The intrinsic stable normal cone

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ABSTRACT

We construct an analog of the intrinsic normal cone of Behrend–Fantechi in the setting of motivic stable homotopy theory. A perfect obstruction theory gives rise to a virtual fundamental class in $\mathcal{E}$ -cohomology for any motivic cohomology theory $\mathcal{E}$; this includes the oriented Chow groups of Barge–Morel and Fasel.

1. Introduction

The various versions of modern enumerative geometry, including Gromov–Witten theory and Donaldson–Thomas theory, are based on two important constructions due to Behrend and Fantechi [BF97]. The first is the construction of the intrinsic normal cone $\mathfrak{C}_Z$ of a Deligne–Mumford (DM) stack Z over some base scheme B . The second, based on the first, is the virtual fundamental class $[Z, [\phi]]^{\text{vir}} \in \text{CH}_r(Z)$ associated to a perfect obstruction theory $[\phi]: E_\bullet \rightarrow \mathbb{L}_{Z/B}$ on Z , with r the virtual rank of $E_\bullet$. In case $r = 0$ and Z is proper over a field k , one has the numerical invariant $\deg_k[Z, [\phi]]^{\text{vir}}$; more generally, one can cut down $[Z, [\phi]]^{\text{vir}}$ to dimension 0 by taking so-called descendants and then taking the degree of the resulting 0-cycle.

A perfect obstruction theory on Z is given by a map $[\phi]: E_\bullet \rightarrow \mathbb{L}_{Z/B}$ in $D^{\text{perf}}(Z)$ such that $E_\bullet$ is locally represented on Z by a two-term complex $F_1 \rightarrow F_0$ in degrees 0, 1 (we use the homological notation) and such that the map $[\phi]$ induces an isomorphism on the sheaf h_0 and a surjection on h_1 .

In case $[\phi]$ admits a global resolution $(F_1 \rightarrow F_0) \rightarrow \mathbb{L}_{Z/B}$, the virtual fundamental class is defined by embedding $\mathfrak{C}_Z$ in the quotient stack $[F^1/F^0]$ ($F^i := \mathbb{V}(F_i)$), pulling back $\mathfrak{C}_Z$ via the quotient map $F^1 \rightarrow [F^1/F^0]$, which gives the subcone $\mathfrak{C}(F_\bullet) \subset F^1$, and then intersecting with the 0-section:

$$[Z, [\phi]]^{\text{vir}} := 0_{F^1}^!([\mathfrak{C}(F_\bullet)]).$$

Here $F^i \rightarrow Z$ is the vector bundle dual to F_i and $[\mathfrak{C}(F_\bullet)]$ is the fundamental class of the closed subscheme $\mathfrak{C}(F_\bullet)$ of F^1 . If one wishes to extend this type of construction to more general cohomology theories, there may be a problem in even defining the fundamental class $[\mathfrak{C}(F_\bullet)]$. For instance, in algebraic cobordism Ω_* , fundamental classes of arbitrary schemes do not exist [Lev03, § 3].

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The main point of this paper is to reinterpret the constructions of the intrinsic normal cone, its fundamental class and the virtual fundamental class associated to a perfect obstruction theory in the setting of motivic homotopy theory. Rather than taking a DM or Artin stack as our basic object, we work in the G -equivariant setting, following the current state of the art in motivic stable homotopy theory, for which unfortunately a suitable theory for stacks is not yet available. We will assume that G is *tame* in the sense of [Hoy17]; this includes the case of a split torus, a finite étale group scheme of order prime to all residue characteristics or a reductive group scheme in characteristic 0. For the full theory, we will also need to assume that the base scheme B is affine and the G -scheme Z carrying the perfect obstruction theory is G -quasi-projective over B .

