_\nu(A)$ generate $\text{gr}_\nu(A)$ as a graded $(\text{gr}_\nu(A))_0$ -algebra. To be meaningful, a generating sequence should come with a formula for computing the values of elements of A and their relations in $\text{gr}_\nu(A)$. In particular, a generating sequence should give the structure of $\text{gr}_\nu(A)$ as a graded $(\text{gr}_\nu(A))_0$ -algebra.

In the case of an inclusion $A \subset B$ of domains and an extension ω of ν to the quotient field of B such that ω has nonnegative value on B , a generating sequence of the extension is an ordered sequence of elements of B whose classes in $\text{gr}_\omega(B)$ generate $\text{gr}_\omega(B)$ as a $\text{gr}_\nu(A)$ -algebra. A generating sequence for an extension should come with a formula for computing the values of elements of B , relative to the values of elements of A , and give their relations in $\text{gr}_\omega(B)$. That is, a generating sequence should give the structure of $\text{gr}_\omega(B)$ as a graded $\text{gr}_\nu(A)$ -algebra.

In this paper, we give a very simple algorithm which allows us to compute a generating sequence and the structure of $\text{gr}_\omega(A[z]/(f(z)))$ in many situations. Throughout this paper, we have the assumption that A is a local domain which contains an algebraically closed field $\mathbf{k}$ such that its residue field A/m_A is $\mathbf{k}$, that ν dominates A and that the residue field R_ν/m_ν of the valuation ring R_ν of ν is $\mathbf{k}$ (ν is a “rational valuation”). This algorithm is derived in Section 4. The algorithm is valid for an arbitrary extension ω of an arbitrary valuation ν dominating A (that is, $m_\nu \cap A = m_A$).

A realization of our algorithm produces a subring of $\text{gr}_\omega(R_\nu[z]/(f(z)))$ which is the quotient C/I of a graded polynomial ring C over $\text{gr}_\nu(R_\nu)$ in either finitely many or countably many variables, and a set of generators of the graded prime ideal I of C . Our algorithm gives an explicit representation of this subring as

$$\text{gr}_\nu(R_\nu)[\bar{\varphi}_1, \dots, \bar{\varphi}_k, \dots]/I,$$

where

$$I = (\bar{\varphi}_1^{n_1} - \bar{c}_1, \bar{\varphi}_2^{n_2} - \bar{c}_2 \bar{\varphi}_1^{j_1(2)}, \dots, \bar{\varphi}_k^{n_k} - \bar{c}_k \bar{\varphi}_1^{j_1(k)} \bar{\varphi}_2^{j_2(k)} \dots \bar{\varphi}_{k-1}^{j_{k-1}(k)}, \dots)$$

with $\bar{c}_1, \dots, \bar{c}_k, \dots \in \text{gr}_\nu(R_\nu)$ homogeneous elements. The elements $\bar{\varphi}_i$ are homogeneous with strictly increasing values. If our algorithm terminates in a finite number of steps k , then elements $\varphi_1, \dots, \varphi_k \in R_\nu[z]$ whose classes are $\bar{\varphi}_1, \dots, \bar{\varphi}_k$ form a generating sequence of $R_\nu[z]/(f(z))$ over R_ν and we have built up the entire associated graded ring

$$\text{gr}_\nu(R_\nu[z]/(f(z))) = \text{gr}_\nu(R_\nu)[\bar{\varphi}_1, \dots, \bar{\varphi}_k]/I,$$

where

$$I = (\bar{\varphi}_1^{n_1} - \bar{c}_1, \bar{\varphi}_2^{n_2} - \bar{c}_2 \bar{\varphi}_1^{j_1(2)}, \dots, \bar{\varphi}_k^{n_k} - \bar{c}_k \bar{\varphi}_1^{j_1(k)} \bar{\varphi}_2^{j_2(k)} \dots \bar{\varphi}_{k-1}^{j_{k-1}(k)}).$$

In this case, $\text{gr}_\nu(R_\nu[z]/(f(z)))$ is a finitely generated and presented $\text{gr}_\nu(R_\nu)$ -module.

When we compare our algorithm to the theory of Vaquié [Vaq07a, Vaq07b, Vaq08] in Subsection 7.1, we conclude in Proposition 7.1 that a realization of our algorithm produces the “first simple admissible family” $\mathcal{S}^{(1)}$ of an “admissible family” $\mathcal{S}$ determining the valuation ω .

In the case of a Noetherian local domain A dominated by R_ν as above, in many situations our algorithm produces a finite sequence of elements of $A[z]$ whose images generate the $\text{gr}_\nu A$ -algebra $\text{gr}_\omega A[z]$. It does this even in cases when there are infinitely many key polynomials. Remarks 8.12 in [Tei14] display a similar phenomenon of finite generation in the presence of an infinity of key polynomials.

More precisely, in Theorem 5.1, we show that if the characteristic p of k does not divide the degree of f , A is a domain as above and ω is the unique extension of ν to a valuation of the quotient field L of $A[z]/(f(z))$, then our algorithm produces a finite generating sequence in $A[z]/(f(z))$. The associated graded ring of $A[z]/(f(z))$ along ω is then a finitely generated and presented module over the associated graded ring of A along ν .

Since the defect $\delta(\omega/\nu)$ is always a power of p , the assumption in Theorem 5.1 that p does not divide the degree of f and the assumption that ω is the unique extension of ν force the defect $\delta(\omega/\nu)$ to be 1 by (1).

We show that if any of the above assumptions are removed, then the conclusions of Theorem 5.1 do not hold (examples of Sections 4 and 11). For instance, in Example 11.3, the assumption that $R_\nu[z]/(f(z))$ is a “hypersurface singularity” is shown to be necessary for finite generation to hold.

To illustrate the power of Theorem 5.1, in Example 5.2 we compute the associated graded ring when $f(z)$ is a quadratic polynomial, k has characteristic not equal to 2 and ω is the unique extension of ν . It has the simple form

$$\text{gr}_\omega(A[z]/(f(z))) \cong \text{gr}_\nu(A)[\bar{\varphi}]/(\bar{\varphi}^2 - \bar{c})$$

for some homogeneous $\bar{c} \in \text{gr}_\nu(A)$. From the classification of associated graded rings of valuations dominating a 2-dimensional regular local ring A (see [Spi90, CV14]), we see that when the extension ω is unique, we are able to completely calculate the associated graded ring along an extended valuation in the local rings of 2-dimensional rational double points. In contrast, if ω is not the unique extension of ν , then $\text{gr}_\omega(A[z]/(f(z)))$ might not be a finitely generated $\text{gr}_\nu(A)$ -module, as shown in Examples 5.2 and 11.4.

In Theorem 8.2, we consider an arbitrary separable extension (with no assumption on the degree) and assume that A is a Nagata local domain. We show that an extension of a rank 1 valuation ν is without defect if and only if there exists a realization of our algorithm with coefficients in a birational extension A_1 of A which constructs ω , as either a valuation or a limit valuation. A birational extension A_1 of A is a localization of a finitely generated A -algebra whose quotient field is K and which is dominated by ν .

An example showing that the conclusions of Theorem 8.2 may not hold if ν has rank larger than 1 is given in Section 10. In Example 8.3, it is shown that the conclusions of Theorem 8.2 may not hold if $f(z)$ is not separable over K .

In Section 9, we analyze our algorithm in a rank 1 example with defect from [CP04] to motivate the necessary condition of Theorem 8.2. We explicitly show that a generating sequence does not exist in $A_1[z]$ for any birational extension A_1 of A which is dominated by ν , and the valuation ω is not realizable as a limit valuation; that is, ω is not realizable as a sequence of approximants, only as a collection of approximants indexed by a more general well-ordered set.

In the final section, Section 11, we give examples showing that the finite generation of extensions of associated graded rings and valuation semigroups ensured by Theorem 5.1 may fail if any of the assumptions of the theorem are removed. The semigroup $S^A(\nu)$ of values of ν on A is

$$S^A(\nu) = \{\nu(g) \mid g \in A \setminus \{0\}\}.$$

In Example 11.3, it is shown that there exist an extension L of the quotient field K of A of degree prime to p and a valuation ν of K which dominates A and has a unique extension to L such that if B is the integral closure of A in L , then $\text{gr}_\omega(B)$ is not a finitely generated $\text{gr}_\nu(A)$ -module and the semigroup $S^B(\omega)$ is not a finitely generated $S^A(\nu)$ -module. In particular, the conclusions of Theorem 5.1 do not hold for this extension. This example shows that we must have the condition that $B = A[z]/(f(z))$ is a “hypersurface singularity” for the conclusions of Theorem 5.1 to be true.

We make use of the theory of Mac Lane [Mac36a, ML36b], which he developed to construct the extensions of a (rank 1) discrete valuation ν of K to a discrete valuation ω of $K[z]$ or of $K[z]/(f(z))$ for some irreducible unitary polynomial $f(z) \in K[z]$. Our algorithm can be viewed as a realization of Mac Lane’s method in the context of a general valuation, in a specific, nice form. Mac Lane’s theory is surveyed in Section 3.

We also make use of Vaquié’s generalization of Mac Lane’s method in [Vaq07a, Vaq07b, Vaq08] to construct extensions of general valuations in $K[z]$ and $K[z]/(f(z))$ in our proof of Theorem 8.2. The essential new concept in Vaquié’s work is that of a “limit key polynomial.” In [Vaq08, Exemple 4.1], he gave an example of infinite sequences of key polynomials due to the nonuniqueness of valuation extension. A survey of Vaquié’s method is given in Section 7, as well as a study of its relationship to our algorithm. In the situation of this paper, we shall meet only finite sequences of limit key polynomials since the number of limit key polynomials is bounded by the degree of $f(z)$. In Section 6, we collect and derive some results about Henselizations of rings and valued fields which we need for the proof of Theorem 8.2.

In this paper, a local ring is a commutative ring with a unique maximal ideal. In particular, we do not require a local ring to be Noetherian. We will denote the maximal ideal of a local ring A by m_A . The quotient field of a domain A will be denoted by $\text{QF}(A)$. We will say that a local ring B dominates a local ring A if $A \subset B$ and $m_B \cap A = m_A$.

We will denote the natural numbers by $\mathbf{N}$ and the positive integers by $\mathbf{Z}_+$.

2. Valuations and pseudovaluations

From here on, we shall consider sequences of valuations which approximate ω . For that reason, we change the notation and denote these sequences by $V_0, V_1, \dots$ as in [Mac36a] and [ML36b]. A general valuation will be denoted by V , and the reader may think of ν as V_0 .

Suppose that V is a valuation on a field K . We will denote the valuation ring of V by R_V and its maximal ideal by m_V . The value group of V will be denoted by G_V .

Suppose that A is a Noetherian local domain with quotient field K and $A \rightarrow A_1$ is an extension of local domains such that A_1 is a domain whose quotient field is K and A_1 is essentially of finite type over A (that is, A_1 is a localization of a finitely generated A -algebra). Then we will say that $A \rightarrow A_1$ is a birational extension.

If A is a domain which is contained in R_V , then the associated graded ring of A along V is $\text{gr}_V(A)$, as defined in the introduction. The initial form $\text{In}_V(g)$ of $g \in A$ is the class of g in $\mathcal{P}_{V(g)}(A)/\mathcal{P}_{V(g)}^+(A)$. The semigroup of V on A has also been defined in the introduction.

A pseudovaluation (or semivaluation) V on a domain A is a surjective map $V: A \rightarrow G_V \cup \{\infty\}$, where G_V is a totally ordered abelian group, and a prime ideal

$$I(V)_\infty = I^A(V)_\infty = \{g \in A \mid V(g) = \infty\}$$

of A_1 such that $V: \text{QF}(A/I(V)_\infty) \setminus \{0\} \rightarrow G_V$ is a valuation.

3. The Mac Lane theory of key polynomials

Suppose that V is a valuation or a pseudovaluation on a domain A . Following Mac Lane in [Mac36a] in the case $A = K[z]$, we can define an equivalence $\sim$ on A defined for $g, h \in A$ by $g \sim h$ in V if $V(g - h) > \min\{V(g), V(h)\}$ or $V(g) = V(h) = \infty$. We say that $g \in A$ is equivalence divisible by h in V , written $h|g$ in V , if there exists an $a \in A$ such that $g \sim ah$ in V . An element g is said to be equivalence irreducible in V if $g|ab$ in V implies $g|a$ or $g|b$ in V .

These conditions can be expressed, respectively, as the statements that $\text{In}_V(h) = \text{In}_V(g)$ in $\text{gr}_V(A)$, that $\text{In}_V(h)$ divides $\text{In}_V(g)$ in $\text{gr}_V(A)$ and that the ideal generated by $\text{In}_V(g)$ in $\text{gr}_V(A)$ is prime.

3.1 Mac Lane's algorithm

We review Mac Lane's algorithm [Mac36a] to construct the extensions of a valuation V_0 of a field K to a valuation or pseudovaluation of the polynomial ring $K[z]$. Mac Lane applied his method to construct extensions of rank 1 discrete valuations of K to $K[z]$. This algorithm has been extended to general valuations by Vaquié [Vaq07a]. Mac Lane constructs “augmented sequences of inductive valuations”

$$V_1, \dots, V_k, \dots, \quad (2)$$

which extend V_0 to $K[z]$. An augmented sequence (2) is constructed from successive inductive valuations

$$V_k = [V_{k-1}; V_k(\varphi_k) = \mu_k] \quad \text{for } 1 \leq k \quad (3)$$

of $K[z]$, where φ_k is a “key polynomial” over V_{k-1} and μ_k is a “key value” of φ_k over V_{k-1} . We always take $\varphi_1 = z$.

We say that $\varphi(z) \in K[z]$ is a key polynomial with key value μ over V_{k-1} if

- 1) $\varphi(z)$ is equivalence irreducible in V_{k-1} ;
- 2) $\varphi(z)$ is minimal in V_{k-1} ; that is, if $\varphi(z)$ equivalence divides $g(z)$ in V_{k-1} , then $\deg_z \varphi(z) \leq \deg_z g(z)$;
- 3) $\varphi(z)$ is unitary and $\deg_z \varphi(z) > 0$;
- 4) $\mu > V_{k-1}(\varphi(z))$.

Following Mac Lane [Mac36a, Definition 6.1], we also assume that

- 5) $\deg_z \varphi_i(z) \geq \deg_z \varphi_{i-1}(z)$ for $i \geq 2$;
- 6) $\varphi_i(z) \sim \varphi_{i-1}(z)$ in V_{i-1} is false; here the equivalence is to be understood for polynomials in $K[z]$.

It follows from [Mac36a, Theorem 9.3] that

$$\text{if } \varphi(z) \text{ is a key polynomial over } V_{k-1}, \text{ then } \deg_z \varphi_{k-1}(z) \text{ divides } \deg_z \varphi(z). \quad (4)$$

The key polynomials $\varphi_k(z)$ can further be assumed to be homogeneous in V_{k-1} , which will be defined after (7).

Mac Lane shows that if V_0 is discrete of rank 1, then the extensions of V_0 to a valuation or pseudovaluation of $K[z]$ are the V_k arising from augmented sequences of finite length (2) and the limit sequences of augmented sequences of infinite length (2) which determine a limit value V_∞ on $K[z]$ defined by

$$V_\infty(g(z)) = \lim_{k \rightarrow \infty} V_k(g(z)) \quad \text{for } g(z) \in K[z].$$

We have that $V_\infty(g(z))$ is well defined whenever V_0 has rank 1 and is a valuation or pseudovaluation, by the argument of [Mac36a, Section 6, p. 372].

Mac Lane's method has been extended by Vaquié [Vaq07a] to eventually construct all extensions of an arbitrary valuation V_0 of K to a valuation or pseudovaluation of $K[z]$. We will discuss Vaquié's method in Section 7.

To compute the “ k th stage” value $V_k(g(z))$ for $g(z) \in K[z]$ by Mac Lane's method, we consider the unique expansion

$$g(z) = g_m(z)\varphi_k^m(z) + g_{m-1}\varphi_k^{m-1}(z) + \cdots + g_0 \quad (5)$$

with $g_i(z) \in K[z]$, $\deg_z g_i(z) < \deg_z \varphi_k(z)$ for all i and $g_m(z) \neq 0$. Then

$$V_k(g(z)) = \min\{V_{k-1}(g_m(z)) + m\mu_k, V_{k-1}(g_{m-1}(z)) + (m-1)\mu_k, \dots, V_{k-1}(g_0(z))\}.$$

Assuming the existence of a unique expansion of the coefficients $g_i(z)$ in terms of the polynomials $\varphi_j(z)$ with $j < k$, this expression suffices to prove by induction that every $g(z) \in K[z]$ has a unique expansion

$$g(z) = \sum_j a_j(z) \varphi_1^{m_{1,j}}(z) \varphi_2^{m_{2,j}}(z) \cdots \varphi_k^{m_{k,j}}(z) \quad (6)$$

with $a_j \in K$ and $0 \leq m_{i,j} < \deg_z \varphi_{i+1} / \deg_z \varphi_i$ for $i = 1, \dots, k-1$. Recall that $\deg_z \varphi_{i+1} / \deg_z \varphi_i$ is a positive integer by (4). Then

$$V_k(g) = \min_j V_k(a_j \varphi_1^{m_{1,j}} \varphi_2^{m_{2,j}} \cdots \varphi_k^{m_{k,j}}). \quad (7)$$

If all terms in (6) have the same values in V_k , then g is said to be homogeneous in V_k .

We shall often, as we just did, simplify the notation by writing g for $g(z)$, etc., when there is no fear of confusion.

Remark 3.1. If A is a subring of K such that $\varphi_i \in A[z]$ for $1 \leq i \leq k$ and $g \in A[z]$, then the coefficients a_j in (6) are all in A .

The polynomial g , with expansion (5), is minimal in V_k if and only if $g_m \in K$ and

$$V_k(g) = V_k(g_m \varphi_k^m), \quad (8)$$

by [ML36b, 2.3] or [Mac36a, Theorem 9.3].

By [ML36b, 3.13] or [Mac36a, Theorem 6.5], for $k > i$,

$$V_k(\varphi_i) = V_i(\varphi_i) \quad \text{and} \quad V_k(g) = V_i(g) \quad \text{whenever} \quad \deg_z g < \deg_z \varphi_{i+1}. \quad (9)$$

Further, by [Mac36a, Theorems 5.1 and 6.4] or [ML36b, 3.11 and 3.12],

$$V_k(g) \geq V_{k-1}(g) \text{ for all } g \in K[z], \text{ with equality if and only if } \varphi_k \nmid g \text{ in } V_{k-1}. \quad (10)$$

3.2 Mac Lane's algorithm in a finite primitive extension

Suppose that $f(z) \in K[z]$ is unitary and irreducible. The extensions of V_0 to valuations of $K[z]/(f(z))$ are the extensions of V_0 to pseudovaluations V of $K[z]$ such that $I(V)_\infty = (f(z))$. In [ML36b, Section 5], Mac Lane gives an explicit explanation of how his algorithm can be applied to construct the pseudovaluations V of $K[z]$ which satisfy $I(V)_\infty = (f(z))$ (when V_0 is discrete of rank 1). In [Vaq07b] and [Vaq08], Vaquié shows how this algorithm can be extended to arbitrary valuations V_0 of K .

Suppose that $V_1, \dots, V_k$ is an augmented sequence of inductive valuations in $K[z]$. Expand

$$f = f_m \varphi_k^m + \dots + f_0$$

as in (5). Define the projection of V_k by $\text{proj}(V_k) = \alpha - \beta$, where α is the largest and β is the smallest among the exponents j for which $V_k(f(z)) = V_k(f_j \varphi_k^j)$. A k th approximant V_k to $f(z)$ over V_0 is a k th stage homogeneous (meaning that the key polynomial φ_i is homogeneous in V_{i-1} for $i \leq k$) inductive valuation which is an extension of V_0 and which has a positive projection [ML36b, Definition 3.3].

First approximants V_1 to f are defined as $V_1 = [V_0; V_1(\varphi_1) = \mu_1]$, where $\varphi_1 = z$ and μ_1 is chosen so that $\text{proj}(V_1) > 0$. In [ML36b, Lemma 3.4], Mac Lane shows that if V_k is a k th approximant to $f(z)$, then so is V_i for $i = 1, \dots, k-1$. Further, $\varphi_k \mid f$ in V_{k-1} and $V_k(f(z)) > V_{k-1}(f(z)) > \dots > V_1(f(z))$. In [ML36b, Theorem 10.1], Mac Lane shows that if V_0 is a discrete valuation of rank 1, then every extension of V_0 to a valuation of $K[z]/(f(z))$ is an augmented sequence of finite length of approximants $V_1, \dots, V_k$ such that $V_k(f(z)) = \infty$ or a limit of an augmented sequence of approximants of infinite length such that $V_\infty(f(z)) = \infty$. If V_0 is not discrete of rank 1, then there is the possibility that the algorithm will have to be continued to construct a pseudovaluation W of $K[z]$ with $W(f(z)) = \infty$. If this last case occurs, then the situation becomes quite complicated, as we must then extend the family $\{V_k \mid k \in \mathbf{Z}_+\}$ to a “simple admissible family” and possibly make some jumps. This is shown by Vaquié in [Vaq07a, Theorem 2.5] and is explained in Section 7. An essential point is that for every construction $V_1, \dots, V_k$ of approximants to f over V_0 by Mac Lane's algorithm, there exists an extension W of V_0 to a pseudovaluation of $K[x]$ such that $I(W)_\infty = (f(z))$ and $W(\varphi_k) = V_k(\varphi_k)$ for all k . (This will be deduced from [Vaq08, Theorem 1] in Theorem 3.4.)