In spite of these restrictions, we gain a great deal of generality. We construct an “intrinsic stable normal cone” $\mathfrak{C}_Z^{\text{st}}$ for each G -quasi-projective B -scheme Z , with $\mathfrak{C}_Z^{\text{st}}$ defined as an object in the equivariant motivic stable homotopy category $\text{SH}^G(B)$ (see Theorem 3.2 and Definition 3.3). The cone $\mathfrak{C}_Z^{\text{st}}$ carries a fundamental class $[\mathfrak{C}_Z^{\text{st}}]$ in co-homotopy $\mathbb{S}_B^{0,0}(\mathfrak{C}_Z^{\text{st}})$ (Definition 4.3). Moreover, for a perfect obstruction theory $[\phi]: E_\bullet \rightarrow \mathbb{L}_{Z/B}$ on Z , we use $[\mathfrak{C}_Z^{\text{st}}]$ to construct a virtual fundamental class in twisted Borel–Moore homology (see Definition 2.5)

$$[Z, [\phi]]^{\text{vir}} \in \mathbb{S}_B^{\text{B.M.}}(Z, \mathbb{V}(E_\bullet)).$$

As we are not relying on a theory in the setting of stacks, the construction (Definition 7.3) relies on a number of choices; Proposition 7.2 provides the crucial independence of these choices.

If $\mathcal{E} \in \text{SH}^G(B)$ is a motivic ring spectrum (that is, a monoid object in $\text{SH}^G(B)$) with unit map $\epsilon_{\mathcal{E}}: \mathbb{S}_B \rightarrow \mathcal{E}$, applying $\epsilon_{\mathcal{E}}$ to $[\mathfrak{C}_Z^{\text{st}}]$ or $[Z, [\phi]]^{\text{vir}}$ gives us elements

$$\begin{aligned} [\mathfrak{C}_Z^{\text{st}}]_{\mathcal{E}} &\in \mathcal{E}^{0,0}(\mathfrak{C}_Z^{\text{st}}), \\ [Z, [\phi]]_{\mathcal{E}}^{\text{vir}} &\in \mathcal{E}^{\text{B.M.}}(Z, \mathbb{V}(E_\bullet)). \end{aligned}$$

For simplicity, we consider the case $G = \{\text{Id}\}$. If we take $\mathcal{E} = H\mathbb{Z}$, the spectrum representing motivic cohomology, then, suitably interpreted, these classes reduce to the classes defined by Behrend–Fantechi. More generally, if $\mathcal{E}$ is orientable, then

$$\mathcal{E}^{\text{B.M.}}(Z, \mathbb{V}(E_\bullet)) \cong \mathcal{E}_{2r,r}^{\text{B.M.}}(Z)$$

with r the virtual rank of $E_\bullet$. We can thus identify $\mathcal{E}^{0,0}(\mathfrak{C}_Z^{\text{st}})$ and $\mathcal{E}^{\text{B.M.}}(Z, \mathbb{V}(E_\bullet))$ as Borel–Moore $\mathcal{E}$ -homology, giving classes

$$\begin{aligned} [\mathfrak{C}_Z^{\text{st}}]_{\mathcal{E}} &\in \mathcal{E}_{2d_M, d_M}^{\text{B.M.}}(\mathfrak{C}_i), \\ [Z, [\phi]]_{\mathcal{E}}^{\text{vir}} &\in \mathcal{E}_{2r,r}^{\text{B.M.}}(Z). \end{aligned}$$

Here $\mathfrak{C}_i$ is the normal cone of Z for a given closed immersion $i: Z \rightarrow M$ with M smooth over B , d_M is the dimension of M over B (which is the same as the dimension of $\mathfrak{C}_i$ over B) and r is the virtual rank of $E_\bullet$. Besides motivic cohomology, this includes such oriented theories as (homotopy invariant) algebraic K -theory or algebraic cobordism.

If we work with theories $\mathcal{E}$ that are not oriented, the identification of the group carrying the virtual fundamental class becomes more complicated. However, there is an interesting class of theories, the SL-oriented theories, which admit a Thom isomorphism for bundles with a trivial determinant. One such theory is cohomology in the sheaf of Milnor–Witt K -groups (see [Mor12]). The part of this theory corresponding to the Chow groups gives the Barge–Morel theory of *oriented* Chow groups (or Chow–Witt groups)

$$\widetilde{\text{CH}}^n(X) := H^n(X, \mathcal{K}_n^{\text{MW}}),$$

a formula reminiscent of Bloch’s formula relating the classical Chow groups with Milnor K -theory. There are also twisted versions of the oriented Chow groups,

$$\widetilde{\mathrm{CH}}^n(X; L) := H^n(X, \mathcal{K}_n^{\mathrm{MW}}(L))$$

for a line bundle L on X . These formulas for the oriented Chow groups are only valid for smooth X , but one has a straightforward extension to the general case using Borel–Moore homology. The general theory gives us classes

$$\begin{aligned} [\mathfrak{C}_Z^{\mathrm{st}}]_{\mathcal{K}_*^{\mathrm{MW}}} &\in \widetilde{\mathrm{CH}}_{d_M}(\mathfrak{C}_i; i^* \omega_{M/B}^{-1}), \\ [Z, [\phi]]_{\mathcal{K}_*^{\mathrm{MW}}}^{\mathrm{vir}} &\in \widetilde{\mathrm{CH}}_r(Z; \det E_\bullet). \end{aligned}$$

In this setting, the push-forward maps on the oriented Chow groups are restricted to *oriented* proper morphisms. This still allows one to achieve a refinement of the usual Gromov–Witten type invariants in case the given perfect deformation theory $E_\bullet$ not only has virtual rank 0 but also has trivial virtual determinant bundle modulo squares. In this case, we have

$$\deg([Z, [\phi]]_{\mathcal{K}_*^{\mathrm{MW}}}^{\mathrm{vir}}) \in \mathrm{GW}(k),$$

where $\mathrm{GW}(k)$ is the Grothendieck–Witt group of the base field k ; applying the rank homomorphism $\mathrm{GW}(k) \rightarrow \mathbb{Z}$ recovers the classical degree. We hope that this approach will prove useful, for example, in studying the enumerative geometry of real varieties, where one takes the signature rather than the rank to obtain the relevant invariant.

Our approach is essentially formal: our construction uses three ingredients beyond some elementary geometry of normal cones.

1. We have the existence of Grothendieck’s six operations for the equivariant motivic stable homotopy category $\mathrm{SH}^G(-): (\mathbf{Sch}^G/B)^{\mathrm{op}} \rightarrow \mathbf{Tr}$. Here $\mathbf{Tr}$ is the 2-category of triangulated categories. In particular, for each G -vector bundle $V \rightarrow X$, we have the autoequivalence $\Sigma^V: \mathrm{SH}^G(X) \rightarrow \mathrm{SH}^G(X)$.
2. For $X \in \mathbf{Sch}^G/B$, we have the path groupoid $\mathcal{V}^G(X)$ of the G -equivariant K -theory space of X . We need the existence of a natural transformation $\Sigma^-: \mathcal{V}^G(-) \rightarrow \mathrm{Aut}(\mathrm{SH}^G(-))$ extending the map $V \mapsto \Sigma^V$ such that the exceptional push-forward and pull-back for a smooth morphism $f: X \rightarrow Y$ is given by $f^! = \Sigma^{T_{X/Y}} \circ f^*$, $f_! = f_\# \circ \Sigma^{-T_{X/Y}}$, where $f_\#$ is the left adjoint to f^* .
3. We have $\mathbb{A}^1$ -homotopy invariance: for $p: V \rightarrow Z$ an affine space bundle, the co-unit of adjunction $p_! p^! \rightarrow \mathrm{Id}_{\mathrm{SH}^G(Z)}$ is an isomorphism.

Presumably other contravariant functors $\mathbf{Sch}^G/B \rightarrow \mathbf{Tr}$ have these three properties.