We will now assume that V_0 has rank 1, so we may assume that G_{V_0} is an ordered subgroup of $\mathbf{R}$. We will now look a little more at the case when we have an infinite sequence of approximants, leading to a limit valuation V_∞ . In this case, there exists a k_0 such that $\varphi_k = \varphi_{k_0} + h_k$ with $\deg_z h_k < \deg_z \varphi_{k_0}$ for $k \geq k_0$. Thus for $k > k_0$,

$$V_k(\varphi_k) > V_{k-1}(\varphi_k) \geq V_{k-1}(\varphi_{k-1}).$$

Thus $\lim_{k \rightarrow \infty} V_k(\varphi_k)$ exists and is either equal to ∞ or an element of $\mathbf{R}$.

LEMMA 3.2. Suppose that V_0 has rank 1 and $V_1, \dots, V_k, \dots$ is an infinite sequence of approximants to f over V_0 . Then the following are equivalent:

- 1) The limit $V_\infty = \lim_{k \rightarrow \infty} V_k$ is a pseudovaluation on $K[z]$ but not a valuation.
- 2) $I^{K[z]}(V_\infty)_\infty = (f(z))$.
- 3) $\lim_{k \rightarrow \infty} V_k(\varphi_k) = \infty$.

Proof. We first prove that assertion 1) implies assertion 3). By assumption, there exists a nonzero $h \in I(V_\infty)_\infty$. There exists a k_0 such that for $k \geq k_0$, we have that $\deg_z \varphi_k = \deg_z \varphi_{k_0}$. Expand

$$h = h_m \varphi_{k_0}^m + h_{m-1} \varphi_{k_0}^{m-1} + \dots + h_0$$

with $\deg_z h_i < \deg_z \varphi_{k_0}$ for all i and $h_m \neq 0$. There exists a $\lambda \in \mathbf{Z}_+$ with $1 \leq \lambda \leq \deg \varphi_{k_0}$ such that $\deg_z z^\lambda h_m = \deg_z \varphi_{k_0}$, and so there exists a nonzero $\alpha \in K$ such that $\alpha z^\lambda h_m = \varphi_{k_0} + \eta_m$ with $\deg \eta_m < \deg \varphi_{k_0}$. This implies that $\alpha z^\lambda h$ has an expansion

$$\alpha z^\lambda h = \varphi_{k_0}^{m+1} + \eta_m \varphi_{k_0}^m + \alpha z^\lambda h_{m-1} \varphi_{k_0}^{m-1} + \dots + \alpha z^\lambda h_{m-j} \varphi_{k_0}^{m-j} + \dots + \alpha z^\lambda h_0$$

with $\deg_z z^\lambda h_{m-j} < 2 \deg_z \varphi_{k_0}$ for all j . Now we can expand each $\alpha z^\lambda h_{m-j} = \eta_{m-j} \varphi_{k_0} + \theta_{m-j}$, with $\deg_z \eta_{m-j}$ and $\deg_z \theta_{m-j}$ less than $\deg_z \varphi_{k_0}$, so that finally we can expand

$$\alpha z^\lambda h = \varphi_{k_0}^{m+1} + h'_m \varphi_{k_0}^m + \dots + h'_{m+1-j} \varphi_{k_0}^{m+1-j} + \dots + h'_0$$

with $\deg_z h'_{m+1-j} < \deg_z \varphi_{k_0}$ for all j . Thus, substituting $\alpha z^\lambda h \in I(V_\infty)_\infty$ for h and continuing to denote by m the degree of its expansion in φ_{k_0} , we may assume that $h_m = 1$. The same argument shows that for $k \geq k_0$, there exist $h_i(k) \in K[z]$ for $i < m$ such that

$$h = \varphi_k^m + h_{m-1}(k) \varphi_k^{m-1} + \dots + h_0(k)$$

with $\deg_z h_j(k) < \deg_z \varphi_k$. Now by the definition of V_k , we have that $V_k(h) \leq m V_k(\varphi_k)$ for $k \geq k_0$, so $\lim_{k \rightarrow \infty} V_k(\varphi_k) = \infty$.

We now prove that assertion 3) implies assertion 2). In the expansion $f = f_m \varphi_k^m + \dots + f_0$ with $\deg_z f_i < \deg_z \varphi_k$, at least two distinct terms have the same value

$$V_k(f(z)) = \min_i \{V_{k-1}(f_i) + i V_k(\varphi_k)\}.$$

Thus $V_k(f(z)) \geq V_k(\varphi_k)$ for all k , which implies $\lim_{k \rightarrow \infty} V_k(f(z)) = \infty$, so that $f \in I(V_\infty)_\infty$. Now $I(V_\infty)_\infty$ is a proper principal ideal in $K[z]$ and f is irreducible in $K[z]$, so $I(V_\infty)_\infty = (f(z))$.

Finally, that assertion 2) implies assertion 1) follows since $I(V_\infty)_\infty \neq (0)$. $\square$

We observe that if the equivalent conditions of Lemma 3.2 hold and $g \in K[z]$ is such that $f \not\sim g$, then there exists a k such that $V_k(g) = V_\infty(g)$. This follows since we can find a φ_k such that $V_k(\varphi_k) = V_\infty(\varphi_k) > V_k(g)$. Then, expanding $g = g_m \varphi_k^m + \dots + g_0$ with $\deg_z g_i < \deg_z \varphi_k$, we have that $V_\infty(g) = V_k(g) = V_k(g_0)$.

For the rest of this section, we will assume that V_0 has arbitrary rank. Mac Lane gives the following explanation of how to find all of the extensions of a $(k-1)$ st stage approximant V_{k-1} to f over V_0 to a k th stage approximant V_k to f over V_0 .

We say that $e \in K[z]$ is an “equivalence unit” for V_k if there exists an “equivalence-reciprocal” $h \in K[z]$ such that $eh \sim 1$ in V_k . It is shown in [ML36b, Section 4] that e is an equivalence unit if and only if e is equivalent in V_k to a polynomial g such that $\deg_z g < \deg_z \varphi_k$.

By [ML36b, Theorem 4.2], the polynomial f has an essentially unique (unique up to equivalence in V_{k-1}) expression

$$f \sim e \varphi_{k-1}^{m_0} \psi_1^{m_1} \dots \psi_t^{m_t} \quad (11)$$

in V_{k-1} , with $m_0 \in \mathbf{N}$ and $m_1, \dots, m_t > 0$. Here e is an equivalence unit for V_{k-1} and $\psi_1, \dots, \psi_t$ are homogeneous key polynomials over V_{k-1} all not equivalent to φ_{k-1} in V_{k-1} and not equivalent to one another in V_{k-1} . We have that $t > 0$ since $\text{proj}(V_{k-1}) > 0$. We have that φ_{k-1} is a homogeneous key polynomial in V_{k-1} by [ML36b, Lemma 4.3].

If f is a homogeneous key polynomial for V_{k-1} , then $V_k = [V_{k-1}; V_k(f(z)) = \infty]$ is a pseudovaluation of $K[z]$ with $I(V)_\infty = (f(z))$.

If f is not a homogeneous key polynomial for V_{k-1} , then none of the ψ_i are equal to f , and we may define a k th stage approximant to f over V_0 which is an inductive valuation of V_{k-1} by $V_k = [V_{k-1}; V_k(\varphi_k) = \mu_k]$, where φ_k is one of the ψ_i . In the expansion (5) of f ,

$$f = f_m \varphi_k^m + \dots + f_0,$$

μ_k must be chosen so that $\text{proj}(V_k) > 0$. All k th stage approximants V_k to f extending V_{k-1} are found by the above procedure.

Let $T = \mathbf{R} \times G_{V_0}$. Given $\alpha, \beta \in G_{V_0}$ and $q \in \mathbf{R}$, we have the line

$$D = \{(x, \gamma) \in T \mid q\gamma + \alpha x + \beta = 0\}$$

in T . When $q \neq 0$, we define the slope of D to be $-\alpha/q \in G_{V_0} \otimes_{\mathbf{Z}} \mathbf{R}$. Associated with D are two half-spaces of T ,

$$H_{\geq}^D = \{(x, \gamma) \in T \mid q\gamma + \alpha x + \beta \geq 0\} \quad \text{and} \quad H_{\leq}^D = \{(x, \gamma) \in T \mid q\gamma + \alpha x + \beta \leq 0\}.$$

Given a subset A of T , the convex closure of A is $\text{Conv}(A) = \cap H$, where H runs over the half-spaces of T which contain A .

The Newton polygon is constructed as in [ML36b, Section 5, p. 500] and [Vaq08, Section 1, p. 2510]. These constructions are equivalent but slightly different. We use the convention of [ML36b]. The possible values μ_k can be conveniently found from the Newton polygon $N(V_{k-1}, \varphi_k)$. This is constructed by taking the convex closure in T of

$$A = \{(m - i, \delta) \mid \delta \geq V_{k-1}(f_i), 0 \leq i \leq m\},$$

where the union is over the i such that $f_i \neq 0$. A segment F of the boundary of $\text{Conv}(A)$ is a subset F of $\text{Conv}(A)$ defined by $F = \text{Conv}(A) \cap D$, where D is a line of T such that $\text{Conv}(A)$ is contained in one of the half-spaces $H_{\geq}^D$ or $H_{\leq}^D$ defined by D and $F = \text{Conv}(A) \cap D$ contains at least two distinct points.

The slopes μ of the segments of $N(V_{k-1}, \varphi_k)$ satisfying $\mu > V_{k-1}(\varphi_k)$ are the possible values of φ_k . The polygon composed of those segments of slope μ with $\mu > V_{k-1}(\varphi_k)$ is called the principal part of the Newton polygon $N(V_{k-1}, \varphi_k)$.

In the proof of Theorem 5.1 of [ML36b], it is shown that for $1 \leq i \leq t$, the principal polygon of $N(V_{k-1}, \psi_i)$ (from (11)) is

$$\{(x, y) \in N(V_{k-1}, \psi_i) \mid x \geq m - m_i\}. \quad (12)$$

Further, m_0 is the smallest exponent i such that in the expansion $f = \sum f_i \varphi_{k-1}^i$ with $\deg_z f_i < \deg_z \varphi_{k-1}$, we have that $V_{k-1}(f_i \varphi_{k-1}^i) = V_{k-1}(f(z))$.

Remark 3.3. If the coefficients of $f(z)$ are all in the valuation ring R_{V_0} of V_0 , then the coefficients of all key polynomials φ_k are also in R_{V_0} , as is established in [ML36b, Theorem 7.1].

The following theorem follows from a criterion of [Vaq08].

THEOREM 3.4. *Suppose that V_k is a k th approximant to f over V_0 . Then there exists a pseudovaluation W of $K[z]$ such that $W|_K = V_0$, $I(W)_\infty = (f(z))$, $W(g) \geq V_k(g)$ for all $g \in K[z]$ and $W(\varphi_i) = V_i(\varphi_i)$ for $1 \leq i \leq k$.*

Proof. As explained in the construction of V_k above, we have that $\varphi_k|f$ in V_{k-1} , and there exists a key polynomial ψ for V_k with ψ not equivalent to φ_k in V_k and such that $\psi|f$ in V_k . The theorem now follows from [Vaq08, Theorem 1]. $\square$

4. An algorithm to construct generating sequences

Let V_0 be a valuation of a field K . Suppose that there exists an algebraically closed field $\mathbf{k}$ such that $\mathbf{k} \subset R_{V_0}$ and $R_{V_0}/m_{V_0} \cong \mathbf{k}$. Let $f(z) \in R_{V_0}[z]$ be an irreducible unitary polynomial.

In this section, we give an inductive construction of a sequence of approximants to f over V_0 such that the constructed key polynomials have a particularly nice form. We will call the sequence of approximants “a realization of the algorithm of Section 4.” We will prove the following theorem by induction on the index k .

THEOREM 4.1. *Suppose that W is a pseudovaluation of $K[z]$ extending V_0 such that $I(W)_\infty = (f(z))$. Then we can construct a sequence of approximants to f over V_0*

$$V_1, \dots, V_k, \dots, \quad (13)$$

where

$$V_i = [V_{i-1}; V_i(\varphi_i) = W(\varphi_i)] \quad (14)$$

for all i such that the key polynomials φ_i satisfy $\varphi_1 = z$ in V_0 and

$$\varphi_i = \varphi_{i-1}^{n_{i-1}} - c_{i-1} \varphi_1^{j_1(i-1)} \dots \varphi_{i-2}^{j_{i-2}(i-1)} \quad (15)$$

in V_{i-1} for $2 \leq i \leq k$ with $c_{i-1} \in R_{V_0}$, $n_{i-1} = [G_{V_{i-1}} : G_{V_{i-2}}]$ and $0 \leq j_l(m) < n_l$ for all l and m . Either the sequence (13) is of finite length k with $\varphi_k = f$ and $V_k(f(z)) = \infty$, or the sequence is infinite.

Observe that $\varphi_i^{n_i} \sim c_i \varphi_1^{j_1(i)} \dots \varphi_{i-1}^{j_{i-1}(i)}$ in V_i for $1 \leq i \leq k-1$ since φ_{i+1} is a key polynomial over V_i .

The proof of the theorem will be given after we have established Lemmas 4.2 and 4.3 and Theorem 4.4.

LEMMA 4.2. *Suppose that $V_1, \dots, V_k$ satisfy the conclusions (14) and (15) of Theorem 4.1 and that we have an equality*

$$n_k V_k(\varphi_k) = V_k(c_k \varphi_1^{j_1(k)} \dots \varphi_{k-1}^{j_{k-1}(k)})$$

in V_k with $c_k \in K$, $n_k = [G_{V_k} : G_{V_{k-1}}]$ and $0 \leq j_l(k) < n_l$ for all l . Then $c_k \in R_{V_0}$.

Proof. In the case $k = 1$, we have that $W(z) \geq 0$ since f is unitary and the coefficients of f are in R_{V_0} . Thus $V_0(c_1) \geq 0$.

Now suppose that $k \geq 2$. Since n_i is the smallest positive integer m such that $mV_i(\varphi_i) \in G_{V_{i-1}}$, by repeated Euclidean division we have that every element $\gamma \in G_{V_k}$ has a unique decomposition as

$$\gamma = \gamma_0 + j_1 \mu_1 + \dots + j_k \mu_k, \quad (16)$$

where $\gamma_0 \in G_{V_0}$, $\mu_i = W(\varphi_i)$ for $1 \leq i \leq k$ and $0 \leq j_i < n_i$ for $1 \leq i \leq k$. From (15),

$$n_i \mu_i < \mu_{i+1} \quad \text{for all } 1 \leq i < k. \quad (17)$$

There is a unique representation $n_l\mu_l = \gamma_0 + j_1\mu_1 + \cdots + j_{l-1}\mu_{l-1}$ of the form of (16). It follows from (17) that $j_1\mu_1 + \cdots + j_{l-1}\mu_{l-1} < n_l\mu_l$. Thus $V_0(c_k) = \gamma_0 > 0$. $\square$

LEMMA 4.3. *Suppose that $V_1, \dots, V_k$ satisfy the conclusions (14) and (15) of Theorem 4.1 with $I(V_i)_\infty = (0)$ for all $i \leq k$. Let A be a local domain whose quotient field is K , and suppose that $f(z) \in A[z]$. Further suppose that A is dominated by V_0 and that A contains $\mathbf{k}$ (so that $A/m_A \cong \mathbf{k}$). Suppose that $c_i \in A$ for $i \leq k-1$. Then we have a graded $\mathbf{k}$ -algebra isomorphism of $\text{gr}_{V_k}(A[z])$ with the quotient $\text{gr}_{V_0}(A)[\bar{\varphi}_1, \dots, \bar{\varphi}_k]/I$ of the graded polynomial ring $\text{gr}_{V_0}(A)[\bar{\varphi}_1, \dots, \bar{\varphi}_k]$ over $\text{gr}_{V_0}(A)$, where*

$$I = (\bar{\varphi}_1^{n_1} - \bar{c}_1, \bar{\varphi}_2^{n_2} - \bar{c}_2\bar{\varphi}_1^{j_1(2)}, \dots, \bar{\varphi}_{k-1}^{n_{k-1}} - \bar{c}_{k-1}\bar{\varphi}_1^{j_1(k-1)} \cdots \bar{\varphi}_{k-2}^{j_{k-2}(k-1)}).$$

Here $\bar{c}_1, \dots, \bar{c}_{k-1}$ are the initial forms of $c_1, \dots, c_{k-1}$ in $\text{gr}_{V_0}(A)$ and $\bar{\varphi}_i$ has weight $V_k(\varphi_i)$ for all i .

Suppose that there exist a $c \in A$ and $j_i \in \mathbf{N}$ for $1 \leq i \leq k-1$ with $0 \leq j_i < n_i$ such that $V_k(\varphi_k^{n_k}) = V_k(c\varphi_1^{j_1} \cdots \varphi_{k-1}^{j_{k-1}})$. Then $(\bar{\varphi}_k^{n_k} - \bar{c}\bar{\varphi}_1^{j_1} \cdots \bar{\varphi}_{k-1}^{j_{k-1}})$ is a prime ideal in $\text{gr}_{V_k}(A[z])$.

Proof. Every $g \in A[z]$ has the unique decomposition of (6) and Remark 3.1,

$$g = \sum_j a_j \varphi_1^{m_{1,j}} \varphi_2^{m_{2,j}} \cdots \varphi_k^{m_{k,j}}$$

with $a_j \in A$, $m_{1,j}, \dots, m_{k,j} \in \mathbf{N}$ and $0 \leq m_{i,j} < n_i$ for $i < k$, and

$$\begin{aligned} V_k(g) &= \min_j \{V_0(a_j) + m_{1,j}V_1(\varphi_1) + \cdots + m_{k,j}V_k(\varphi_k)\} \\ &= \min_j \{V_k(a_j) + m_{1,j}V_k(\varphi_1) + \cdots + m_{k,j}V_k(\varphi_k)\} \end{aligned}$$

by (9).

Since $\text{gr}_{V_k}(A[z])$ is generated by the initial forms of elements of $A[z]$, the natural graded $\text{gr}_{V_0}(A)$ -algebra map

$$\Psi: \text{gr}_{V_0}(A)[\bar{\varphi}_1, \dots, \bar{\varphi}_k] \rightarrow \text{gr}_{V_k}(A[z])$$

is a surjection and I is contained in the kernel. A homogeneous element G of $\text{gr}_{V_0}[\bar{\varphi}_1, \dots, \bar{\varphi}_k]$ has a unique representation $G \equiv \bar{c}\bar{\varphi}_1^{j_1} \cdots \bar{\varphi}_{k-1}^{j_{k-1}} \bar{\varphi}_k^{j_k} \pmod{I}$ with $c \in A$, $j_1, \dots, j_k \in \mathbf{N}$ and $0 \leq j_i < n_i$ for $i < k$. Now $\Psi(G) = 0$ implies that $c = 0$, which implies that $G \equiv 0 \pmod{I}$. Thus Ψ is an isomorphism, and the first statement of the lemma follows.

We now prove the second statement. Let

$$\psi = \bar{\varphi}_k^{n_k} - \bar{c}\bar{\varphi}_1^{j_1} \cdots \bar{\varphi}_{k-1}^{j_{k-1}}.$$

We have that $\text{gr}_{V_k}(A[z]) \cong B[\bar{\varphi}_k]$ is a graded polynomial ring over the domain

$$B = \text{gr}_{V_0}(A)[\bar{\varphi}_1, \dots, \bar{\varphi}_{k-1}]/(\bar{\varphi}_1^{n_1} - \bar{c}_1, \dots, \bar{\varphi}_{k-1}^{n_{k-1}} - \bar{c}_{k-1}\bar{\varphi}_1^{j_1(k-1)} \cdots \bar{\varphi}_{k-2}^{j_{k-2}(k-1)}).$$

Let L be an algebraic closure of the quotient field of B . Choose $t \in L$ such that $t^{n_k} = \bar{c}\bar{\varphi}_1^{j_1} \cdots \bar{\varphi}_{k-1}^{j_{k-1}}$. Then giving t the weight $V_k(\varphi_k)$, we have that $B[t]$ is a graded domain which is a free B -module of rank n_k since $V_k(\varphi_k)$ has order n_k in G_k/G_{k-1} , and so $1, t, \dots, t^{n_k-1}$ is a B -basis of $B[t]$. We have a natural surjection of graded B -modules

$$B[\bar{\varphi}_k]/(\psi) \rightarrow B[t]. \quad (18)$$

Now $B[\bar{\varphi}_k]/(\psi)$ is also a free B -module of rank n_k , as $1, \bar{\varphi}_k, \dots, \bar{\varphi}_k^{n_k-1}$ is a B -basis. Thus (18) is an isomorphism, and so $B[\bar{\varphi}_k]/(\psi)$ is a domain. $\square$

Suppose that G is a totally ordered abelian group. Let $U = G \otimes_{\mathbf{Z}} \mathbf{R}$, $d \in \mathbf{Z}_+$ and $\gamma \in G$. Since $\mathbf{Z}$ is a principal ideal domain,

$$\left(\frac{1}{d}\mathbf{Z}\gamma\right) \cap G = \frac{1}{\bar{m}}\mathbf{Z}\gamma \quad \text{for some } \bar{m} \in \mathbf{Z}_+. \quad (19)$$

Indeed, we must have that

$$\left(\frac{1}{d}\mathbf{Z}\gamma\right) \cap G = \frac{a}{d}\mathbf{Z}\gamma \quad (20)$$

for some $a \in \mathbf{Z}_+$. Now $\gamma \in (a/d)\mathbf{Z}\gamma$ implies that $a|d$, and so there exists an $\bar{m} \in \mathbf{Z}_+$ such that $1/\bar{m} = a/d$.