A construction of the fundamental class of the normal cone $\mathfrak{C}_{Z \subset M}$ in algebraic cobordism was communicated to us by Parker Lowrey some years ago. Our construction of the fundamental class may be viewed as a generalization of this method; see Example 8.1 for further details. Déglise, Jin and Khan [DJK21] have generalized aspects of the work of Lowrey–Schürg [LS16], constructing fundamental classes of quasi-smooth derived schemes in a motivic stable homotopy category of derived schemes; we expect that there is a suitable dictionary translating between some of their constructions and some of the ones given here. For instance, Khan [Kha19b] has constructed a virtual fundamental class associated to a quasi-smooth morphism $f: \mathcal{X} \rightarrow \mathcal{Y}$ of derived higher stacks. Due to his use of (higher) stacks, Khan’s construction takes place in the framework of the motivic stable homotopy category for the étale topology, which places the virtual fundamental class in a Borel–Moore homology for a theory that satisfies étale descent

and thus is not applicable for getting invariants in more general theories such as those closely related to the Grothendieck–Witt ring.¹

It seems one can restrict Khan’s construction to the setting of derived schemes, for which one does have a good theory comparable to that of $\mathrm{SH}(B)$ for B a scheme; in fact, a central result of Khan [Kha19a] states that for X the underlying classical scheme of a (connective) derived scheme $\mathcal{X}$, the restriction map $\mathrm{SH}(\mathcal{X}) \rightarrow \mathrm{SH}(X)$ is an equivalence. A quasi-smooth map of derived schemes $f: \mathcal{X} \rightarrow \mathcal{Y}$ does give rise to a (relative) perfect obstruction theory on the underlying classical scheme of $\mathcal{X}$, and we expect that Khan’s classes agree with the ones constructed here. However, Schürg [Sch13] has found obstructions for a given perfect obstruction theory to arise this way, so the approach using derived schemes may not always be applicable. These issues of relating the derived theory with the one presented here are being investigated by D’Angelo [D’An20].

In Section 2, we recall the necessary background from motivic homotopy theory. We construct the intrinsic stable normal cone in Section 3 and its fundamental class in Section 4. The construction of the virtual fundamental class in a special case (for a *reduced, normalized representative* of a perfect obstruction theory) is given in Section 5, where the formula is completely analogous to the one of Behrend–Fantechi.

In Section 6, we show how a Jouanolou cover $p_Z: \tilde{Z} \rightarrow Z$ and a perfect obstruction theory $[\phi]: E_\bullet \rightarrow \mathbb{L}_{Z/B}$ give rise to an *induced* perfect obstruction theory $p_Z^![\phi]: p_Z^!E_\bullet \rightarrow \mathbb{L}_{\tilde{Z}/B}$, which admits a reduced, normalized representative.

The general case is handled in Section 7. Roughly speaking, one uses Jouanolou’s trick and $\mathbb{A}^1$ -homotopy invariance properties and the results of Section 6 to reduce the general case to the case handled in Section 5. The main point is to show that the resulting class is independent of the various choices made along the way.

We compare our constructions with those of Behrend–Fantechi and discuss variants in Section 8. We conclude with some explicit computations in Section 9, looking at critical loci and relating the virtual fundamental class of a local complete intersection to the $\mathbb{A}^1$ -local degree defined in [KW19].

2. Background on motivic homotopy theory

We begin by recalling some of the aspects of the six operations for the motivic stable homotopy category. We refer the reader to [Ayo07, CD19, Jar00, MV99, Voe98] for details on the non-equivariant case and [Hoy17] for the extension to the equivariant setting.

Fix a noetherian affine scheme U with a flat, finitely presented linearly reductive group scheme G_0 over U (see [Hoy17, Definition 2.14]).