This implies that

$$\frac{1}{\bar{m}}(d, \gamma) \in \mathbf{Z} \oplus G. \quad (21)$$

We shall need the following fact:

For $n, q \in \mathbf{Z}_+$, $(q/n)(d, \gamma) \in \mathbf{Z} \oplus G$ if and only if n divides qd and $q/n = e/\bar{m}$ for some $e \in \mathbf{Z}_+$. (22)

For the reader's convenience, we give a proof of (22). Suppose that $(q/n)(d, \gamma) \in \mathbf{Z} \oplus G$. Then n divides qd , and writing $qd = rn$, we see that $(r/d)\mathbf{Z}\gamma \subset G$, so that it follows from (19) that $r/d = q/n$ is an integral multiple of $1/\bar{m}$. The converse follows from (21).

THEOREM 4.4. *Suppose that we have constructed approximants $V_i = [V_{i-1}, V_i(\varphi_i) = W(\varphi_i)]$ to f over V_0 for $1 \leq i \leq k-1$ satisfying the conclusions of Theorem 4.1, that $V_{k-1}(\varphi_{k-1}) < \infty$ and that in V_{k-1} we have an equivalence*

$$f \sim e\varphi_{k-1}^{m_0}\bar{\psi}_1^{m_1}\cdots\bar{\psi}_t^{m_t} \quad (23)$$

of the form of (11) with $m_0 \in \mathbf{N}$ and $m_1, \dots, m_t \in \mathbf{Z}_+$ such that e is an equivalence unit for V_{k-1} and that $\bar{\psi}_1, \dots, \bar{\psi}_t$ are homogeneous key polynomials over V_{k-1} such that there are expressions

$$\bar{\psi}_i = \varphi_{k-1}^{n_{k-1}} - \varepsilon_{k-1,i}c_{k-1}\varphi_1^{j_1(k-1)}\cdots\varphi_{k-2}^{j_{k-2}(k-1)}$$

with $c_{k-1} \in R_{V_0}$ nonzero, $\varepsilon_{k-1,i} \in \mathbf{k}$ distinct and nonzero and $0 \leq j_i(k-1) < n_i$ for all i . The $\bar{\psi}_i$ define approximants to f as explained after (11).

Then there exists a unique $\bar{\psi}_i$ such that $W(\bar{\psi}_i) > V_{k-1}(\bar{\psi}_i)$, and, for $\varphi_k = \bar{\psi}_i$, there exists a unique segment S of the principal part of the Newton polygon $N(V_{k-1}, \varphi_k)$ which has slope $s = W(\varphi_k)$.

Defining $V_k = [V_{k-1}, V_k(\varphi_k) = W(\varphi_k)]$, we have that V_k is an approximant to f over V_0 such that the approximants $V_1, \dots, V_k$ satisfy the conclusions of Theorem 4.1.

Now suppose that $V_k(\varphi_k) < \infty$. The Newton polygon $N(V_{k-1}, \varphi_k)$ is computed from the expansion

$$f = \sum f_i \varphi_k^i \quad (24)$$

with $\deg_z f_i < \deg_z \varphi_k$. Let $(m-i_1, \beta_1)$ be the lowest point on the segment S , and let $(m-i_0, \beta_0)$ be the highest point. Let

$$F_{k,s}(\varphi_k) = \sum f_i \varphi_k^i, \quad (25)$$

where the sum is restricted to i such that $(m-i, V_{k-1}(f_i))$ is on S . Then there exists a polynomial in φ_k

$$G_{k,s}(\varphi_k) = \sum g_i \varphi_k^i \quad (26)$$

with $g_i \in K[z]$ such that the i such that g_i is not zero are exactly the i such that f_i is a coefficient of $F_{k,s}$, and $g_i \sim f_i$ in V_{k-1} for all such i . Further, we can factor the right side of (26) as a polynomial in φ_k ,

$$G_{k,s}(\varphi_k) = f_{i_1}^{\overline{m}+1} \varphi_k^{i_0} \psi_1^{a_1} \dots \psi_t^{a_t}, \quad (27)$$

where

$$\psi_i = \varphi_k^{n_k} - \varepsilon_{k,i} c_k \varphi_1^{j_1(k)} \dots \varphi_{k-1}^{j_{k-1}(k)} \quad (28)$$

with $c_k \in R_{V_0}$ nonzero, $\varepsilon_{k,i} \in \mathbf{k}$ distinct and nonzero and $0 \leq j_i(k) < n_i$ for all i . Moreover, $f_{i_1}^{\overline{m}+1}$ is an equivalence unit in V_k , $n_k = [G_k : G_{k-1}]$ and the ψ_i are homogeneous key polynomials in V_k . Also, there is a V_k -equivalence

$$f \sim G_{k,s}(\varphi_k). \quad (29)$$

Proof. The fact that there exists a $\overline{\psi}_i$ such that $W(\overline{\psi}_i) > V_{k-1}(\overline{\psi}_i)$ follows from the equivalence relation (23) since $W(f(z)) = \infty$ and $W(e\varphi_{k-1}^{m_0}) = V_{k-1}(e\varphi_{k-1}^{m_0})$. The uniqueness of $\overline{\psi}_i$ follows since the $\varepsilon_{k-1,i}$ are distinct. The existence of a segment S of the principal part of the Newton polygon $N(V_{k-1}, \varphi_k)$ with slope $s = W(\varphi_k)$ follows from Theorem 3.4 and the discussion of Section 3.2. The fact that upon setting $\varphi_k = \overline{\psi}_i$, we have that $V_k = [V_{k-1}, V_k(\varphi_k) = W(\varphi_k)]$ is an approximate to f over V_0 then follows since $\text{proj}(V_k)$ is positive, as $W(f(z)) = \infty$, and the fact that the approximants $V_1, \dots, V_k$ satisfy the conclusions of Theorem 4.1 follows from our assumptions on the φ_i for $i \leq k$.

Let $y = sx + r$ be the equation of the line containing the segment S , so that

$$s = \frac{\beta_0 - \beta_1}{i_1 - i_0}.$$

Let $\overline{m}$ be the largest positive integer such that

$$\frac{1}{\overline{m}}(i_1 - i_0, \beta_0 - \beta_1) \in \mathbf{Z} \oplus G_{k-1}. \quad (30)$$

Here $\overline{m}$ is as defined in (19), with $d = i_1 - i_0$, $\gamma = \beta_0 - \beta_1$ and $G = G_{k-1}$. Let $(\overline{b}, \overline{c}) = (1/\overline{m})(i_1 - i_0, \beta_0 - \beta_1)$. If $V_{k-1}(f_i) - s(m - i) = r$, then

$$(m - i, V_{k-1}(f_i)) = (m - i_1, \beta_1) + \lambda(\overline{b}, \overline{c}) = \left(m - i_1 + \lambda\overline{b}, \beta_1 + \lambda \left(\frac{\beta_0 - \beta_1}{\overline{m}} \right) \right)$$

for some $\lambda \in \mathbf{N}$ with $0 \leq \lambda \leq \overline{m}$ (this follows from (22)). Using the relations (15) for $2 \leq i \leq k$, we see that there exists an

$$h = c_k \varphi_1^{j_1(k)} \dots \varphi_{k-1}^{j_{k-1}(k)} \in K[z]$$

with $c_k \in K$ and $0 \leq j_l(k) < n_l$ for $1 \leq l < k$ such that $V_{k-1}(h) = (\beta_0 - \beta_1)/\overline{m}$. We have that

$$F_{k,s} = \sum_{\tau=0}^{\overline{m}} f_{i_0+\tau\overline{b}} \varphi_k^{i_0+\tau\overline{b}} = \varphi_k^{i_0} \left(\sum_{\tau=0}^{\overline{m}} f_{i_0+\tau\overline{b}} \varphi_k^{\tau\overline{b}} \right),$$

where

$$\begin{aligned} V_{k-1}(f_{i_0+\tau\overline{b}}) &= s(m - (i_0 + \tau\overline{b})) + r = -\tau \left(\frac{\beta_0 - \beta_1}{\overline{m}} \right) + \beta_0 \\ &= (\overline{m} - \tau) \left(\frac{\beta_0 - \beta_1}{\overline{m}} \right) + \beta_1 = V_{k-1}(h^{\overline{m}-\tau}) + V_{k-1}(f_{i_1}). \end{aligned} \quad (31)$$

By (31), and since V_0 is rational ($R_{V_0}/m_{V_0} = \mathbf{k}$), there exist $\gamma_\tau \in \mathbf{k}$ such that

$$\gamma_\tau \text{In}(h^{\overline{m}-\tau}) \text{In}(f_{i_1}) = \text{In}(f_{i_0+\tau\overline{b}}) \quad (32)$$

in $\text{gr}_{V_{k-1}}(K[z])$. Define $G_{k,s}(\varphi_k)$ by

$$\begin{aligned} G_{k,s}(\varphi_k) &= \varphi_k^{i_0} \left(\sum_{\tau=0}^{\bar{m}} \gamma_\tau h^{\bar{m}-\tau} f_{i_1} \varphi_k^{\tau \bar{b}} \right) = f_{i_1}^{\bar{m}+1} \varphi_k^{i_0} h^{\bar{m}} \left(\sum_{\tau=0}^{\bar{m}} \gamma_\tau (h^{-1} \varphi_k^{\bar{b}})^\tau \right) \\ &= f_{i_1}^{\bar{m}+1} \varphi_k^{i_0} h^{\bar{m}} \prod_{j=1}^{\bar{m}} ((h^{-1} \varphi_k^{\bar{b}}) - \alpha_j) = f_{i_1}^{\bar{m}+1} \varphi_k^{i_0} \prod_{j=1}^{\bar{m}} (\varphi_k^{\bar{b}} - \alpha_j h) \end{aligned}$$

for suitable nonzero $\alpha_j \in \mathbf{k}$.

We will compute the order $n_k = [G_k : G_{k-1}] = [(G_{k-1} + s\mathbf{Z}) : G_{k-1}]$. We will show that it equals $\bar{b} = (i_1 - i_0)/\bar{m}$. Since $s = (\beta_0 - \beta_1)/(i_1 - i_0)$,

$$\bar{b}s = \frac{\beta_0 - \beta_1}{\bar{m}} \in G_{k-1}.$$

With a as defined in (20), with $d = i_1 - i_0$, $\gamma = \beta_0 - \beta_1$ and $G = G_{k-1}$, we have that $a = d/\bar{m} = \bar{b}$.

Suppose that $n \in \mathbf{Z}_+$ and $ns \in G_{k-1}$. Now

$$ns = n \left(\frac{\beta_0 - \beta_1}{i_1 - i_0} \right) = \frac{n}{d} \gamma \in G_{k-1},$$

which implies that $a = \bar{b} \mid n$. Thus $n_k = \bar{b} = [G_k : G_{k-1}]$. Hence $c_k \in R_{V_0}$ by Lemma 4.2.

The fact that $f \sim G_{k,s}$ in V_k follows since $\gamma_\tau h^{\bar{m}-\tau} f_{i_1} \sim f_{i_0+\tau\bar{b}}$ in V_k , which follows from (32), the facts that by (9),

$$V_k(h) = V_k(c_k) + \sum_{i=1}^{k-1} j_i(k) V_k(\varphi_i) = V_0(c_k) + \sum_{i=1}^{k-1} j_i(k) V_i(\varphi_i) = V_{k-1}(h)$$

and $V_k(f_i) = V_{k-1}(f_i)$ for all i since $\deg_z f_i < \deg_z \varphi_k$.

We know that φ_k is a key polynomial in V_k , as discussed after (11). Finally, we verify that each $\psi_i = \varphi_k^{n_k} - \varepsilon_{k,i} c_k \varphi_1^{j_1(k)} \cdots \varphi_{k-1}^{j_{k-1}(k)}$ is a key polynomial in V_k . By Lemma 4.3, the ideal

$$(\text{In}(\psi_i)) = (\bar{\varphi}_k^{n_k} - \varepsilon_{k,i} \bar{c}_k \bar{\varphi}_1^{j_1(k)} \cdots \bar{\varphi}_{k-1}^{j_{k-1}(k)})$$

is a prime ideal in $\text{gr}_{V_k}(R_{V_0}[z])$, where $\bar{c}_k = \text{In}(c_k)$ and $\bar{\varphi}_i = \text{In}(\varphi_i)$. Thus ψ_i is equivalence irreducible in V_k as a polynomial in $R_{V_0}[z]$. Since every nonzero element of $R_{V_0}[z]$ is a unit in K , this implies that it is equivalence irreducible in V_k as a polynomial in $K[z]$. It is minimal in V_k by (8). Since ψ_i has the leading coefficient 1 and $\deg_z \psi_i > 0$, we have that ψ_i is a key polynomial over V_k . $\square$

Constructions similar to those used in the proof of Theorem 4.4 can be found in [HOMS14, Definition 3.23, pp. 17–18].

We now give the proof of Theorem 4.1. Set $\varphi_1 = z$ and $V_1 = [V_0; V_1(\varphi_1) = W(\varphi_1)]$, which is an approximant to f over V_0 since $W(f(z)) = \infty$. By a simplification of the proof of Theorem 4.4, we have that $f \sim e z^{m_0} \bar{\psi}_1^{m_1} \cdots \bar{\psi}_t^{m_t}$ in V_1 , where e is an equivalence unit in V_1 and $\bar{\psi}_i = z^{n_1} - \varepsilon_{1,i} c_1$ with $c_1 \in R_{V_0}$ and nonzero and distinct $\varepsilon_{1,i} \in k$.

Now the conclusions of Theorem 4.1 follow by induction using Theorem 4.4.

As pointed out by the referee, another point of view on Theorem 4.1 can be obtained from [HOMS14, formula (3.8)] applied to our situation; note here that one should prove that $c_i \in R_{V_0}$.

PROPOSITION 4.5. *Suppose that there is a unique extension of V_0 to a pseudovaluation W of $K[z]$ with $I(W)_\infty = (f(z))$ and that we have constructed a finite or infinite sequence of approximants $V_1, \dots, V_k, \dots$ to f over V_0 satisfying the conclusions of Theorem 4.1. Then for $k \geq 2$, with notation as in (27), setting $e_k = i_0$, we have that*

$$f \sim \varphi_k^{e_k} \text{ in } V_{k-1}, \quad (33)$$

where

$$\varphi_{k+1} = \varphi_k^{n_k} - c_k \varphi_1^{j_1(k)} \cdots \varphi_{k-1}^{j_{k-1}(k)} \quad (34)$$

with $c_k \in R_{V_0}$ nonzero, $0 \leq j_i(k) < n_i$ for all i and

$$\deg_z f = e_k \deg_z \varphi_k. \quad (35)$$

Proof. We use the notation of the statement and proof of Theorem 4.4. By Theorem 3.4, every realization of the algorithm to construct a k th stage approximant V_k to f over V_0 extends to the construction of a pseudovaluation U extending V_0 with $I(U)_\infty = (f(z))$. Since W is unique, every realization of the algorithm must extend to the construction of $U = W$.

We will prove the relations

$$f \sim \varphi_k^{e_k} \text{ in } V_{k-1} \text{ with } \deg_z f = e_k \deg_z \varphi_k \quad (36)$$

and for all $k \geq 2$, the polynomial (26) of Theorem 4.4 satisfies

$$f \sim \psi_1^{a_1} \cdots \psi_t^{a_t} \text{ in } V_k \text{ with } \deg_z f = a_1 \deg_z \psi_1 + \cdots + a_t \deg_z \psi_t. \quad (37)$$

We will establish (36) and (37) for $k = 2$. Since the extension is unique, every realization of the algorithm must extend to the construction of W , so $N(V_0; \varphi_1 = z)$ has a unique segment. Let $\mu_1 = s = W(z)$ be the slope of this segment, so that $V_1 = [V_0; V_1(\varphi_1) = \mu_1]$. Expand $f = z^d + f_{d-1}z^{d-1} + \cdots + f_0$ with $f_i \in K$. Since $N(V_0, \varphi_1)$ has a unique segment, $i_0 = 0$, $i_1 = d$ and $f_{i_1} = 1$ in (27) for $k = 1$, so by (29) and (27), for $k = 1$,

$$f \sim G_{1,s}(\varphi_1) = \psi_1^{a_1} \cdots \psi_t^{a_t} \quad (38)$$

in V_1 , where

$$\psi_i = \varphi_1^{n_1} - \varepsilon_{1,i} c_1 \quad (39)$$

from (28). Suppose that $t > 1$. Any choice of ψ_i is a key polynomial for V_1 , and if $W_2 = [V_1; V_2(\psi_i) = \mu_2]$ is an approximant extending V_1 , then since every realization of the algorithm must extend to the construction of W , as observed in the first part of the proof, we have that

$$W(\psi_i) = W_2(\psi_i) = \mu_2 > n_1 V_1(\varphi_1) = V_0(c_1).$$

For $j \neq i$, we have that $\psi_j = \psi_i + (\varepsilon_{1,i} - \varepsilon_{1,j})c_1$ so for $j \neq i$,

$$W(\psi_j) = W(\psi_i + (\varepsilon_{1,i} - \varepsilon_{1,j})c_1) = V_0(c_1).$$

The resulting contradiction shows that $t = 1$ in (38) and so $f \sim \varphi_2^{e_2}$ in V_1 with $\deg_z f = e_2 \deg_z \varphi_2$, establishing (36) for $k = 2$.

From (36) for $k = 2$, there is an expression

$$f = \varphi_2^{e_2} + f_{e_2-1} \varphi_2^{e_2-1} + \cdots + f_0$$

with $\deg_z f_i < \deg_z \varphi_2$ for all i . From (12), we then have that the principal part of the Newton polygon $N(V_1, \varphi_2)$ is the entirety of $N(V_1, \varphi_2)$. Further, by the uniqueness of the extension of V_0 , we have that $N(V_1, \varphi_2)$ has a unique segment, so $i_0 = 0$, $i_1 = e_2$ and $f_{i_1} = 1$ in (27) for $k = 2$, so

$$f \sim G_{2,s}(\varphi_2) = \psi_1^{a_1} \cdots \psi_t^{a_t}$$

in V_2 with the ψ_i given by (28) for $k = 2$, establishing (37) in V_2 for $k = 2$, with $\deg_z f = a_1 \deg_z \psi_1 + \dots + a_t \deg_z \psi_t$.

Now by induction on k , repeating the argument for the case $k = 2$ with the application of Theorem 4.4, we obtain the conclusions of Proposition 4.5. $\square$

Formulas (33) and (35) also follow from [Vaq08, Theorem 3.1], and then formula (34) follows from Theorem 4.4.

5. When the degree is prime to p and the extension is unique

THEOREM 5.1. *Suppose that A is a local domain which contains an algebraically closed field $\mathbf{k}$ such that $A/m_A \cong \mathbf{k}$. Let K be the quotient field of A , and suppose that V_0 is a valuation of K which dominates A such that the residue field of the valuation ring of V_0 is $\mathbf{k}$. Suppose that $f(z) \in A[z]$ is unitary and irreducible, that there is a unique extension of V_0 to a valuation ω of $K[z]/(f(z))$ and that the characteristic p of $\mathbf{k}$ does not divide $\deg_z f$. Let W be the associated pseudovaluation of $K[z]$ such that $I(W)_\infty = (f(z))$ in $K[z]$. Then there exists a realization of the algorithm of Section 4 constructing approximants $V_1, \dots, V_k$ to f over V_0 satisfying equations (14) and (15) for all $i \leq k$ such that $W = V_k$. We have that*

$$\deg_z f = [G_\omega : G_{V_0}] = [G_{V_k} : G_{V_0}].$$

Further, with the notation of (15), we have that $c_i \in A$ for all $1 \leq i \leq k$, and

$$\text{gr}_\omega(A[z]/(f(z))) \cong \text{gr}_{V_0}(A)[\bar{\varphi}_1, \dots, \bar{\varphi}_{k-1}]/I,$$

where

$$I = (\bar{\varphi}_1^{n_1} - \bar{c}_1, \bar{\varphi}_2^{n_2} - \bar{c}_2 \bar{\varphi}_1^{j_1(2)}, \dots, \bar{\varphi}_{k-1}^{n_{k-1}} - \bar{c}_{k-1} \bar{\varphi}_1^{j_1(k-1)} \bar{\varphi}_2^{j_2(k-1)} \dots \bar{\varphi}_{k-2}^{j_{k-2}(k-1)})$$

is a finitely generated and presented $\text{gr}_{V_0}(A)$ -module.

Proof. Suppose by induction on i that we have constructed approximants $V_1, \dots, V_i$ to f over V_0 satisfying equations (14) and (15) with $c_1, \dots, c_{i-1} \in A$ and that φ_i is not equal to f . By Theorem 4.4 and Proposition 4.5, we have that $f \sim G_i = \varphi_{i+1}^{e_{i+1}}$ in V_i , with φ_{i+1} a key polynomial over V_i such that

$$\varphi_{i+1} = \varphi_i^{n_i} - c_i \varphi_1^{j_1(i)} \dots \varphi_{i-1}^{j_{i-1}(i)} \quad \text{and} \quad \deg_z f = e_{i+1} \deg_z \varphi_{i+1} \quad (40)$$

for some nonzero $c_i \in R_{V_0}$. Expanding

$$f = \sum f_j \varphi_i^j \quad (41)$$

in $K[z]$, with $\deg_z f_j < \deg_z \varphi_i$, let $F = \sum f_j \varphi_i^j$, where the sum is restricted to f_j such that $V_{i-1}(f_j) + j\mu_i$ (with $\mu_i = W(\varphi_i)$) is minimal. Expanding G_i as a polynomial in φ_i , we see that the coefficients of $G_i = \varphi_i^{n_i e_{i+1}} - e_{i+1} c_i \varphi_1^{j_1(i)} \dots \varphi_{i-1}^{j_{i-1}(i)} \varphi_i^{n_i(e_{i+1}-1)} + \dots$ as a polynomial in φ_i and the coefficients f_j in the expansion $F = \sum f_j \varphi_i^j$ must be equivalent in V_{i-1} by Theorem 4.4.

Now $e_{i+1} n_i \deg_z \varphi_i = \deg_z f$, so since we assume that p does not divide $\deg_z f$, we have that p does not divide e_{i+1} . Comparing the expansions of F and G_i , we see that

$$0 \neq f_{n_i(e_{i+1}-1)} \sim g_{n_i(e_{i+1}-1)} = -e_{i+1} c_i \varphi_1^{j_1(i)} \dots \varphi_{i-1}^{j_{i-1}(i)}$$

in V_{i-1} . Since $\deg_z f_{n_i(e_{i+1}-1)} < \deg_z \varphi_i$ and $c_1, \dots, c_{i-1} \in A$ by induction, by (6) and Remark 3.1,

the polynomial $f_{n_i(e_{i+1}-1)}$ has a unique expansion (with only finitely many terms)

$$f_{n_i(e_{i+1}-1)} = \sum_{\alpha \geq 1} a_{\sigma_1(\alpha), \dots, \sigma_{i-1}(\alpha)} \varphi_1^{\sigma_1(\alpha)} \cdots \varphi_{i-1}^{\sigma_{i-1}(\alpha)} \quad (42)$$

with

$$W(a_{\sigma_1(\alpha), \dots, \sigma_{i-1}(\alpha)} \varphi_1^{\sigma_1(\alpha)} \cdots \varphi_{i-1}^{\sigma_{i-1}(\alpha)}) < W(a_{\sigma_1(\alpha+1), \dots, \sigma_{i-1}(\alpha+1)} \varphi_1^{\sigma_1(\alpha+1)} \cdots \varphi_{i-1}^{\sigma_{i-1}(\alpha+1)})$$

for all α , $0 \leq \sigma_l(\alpha) < n_l$ for $1 \leq l \leq i-1$ and $a_{\sigma_1(\alpha), \dots, \sigma_{i-1}(\alpha)} \in A$. Thus the minimum value term in V_{i-1} in this expansion is

$$a_{\sigma_1(1), \dots, \sigma_{i-1}(1)} \varphi_1^{\sigma_1(1)} \cdots \varphi_{i-1}^{\sigma_{i-1}(1)},$$

and so $j_l(i) = \sigma_l(1)$ for $1 \leq l \leq i-1$ and $-e_{i+1}c_i \sim a_{\sigma_1(1), \dots, \sigma_{i-1}(1)}$ in V_0 . Replacing c_i with $-(1/e_{i+1})a_{\sigma_1(1), \dots, \sigma_{i-1}(1)}$ in (40), we have that $c_i \in A$.

Suppose that $n_i = 1$, so that $e_{i+1} = e_i$. Then substituting (40) and (42) into (41), we obtain

$$f = \varphi_{i+1}^{e_i} + \left(\sum_{k \geq 2} a_{\sigma_1(k), \dots, \sigma_{i-1}(k)} \varphi_1^{\sigma_1(k)} \cdots \varphi_{i-1}^{\sigma_{i-1}(k)} \right) \varphi_{i+1}^{e_i-1} + \sum_{j=2}^{e_i-2} f'_j \varphi_{i+1}^j,$$

where $\deg_z f'_j < \deg_z \varphi_{i+1} = \deg_z \varphi_i$ for all j . Since (42) is a finite sum, we can only have $n_i = 1$ for finitely many consecutive i .

Since $\deg_z f = e_i n_1 \cdots n_{i-1}$ for all i , the algorithm must terminate in a finite number of iterations k . We then have that $\varphi_k = f$ and $W = V_k$.

The final statement on the structure of $\text{gr}_\omega(A[z]/(f(z)))$ now follows from Lemma 4.3. $\square$

As an immediate consequence of Theorem 5.1, we have the following example, which allows us to easily compute the associated graded rings and valuation semigroups of many examples, including the rational double point singularities in dimension 2, since the semigroups of valuations dominating 2-dimensional regular local rings are completely known [Spi90, CV14].

Example 5.2. Let $\mathbf{k}$ be an algebraically closed field of characteristic $p \neq 2$ and $A = \mathbf{k}[[x_1, \dots, x_n]]$ be a power series ring over $\mathbf{k}$. Let $f(z) = z^2 + az + b$ with $a, b \in m_A$ be irreducible, and let $B = A[z]/(f(z))$. Suppose that ν is a valuation of the quotient field of A which dominates A and is such that $R_\nu/m_\nu = \mathbf{k}$.

Suppose that ν has a unique extension ω to the quotient field of B which dominates B . Then there exists a $g \in m_A$ such that for $\bar{z} = z - g$,

- 1) $\omega(\bar{z})$ is a generator of $G_\omega/G_\nu \cong \mathbf{Z}/2\mathbf{Z}$ and
- 2) $\text{gr}_\omega(B) = \text{gr}_\nu(A)[\text{in}(\bar{z})] \cong \text{gr}_\nu(A)[\bar{\varphi}]/(\bar{\varphi}^2 - \bar{c})$ for some $\bar{c} \in \text{gr}_\nu(A)$.

In contrast, if ν does not have a unique extension to the quotient field of B which dominates B , then it can happen that $\text{gr}_\omega(B)$ is not a finitely generated $\text{gr}_\nu(A)$ -module (as will follow from Example 11.4).

The good conclusions of Theorem 5.1 may fail if either the extension is not unique or p divides $\deg_z f$. In [Tei14, Example 8.1], an example of G. Rond is presented which shows that the conclusions of Theorem 5.1 may fail if the extension of V_0 to a valuation of $K[z]/(f(z))$ is not unique and $p \nmid \deg_z f$.

Example 5.3. The conclusions of Theorem 5.1 may fail if the characteristic p of the field $\mathbf{k}$ divides the degree of $f(z)$. In our example, $f(z)$ is separable and V_0 has a unique extension to $K[z]/(f(z))$.

We now give the construction of the example. Let $\mathbf{k}$ be an algebraically closed field of characteristic 2, and let $A = \mathbf{k}[x_1, x_2]_{(x_1, x_2)}$ be a localization of a 2-dimensional polynomial ring over $\mathbf{k}$. Let K be the quotient field of A . Let V_0 be the rank 1 valuation on K defined by $V_0(x_1) = 1$ and $V_0(x_2) = \sqrt{37}$, so that $G_{V_0} = \mathbf{Z} + \sqrt{37}\mathbf{Z}$. Let

$$f(z) = z^4 + x_1^{317}z + x_1^4x_2^2 + x_2^{31}.$$

This is an irreducible, separable polynomial in $K[z]$.

For $\varphi_1 = z$, the Newton polygon $N(V_0, \varphi_1)$ has only one segment, from $(0, 0)$ to $(4, 4 + 2\sqrt{37})$. The slope of this segment is $1 + \frac{1}{2}\sqrt{37}$, giving the first step approximant to f over V_0 , namely $V_1 = [V_0; V_1(\varphi_1) = 1 + \frac{1}{2}\sqrt{37}]$. We have that $G_{V_1} = \mathbf{Z} + \frac{1}{2}\sqrt{37}\mathbf{Z}$.

Now $f \sim (z^2 + x_1^2x_2)^2$ in V_1 and $V_1(z) \notin G_{V_0}$, so $\varphi_2 = z^2 + x_1^2x_2$ is a key polynomial over V_1 . We have that

$$f = \varphi_2^2 + x_1^{317}z + x_2^{31},$$

so the principal part of $N(V_1, \varphi_2)$ is equal to $N(V_1, \varphi_2)$, which has only one segment, from $(0, 0)$ to $(2, 31\sqrt{37})$. The slope is $\frac{1}{2} \cdot 31\sqrt{37}$, giving the second step approximant to f over V_0 , namely $V_2 = [V_1; V_2(\varphi_2) = \frac{1}{2} \cdot 31\sqrt{37}]$, with $G_{V_2} = G_{V_1}$. We have that

$$f = (\varphi_2 + zx_1^{-1}x_2^{15})^2 + \varphi_2x_1^{-2}x_2^{30} + x_1^{317}z,$$

so that $f \sim (\varphi_2 + zx_1^{-1}x_2^{15})^2$ in V_2 . Thus

$$\varphi_3 = \varphi_2 + zx_1^{-1}x_2^{15} \tag{43}$$

is a key polynomial for V_2 . We have that

$$f = \varphi_3^2 + x_1^{-2}x_2^{30}\varphi_3 + x_1^{-3}x_2^{45}z + x_1^{317}z,$$

so the principal part of $N(V_2, \varphi_3)$ is equal to $N(V_2, \varphi_3)$ which has only one segment, from $(0, 0)$ to $(2, \frac{1}{2} \cdot 91\sqrt{37} - 2)$. The slope is $\frac{1}{4} \cdot 91\sqrt{37} - 1$, giving the third stage approximant to f over V_0 , namely $V_3 = [V_2; V_3(\varphi_3) = \frac{1}{4} \cdot 91\sqrt{37} - 1]$, with

$$G_{V_3} = G_{V_1} + \left(\frac{91}{4}\sqrt{37} - 1\right)\mathbf{Z} = \mathbf{Z} + \frac{\sqrt{37}}{4}\mathbf{Z}.$$

Now $f \sim \varphi_3^2 + x_1^{-3}x_2^{45}z$ in V_3 and $V_3(\varphi_3) \notin G_{V_1}$, so $\varphi_4 = \varphi_3^2 + x_1^{-3}x_2^{45}z$ is a key polynomial over V_3 . We have that

$$f = \varphi_4 + x_1^{-2}x_2^{30}\varphi_3 + x_1^{317}z,$$

so the principal part of $N(V_3, \varphi_4)$ is $N(V_3, \varphi_4)$, which has only one segment, from $(0, 0)$ to $(1, -3 + \frac{1}{4} \cdot 131\sqrt{37})$. The slope is $-3 + \frac{1}{4} \cdot 131\sqrt{37}$, giving the fourth stage approximant to f over V_0 , namely $V_4 = [V_3; V_4(\varphi_4) = -3 + \frac{1}{4} \cdot 131\sqrt{37}]$. We have that $G_{V_4} = G_{V_3}$. Now $f \sim \varphi_4 + x_1^{-2}x_2^{30}\varphi_3$ in V_4 , so

$$\varphi_5 = \varphi_4 + x_1^{-2}x_2^{30}\varphi_3$$

is a key polynomial over V_4 . We have that $f = \varphi_5 + x_1^{317}z$, so the principal part of $N(V_4, \varphi_5)$ is $N(V_4, \varphi_5)$, which has only one segment, from $(0, 0)$ to $(1, 318 + \frac{1}{2}\sqrt{37})$. The slope is $318 + \frac{1}{2}\sqrt{37}$, giving the fifth stage approximant to f over V_4 , namely $V_5 = [V_4; V_5(\varphi_5) = 318 + \frac{1}{2}\sqrt{317}]$. We have that $G_{V_5} = G_{V_3}$.

Now $f = \varphi_5 + x_1^{317}z$ is a key polynomial for V_5 , so $V_6 = [V_5; V_6(f(z)) = \infty]$ is a pseudoevaluation with $I(V_6)_\infty = (f(z))$.

Let ω be the induced extension of V_0 to $K[z]/(f(z))$. We have that $G_\omega = G_{V_3}$, and thus

$$[G_\omega : G_{V_0}] = 4 = \deg_z f = [L : K],$$

showing that ω is the unique extension of V_0 to a valuation of L and that $\delta(\omega/V_0) = 1$, so that the extension is defectless (Section 8). Observe that we cannot avoid substitutions like (43), leaving the ring A in any realization of the algorithm. Notice that the conclusions of Theorem 5.1 are verified if we take A_1 to be a birational extension of A containing $x_1^{-1}x_2^{15}$.

Remark 5.4. In the example, the valuation V_0 is an Abhyankar valuation, which means that there is equality in the fundamental inequality of S. Abhyankar [Abh56, Theorem 1],

$$\dim_{\mathbf{Q}} G_{V_0} \otimes_{\mathbf{Z}} \mathbf{Q} + \text{trdeg}_{A/m_A} R_{V_0}/m_{V_0} = \dim A.$$

It is known [Kuh10b, Theorem 1] that Abhyankar valuations have “no defect,” a fact which plays a role in this example. We will come back to the study of the effect of defect in Sections 6–9.

6. Henselization and completion

A valued field (K, ν) is Henselian if for all algebraic extensions L of K , there exists a unique valuation ω of L which extends ν . Some references on the theory of Henselian fields are [Kuh00, End72, Rib68] and [Vaq07b].

An extension (K^h, ν^h) of a valued field (K, ν) is called a Henselization of (K, ν) if (K^h, ν^h) is Henselian and for all Henselian valued fields (L, ω) and all embeddings $\lambda: (K, \nu) \rightarrow (L, \omega)$, there exists a unique embedding $\tilde{\lambda}: (K^h, \nu^h) \rightarrow (L, \omega)$ which extends λ .

A Henselization (K^h, ν^h) of (K, ν) can be constructed by choosing an extension ν^s of ν to a separable closure K^{sep} of K , letting K^h be the fixed field of the decomposition group

$$\{\sigma \in G(K^{\text{sep}}/K) \mid \nu^s \circ \sigma = \nu^s\}$$

of ν^s and defining ν^h to be the restriction of ν^s to K^h (see [End72, Theorem 17.11]).

LEMMA 6.1. *Suppose that (K, ν) is a valued field, and let (K^h, ν^h) be a Henselization of (K, ν) . Suppose that $f(z) \in K[z]$ is unitary, irreducible and separable. Then $f(z)$ is reduced in $K^h[z]$. Let $f(z) = f_1(z)f_2(z) \cdots f_r(z)$ be the factorization of $f(z)$ into irreducible unitary factors in $K^h[z]$. If the coefficients of $f(z)$ are in R_ν , then the coefficients of the $f_i(z)$ are in R_{ν^h} .*

Let ν_i^h be the (unique) extension of ν^h to $K^h[z]/(f_i)$. Then the distinct extensions of ν to $K[z]/(f(z))$ are the r restrictions ν_i of ν_i^h to $K[z]/(f(z))$, under the natural inclusions $K[z]/(f(z)) \rightarrow K^h[z]/(f_i(z))$.

Proof. The polynomial $f(z)$ is reduced in $K^h[z]$ since the separable polynomial $f(z)$ is reduced in $K^{\text{sep}}[z]$, where K^{sep} is a separable closure of K .

Let $\bar{z}$ be a root of $f_i(z)$ in K^{sep} . Then $f(z)$ is the minimal polynomial of $\bar{z}$ in $K[z]$, and $K[z]/(f(z)) \cong K[\bar{z}]$. If $\bar{z}$ is integral over R_ν , then $\bar{z}$ is integral over R_{ν^h} . Thus the coefficients of f_i are in R_{ν^h} since R_{ν^h} is normal [ZS58, V.3, Theorem 5]. If L is a finite separable extension of K , then we have two associated sets,

$$\text{Mon}(L, K) = K\text{-embeddings of } L \text{ in } K^{\text{sep}},$$

$$\mathcal{E}(L, \nu) = \text{Extensions of } \nu \text{ to a valuation of } L.$$

By [Vaq07b, Lemma 1.4] or [End72, Section 17], the map $\Phi: \text{Mon}(L, K) \rightarrow \mathcal{E}(L, \nu)$ defined by $\Phi(\lambda) = \nu^s \circ \lambda$ is surjective, with $\Phi(\lambda) = \Phi(\lambda')$ if and only if $\lambda \sim_{K^h} \lambda'$. The equivalence $\sim_{K^h}$

is defined by $\lambda \sim_{K^h} \lambda'$ if and only if there exists a K^h -isomorphism $\sigma: K^{\text{sep}} \rightarrow K^{\text{sep}}$ such that $\lambda' = \sigma \circ \lambda$.

The valuation $\nu^s \circ \lambda$ is obtained from the embedding

$$L \cong \lambda(L) \rightarrow \lambda(L) \cdot K^h$$

into the join of $\lambda(L)$ and K^h in K^{sep} and the restriction of the valuation $\nu^s|_{\lambda(L) \cdot K^h}$ to L .

Let $L = K[z]/(f(z))$. The elements $\lambda \in \text{Mon}(L, K)$ are in one-to-one correspondence with the distinct roots α_λ of $f(z)$ in K^{sep} . We have that $\lambda(L) \cdot K^h = K^h[\alpha_\lambda]$. Thus $\lambda(L) \cdot K^h \cong K^h[z]/(f_i)$ for some i . Further, $\lambda \sim_{K^h} \lambda'$ if and only if α_λ and $\alpha_{\lambda'}$ have the same minimal polynomial f_i in $K^h[z]$.

Since K^h is Henselian, for each i there is a unique extension of ν^h to $K^h[z]/(f_i)$, and so the last assertion of the lemma follows. $\square$

Suppose that A is a local ring and $g(z) \in A[z]$ is a polynomial. Let $\bar{g}(z) \in A/m_A[z]$ be the polynomial obtained by reducing the coefficients of $g(z)$ mod m_A .

A local ring A is a Henselian local ring if it has the following property: Let $f(z) \in A[z]$ be a unitary polynomial of degree n . If $\alpha(z)$ and $\alpha'(z)$ are relatively prime unitary polynomials in $A/m_A[z]$ of degrees r and $n-r$, respectively, such that $\bar{f}(z) = \alpha(z)\alpha'(z)$, then there exist unitary polynomials $g(z)$ and $g'(z)$ in $A[z]$ of degrees r and $n-r$, respectively, such that $\bar{g}(z) = \alpha(z)$, $\bar{g}'(z) = \alpha'(z)$ and $f(z) = g(z)g'(z)$.

If A is a local ring, a local ring A^h which dominates A is called a Henselization of A if any local homomorphism from A to a Henselian local ring can be uniquely extended to A^h . A Henselization always exists [Nag62, Theorem 43.5]. The construction is particularly nice when A is a normal local ring. Let K be the quotient field of A , and let K^{sep} be a separable closure of K . Let $\bar{A}$ be the integral closure of A in K^{sep} , and let $\bar{m}$ be a maximal ideal of $\bar{A}$.

Let H be the decomposition group

$$H = G^s(\bar{A}_{\bar{m}}/A) = \{\sigma \in G(K^{\text{sep}}/K) \mid \sigma(\bar{A}_{\bar{m}}) = \bar{A}_{\bar{m}}\}.$$

Then $A^h = (\bar{A}_{\bar{m}})^H$ is the fixed ring of the action of H on $\bar{A}_{\bar{m}}$. We have that

$$A^h = (\bar{A} \cap K^H)_{\bar{m} \cap (\bar{A} \cap K^H)} = \bar{A}_{\bar{m}} \cap K^H = (\tilde{A})_{\bar{m} \cap \tilde{A}},$$

where $\tilde{A}$ is the integral closure of A in K^H .

Nagata rings are defined and their basic properties are developed in [Mat80, Chapter 12]. Nagata rings are called universally Japanese in [GD65]. Their basic properties are established in [GD65, IV.7.2.2].

We remark that if A is a Nagata local domain with quotient field K and ν is a valuation of K which dominates A , then there exists a directed system of normal birational extensions A_i of A such that $\bigcup_i A_i = R_\nu$.

LEMMA 6.2. *Continuing the assumptions of Lemma 6.1, suppose that A is a Nagata local domain with quotient field K such that ν dominates A and that A_i is a directed system of birational extensions of A such that the A_i are normal local domains which are dominated by ν and $\bigcup_i A_i = R_\nu$. Then there are natural equalities*

$$R_{\nu^h} = (R_\nu)^h = \bigcup_i A_i^h.$$

Proof. Let ν^s be an extension of ν to K^{sep} and

$$H = \{ \sigma \in \text{Gal}(K^{\text{sep}}/K) \mid \nu^s \circ \sigma = \nu^s \},$$

so that $K^h = (K^{\text{sep}})^H$. Let $\bar{V}$ be the integral closure of R_ν in K^{sep} , and let $m = \bar{V} \cap m_{\nu^s}$, a maximal ideal in $\bar{V}$. Since K^{sep} is algebraic over K , we have that $R_{\nu^s} = \bar{V}_m$ by [ZS60, VI.7, Theorem 12]. Now, as is shown in [ZS60, VI.12, bottom of p. 68], the group H is the decomposition group

$$H = G^s(R_{\nu^s}/R_\nu) = \{ \sigma \in G(K^{\text{sep}}/K) \mid \sigma(R_{\nu^s}) = R_{\nu^s} \},$$

so that

$$(R_\nu)^h = \bar{V}_m \cap K^h = R_{\nu^s} \cap K^h = R_{\nu^h},$$

establishing the first assertion of the lemma.

Suppose that A is a normal local ring with quotient field K . Let $\tilde{A}$ be the integral closure of A in K^h . If A is dominated by $V = R_\nu$, then $\tilde{A}_{m_{\nu^s} \cap \tilde{A}}$ is dominated by $\tilde{V}_{m_{\nu^s} \cap \tilde{V}}$ (where $\tilde{V}$ is the integral closure of V in K^h). Suppose that $g, h \in \tilde{V}$ with $h \notin m_{\nu^s} \cap \tilde{V}$. Since $\tilde{A}_i$ is a directed system, there exists an i such that $g, h \in \tilde{A}_i$, so $h \notin m_{\nu^s} \cap \tilde{A}_i$ and $g/h \in (\tilde{A}_i)_{m_{\nu^s} \cap \tilde{A}_i}$. Thus

$$\bigcup_i (\tilde{A}_i)_{m_{\nu^s} \cap \tilde{A}_i} = R_\nu^h.$$

Let $\bar{A}_i$ be the integral closure of A_i in K^{sep} . By [Cut18, Lemma 3.3], we have inclusions of decomposition groups

$$G^s(R_{\nu^s}/R_\nu) \subset G^s((\bar{A}_i)_{m_{\nu^s} \cap \bar{A}_i}/A_i)$$

for all i , and by [Cut18, Lemma 3.4], there exists an i_0 such that

$$G^s(R_{\nu^s}/R_\nu) = G^s((\bar{A}_i)_{m_{\nu^s} \cap \bar{A}_i}/A_i)$$

for $i \geq i_0$. Thus $A_i^h \subset (\tilde{A}_i)_{m_{\nu^s} \cap \tilde{A}_i}$ for all i and $A_i^h = (\tilde{A}_i)_{m_{\nu^s} \cap \tilde{A}_i}$ for $i \gg 0$. The last assertion of the lemma now follows. $\square$

Let (K, ν) be a valued field such that ν has rank 1. The completion $(\hat{K}, \hat{\nu})$ (when ν has rank 1) is defined in Section 2 of [End72]. The completion $\hat{K}$ is defined to be the ring of ν -Cauchy sequences in K modulo the maximal ideal of ν -null sequences (that is, ν -Cauchy sequences whose limit is ∞). The extension $\hat{\nu}$ of ν is defined by $\hat{\nu}(h) = \lim_{i \rightarrow \infty} \nu(h_i)$ if (h_i) is a ν -Cauchy sequence in K which converges to h . The field $\hat{K}$ is Henselian [End72, Lemma 16.7]. The following lemma is proven in [End72, Theorem 2.12].