We fix a quasi-projective U -scheme $B \rightarrow U$ (with trivial G_0 -action) as base scheme and let $G = G_0 \times_U B$. A G -equivariant morphism $q: Y \rightarrow X$ of G -schemes over B is called *G -quasi-projective* if there is a G -vector bundle $V \rightarrow X$ that admits a G -equivariant locally closed immersion $i: Y \rightarrow \mathbb{P}(V)$ of X -schemes. We let $\mathbf{Sch}^G/B$ be the full subcategory of G -schemes over B with objects the G -quasi-projective B -schemes and let $\mathbf{Sm}^G/B$ be the full subcategory of smooth B -schemes in $\mathbf{Sch}^G/B$. We will usually denote the structure morphism for $Z \in \mathbf{Sch}^G/B$

¹*Added in print:* A recent paper by Khan and Ravi, [KR21], and the Ph.D. thesis of Chirantan Chowdhury (2021, Duisburg-Essen) both construct theories of a motivic stable homotopy of stacks that may be suitable for detecting such invariants as the Grothendieck–Witt ring.

by $\pi_Z: Z \rightarrow B$.

For $X \in \mathbf{Sch}^G/B$, we have the category QCoh_X^G of quasi-coherent $\mathcal{O}_X$ -modules with G -action and the full subcategory Coh_X^G of coherent sheaves. We call $\mathcal{F}$ in Coh_X^G locally free if $\mathcal{F}$ is locally free as an $\mathcal{O}_X$ -module. We let $D_G^b(X)$ denote the bounded derived category of coherent G -sheaves, $D_G(X)$ the unbounded derived category of quasi-coherent G -sheaves and $D_G^{\mathrm{perf}}(X)$ the full subcategory of $D_G(X)$ of complexes isomorphic in $D_G(X)$ to a bounded complex of locally free coherent sheaves. Such a complex is called a *perfect complex*.

We will use the homological notation for complexes: for a homological complex $C_\bullet$, the complex $\tau_{\leq n} C_\bullet$ is the quotient complex which is C_m in degree $m < n$, 0 in degree $m > n$ and $C_n/\partial(C_{n+1})$ in degree n .

We will assume that U has the G_0 -resolution property, namely, that each $\mathcal{F} \in \mathrm{Coh}_U^{G_0}$ admits a surjection $\mathcal{E} \rightarrow \mathcal{F}$ from a locally free $\mathcal{E}$ in $\mathrm{Coh}_U^{G_0}$ (see [Hoy17, Definition 2.7]). This implies that the group scheme G over B is *tame* in the sense of [Hoy17, Definition 2.26]. Examples of linearly reductive G_0 such that U has the G_0 -resolution property include

- G_0 finite locally free of order invertible on U ,
- G_0 of multiplicative type and isotrivial,
- U of characteristic 0 and G_0 reductive with isotrivial radical and coradical (for example, G_0 is semisimple).

See [Hoy17, Examples 2.8, 2.16, 2.27].

Hoyois shows [Hoy17, Lemma 2.11] that each $Z \in \mathbf{Sch}^G/B$ has the G -resolution property. In addition, if $Z \in \mathbf{Sch}^G/B$ is affine, then a locally free coherent G -sheaf $\mathcal{F}$ on Z is projective in QCoh_X^G ; see [Hoy17, Lemma 2.17]. This also implies that a complex $E_\bullet$ in $D_G(Z)$ that is locally (on Z_{Zar}) a perfect complex is in fact a perfect complex on Z . Similarly, if $E_\bullet \in D_G(X)$ is locally isomorphic to a complex of locally free coherent sheaves that is 0 in degrees outside a given interval $[a, b]$, then $E_\bullet$ is isomorphic to a complex of coherent locally free G -sheaves on X that is 0 in degrees outside $[a, b]$. Such a complex is called a *perfect complex supported in $[a, b]$* .

For $E \in \mathrm{Coh}_X^G$ locally free, we have the associated vector bundle $p: \mathbb{V}(E) \rightarrow X$, with $\mathbb{V}(E) := \mathrm{Spec}_{\mathcal{O}_X} \mathrm{Sym}^* E$. The G -action on E gives $\mathbb{V}(E)$ a G -action, with $p: \mathbb{V}(E) \rightarrow X$ a G -equivariant morphism.

We will often drop the “ G ” in our notation, speaking of B -morphisms for G -equivariant B -morphisms, vector bundles $V \rightarrow X$ for G -vector bundles, etc.