LEMMA 6.3. *Suppose that (K, ν) is a rank 1 valued field and $(\hat{K}, \hat{\nu})$ is a completion of (K, ν) . Suppose that $f(z) \in K[z]$ is unitary, irreducible and separable, so that $f(z)$ is reduced in $\hat{K}[z]$. Let $f(z) = f_1(z)f_2(z) \cdots f_r(z)$ be the factorization of f into irreducible unitary factors in $\hat{K}[z]$.*

Let $\hat{\nu}_i$ be the (unique) extension of $\hat{\nu}$ to $\hat{K}[z]/(f_i)$. Then the distinct extensions of ν to $K[z]/(f(z))$ are the r restrictions ν_i of $\hat{\nu}_i$ to $K[z]/(f(z))$, under the natural inclusions $K[z]/(f(z)) \rightarrow \hat{K}[z]/(f_i)$.

LEMMA 6.4. *Let the notation be as in the statement of Lemma 6.3. We then have a factorization $K \rightarrow K^h \rightarrow \hat{K}$ of valued fields. Further, the factorizations of $f(z)$ into products of unitary irreducible polynomials in $K^h[z]$ and $\hat{K}[z]$ are the same.*

Proof. We have a natural inclusion of K^h into $\hat{K}$ since $\hat{K}$ is a Henselian field. The irreducible factors of $f(z)$ in $K^h(z)$ remain irreducible in $\hat{K}[z]$ since there is a one-to-one correspondence of the irreducible factors of $f(z)$ in $\hat{K}[z]$ with the distinct extensions of ν to $L = K[z]/(f(z))$ by Lemma 6.3 and there is a one-to-one correspondence of the irreducible factors of $f(z)$ in $K^h[z]$ with the distinct extensions of ν to L by Lemma 6.1. $\square$

Some references on the defect of a finite field extension are [Kuh00, End72, Rib68] and [Vaq07b].

Suppose that $(K, \nu) \rightarrow (L, \omega)$ is a finite separable extension of valued fields. Let K^{sep} be a separable closure of K with an embedding of L in K^{sep} . Let ν^s be an extension of ω to a valuation of K^{sep} . As discussed above, we can use ν^s to define the Henselization K^h of (K, ν) , with valuation $\nu^h = \nu^s|_{K^h}$, and then $L^h = L \cdot K^h$, the join of L and K^h in K^{sep} , is a Henselization of (L, ω) with valuation $\omega^h = \nu^s|_{L^h}$ (see [Vaq07b, Lemma 1.3], [Kuh00], [End72, (17.16)]). The defect of ω over ν is defined as

$$\delta(\omega/\nu) = [L^h : K^h]/e(\omega^h/\nu^h)f(\omega^h/\nu^h) = [L^h : K^h]/e(\omega/\nu)f(\omega/\nu). \quad (44)$$

The defect is a power of the residue characteristic p of the valuation ring of ν by Ostrowski's lemma [Kuh00, Theorem 8.2].

7. Vaquié's algorithm

Suppose that K is a field, $f(z) \in K[z]$ is unitary and irreducible, ν is a valuation of K and μ is a pseudovaluation of $K[z]$ which extends ν such that $I(\mu)_\infty = (f(z))$. In [Vaq07a, Theorem 2.5], Vaquié shows that there exists a “finite admissible family of valuations” $\mathcal{S}$ which determines μ . This result follows from [Vaq07a, Proposition 2.3], which gives an algorithm for constructing such a family. We will take the last element of $\mathcal{S}$ to be the pseudovaluation μ .

We summarize the definition of an “admissible family of valuations” approximating μ (from [Vaq07a, Section 2.1]), which takes the following form since $I(\mu)_\infty = (f(z)) \neq 0$. A family $\mathcal{S}$ of iterated augmented valuations is called a “simple admissible family” if it is of the form $\mathcal{S} = (\mu_i)_{i \in I}$, where the set of indices I is a disjoint union $I = B \amalg A$ with B a finite set and A a totally ordered set, where all elements of A are larger than all elements of B and A does not have a largest element.

A family of valuations $\mathcal{A} = (\mu_i)_{i \in I}$ is called an “admissible family” for μ (defined in [Vaq07a, 2.2, p. 3473]) if it is a finite or countable union of simple admissible families $\mathcal{S}^{(t)} = (\mu_i^{(t)})_{i \in I^{(t)}}$. The first valuation of $\mathcal{S}^{(1)}$ is an inductive valuation of the form $\mu_1^{(1)} = [\mu_0; \mu_1^{(1)}(\varphi_1^{(1)}) = \gamma_1]$, where $\mu_0 = \nu$ is the given valuation of K and $\varphi_1^{(1)}$ is a polynomial of degree 1. For $t \geq 2$, the first valuation $\mu_1^{(t)}$ of $\mathcal{S}^{(t)}$ is a “limit augmented valuation” for the family $(\mu_{\alpha^{(t-1)}})_{\alpha^{(t-1)} \in A^{(t-1)}}$. The construction of limit augmented valuations will be explained below.

Write $I^{(t)} = B^{(t)} \amalg A^{(t)}$ as above, and write $B^{(t)} = \{1, \dots, n^{(t)}\}$. Then for $i \geq 2$ in $B^{(t)}$, $\mu_i^{(t)} = [\mu_{i-1}^{(t)}; \mu_i^{(t)}(\varphi_i^{(t)}) = \gamma_i^{(t)}]$ is an inductive valuation (see Section 3). For $\alpha \in A^{(t)}$, we have that $\mu_\alpha^{(t)} = [\mu_{n^{(t)}}^{(t)}; \mu_\alpha^{(t)}(\varphi_\alpha^{(t)}) = \gamma_\alpha^{(t)}]$ is an inductive valuation, where $\deg_z \varphi_\alpha^{(t)} = \deg_z \varphi_{n^{(t)}}^{(t)}$.

Vaquié requires that $\deg_z \varphi_{i-1}^{(t)} < \deg_z \varphi_i^{(t)}$ for $i \geq 2$ in $B^{(t)}$, but we do not assume this. By the definition of an inductive value, we do have that $\deg_z \varphi_{i-1}^{(t)} \leq \deg_z \varphi_i^{(t)}$ for $i \geq 2$ in $B^{(t)}$. By the construction of limit key polynomials, we have that $\deg_z \varphi_{n^{(t)}}^{(t)} < \deg_z \varphi_1^{(t+1)}$ for all t .

We require that for $g \in K[z]$ and $i < j \in I$,

$$\mu_i(g) \leq \mu_j(g) \leq \mu(g). \quad (45)$$

Further, $\mu_i(\varphi_i) = \mu(\varphi_i)$ for all i .

We now discuss the construction of limit augmented valuations.

Suppose that $\mathcal{A} = (\mu_\alpha)_{\alpha \in A}$ is an admissible family of valuations for μ . Define [Vaq07a, 2.2, p. 3473]

$$\tilde{\Sigma}(\mathcal{A}) = \{g \in K[z] \mid \mu_\alpha(g) < \mu(g) \text{ for all } \mu_i \in \mathcal{A}\}.$$

Define $d(\mathcal{A}) = \infty$ if $\tilde{\Sigma}(\mathcal{A}) = \emptyset$ and

$$d(\mathcal{A}) = \inf \{ \deg_z \varphi \mid \varphi \in \tilde{\Sigma}(\mathcal{A}) \}$$

if $\tilde{\Sigma}(\mathcal{A}) \neq \emptyset$. Now define

$$\Sigma(\mathcal{A}) = \{ \varphi \in \tilde{\Sigma}(\mathcal{A}) \mid \varphi \text{ is unitary and } \deg_z \varphi = d(\mathcal{A}) \}$$

and

$$\Lambda(\mathcal{A}) = \{ \mu(\varphi) \mid \varphi \in \Sigma(\mathcal{A}) \}. \quad (46)$$

Suppose that $\Lambda(\mathcal{A})$ does not have a largest element. We then define a totally ordered index set C which does not have a largest element, such that

$$\Lambda(\mathcal{A}) = \{ \gamma_\alpha \mid \alpha \in C \},$$

where $\alpha < \beta$ if and only if $\gamma_\alpha < \gamma_\beta$.

A “limit key polynomial” φ for $\mathcal{A}$ is defined in [Vaq07a, Definition before Proposition 1.21]. It satisfies the three properties that φ is $\mathcal{A}$ -minimal, φ is $\mathcal{A}$ -irreducible and φ is unitary. The elements of $\Sigma(\mathcal{A})$ are limit key polynomials for $\mathcal{A}$ by [Vaq07a, Proposition 1.21]. Choose $\varphi_\alpha \in \Sigma(\mathcal{A})$ for each $\alpha \in C$ such that $\mu(\varphi_\alpha) = \gamma_\alpha$. We then have a limit augmented valuation $\mu_\alpha = [\mathcal{A}; \mu_\alpha(\varphi_\alpha) = \gamma_\alpha]$ (see [Vaq07a, Proposition 1.22]), which is defined by

$$\mu_\alpha(g) = \max_{j \in A} \left\{ \min_i \{ \mu_j(g_i) + i \mu(\varphi_\alpha) \} \right\} \quad (47)$$

for $g \in K[z]$, where $g = \sum g_i \varphi_\alpha^i$ with $\deg_z g_i < \deg_z \varphi_\alpha$.

The “family of iterated augmented valuations associated” with $\mathcal{A}$ is

$$(\mu_\alpha)_{\alpha \in C}. \quad (48)$$

We will explain here how the algorithm proceeds if we are given a discrete simple admissible family $\mathcal{S} = \{\mu_1, \dots, \mu_n\}$ such that $\Sigma(\mu_n)$ is nonempty. We will produce an admissible family of valuations $\mathcal{B}$ such that $d(\mathcal{B}) > d(\mu_n)$.

All elements of $\Sigma(\mu_n)$ are key polynomials for μ_n by [Mac36a, Theorem 8.1] or [Vaq07a, Theorem 1.15].

First suppose that the set of values $\Lambda(\mu_n)$ has a largest element γ' (which could be ∞). Then we can define $\mu' = [\mu_n; \nu'(\varphi') = \gamma']$, where $\varphi' \in \Sigma(\mu_n)$ satisfies $\mu(\varphi') = \gamma'$. We then have two cases, depending on whether $\deg_z \varphi' > \deg_z \varphi_n$ or $\deg_z \varphi' = \deg_z \varphi_n$.

Assume that $\deg_z \varphi' > \deg_z \varphi_n$. Set $\varphi_{n+1} = \varphi'$, $\gamma_{n+1} = \gamma'$ and

$$\mu_{n+1} = \mu' = [\mu_n; \mu_{n+1}(\varphi_{n+1}) = \gamma_{n+1}].$$

Then define $\mathcal{B} = \{\mu_1, \dots, \mu_n, \mu_{n+1}\}$, which is a discrete simple admissible family with $d(\mathcal{B}) \geq \deg_z(\varphi_{n+1}) > \deg_z \varphi_n$.

Now assume that $\deg_z \varphi' = \deg_z \varphi_n$. Then define $\mathcal{B} = \{\mu_1, \dots, \mu_n, \mu'\}$, which is again a discrete simple admissible family with $d(\mathcal{B}) > d(\mu')$ (by [Mac36a, Lemma 15.1] or [Vaq07a, Corollary to Proposition 1.8]).

The last case is when $\Lambda(\mu_n)$ does not have a largest element. Define the associated family of iterated augmented valuations $(\mu_\alpha)_{\alpha \in C}$ of (48) for μ_n . For all $\gamma_\alpha \in \Lambda(\mu_n)$, define $\mu_\alpha = [\mu_n; \mu_\alpha(\varphi_\alpha) = \gamma_\alpha]$. Define $\mathcal{S}^{(1)}$ by adding to $\mathcal{S}$ the family $\mathcal{C} = (\mu_\alpha)_{\alpha \in C}$, so $\mathcal{S}^{(1)}$ is indexed by $I' = \{1, \dots, n\} \amalg C$ (which does not have a largest element). Then $\mathcal{S}^{(1)}$ is a simple admissible family. The family $\mathcal{C}$ is an “exhaustive, continuous family of iterated augmented valuations” with the property that $\deg_z \varphi_\alpha = d(\mu_n)$ for all $\alpha \in C$. We have that $f \notin \Sigma(\mu_n)$ since C does not have a largest element. Thus $\tilde{\Sigma}(\mathcal{C}) \neq \emptyset$. By [Vaq07a, Proposition 1.21], all polynomials of $\Sigma(\mathcal{C})$ are limit key polynomials for the family $\mathcal{C}$. We now choose a polynomial $\varphi_1^{(2)} \in \Sigma(\mathcal{C})$ and define the “limit augmented valuation” $\mu_1^{(2)} = [(\mu_\alpha)_{\alpha \in C}; \nu_1^{(2)}(\varphi_1^{(2)}) = \mu(\varphi_1^{(2)})]$ (by the definition before [Vaq07a, Proposition 1.21] and [Vaq07a, Proposition 1.22] and as explained in (47)) and the discrete, simple admissible family $\mathcal{S}^{(2)} = \{\mu_1^{(2)}\}$. By [Vaq07a, Proposition 1.27], $\deg_z \varphi_1^{(2)}$ is greater than the degree of the polynomials in $\Sigma(\mu_n)$. Define the admissible family $\mathcal{B} = \mathcal{S}^{(1)} \cup \mathcal{S}^{(2)}$, which is indexed by $I'' = I' \amalg \{1^{(2)}\}$ (where $1^{(2)}$ is larger than every element of I').

7.1 Comparison of the algorithms of Section 4 and Vaquié

Suppose that W is a pseudovaluation of $K[z]$ which extends a valuation V_0 of K , such that $I(W)_\infty = (f(z))$, where f is unitary and $f(z) \in R_{V_0}[z]$. Let

$$V_1, \dots, V_k, \dots \quad (49)$$

be a sequence of approximants to f over V_0 , constructed by the algorithm of Section 4, which satisfy (45) (with $\mu_j = V_j$ and $\mu = W$).

Then either $\varphi_k = f$, or $V_0, V_1, \dots, V_k, \dots$ is infinite with $\deg_z \varphi_k = \deg_z \varphi_{k_0}$ for $k \geq k_0$. In the first case, $\mathcal{S} = \{V_1, \dots, V_k\}$ is a discrete simple admissible family of valuations which determines W . In the second case, $V_1, \dots, V_k, \dots$ is infinite. Then $\varphi_k \in \Sigma(V_{k_0})$ for $k > k_0$, and so $d(V_{k_0}) = \deg_z \varphi_{k_0}$.

If $\Lambda(V_{k_0})$ has a maximal element γ and $\varphi' \in \Sigma(V_{k_0})$ is a key polynomial with $W(\varphi') = \gamma$ and corresponding valuation $\mu' = [V_{k_0}; \mu'(\varphi') = W(\varphi')]$, then $\{V_1, \dots, V_{k_0}, \mu'\}$ is the first part of the discrete part of $\mathcal{S}^{(1)}$ constructed by Vaquié’s algorithm. If $W(\varphi') = \infty$, then $\mathcal{S} = \mathcal{S}^{(1)} = \{V_1, \dots, V_{k_0}, \mu' = W\}$ is an admissible family of valuations which determines W .

Suppose that $\Lambda(V_{k_0})$ does not have a largest element. Let $\mathcal{C} = (\mu_\alpha)_{\alpha \in C}$ be the family of iterated augmented valuations of (48) associated with V_{k_0} . Choose a limit key polynomial $\varphi_1^{(2)}$ for $\mathcal{C}$. The next step in Vaquié’s algorithm is to construct $\mathcal{S} = \mathcal{S}^{(1)} \cup \mathcal{S}^{(2)}$, where $\mathcal{S}^{(1)} = \{V_1, \dots, V_{k_0}\} \cup \mathcal{C}$ and $\mathcal{S}^{(2)} = \{\mu_1^{(2)} = [\mathcal{C}; V_1^{(2)}(\varphi_1^{(2)}) = W(\varphi_1^{(2)})]\}$.

Looking again at the case when $\Lambda(V_{k_0})$ has a maximal element γ and $\varphi' \in \Sigma(V_{k_0})$ is the corresponding key polynomial, we have an expression $\varphi' = \varphi_{k_0} + h$, where $h \in K[z]$ has $\deg_z h < \deg_z \varphi_{k_0}$. We further have that $h \in R_{V_0}[z]$ by Remark 3.3. We have an expression (for some r)

$$h = \sum_{j=1}^r a_j \varphi_1^{\sigma_1(j)} \cdots \varphi_{k_0-1}^{\sigma_{k_0-1}(j)}$$

with $a_j \in R_{V_0}$, $0 \leq \sigma_i(j) < n_i = [G_{V_i} : G_{V_{i-1}}]$ for all i and j and

$$W(a_i \varphi_1^{\sigma_1(i)} \cdots \varphi_{k_0-1}^{\sigma_{k_0-1}(i)}) < W(a_j \varphi_1^{\sigma_1(j)} \cdots \varphi_{k_0-1}^{\sigma_{k_0-1}(j)})$$

if $i < j$. Let

$$\psi_i = \varphi_{k_0} + a_1 \varphi_1^{\sigma_1(1)} \cdots \varphi_{k_0-1}^{\sigma_{k_0-1}(1)} + \cdots + a_i \varphi_1^{\sigma_1(i)} \cdots \varphi_{k_0-1}^{\sigma_{k_0-1}(i)} \quad (50)$$

for $1 \leq i \leq r$. Then (for instance by the criterion of [Vaq07a, Proposition 1.9])

$$V_1, \dots, V_{k_0}, V'_{k_0+1}, \dots, V'_{k_0+r} \quad (51)$$

is a $(k_0 + r)$ th stage approximant to f over V_0 , where

$$V'_{k_0+1} = [V_{k_0}; V'_{k_0+1}(\psi_1) = W(\psi_1)] \text{ and } V'_{k_0+i} = [V'_{k_0+i-1}; V'_{k_0+i}(\psi_i) = W(\psi_i)] \text{ for } 2 \leq i \leq r.$$

Further, either $W(\varphi') < \infty$ and

$$d(\{V_1, \dots, V_{k_0}, V'_{k_0+1}, \dots, V'_{k_0+r}\}) > \deg_z \varphi_{k_0},$$

or $W(\varphi') = \infty$, in which case $f = \varphi'$ (since f and φ' are unitary in z of the same degree) and $\psi_r = f$.

We may now continue the algorithm of Section 4 to construct higher-stage approximants, starting from V'_{k_0+r} . After a finite number of iterations of this procedure, we construct a sequence of approximants to f ,

$$V_1, \dots, V_{k_1}, \dots, \quad (52)$$

such that $\deg_z \varphi_i \leq \deg_z \varphi_{i+1}$ if $i < k_1$ and $\deg_z \varphi_i = \deg_z \varphi_i$ for $i \geq k_1$. Either this is of finite length k_1 , so that $V_{k_1} = W$, or there is a jump ($t > 1$) in the construction of the admissible family $\mathcal{S} = \mathcal{S}^{(1)} \cup \cdots \cup \mathcal{S}^{(t)}$ determining W .

Suppose that (52) is infinite and the equivalent conditions of Lemma 3.2 hold for (52). Let $\mathcal{C} = (\mu_\alpha)_{\alpha \in C}$ be the family of (48) associated with V_{k_1} . Suppose that $g \in K[z]$ and $W(g) < \infty$ and that k is so large that $W(\varphi_k) > W(g)$. Write $g = g_m \varphi_k^m + \cdots + g_0$ with $\deg_z g_i < \deg_z \varphi_k$ for all i . Then

$$V_k(g) = V_k(g_0) = V_{k_0-1}(g_0) = W(g_0) = W(g).$$

Thus $g \notin \tilde{\Sigma}(\mathcal{C})$, and so $\deg_z f$ is the smallest degree of an element of $\tilde{\Sigma}(\mathcal{C})$. Thus $\mathcal{S} = \mathcal{S}^{(1)} \cup \mathcal{S}^{(2)}$, where $\mathcal{S}^{(1)} = \{V_1, \dots, V_{k_1}\} \cup \mathcal{C}$, and $\mathcal{S}^{(2)} = \{V_1^{(2)}\}$, where $V_1^{(2)} = [\mathcal{C}; V_1^{(2)}(f(z)) = \infty]$.

The following proposition results from our analysis.

PROPOSITION 7.1. *Suppose that V_0 has finite rank. Then there exists a realization of the algorithm of Section 4 which produces the first simple admissible family $\mathcal{S}^{(1)}$ of an admissible family $\mathcal{S} = \mathcal{S}^{(1)} \cup \cdots \cup \mathcal{S}^{(t)}$ determining W , where all key polynomials are in $R_\nu[z]$.*

7.2 Invariants of ramification and jumps

Suppose that W is an extension of a valuation $V = V_0$ of K to a pseudovaluation of $K[z]$ with $I(W)_\infty = (f(z))$ in $K[z]$ with f unitary. Let ω be the induced valuation on $L = K[z]/(f(z))$.

The jumps $s^{(j-1)}(\mathcal{S})$ in a family $\mathcal{S} = \mathcal{S}^{(1)} \cup \cdots \cup \mathcal{S}^{(t)}$ realizing W are defined by the equations

$$\deg_z \varphi_1^{(j)} = s^{(j-1)}(\mathcal{S}) \deg_z \varphi_\alpha^{(j-1)}, \quad (53)$$

where $\varphi_\alpha^{(j-1)}$ is a key polynomial of a member of the continuous family $\mathcal{C}^{(j-1)}$ associated with

$\mathcal{S}^{(j-1)}$. The total jump of the family $\mathcal{S}$ is

$$s^{\text{tot}}(\mathcal{S}) = \prod_{j=2}^t s^{(j-1)}(\mathcal{S}).$$

By [Vaq07b, Lemma 2.11 and Corollary 2.10],

$$\deg_z f = [L : K] = e(\omega/V) f(\omega/V) s^{\text{tot}}(\mathcal{S}). \quad (54)$$

We have that $s^{\text{tot}}(\mathcal{S}) = 1$ if and only if there are no jumps in the construction of approximants. Here $e(\omega/V) = [G_\omega : G_V]$, where G_ω and G_V are the respective value groups of ω and V , and $f(\omega/V)$ is the index of the respective residue fields of the valuation rings of ω and V .

In the case when ω is the unique extension of V to a valuation of L , by Ostrowski's lemma,

$$[L : K] = e(\omega/V) f(\omega/V) \delta(\omega/V), \quad (55)$$

where the defect $\delta(\omega/V)$ is a power of the residue characteristic p of V . Comparing with (54), we see that in this case, $s^{\text{tot}}(\mathcal{S}) = \delta(\omega/V)$. Thus (assuming that ω is the unique extension of V) there is no jump if and only if there is no defect, and in this case,

$$[L : K] = e(\omega/V) f(\omega/V). \quad (56)$$

In contrast to the good property of key polynomials of (4), we have examples of the following type for limit key polynomials.

Example 7.2. The jumps $s^{(i)}(\mathcal{S})$ and total jump $s^{\text{tot}}(\mathcal{S})$ can be rational numbers which are not integers.

We now construct such an example. Let $\mathbf{k}$ be an algebraically closed field and $K = \mathbf{k}(x)$ be a rational function field in one variable over $\mathbf{k}$. Let ν be the valuation of K with valuation ring $R_\nu = \mathbf{k}[x]_{(x)}$ and such that $\nu(x) = 1$. Let $L = K[z]/(z^3 - z^2 - x) \cong \mathbf{k}(z)$. Let ω be the extension of ν to L with valuation ring $R_\omega = \mathbf{k}[z]_{(z)}$ and $\omega(z) = \frac{1}{2}$. Then $e(\omega/\nu) = 2$ and $f(\omega/\nu) = 1$. Thus by (54),

$$s^{\text{tot}}(\mathcal{S}) = \frac{\deg_z (z^3 - z^2 - x)}{e(\omega/\nu) f(\omega/\nu)} = \frac{3}{2}.$$

8. Defectless extensions

LEMMA 8.1. Suppose that (K, ν) is a valued field containing an algebraically closed field $\mathbf{k}$ such that $R_\nu/m_\nu \cong \mathbf{k}$ and $f(z) \in R_\nu[z]$ is unitary, irreducible and separable. Let $L = K[z]/(f(z))$, and let ω be an extension of ν to L . Let W be the induced pseudovaluation on $K[z]$. Let $\bar{f}(z) \in K^h[z]$ be the irreducible factor of $f(z)$ which induces ω (by Lemma 6.1), and let ω^h be the (unique) extension of ν^h to $K^h[z]/(\bar{f}(z))$. Let $\bar{W}$ be the induced pseudovaluation on $K^h[z]$. Let $V_0 = \nu$ and $W_0 = \nu^h$. Then the following hold:

- 1) $\text{gr}_{\nu^h}(R_{\nu^h}) = \text{gr}_\nu(R_\nu)$.
- 2) Set $\varphi_1 = z$, let $V_1 = [V_0; V_1(\varphi_1) = W(\varphi_1)]$, and let $W_1 = [\nu^h; W_1(\varphi_1) = W(\varphi_1)]$. Then $\text{gr}_{W_1}(R_{\nu^h}[z]) = \text{gr}_{V_1}(R_\nu[z])$.
- 3) Suppose that $V_i = [V_{i-1}; V_i(\varphi_i) = W(\varphi_i)]$ for $1 \leq i \leq k$ is a realization of the algorithm of Section 4 in $R_\nu[z]$ such that $W_i = [W_{i-1}; W_i(\varphi_i) = W(\varphi_i)]$ for $1 \leq i \leq k$ is a realization of

the algorithm of Section 4 in $R_{\nu^h}[z]$ and

$$\mathrm{gr}_{W_i}(R_{\nu^h}[z]) = \mathrm{gr}_{V_i}(R_{\nu}[z]) \quad \text{for } 1 \leq i \leq k. \quad (57)$$

Suppose that $\varphi_{k+1} \in R_{\nu}[z]$ has an expression $\varphi_{k+1} = \varphi_k^n - c_k \varphi_1^{j_1} \cdots \varphi_{k-1}^{j_{k-1}}$ of the form of (15), φ_{k+1} is a key polynomial for W_k and $W_{k+1} = [W_k; W_{k+1}(\varphi_{k+1}) = W(\varphi_{k+1})]$ is a $(k+1)$ st approximant of $\bar{f}$ over W_0 . Then φ_{k+1} is a key polynomial for V_k and $V_{k+1} = [V_k; V_{k+1}(\varphi_{k+1}) = W(\varphi_{k+1})]$ is a $(k+1)$ st approximant of f over V_0 . Further, $\mathrm{gr}_{W_{k+1}}(R_{\nu^h}[z]) = \mathrm{gr}_{V_{k+1}}(R_{\nu}[z])$.

Proof. Statement 1) holds since $G_{\nu^h} = G_{\nu}$ and $R_{\nu^h}/m_{\nu^h} = R_{\nu}/m_{\nu}$ by [End72, Theorem 17.19]. Statement 2) holds since

$$\mathrm{gr}_{W_1}(R_{\nu^h}[z]) = \mathrm{gr}_{\nu^h}(R_{\nu^h})[\mathrm{in}_{W_1}(\varphi_1)] = \mathrm{gr}_{\nu}(R_{\nu})[\mathrm{in}_{V_1}(\varphi_1)] = \mathrm{gr}_{V_1}(R_{\nu}[z]).$$

Now we will prove statement 3). To show that φ_{k+1} is a key polynomial over V_k , we must verify that conditions 1)–6) of the definition of a key polynomial, given in Subsection 3.1 after (3), hold for φ_{k+1} over V_k . This holds since these conditions hold for φ_{k+1} over W_k . The fact that W_{k+1} is a $(k+1)$ st approximant to $\bar{f}$ over W_k implies that φ_{k+1} equivalence divides $\bar{f}$ in W_k . Thus $\mathrm{in}_{W_k}(\varphi_{k+1})$ divides $\mathrm{in}_{W_k}(\bar{f})$ in $\mathrm{gr}_{W_k}(R_{\nu^h}[z])$. Now $\mathrm{in}_{W_k}(\bar{f})$ divides $\mathrm{in}_{W_k}(f(z))$ in $\mathrm{gr}_{W_k}(R_{\nu^h}[z])$. So $\mathrm{in}_{W_k}(\varphi_{k+1})$ divides $\mathrm{in}_{W_k}(f(z))$ in $\mathrm{gr}_{W_k}(R_{\nu^h}[z]) = \mathrm{gr}_{V_k}(R_{\nu}[z])$. Thus φ_{k+1} equivalence divides $f(z)$ in V_k , and so V_{k+1} is a $(k+1)$ st approximant to $f(z)$ over V_k . As $G_{W_{k-1}} = G_{V_{k-1}}$, we have that $n = [G_{W_k} : G_{W_{k-1}}] = [G_{V_k} : G_{V_{k-1}}]$. Finally, we have that $\mathrm{gr}_{V_{k+1}}(R_{\nu}[z]) = \mathrm{gr}_{W_{k+1}}(R_{\nu^h}[z])$ by Lemma 4.3. $\square$

THEOREM 8.2. *Suppose that A is a Nagata local domain which contains an algebraically closed field $\mathbf{k}$ such that $A/m_A \cong \mathbf{k}$. Let K be the quotient field of A , and suppose that $V_0 = \nu$ is a rank 1 valuation of K which dominates A and is such that the residue field of the valuation ring of V_0 is $\mathbf{k}$. Suppose that $f(z) \in A[z]$ is unitary, irreducible and separable and W is a pseudovaluation of $K[z]$ such that $I(W)_{\infty} = (f(z))$ in $K[z]$, which extends V_0 . Let ω be the induced valuation of $L = K[z]/(f(z))$. Then ω is defectless over ν ($\delta(\omega/\nu) = 1$) if and only if there is a normal birational extension A_1 of A which is dominated by ν such that there exists a realization*

$$V_1, \dots, V_k, \dots$$

of the algorithm of Section 4 in $A_1[z]$, satisfying (14) and (15) for all k with $c_k \in A_1$ for all $k \geq 1$, such that $W = V_k$ for some finite k or $W = \lim_{k \rightarrow \infty} V_k$.

If these equivalent conditions hold, then there exists a positive integer k such that

$$\mathrm{gr}_{\omega}(A_1[z]/(f(z))) \cong \mathrm{gr}_{\nu}(A_1)[\bar{\varphi}_1, \dots, \bar{\varphi}_k]/I,$$

where

$$I = (\bar{\varphi}_1^{n_1} - \bar{c}_1, \bar{\varphi}_2^{n_2} - \bar{c}_2 \bar{\varphi}_1^{j_1(2)}, \dots, \bar{\varphi}_k^{n_k} - \bar{c}_k \bar{\varphi}_1^{j_1(k)} \bar{\varphi}_2^{j_2(k)} \cdots \bar{\varphi}_{k-1}^{j_{k-1}(k)})$$

is a finitely generated and presented $\mathrm{gr}_{\nu}(A_1)$ -module.

An example showing that the conclusions of Theorem 8.2 may not hold if ν has rank larger than 1 will be given in Section 10. In Example 8.3, it will be shown that the conclusions of Theorem 8.2 may not hold if $f(z)$ is not separable over K .

Proof. First suppose that $\delta(\omega/\nu) = 1$. Let the notation be as in Section 6. By Lemma 6.1, there exists an extension $\bar{W}$ of ν^h to a pseudovaluation of $K^h[z]$ such that $I(\bar{W})_{\infty} = (\bar{f})$, where $\bar{f}(z)$ is an irreducible factor of $f(z)$ in $K^h[z]$ and $\bar{W}$ is an extension of W .

We will construct a special sequence of approximants $W_1, \dots, W_{k_0}$ to $\bar{f}$ over ν^h such that $\bar{W} = W_{k_0}$. In particular, $W_{k_0} = [W_{k_0-1}; W_{k_0}(\varphi_{k_0}) = \infty]$, where $\varphi_{k_0} = \bar{f}$.

Set $\varphi_1 = z$, and let $W_1 = [\nu^h; W_1(\varphi_1) = W(\varphi_1)]$. Suppose by induction on k that we have constructed a sequence $W_1, \dots, W_k$ of approximants to $\bar{f}$ over ν^h giving a realization of the algorithm of Section 4, such that expressions

$$\varphi_i = \varphi_{i-1}^{n_{i-1}} - c_{i-1} \varphi_1^{j_1(i-1)} \dots \varphi_{i-2}^{j_{i-2}(i-1)}$$

of the form of (15) hold for $i \leq k$ with $c_i \in R_\nu$ for $i \leq k-1$. After replacing A with a birational extension A_1 of A , we may suppose that $c_i \in A$ for $i \leq k-1$.

If $\Lambda(W_k)$ does not have a largest element, then we have a jump $s^{(1)} > 1$ by (53) and the analysis of this case in Subsection 7.1. But by (54) and (55), there cannot be a jump, and we have a contradiction, showing that $\Lambda(W_k)$ has a largest element.

Suppose that we are in the case when $\Lambda(W_k)$ has a maximal element $\gamma \neq \infty$, and let $\varphi' \in \Sigma(W_k)$ be a corresponding key polynomial. We modify the resulting sequence (51) of the analysis in Section 7.1, which we write as

$$W_1, \dots, W_k, W'_{k+1}, \dots, W'_{k+r}, \quad (58)$$

by modifying the ψ_i of (50), replacing the a_i with suitable $b_i \in R_\nu$ for $1 \leq i \leq r$. With the notation of Lemma 6.2, since $a_1, \dots, a_r \in R_{\nu^h}$, there exists an A_l such that $a_i \in A_l^h$ for $1 \leq i \leq r$ and $\varphi_1, \dots, \varphi_{k_0} \in A_l[z]$. Thus, since $\bar{W}$ induces a rank 1 valuation on $K^h[z]/(\bar{f}(z))$, there exists an $n \in \mathbf{Z}_+$ such that $nV_0(m_{A_l^h}) > \bar{W}(\varphi') = \gamma$. Now $A_l \rightarrow A_l^h$ is unramified with no residue field extension, so there exist $b_i \in A_l$ such that $a_i - b_i \in m_{A_l^h}^n$ for $1 \leq i \leq r$. Thus $V_0(b_i) - \nu^h(a_i) > \bar{w}(\varphi')$ for $1 \leq i \leq r$, and we can replace ψ_i with $\psi_{i-1} + b_i \varphi_1^{j_1(i)} \dots \varphi_{k-1}^{j_{k-1}(i)}$ in (50) for $1 \leq i \leq r$ to produce a sequence (58) with $\psi_i \in R_\nu[z]$ for all i . Then by Lemma 8.1, we have a sequence

$$V_1, \dots, V_k, V'_{k+1}, \dots, V'_{k+r}$$

of approximants to f over V_0 corresponding to (58).

Now using the algorithm of Section 4, we can continue applying the above argument as necessary until we reach a W_k such that the maximal element of $\Lambda(W_k)$ is ∞ , so that $\bar{f} \in \Sigma(W_k)$.

With this assumption, there exists an l (with the notation of Lemma 6.2) such that the coefficients of $\bar{f}$ are in A_l^h and the coefficients of $\varphi_1, \dots, \varphi_k$ are in A_l . We have $\bar{f} = \varphi_k + h$, where $h \in A_l^h[z]$ and $\deg_z h < \deg_z \varphi_k$. Set $\psi_0 = \varphi_k$. By induction, we may construct a sequence $\psi_i \in A_l[z]$ of monic polynomials with $\deg_z \psi_i = \deg_z \varphi_k$ such that for all i , we have that $\bar{f} = \psi_i + h_i$ with $h_i \in (A_l)^h[z]$ a polynomial of degree strictly less than $\deg_z \varphi_k$ and

$$\psi_{i+1} = \psi_i + b_i \varphi_1^{\sigma_1(i)} \dots \varphi_{k_0-1}^{\sigma_{k_0-1}(i)}$$

with $b_i \in A_l$ and $0 \leq \sigma_j(i) < n_j$ for $1 \leq j \leq k_0 - 1$ and such that $\bar{W}(\psi_{i+1}) > \bar{W}(\psi_i)$ for all i . Since A_l is Noetherian and $\bar{W}$ induces a rank 1 valuation on $K^h[z]/(\bar{f}(z))$, on $A_l[z]$, the extension $\bar{W}$ takes only a finite number of values which are less than or equal to a given finite upper bound. Thus we obtain either that $\psi_i = \bar{f}(z)$ for some i or that

$$\lim_{i \rightarrow \infty} W(\psi_i) = \lim_{i \rightarrow \infty} \bar{W}(\psi_i) = \infty.$$

Using Lemma 8.1, inductively defining $V_i = [V_{i-1}; V_i(\varphi_i) = W(\varphi_i)]$ for $1 \leq i \leq k$ and $V_{i+k} = [V_{i+k-1}; V_i(\psi_i) = W(\psi_i)]$ for $k < i$, we construct a sequence $V_1, \dots, V_k, \dots$ of approximants to $f(z)$ over V_0 such that $\lim_{i \rightarrow \infty} V_i(\varphi_i) = \infty$, so that $W = \lim_{i \rightarrow \infty} V_i$ by Lemma 3.2.

Now suppose that there exist a normal birational extension A_1 of A and a realization $V_1, \dots, V_k, \dots$ of the algorithm of Section 4 as in the statement of the theorem. We will show that the defect $\delta(\omega/\nu)$ is 1.

First suppose that the sequence is of finite length, terminating with $V_k = W$, so that the last key polynomial is $\varphi_k = f$ (with $V_k(\varphi_k) = \infty$). We have that $\deg_z \varphi_1 = 1$ and $\deg_z \varphi_i = n_{i-1} \deg_z \varphi_{i-1}$ for $i \geq 2$. Thus

$$[G_\omega : G_{V_0}] \delta(\omega/V_0) \leq \deg_z f = n_1 n_2 \cdots n_{k-1} = [G_\omega : G_{V_0}],$$

so that $\delta(\omega/\nu) = 1$.

Now suppose that $V_1, \dots, V_k, \dots$ is of infinite length. We have (by Lemma 6.4) natural extensions of valued fields

$$(K, \nu) \rightarrow (K^h, \nu^h) \rightarrow (\hat{K}, \hat{\nu}).$$

Let $\bar{f}(z)$ be the irreducible factor of $f(z)$ in $K^h[z]$ which induces ω (from Lemma 6.1). Then $\bar{f}(z)$ is irreducible in $\hat{K}[z]$ (by Lemma 6.4) and so is the irreducible factor of $f(z)$ in $\hat{K}[z]$ which induces ω (by Lemma 6.3). Thus the pseudovaluation W extends to a pseudovaluation W^h of $K^h[z]$ and to a pseudovaluation $\hat{W}$ of $\hat{K}[z]$ such that $I(W^h)_\infty = (\bar{f}(z))$ in $K^h[z]$ and $I(\hat{W})_\infty = (\bar{f}(z))$ in $\hat{K}[z]$. By (44),

$$\delta(\omega/\nu) = [L^h : K^h] / [G_\omega : G_\nu] = \deg_z \bar{f} / [G_\omega : G_\nu]. \quad (59)$$

There exists a k_0 such that $\deg_z \varphi_k = \deg_z \varphi_{k_0}$ for $k \geq k_0$. There exist $a_i \in A_1$ and $j_1(i), \dots, j_{k_0-1}(i)$ with $0 \leq j_l(i) < n_l$ for $1 \leq l \leq k_0 - 1$ such that $\varphi_{k_0+i+1} = \varphi_{k_0+i} - a_i \varphi_1^{j_1(i)} \cdots \varphi_{k_0-1}^{j_{k_0-1}(i)}$ for $i \geq 0$. Now

$$W(\varphi_{k_0+i}) = W(a_i \varphi_1^{j_1(i)} \cdots \varphi_{k_0-1}^{j_{k_0-1}(i)})$$

for $i > 0$ and

$$W(\varphi_{k_0+i}) \rightarrow \infty \quad \text{as } i \rightarrow \infty \quad (60)$$

by Lemma 3.2. Thus $\nu(a_i) \rightarrow \infty$ as $i \rightarrow \infty$.

For fixed $(b_1, \dots, b_{k_0-1})$ such that $0 \leq b_l < n_l$ for $1 \leq l \leq k_0 - 1$, define

$$c_l(b_1, \dots, b_{k_0-1}) = \sum a_i,$$

where the sum is over the $i < l$ such that $(j_1(i), \dots, j_{k_0-1}(i)) = (b_1, \dots, b_{k_0-1})$. Let

$$\tau_i = \sum_{b_1, \dots, b_{k_0-1}} c_i(b_1, \dots, b_{k_0-1}) \varphi_1^{b_1} \cdots \varphi_{k_0-1}^{b_{k_0-1}},$$

where the sum is over the $b_1, \dots, b_{k_0-1}$ such that $0 \leq b_j < n_j$ for $1 \leq j \leq k_0 - 1$. We have that $\varphi_{k_0+i} = \varphi_{k_0} - \tau_i$. Thus

$$W(\tau_j - \tau_i) = W(\varphi_{k_0+i} - \varphi_{k_0+j}) \geq \min\{W(\varphi_{k_0+i}), W(\varphi_{k_0+j})\},$$

so $W(\tau_j - \tau_i) \rightarrow \infty$ as $j \geq i \rightarrow \infty$. We have that

$$W(\tau_j - \tau_i) = \min \{ \nu(c_i(b_1, \dots, b_{k_0-1}) - c_j(b_1, \dots, b_{k_0-1})) + W(\varphi_1^{b_1} \cdots \varphi_{k_0-1}^{b_{k_0-1}}) \},$$

where the minimum is over the $b_1, \dots, b_{k_0-1}$ with $0 \leq b_j < n_j$ for $1 \leq j \leq k_0 - 1$. So for all $b_1, \dots, b_{k_0-1}$, we have that $\nu(c_i(b_1, \dots, b_{k_0-1}) - c_j(b_1, \dots, b_{k_0-1})) \rightarrow \infty$ as $j \geq i \rightarrow \infty$. Thus for each $b_1, \dots, b_{k_0-1}$, the $(c_i(b_1, \dots, b_{k_0-1}))$ is a ν -Cauchy sequence.

Thus these sequences have limits in $\hat{K}$, and so (φ_{k_0+i}) is a ν -Cauchy sequence in $K[z]$ which has a nonzero limit φ_∞ in $\hat{K}[z]$ (note that φ_∞ is necessarily unitary of degree equal to $\deg_z \varphi_{k_0}$).