Let $\mathbf{Tr}$ be the 2-category of triangulated categories. Following [Hoy17, §6, Theorem 6.18], we have the motivic stable homotopy category

$$\mathrm{SH}^G(-): \mathbf{Sch}^G/B^{\mathrm{op}} \rightarrow \mathbf{Tr};$$

for $f: Y \rightarrow X$ in $\mathbf{Sch}^G/B$, we have the exact functor $f^*: \mathrm{SH}^G(X) \rightarrow \mathrm{SH}^G(Y)$ with right adjoint $f_*: \mathrm{SH}^G(Y) \rightarrow \mathrm{SH}^G(X)$ and the exceptional pull-back $f^!: \mathrm{SH}^G(X) \rightarrow \mathrm{SH}^G(Y)$ with left adjoint $f_!: \mathrm{SH}^G(X) \rightarrow \mathrm{SH}^G(Y)$. If f is a smooth morphism, f^* admits the left adjoint $f_\#$. The category $\mathrm{SH}^G(X)$ is a closed symmetric monoidal triangulated category with product denoted by $\wedge_X$, unit 1_X (we often write $\mathbb{S}_B$ for 1_B) and internal Hom $\mathrm{Hom}_X(-, -)$; the functor f^* is symmetric and monoidal, and f_* and $f_!$ satisfy projection formulas, that is, for $f: Y \rightarrow X$, the maps f_* and $f_!$ are $\mathrm{SH}^G(X)$ -module maps, and the same holds for $f_\#$ if f is smooth. There is a natural transformation $\eta_{f!}^f: f_! \rightarrow f_*$ which is an isomorphism if f is proper. See also the earlier

treatments [Ayo07] and [CD19] for the non-equivariant case.

For the pair of adjoint functors $a_! \dashv a^!$, we let $e_a: a_! a^! \rightarrow \text{Id}$ denote the co-unit. For the pair of adjoint functor $a^* \dashv a_*$, we let $u_a: \text{Id} \rightarrow a_* a^*$ denote the unit. We will use the analogous notation for other adjoint pairs, leaving the context to make the meaning clear.

For $p: V \rightarrow X$ a vector bundle with 0-section $s: X \rightarrow V$, we have the V -suspension and $-V$ -suspension operators $\Sigma^{\pm V}: \text{SH}^G(X) \rightarrow \text{SH}^G(X)$ defined as

$$\Sigma^V := p_{\#} \circ s_* = p_{\#} \circ s_!, \quad \Sigma^{-V} := s^! \circ p^*,$$

with Σ^V the left adjoint to Σ^{-V} . These endofunctors are in fact inverse equivalences [Hoy17, Proposition 6.5].

For a locally free sheaf $E \in \text{Coh}_X^G$, we thus have the autoequivalence $\Sigma^{\mathbb{V}(E)}$ of $\text{SH}^G(X)$. Ayoub [Ayo07, Théorème 1.5.18] and Riou [Rio10, Proposition 4.1.1] have shown that for $G = \{\text{Id}\}$, the association $E \mapsto \Sigma^{\mathbb{V}(E)}$ extends to a functor

$$\Sigma^-: \mathcal{V}(X) \rightarrow \text{Aut}(\text{SH}(X)).$$

Here $\mathcal{V}(X)$ is the path groupoid of the K -theory space of X , and $\text{Aut}(\text{SH}(X))$ is the category with objects the autoequivalences of $\text{SH}(X)$ and morphisms the natural isomorphisms.

The functor Σ^- is natural with respect to f^* , $f^!$, f_* and $f_!$ for morphisms $f: Y \rightarrow X$:

$$\Sigma^{f^*\mathbb{V}(E)} \circ f^? \cong f^? \circ \Sigma^{\mathbb{V}(E)}, \quad f^? \circ \Sigma^{f^*\mathbb{V}(E)} \cong \Sigma^{\mathbb{V}(E)} \circ f^?,$$

for $? = *$ or $? = !$. This extends without problem to the equivariant case to give a functor $\Sigma^-: \mathcal{V}^G(X) \rightarrow \text{Aut}(\text{SH}^G(X))$, with $\mathcal{V}^G(X)$ the path groupoid of the G -equivariant K -theory space of X , having the properties listed above.