Now $\varphi_\infty \in I(\hat{W})_\infty = (\bar{f})$ by (60). Thus $\deg_z \varphi_\infty \geq \deg_z \bar{f}$. Now

$$\deg_z \bar{f} \leq \deg_z \varphi_\infty = \deg_z \varphi_{k_0} = [G_{V_{k_0}} : G_{V_0}] = [G_\omega : G_\nu].$$

Thus $\deg_z \bar{f} = [G_\omega : G_\nu]$ and $\delta(\omega/\nu) = 1$ by (59). $\square$

Example 8.3. The conclusions of Theorem 8.2 may fail if $f(z)$ is not separable over K .

An example of F.K. Schmidt of a discrete valuation ring (with value group $\mathbf{Z}$) and an inseparable extension of its quotient field which has defect is explained in [Kuh11, Example 3.1]. The example is as follows. Let $\mathbf{k}$ be an algebraically closed field of characteristic $p > 0$, let $A = \mathbf{k}[x, y]_{(x, y)}$ be the localization of a polynomial ring in two variables, and let K be the quotient field of A . Let $\mathbf{k}[[t]]$ be a power series ring, and let $s \in \mathbf{k}[[t]]$ be transcendental over $\mathbf{k}(t)$ and such that $\text{ord}_t(s) > 0$. The $\mathbf{k}$ -algebra embedding $K \rightarrow \mathbf{k}[[t]]$ defined by $x \mapsto t$ and $y \mapsto s^p$ induces a valuation ν on K which dominates A by $\nu(g(x, y)) = \text{ord}_t(g(t, s^p))$. We have that $G_\nu = \mathbf{Z}$ and $R_\nu/m_\nu = \mathbf{k}$. Let $f(z) = z^p - y \in K[z]$. There is a unique extension of ν to a valuation ω of $L = K[z]/(f(z))$ (since L is purely inseparable over K) which is an immediate extension of ν ($G_\omega = G_\nu$ and $R_\omega/m_\omega = R_\nu/m_\nu$). Thus we have defect $\delta(\omega/\nu) = \deg_z f = p$ by Ostrowski's lemma (1). Since ν is a rank 1 discrete valuation, by Mac Lane's theorem (Section 3), the valuation ω is a limit valuation which is realized by his algorithm. We will give an explicit construction.

Let W be the pseudovaluation induced by ω on $K[z]$, and let $V_0 = \nu$. We will construct a sequence of approximants $V_1, \dots, V_i, \dots$ to f over V_0 which realize W .

Expand $s = \sum_{i=1}^{\infty} a_i t^i$ with $a_i \in \mathbf{k}$. We have that $s^p = \sum_{i=1}^{\infty} a_i^p t^{ip}$. Define

$$\sigma(1) = \text{ord}_t(s) = \min\{i \mid a_i \neq 0\}$$

and for $j > 1$,

$$\sigma(j) = \min\{i \mid \sigma(j-1) < i \text{ and } a_i \neq 0\}.$$

The first approximant is $V_1 = [V_0; V_1(\varphi_1) = \sigma(1)]$, where $\varphi_1 = z$. For $i \geq 1$, the approximant V_{i+1} is defined by $V_{i+1} = [V_i; V_{i+1}(\varphi_{i+1}) = \sigma(i+1)]$, where $\varphi_{i+1} = \varphi_i - a_{\sigma(i+1)} x^{\sigma(i+1)}$. Then $\lim_{i \rightarrow \infty} V_i(\varphi_i) = \infty$ and so W is the limit valuation $W = \lim_{i \rightarrow \infty} V_i$ by Lemma 3.2.

9. A rank 1 separable example with defect

We consider an example from [CP04, Theorem 7.38] with regard to the algorithm of Section 4. Let $\mathbf{k}$ be an algebraically closed field of characteristic $p > 0$. Let $K = \mathbf{k}(u, v)$ be a 2-dimensional rational function field over $\mathbf{k}$, and, using the method of [Spi90] and [CV14], define a valuation ν of K by the following generating sequence:

$$P_0 = u, \quad P_1 = v, \quad P_2 = v^{p^2} - u \quad \text{and} \quad P_{i+1} = P_i^{p^2} - u^{p^{2i-2}} P_{i-1} \quad \text{for } i \geq 2.$$

We normalize the valuation ν so that $\nu(u) = 1$. We have the defining relations $v^{p^2} \sim u$ in ν and $P_i^{p^2} \sim u^{p^{2i-2}} P_{i-1}$ in ν for $i \geq 2$. As shown in [CP04], the value group G_ν equals $(1/p^\infty)\mathbf{Z} = \bigcup_{i \geq 1} (1/p^i)\mathbf{Z}$. Let $f = x^p + ux^{p-1} - u \in K[x]$. By [CP04, Theorem 7.38], the valuation ν has a unique extension to a valuation ω of $L = K[x]/(f(z))$. Further, ω is an immediate extension, so it is a defect extension with $[L : K] = \delta(\omega/\nu) = [L : K] = p$.

Let W be the pseudovaluation induced by ω on $K[x]$. We will construct a realization of the algorithm of Section 4, giving an infinite sequence $V_1, \dots, V_k, \dots$ of approximants to f over $V_0 = \nu$ satisfying (14) and (15) with $c_i \in R_\nu$ for all i .

Setting $\varphi_1 = x$, we have that $N(V_0, \varphi_1)$ has a single segment, which has slope $V_0(u)/p = 1/p$. Thus the first approximant to f over V_0 is $V_1 = [V_0; V_1(\varphi_1) = 1/p]$.

We use the following observation when constructing our sequence of approximants. Suppose that we have constructed the sequence $V_1, \dots, V_k$ of approximants, where $\deg_x \varphi_i = 1$ for all i . Then for $i \leq k$, we have that $\varphi_i = \varphi_{i-1} + a_{i-1}$ with $a_{i-1} \in R_\nu$ and

$$W(\varphi_i) > W(\varphi_{i-1}) = \nu(a_{i-1}).$$

Expanding

$$ux^{p-1} = g_{p-1}\varphi_k^{p-1} + g_{p-2}\varphi_k^{p-2} + \dots + g_0$$

with $g_i \in R_\nu$, we have that

$$V_0(g_0) \geq \min \{W(g_i \varphi_k^i)\} = V_k(ux^{p-1}) = W(ux^{p-1}) = 1 + \frac{p-1}{p} > \frac{p^4}{p^4-1}. \quad (61)$$

Now $f \sim \varphi_1^p - u$ in V_1 and $u = v^{p^2} - P_2 \sim v^{p^2}$ in V_0 . Thus $f \sim (\varphi_1 - v^p)^p$ in V_1 , and we take our second key polynomial to be $\varphi_2 = \varphi_1 - v^p = \varphi_1 - P_1^p$. Thus the second approximant is $V_2 = [V_1; V_2(\varphi_2) = W(\varphi_2)]$. Expanding $f = \varphi_2^p + f_{p-1}\varphi_2^{p-1} + \dots + f_1\varphi_2 + f_0$ with $f_i \in R_\nu$, by (61) with $k = 2$, we have that

$$f_0 = v^{p^2} - u + \text{terms of value} > \frac{p^4}{p^4-1}$$

and $v^{p^2} - u = P_2$. Now ω is the unique extension of ν , so the principal part of $N(V_1; \varphi_2)$ is $N(V_1, \varphi_2)$ and $N(V_1, \varphi_2)$ has a single segment, which has slope $V_0(f_0)/p = V_0(P_2)/p = 1/p + 1/p^5$, which is less than $p^4/p^4 - 1$. The second approximant is $V_2 = [V_1; V_2(\varphi_2) = 1/p + 1/p^5]$.

By (61), we have that $f \sim \varphi_2^p + P_2$ in V_2 . Now $P_2 \sim P_3^p/u^{p^4}$ in V_0 , so

$$f \sim \left(\varphi_2 + \frac{P_3^p}{u^{p^3}} \right)^p \quad \text{in } V_2,$$

and so $\varphi_3 = \varphi_2 + P_3^p/u^{p^3}$ is a key polynomial for V_2 . Thus the third approximant is $V_3 = [V_2; V_3(\varphi_3) = W(\varphi_3)]$. We expand $f = \varphi_3^p + f_{p-1}\varphi_3^{p-1} + \dots + f_1\varphi_3 + f_0$ with $f_i \in R_\nu$. By (61), we have that

$$f_0 = -\frac{P_3^{p^2}}{u^{p^4}} + P_2 + \text{terms of value} > \frac{p^4}{p^4-1}.$$

Also, $-P_3^{p^2}/u^{p^4} + P_2 = -P_4/u^{p^4}$. Now ω is the unique extension of ν , so the principal part of $N(V_2, \varphi_3)$ is $N(V_2, \varphi_3)$ and $N(V_2, \varphi_3)$ has a single segment, which has slope

$$\frac{V_0(f_0)}{p} = \frac{1}{p} V_0 \left(-\frac{P_4}{u^{p^4}} \right) = \frac{1}{p} + \frac{1}{p^5} + \frac{1}{p^9} < \frac{p^4}{p^4-1}.$$

The third approximant is $V_3 = [V_2; V_3(\varphi_3) = 1/p + 1/p^5 + 1/p^9]$.

Continuing in this way, we construct an infinite sequence $V_1, \dots, V_k, \dots$ of approximants to f over V_0 with $V_k = [V_{k-1}; V_k(\varphi_k) = \mu_k]$ and for $k \geq 3$,

$$\varphi_k = \varphi_{k-1} + (-1)^{k-1} \frac{P_{2(k-2)+1}^p}{u^{p^3+p^7+\dots+p^{4(k-3)+3}}}$$

with

$$\mu_k = \frac{1}{p} + \frac{1}{p^5} + \frac{1}{p^9} + \dots + \frac{1}{p^{4(k-1)+1}}.$$

We have that

$$\lim_{k \rightarrow \infty} \mu_k = \frac{p^4}{p(p^4 - 1)}. \quad (62)$$

In particular, by Lemma 3.2, the limit valuation $V_\infty = \lim_{k \rightarrow \infty} V_k$ is a valuation and thus is not equal to W .

We observe that there does not exist a birational extension A_1 of $\mathbf{k}[u, v]_{(u, v)}$ dominated by V_0 such that $\varphi_i \in A_1[x]$ for all i , as there can only be finitely many values of elements in a Noetherian local ring dominated by a rank 1 valuation that are less than a fixed finite bound.

We now analyze the extension W of ν in the context of Vaquié's algorithm. We will construct an admissible family of valuations $\mathcal{S}$ which determines W .

In the above realization of the algorithm of Section 4, we started by defining $\varphi_1 = x$ and $V_1 = [V_0; V_1(\varphi_1) = 1/p]$. With the notation of Section 7, we have

$$\Sigma(V_1) = \{x - g \mid g \in K \text{ and } W(x - g) > W(x)\}$$

and

$$\Lambda(V_1) = \{W(\varphi) \mid \varphi \in \Sigma(V_1)\}.$$

Let $(\mu_\alpha)_{\alpha \in C}$ be the family of iterated augmented valuations of (48) associated with $\mathcal{A} = \{V_1\}$.

The concept of distance of an element of L from K and the concepts of dependent and independent Artin–Schreier extensions are introduced in [Kuh10a]. In [EG16], our extension ω of ν is analyzed, and it is shown that it is a dependent Artin–Schreier extension. We use a calculation in their proof to determine $\limsup\{\Lambda(V_1)\}$. Suppose that $g \in \Sigma(V_1)$. Then $W(x - g) > W(x)$, so $\nu(g) = W(x) = 1/p$. Thus $g \in R_\nu$, and by part 2) of [EG16, Theorem 4.4],

$$W(g^p - x^p) \leq 1 + \frac{1}{p^4} + \cdots + \frac{1}{p^{4(k+1)}}$$

for some $k \geq 0$. Thus

$$W(x - g) = \frac{1}{p}W(g^p - x^p) \leq \frac{1}{p} + \frac{1}{p^5} + \cdots + \frac{1}{p^{4(k+1)+1}} < \frac{p^4}{p(p^4 - 1)}.$$

By (62), we have that $\limsup\{\Lambda(V_1)\} = p^4/p(p^4 - 1)$ and $p^4/p(p^4 - 1) \notin \Lambda(V_1)$. In particular, $\Lambda(V_1)$ does not have a largest element. Thus the first simple admissible family associated with W is

$$\mathcal{S}^{(1)} = \{V_1\} \cup \{(\mu_\alpha)_{\alpha \in C}\},$$

and $\mathcal{S}$ is the union of $t > 1$ simple admissible families. Since ω is an immediate extension of ν , by (54),

$$p = \deg_x f = s^{\text{tot}}(\mathcal{S}) = \prod_{j=2}^t s^{(j-1)}(\mathcal{S}). \quad (63)$$

Let $\psi_\alpha = x - \varphi_\alpha \in K$ for $\alpha > 1$. We have that

$$\nu(\psi_\sigma - \psi_\rho) = W(\varphi_\sigma - \varphi_\rho) = W(\varphi_\rho) = \mu_\rho < \mu_\sigma = W(\varphi_\sigma) = \nu(\psi_\tau - \psi_\sigma)$$

for $\rho < \sigma < \tau$. Thus $\{\psi_\alpha\}$ is a pseudoconvergent set in K in the sense of Kaplansky [Kap42].

Let $g(x)$ be a limit key polynomial for $\{\varphi_\alpha\}$ (defined in Section 7). As explained in [NS18, Section 3], the polynomial $g(x)$ is a polynomial of smallest degree such that $g(\psi_\alpha) < g(\psi_\beta)$ for $\alpha < \beta$. By [Kap42, Lemma 10], the degree of g is a power of p . By (63), the degree of g is p , and

so f is a limit key polynomial for $\{\varphi_\alpha\}$. Thus $\varphi_1^{(2)} = f$ and so $\mathcal{S}^{(2)} = \{\mu_1^{(2)}\}$, where $\mu_1^{(2)}$ is the limit augmented value

$$\mu_1^{(2)} = [(\mu_\alpha)_{\alpha \in B}; \mu_1^{(2)}(f) = \infty].$$

In summary, our admissible family of valuations $\mathcal{S}$ which determine W is $\mathcal{S} = \mathcal{S}^{(1)} \cup \mathcal{S}^{(2)}$, where $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$ are as described above.

We now consider the key polynomials φ_i and valuations V_i constructed in our realization of Mac Lane's algorithm. Since

$$\limsup W(\varphi_i) = \frac{p^4}{p(p^4 - 1)} = \limsup \Lambda(V_1)$$

and $\varphi_i \in \Sigma(V_1)$ for $i > 1$, by [Vaq07a, Proposition 1.9], the limit valuations $V_\infty = \lim_{i \rightarrow \infty} V_i$ and $\lim_{\alpha \in B} \mu_\alpha$ are equal. Thus the pseudovaluation W satisfies

$$W(g) = \begin{cases} \infty & \text{if } f|g \text{ in } K[x], \\ V_\infty(g) & \text{if } f \nmid g \text{ in } K[x]. \end{cases}$$

10. A defectless extension of a rank 2 valuation with many jumps

In this section, we construct the following example, which shows that the conclusions of Theorem 8.2 may not hold if ν has rank larger than 1.

Example 10.1. Let $\mathbf{k}$ be an algebraically closed field of characteristic not equal to 2, and let $\mathbf{k}[x, y]$ be a polynomial ring in two variables over $\mathbf{k}$. Let $K = \mathbf{k}(x, y)$, and let ν be the rank 2 valuation on K defined by $\nu(x) = (0, 1)$, $\nu(y) = (1, 0) \in (\mathbf{Z}_2)_{\text{lex}}$ and $\nu(\mathbf{k} \setminus 0) = 0$. Let

$$f = ((z^2 - x^2 - x^3)^2 - y^2(x^2 + 2x^3))^2 - (y^6 + y^7), \quad (64)$$

and let ω be an extension of ν to $K[z]/(f(z))$. Let W be the induced pseudovaluation of $K[z]$. Then a family $\mathcal{S} = \mathcal{S}^{(1)} \cup \dots \cup \mathcal{S}^{(t)}$ (with the notation of Section 7) realizing W has at least three jumps; that is, $t \geq 3$.

We first establish that f is irreducible in $K[z]$. Setting $x = 0$ in f , we obtain the reduction $\tilde{f} = z^8 - (y^6 + y^7) \in \mathbf{k}(y)[z]$. We have that

$$\tilde{f} = \prod_{j=0}^7 (z - \tau^j y^{3/4} (1 + y)^{1/8})$$

over an algebraic closure of $\mathbf{k}(y)$, where τ is a primitive eighth root of unity in $\mathbf{k}$. A unitary factor of $\tilde{f}$ of degree r must have constant term $\tau^s (y^{3/4} (1 + y)^{1/8})^r$ for some $s \in \mathbf{N}$. But $(y^{3/4} (1 + y)^{1/8})^r$ is in $\mathbf{k}(y)$ only if $r = 8$, so $\tilde{f}$ is irreducible in $\mathbf{k}(y)[z]$, and thus f is irreducible in $K[z]$.

Henselization is discussed in Section 6.

LEMMA 10.2. *The polynomial f factors into a product of linear unitary polynomials in $K^h[z]$, where (K^h, ν^h) is a Henselization of (K, ν) .*

Proof. We will solve the equation $f(z) = 0$ in R_{ν^h} . Let

$$Q = z^2 - (x^2 + x^3), \quad U = Q^2 - y^2(x^2 + 2x^3).$$

With these substitutions, the equation $f(z) = 0$ becomes $U^2 = (y^6 + y^7)$. Let $(1 + y)^{1/2}$ be a square root of $1 + y$ in the Henselization A^h of $A = \mathbf{k}[x, y]_{(x, y)}$. Then $U = y^3(1 + y)^{1/2}$ in A^h .

Thus

$$Q^2 = y^3(1+y)^{\frac{1}{2}} + y^2(x^2 + 2x^3) = y^2(x^2 + 2x^3 + y(1+y)^{1/2}).$$

Set $x = x_1$ and $y = x_1^2 y_1$. We have that $x_1, y_1 \in R_\nu$. Then

$$Q^2 = x_1^4 y_1^2 (x_1^2 + 2x_1^3 + x_1^2 y_1 (1 + x_1^2 y_1)^{1/2}) = x_1^6 y_1^2 (1 + 2x_1 + y_1 (1 + x_1^2 y_1)^{1/2}).$$

Let $(1 + 2x_1 + y_1 (1 + x_1^2 y_1)^{1/2})^{1/2}$ be a square root of $1 + 2x_1 + y_1 (1 + x_1^2 y_1)^{1/2}$ in the Henselization A_1^h of $A_1 = \mathbf{k}[x_1, y_1]_{(x_1, y_1)}$. Then

$$Q = x_1^3 y_1 (1 + 2x_1 + y_1 (1 + x_1^2 y_1)^{1/2})^{1/2}$$

in A_1^h . We now have that

$$\begin{aligned} z^2 &= Q + x^2 + x^3 \\ &= x_1^3 y_1 (1 + 2x_1 + y_1 (1 + x_1^2 y_1)^{1/2})^{1/2} + x_1^2 + x_1^3 \\ &= x_1^2 (1 + x_1 + x_1 y_1 (1 + 2x_1 + y_1 (1 + x_1^2 y_1)^{1/2})^{1/2}). \end{aligned}$$

Let $(1 + x_1 + x_1 y_1 (1 + 2x_1 + y_1 (1 + x_1^2 y_1)^{1/2})^{1/2})^{1/2}$ be a square root of

$$1 + x_1 + x_1 y_1 (1 + 2x_1 + y_1 (1 + x_1^2 y_1)^{1/2})^{1/2}$$

in A_1^h . Then

$$z = x_1 (1 + x_1 + x_1 y_1 (1 + 2x_1 + y_1 (1 + x_1^2 y_1)^{1/2})^{1/2})^{1/2} \in A_1^h \subset R_{\nu^h}$$

by Lemma 6.2.

Since all eight roots of $f(z)$ can be found this way, by making different choices of square roots, we have the desired factorization of $f(z)$ in $K^h[z]$ into a product of linear polynomials. $\square$

By Lemma 6.1, the valuation ω is the restriction to $K[z]/(f(z))$ of the extension of ν^h to a valuation ω^h of $K^h[z]/(\bar{f})$ for some factor $\bar{f}$ of f in $K^h[z]$. Since $\bar{f}$ is a linear polynomial by Lemma 10.2, by (54), we have that

$$[G_\omega : G_\nu] = [G_{\omega_h} : G_{\nu^h}] = \deg_z \bar{f} = 1. \quad (65)$$

We will require the following remark.

Remark 10.3. An element $g \in \mathbf{k}(z)$ is a square of an element of $\mathbf{k}(z)$ if and only if all zeros and poles of $g(z)$ in $\mathbf{A}_\mathbf{k}^1$ have even order.

The remark holds since every element $g(z)$ of $\mathbf{k}(z)$ has a unique factorization

$$g(z) = c(z - a_1)^{n_1} \cdots (z - a_t)^{n_t}$$

with $c \in \mathbf{k}$, $a_1, \dots, a_t$ distinct elements of $\mathbf{k}$ and $n_1, \dots, n_t$ nonzero integers.

We now turn to the construction of the family $\mathcal{S}$. We will use the notation of Section 7. To begin with, we observe that the total jump $s^{\text{tot}}(\mathcal{S})$ of $\mathcal{S}$ satisfies

$$s^{\text{tot}}(\mathcal{S}) = \deg_z f(z) = 8 \quad (66)$$

by (54) and (65).

Let $V_0 = \nu$. Since $W(f(z)) = \infty$, we have that $W(z) = (0, 1)$, and so the first approximant is $V_1 = [V_0; V_1(z) = (0, 1)]$. As above, let $Q = z^2 - (x^2 + x^3)$. Since $W(f(z)) = \infty$, we have that

$W(Q) = (1, 1)$. Let $\sum_{i=1}^{\infty} \alpha_i x^i$ with $\alpha_i \in k$ be a square root of $x^2 + x^3 = x^2(1+x)$ in $k[[x]]$. Let $\bar{z} = z - (\alpha_1 x + \cdots + \alpha_n x^n)$ for some $n \in \mathbf{Z}_+$. Then

$$\begin{aligned} Q &= (\bar{z} + \alpha_1 x + \cdots + \alpha_n x^n)^2 - (x^2 + x^3) \\ &= \bar{z}^2 + 2(\alpha_1 x + \cdots + \alpha_n x^n)\bar{z} + (\alpha_1 x + \cdots + \alpha_n x^n)^2 - (x^2 + x^3), \end{aligned}$$

so that $W(\bar{z}(\bar{z} + 2(\alpha_1 x + \cdots + \alpha_n x^n))) > (0, n)$. Thus

$$W(z - \alpha_1 x - \cdots - \alpha_n x^n) > \left(0, \frac{n}{2}\right) \quad \text{or} \quad W(z + \alpha_1 x + \cdots + \alpha_n x^n) > \left(0, \frac{n}{2}\right). \quad (67)$$

Thus $d(V_1) = 1$, and so

$$\Sigma(V_1) = \{\varphi = z + h \mid h \in K \text{ and } V_1(\varphi) < W(\varphi)\}.$$

We will show that

$$\Lambda(V_1) = \{W(\varphi) \mid \varphi \in \Sigma(V_1)\} \subset \{0\} \times \mathbf{Z}_+. \quad (68)$$

Suppose that there exists an $h \in K$ such that setting $\varphi = z + h$, we have that $W(\varphi) \geq (1, 0)$. Then

$$W(h) = (0, 1). \quad (69)$$

Substituting this into Q gives $Q = \varphi^2 - 2h\varphi + h^2 - (x^2 + x^3)$. Now $W(Q) = (1, 1)$ implies that

$$W(h^2 - (x^2 + x^3)) \geq (1, 0). \quad (70)$$

By (69), we have an expression

$$h = \frac{\alpha_0(x) + y\Omega_1}{\beta_0(x) + y\Omega_2}$$

with nonzero $\alpha(x), \beta(x) \in \mathbf{k}[x]$ and $\Omega_1, \Omega_2 \in \mathbf{k}[x, y]$. Substituting this into (70) gives

$$W((\alpha_0(x) + y\Omega_1)^2 + (x^2 + x^3)(\beta_0(x) + y\Omega_2)^2) \geq (1, 0),$$

which implies

$$W\left(\left(\frac{\alpha_0(x)}{\beta_0(x)}\right)^2 - (x^2 + x^3)\right) \geq (1, 0),$$

so that $(\alpha_0(x)/\beta_0(x))^2 = x^2 + x^3$, which leads to a contradiction by Remark 10.3. Thus (68) holds.

Let

$$\mathcal{A} = \{\mu_\alpha = [V_1; \mu_\alpha(\varphi_\alpha) = W(\varphi_\alpha) \mid \varphi_\alpha \in \Sigma(V_1)]\}.$$

By (67) and (68), we have that $(1, 0)$ is the least upper bound of $\Lambda(\mathcal{A})$ in $(\mathbf{Z}^2)_{\text{lex}}$ but $(1, 0) \notin \Lambda(\mathcal{A})$. Thus $\mathcal{A}$ does not have a maximal element.

Suppose that $\mu_\alpha \in \mathcal{A}$. Then $\mu_\alpha = [V_1; \mu_\alpha(\varphi_\alpha) = W(\varphi_\alpha)]$ with $\varphi_\alpha = z + h$ for some $h \in K$. Expand

$$Q = \varphi_\alpha^2 - 2h\varphi_\alpha + (h^2 - (x^2 + x^3)),$$

so that

$$\mu_\alpha(Q) \leq 2\mu_\alpha(\varphi_\alpha),$$

and $\mu_\alpha(Q) < (1, 0)$ by (68). Thus $Q \in \tilde{\Sigma}(\mathcal{A})$, and since Q has the smallest possible degree that a polynomial in $\tilde{\Sigma}(\mathcal{A})$ can have (it must have degree greater than $1 = d(V_1)$), we have that $d(\mathcal{A}) = 2$ and $Q \in \Sigma(\mathcal{A})$, and so Q is a limit key polynomial for $\mathcal{A}$. Let $V_2 = [\mathcal{A}; V_2(Q) = (1, 1)]$.

Then the first simple admissible family in $\mathcal{S}$ is $\mathcal{S}^{(1)} = \{V_1\} \cup \{\mathcal{A}\}$, and the second admissible family $\mathcal{S}^{(2)}$ begins with V_2 . Thus the first jump in $\mathcal{S}$ is

$$s^{(1)}(\mathcal{S}) = \frac{\deg_z Q}{\deg_z z} = 2.$$

We have that $f = (Q^2 - y^2(x^2 + 2x^3))^2 - (y^6 + y^7)$. Let

$$U = Q^2 - y^2(x^2 + 2x^3)$$

as above. We have that $W(U) = (3, 0)$ since $W(f(z)) = \infty$. Let $\sum_{i=1}^{\infty} \beta_i x^i$ with $\beta_i \in \mathbf{k}$ be a square root of $x^2 + 2x^3 = x^2(1+x)$ in $\mathbf{k}[[x]]$. For $n \in \mathbf{Z}_+$, let $\overline{Q} = Q - y(\beta_1 x + \cdots + \beta_n x^n)$. Then

$$U = \overline{Q}^2 + 2y(\beta_1 x + \cdots + \beta_n x^n)\overline{Q} + y^2(\beta_1 x + \cdots + \beta_n x^n)^2 - y^2(x^2 + 2x^3),$$

so that $W(\overline{Q}(\overline{Q} + 2y(\beta_1 x + \cdots + \beta_n x^n))) > (2, n)$. Thus

$$W(Q - y(\beta_1 x + \cdots + \beta_n x^n)) > \left(1, \frac{n}{2}\right) \quad \text{or} \quad W(Q + y(\beta_1 x + \cdots + \beta_n x^n)) > \left(1, \frac{n}{2}\right). \quad (71)$$

Thus $d(V_2) = 2$, and so $\Sigma(V_2) = \{\varphi = Q + Az + B \mid A, B \in K \text{ and } V_2(\varphi) < W(\varphi)\}$. We will show that

$$\Lambda(V_2) = \{W(\varphi) \mid \varphi \in \Sigma(V_2)\} \subset \{1\} \times \mathbf{Z}_+. \quad (72)$$

Suppose that there exist $A, B \in K$ such that setting $\varphi = Q + Az + B$, we have that $W(\varphi) \geq (2, 0)$. We have that $W(Q) = W(Az + B)$. Expand

$$U = \varphi^2 - 2(Az + B)\varphi + (Az + B)^2 - y^2(x^2 + 2x^3).$$

Now $W(\varphi^2) \geq (4, 0)$ and $W((Az + B)\varphi) > (3, 0)$. Since $W(U) = (3, 0)$, we have that

$$W((Az + B)^2 - y^2(x^2 + 2x^3)) \geq (3, 0).$$

Thus

$$(1, 1) = W((Az + B)) = \min\{W(A) + (0, 1), W(B)\}.$$

We can thus write

$$A = y \left(\frac{\alpha_0(x) + y\Omega_1}{\gamma_0(x) + y\Omega_3} \right), \quad B = y \left(\frac{\beta_0(x) + y\Omega_2}{\gamma_0(x) + y\Omega_3} \right)$$

with $\Omega_1, \Omega_2, \Omega_3 \in \mathbf{k}[x, y]$, $\gamma_0(x) \neq 0$ and at least one of $\alpha_0(x), \beta_0(x)$ nonzero. Thus

$$W([(\alpha_0(x) + y\Omega_1)z + (\beta_0(x) + y\Omega_2)]^2 - (\gamma_0(x) + y\Omega_3)^2(x^2 + 2x^3)) \geq (1, 0),$$

and so

$$\begin{aligned} (1, 0) &\leq W((\alpha_0(x)z + \beta_0(x))^2 - \gamma_0(x)^2(x^2 + 2x^3)) \\ &= W(\alpha_0(x)^2 z^2 + 2\alpha_0(x)\beta_0(x)z + \beta_0(x)^2 - \gamma_0(x)^2(x^2 + 2x^3)) \\ &= W(\alpha_0(x)^2 Q + 2\alpha_0(x)\beta_0(x)z + (\alpha_0(x)^2(x^2 + x^3) + \beta_0(x)^2 - \gamma_0(x)^2(x^2 + 2x^3))). \end{aligned}$$

Thus

$$W(2\alpha_0(x)\beta_0(x)z + (\alpha_0(x)^2(x^2 + x^3) + \beta_0(x)^2 - \gamma_0(x)^2(x^2 + 2x^3))) \geq (1, 0).$$

But this implies that

$$\alpha_0(x)\beta_0(x) = 0 \quad (73)$$

by (68), and thus

$$\alpha_0(x)^2(x^2 + x^3) + \beta_0(x)^2 - \gamma_0(x)^2(x^2 + 2x^3) = 0. \quad (74)$$

We have that $\alpha_0(x) = 0$ or $\beta_0(x) = 0$ by (73). If $\alpha_0(x) = 0$, then (74) becomes

$$\left(\frac{\beta_0(x)}{\gamma_0(x)}\right)^2 = x^2 + 2x^3,$$

which is not a square in $\mathbf{k}(x)$ by Remark 10.3, giving a contradiction. If $\beta_0(x) = 0$, then (74) becomes

$$\left(\frac{\alpha_0(x)}{\gamma_0(x)}\right)^2 = \frac{x+2}{x+1},$$

again giving a contradiction by Remark 10.3. Thus (72) holds.

Set

$$\mathcal{B} = \{\nu_\beta = [V_2; \nu_\beta(\varphi_\beta) = W(\varphi_\beta)] \mid \varphi_\beta \in \Sigma(V_2)\}.$$

Suppose that $\nu_\beta \in \mathcal{B}$. Then $\nu_\beta = [V_2; \nu_\beta(\varphi_\beta) = W(\varphi_\beta)]$ with $\varphi_\beta = Q + Az + B$ for some $A, B \in K$. Expand

$$U = \varphi_\beta^2 - 2(Az + B)\varphi_\beta + (Az + B)^2 - y^2(x^2 + 2x^3)$$

to see that $\nu_\beta(U) \leq 2\nu_\beta(\varphi_\beta)$, and thus $\nu_\beta(U) < (3, 0)$ by (72). Thus $U \in \tilde{\Sigma}(\mathcal{B})$. Thus $d(\mathcal{B}) = 4$ or $d(\mathcal{B}) = 3$.

Let $\psi \in \Sigma(\mathcal{B})$, and define $V_3 = [\mathcal{B}; V_3(\psi) = W(\psi)]$. Then the second admissible family in $\mathcal{S}$ begins with V_3 . Thus the second jump is

$$s^{(2)}(\mathcal{S}) = \frac{\deg_z \psi}{\deg_z Q} = \frac{3}{2} \text{ or } 2.$$

Thus

$$s^{(1)}(\mathcal{S})s^{(2)}(\mathcal{S}) \leq 4 < 8 = s^{\text{tot}}(\mathcal{S}),$$

so there must be at least one more jump in the construction of $\mathcal{S}$, so that $t \geq 3$.

11. Extensions of associated graded rings and semigroups

In this section, we will consider the conditions of finite generation of extensions of associated graded rings along a valuation and relative finite generation of extensions of valuation semigroups.

In this section, we will have the following assumptions. Suppose that A is a Noetherian local domain which contains an algebraically closed field $\mathbf{k}$ such that $A/m_A \cong \mathbf{k}$. Let K be the quotient field of A , and suppose that ν is a rank 1 valuation of K which dominates A , such that the residue field of the valuation ring of ν is $\mathbf{k}$.

Suppose that S is a subsemigroup of a semigroup T . We say that T is a finitely generated module over S if there exists a finite number of elements $t_1, \dots, t_r$ of T such that

$$T = (t_1 + S) \cup \dots \cup (t_r + S).$$

With our assumptions, $\text{gr}_\nu(A)$ is isomorphic to the semigroup algebra $\mathbf{k}[t^{S^A(\nu)}]$. Thus if $A \rightarrow B$ is an inclusion of domains and ω is an extension of ν to the quotient field of B which is nonnegative on B such that the residue field of ω is $\mathbf{k}$, then $\text{gr}_\omega(B)$ is a finitely generated $\text{gr}_\nu(A)$ -module if and only if $S^B(\omega)$ is a finitely generated module over $S^A(\nu)$.

We have the following immediate corollary of Theorem 5.1.

COROLLARY 11.1. *Suppose that $f(z) \in A[z]$ is unitary and irreducible, that there is a unique extension of ν to a valuation ω of $K[z]/(f(z))$ and that the characteristic p of $\mathbf{k}$ does not divide*

$\deg_z f(z)$. Then $\text{gr}_\omega(A[z]/(f(z)))$ is a finitely generated $\text{gr}_\nu(A)$ -module and $S^{A[z]/(f(z))}(\omega)$ is a finitely generated module over the semigroup $S^A(\nu)$.

The following corollary addresses the case when the extension of valuations is not unique. It is an immediate corollary of Theorem 8.2.

COROLLARY 11.2. *Further suppose that A is a Nagata ring. Suppose that $f(z) \in A[z]$ is unitary, irreducible and separable and ω is a valuation of $K[z]/(f(z))$ which extends ν and that there is no defect in the extension (that is, $\delta(\omega/\nu) = 1$). Then there exists a birational extension A_1 of A which is dominated by ν such that $\text{gr}_\omega(A_1[z]/(f(z)))$ is a finitely generated $\text{gr}_\nu(A_1)$ -module and $S^{A_1[z]/(f(z))}(\omega)$ is a finitely generated module over the semigroup $S^{A_1}(\nu)$.*

If we remove any of the assumptions of Corollary 11.1, then the conclusions of the corollary are false, as is shown in the following three examples. We consider finite extensions $A \rightarrow B$ where A and B are excellent, B is a domain with quotient field L and ω is an extension of ν to L which dominates B .

Example 11.3. There exists a finite extension $A \rightarrow B$ such that ω is the unique extension of ν to $L = \text{QF}(B)$ and p does not divide $[L : K]$ but $\text{gr}_\omega(B)$ is not a finitely generated $\text{gr}_\nu(A)$ -module and $S^B(\omega)$ is not a finitely generated module over the semigroup $S^A(\nu)$.

In particular, the representation of B as a “hypersurface singularity” over A is essential to the conclusions of Theorem 5.1 and Corollary 11.1.

Example 11.4. There exists an extension $A \rightarrow B = A[z]/(f(z))$, where $f(z)$ is unitary and irreducible and p does not divide $\deg_z f(z)$ but the extension ω of ν to a valuation of $L = \text{QF}(B)$ is not unique, such that $\text{gr}_\omega(B)$ is not a finitely generated $\text{gr}_\nu(A)$ -module and $S^B(\omega)$ is not a finitely generated module over the semigroup $S^A(\nu)$.

Example 11.4 shows that the condition that ω is the unique extension of ν is necessary in Theorem 5.1 and Corollary 11.1 and that the birational extension $A \rightarrow A_1$ in the conclusions of Corollary 11.2 is necessary.

Example 11.5. There exists an extension $A \rightarrow B = A[z]/(f(z))$ where $f(z)$ is unitary and irreducible and the extension ω of ν to a valuation of $L = \text{QF}(B)$ is unique but p divides $\deg_z f(z)$, such that $\text{gr}_\omega(B)$ is not a finitely generated $\text{gr}_\nu(A)$ -module and $S^B(\omega)$ is not a finitely generated module over $S^A(\nu)$. In the example, $\delta(\omega/\nu) = 1$.

Example 11.5 shows that the condition that $p \nmid \deg_z f(z)$ is necessary in Corollary 11.1.

In the remainder of this section, we will construct these three examples.

Examples 11.3 and 11.4 will be obtained from Example 9.3 of [CV14], where the field $\mathbf{k}$ is arbitrary. We will make the further restriction that $\mathbf{k}$ is an algebraically closed field of characteristic $p > 2$. Let T be $T = \mathbf{k}[x, y]_{(x, y)}$, a localization of a polynomial ring in two variables, and let R be the subring $R = \mathbf{k}[x^2, xy, y^2]_{(x^2, xy, y^2)}$. Let ω be the rational rank 1 valuation dominating T which is determined by the generating sequence

$$P_0 = x, \quad P_1 = y, \quad P_2 = y^3 - x^5, \quad P_{i+1} = P_i^3 - x^{a_i} P_{i-1} \quad \text{for } i \geq 2,$$

where a_i is even and chosen so that $S^T(\omega)$ is not a finitely generated module over $S^R(\nu)$, where ν is the restriction of ω to the quotient field M of R . Let N be the quotient field of T .

Since the characteristic of $\mathbf{k}$ is not equal to 2, the field N is Galois over M , and the Galois group is generated by the involution σ defined by $\sigma(x) = -x$ and $\sigma(y) = -y$. Given $0 \neq g \in T$, we expand

$$g = \sum \alpha_{i_0, i_1, \dots, i_r} P_0^{i_0} P_1^{i_1} \dots P_r^{i_r}$$

with $\alpha_{i_0, i_1, \dots, i_r} \in \mathbf{k}$, $i_0 \in \mathbf{N}$ and $0 \leq i_j < 3$ for $1 \leq j$, so that

$$\omega(g) = \min\{i_0\omega(P_0) + i_1\omega(P_1) + \dots + i_r\omega(P_r) \mid \alpha_{i_0, i_1, \dots, i_r} \neq 0\}.$$

Then

$$\sigma(g) = \sum \alpha_{i_0, i_1, \dots, i_r} (-1)^{i_0 + i_1 + \dots + i_r} P_0^{i_0} P_1^{i_1} \dots P_r^{i_r},$$

and thus $\omega(\sigma(g)) = \omega(g)$. Since the extensions of a valuation in a finite Galois extension are conjugate [ZS60, VI.7, Corollary 3 to Theorem 12], we have that ω is the unique extension of ν to N .

We now give a direct verification that T is not isomorphic to $R[z]/(f(z))$ for some $f(z) \in R[z]$. This holds since for a maximal ideal m in $R[z]/(f(z))$, we have that

$$\dim_{\mathbf{k}} m/m^2 \geq 3 > 2 = \dim_{\mathbf{k}} m_T/m_T^2.$$

Thus $R \rightarrow T$ gives Example 11.3.

In [CV14, Example 9.4], it is shown that in the natural extension $S \rightarrow T$, where $S = \mathbf{k}[u, v]_{(u, v)}$ and $u = x^2$, $v = y^2$, with valuation μ obtained by restricting ω to the quotient field of S , the semigroup $S^T(\omega)$ is not a finitely generated $S^S(\mu)$ -module. We have a factorization of our extension $S \rightarrow U \rightarrow T$, where $U = \mathbf{k}[x, v]_{(x, v)}$. Now $U \cong S[z]/(z^2 - u)$ and $T \cong U[z]/(z^2 - v)$. Let τ be the restriction of ω to the quotient field L of U .

We must have that $S^U(\tau)$ is not a finitely generated $S^S(\mu)$ -module or $S^T(\omega)$ is not a finitely generated $S^U(\tau)$ -module since $S^T(\omega)$ is not a finitely generated $S^S(\mu)$ -module.

By Corollary 11.1, we necessarily have that either τ is not the unique extension of μ to L or ω is not the unique extension of τ to N , giving Example 11.4.

In [Dut20], a general theory of eigenfunctions for a valuation is developed for 2-dimensional quotient singularities, and a complete characterization is given of when the resulting extension of associated graded rings along the valuation is finite.

We now construct Example 11.5. Let $A = \mathbf{k}[u, v]_{(u, v)}$ with quotient field K , and let ν be the valuation of K dominating A constructed in [CP04, Theorem 7.38] and analyzed in Section 9. Let $f(x) = x^p + ux^{p-1} - u$. It is shown in [CP04, Theorem 7.38] that there is a unique extension of ν to a valuation ω of $L = K[x]/(f(x))$. The extension is immediate, with defect $\delta(\omega/\nu) = p$. Let $B = A[x]/(f(x))$.

We see from the generating sequence $P_0, \dots, P_i, \dots$ recalled at the beginning of Section 9 that $\text{gr}_{\nu}(A) \cong \mathbf{k}[\overline{P}_0, \overline{P}_1, \dots]/I$, where $I = (\overline{P}_1^{p^2} - \overline{P}_0, \overline{P}_i^{p^2} - \overline{P}_0^{p^{2i-2}} \overline{P}_{i-1} \text{ for } i \geq 2)$. It is shown in formulas (35) and (36) of [Cut16b] that

$$U_0 = x, \quad U_1 = v, \quad U_2 = v^p - x$$

and for $j \geq 2$,

$$U_{j+1} = \begin{cases} U_j^p - x^{p^{2j-2}} U_{j-1} & \text{if } j \text{ is odd,} \\ U_j^{p^3} - x^{p^{2j-1}} U_{j-1} & \text{if } j \text{ is even} \end{cases}$$

is a generating sequence for ω in B . Thus $\mathrm{gr}_\nu(B) \cong \mathbf{k}[\overline{U}_0, \overline{U}_1, \dots]/J$, where

$$J = (\overline{U}_1^p - \overline{U}_0, \overline{U}_i^p - \overline{U}_0^{p^{2i-2}} \overline{U}_{i-1} \text{ for } i \geq 2 \text{ odd}, \overline{U}_i^{p^3} - \overline{U}_0^{p^{2i-1}} \overline{U}_{i-1} \text{ for } i \geq 2 \text{ even}).$$

Thus $\overline{U}_n^p = \overline{P}_n$ if n is even and $\overline{U}_n = \overline{P}_n$ if n is odd, and so $\mathrm{gr}_\omega(B)$ is not a finitely generated $\mathrm{gr}_\nu(A)$ -module and $S^\omega(B)$ is not a finitely generated $S^\nu(A)$ -module.

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