We write $\Sigma^{\mathbb{V}(E_{\bullet})}$ for the image of a perfect complex $E_{\bullet}$ under this functor.² For each distinguished triangle $E_{\bullet}^1 \rightarrow E_{\bullet} \rightarrow E_{\bullet}^2 \rightarrow$ in $D_G^{\text{perf}}(X)$, there is an isomorphism

$$\Sigma^{\mathbb{V}(E_{\bullet})} \cong \Sigma^{\mathbb{V}(E_{\bullet}^1)} \circ \Sigma^{\mathbb{V}(E_{\bullet}^2)},$$

natural with respect to isomorphisms of distinguished triangles, and for each $E_{\bullet}$, there is an isomorphism $\Sigma^{\mathbb{V}(E_{\bullet}[1])} \cong (\Sigma^{\mathbb{V}(E_{\bullet})})^{-1}$. Thus, if $E_{\bullet}$ is $(E_n \rightarrow \cdots \rightarrow E_m)$, supported in $[m, n]$, then $\Sigma^{\mathbb{V}(E_{\bullet})}$ is canonically isomorphic to $\Sigma^{(-1)^n \mathbb{V}(E_n)} \circ \cdots \circ \Sigma^{(-1)^m \mathbb{V}(E_m)}$.

For $f: Y \rightarrow X$ smooth, there are canonical isomorphisms [Hoy17, Theorem 6.9]

$$f_! \cong f_{\#} \circ \Sigma^{-T_f}, \quad f^! \cong \Sigma^{T_f} \circ f^*,$$

with $T_f \rightarrow Y$ the relative tangent bundle, giving the canonical isomorphism

$$f_! f^! \cong f_{\#} f^*.$$

If $f: V \rightarrow X$ is an affine space bundle over X , then the $\mathbb{A}^1$ -homotopy property shows that the co-unit $e_f: f_{\#} f^* \rightarrow \text{Id}$ of the adjunction $f_{\#} \dashv f^*$ is a natural isomorphism.

Besides the equivariant stable motivic homotopy category, Hoyois has defined an equivariant unstable motivic category $\mathcal{H}_{\bullet}^G(X)$ for $X \in \mathbf{Sch}^G/B$, see [Hoy17, § 5], generalizing the constructions of Morel–Voevodsky [MV99] in the non-equivariant case. There is an infinite T -suspension functor $\Sigma_T^{\infty}: \mathcal{H}_{\bullet}^G(X) \rightarrow \text{SH}^G(X)$; we often simply write $\mathcal{X}$ for $\Sigma_T^{\infty} \mathcal{X}$ when the context makes the

²Many authors, for example [Hoy17], use the notation $\Sigma^{E_{\bullet}}$ for our notation $\Sigma^{\mathbb{V}(E_{\bullet})}$. For example, for X smooth over S and $E = \Omega_{X/S}$, we have $\mathbb{V}(E) = T_{X/S}$, the relative tangent bundle, and the operator denoted by $\Sigma^{\Omega_{X/S}}$ in [Hoy17] will be written here as $\Sigma^{T_{X/S}}$.

meaning clear. The category $\mathcal{H}_\bullet^G(X)$ is a symmetric monoidal category with unit $S_X^0 := X_+$, and the unit $1_X \in \mathrm{SH}^G(X)$ is $\Sigma_T^\infty(S_X^0)$.

The functors f^* and f_* are T -stabilizations of functors $f^*: \mathcal{H}_\bullet(X) \rightarrow \mathcal{H}_\bullet(Y)$ and $f_*: \mathcal{H}_\bullet(Y) \rightarrow \mathcal{H}_\bullet(X)$, with f^* left adjoint to f_* . If $f: Y \rightarrow X$ is smooth, then $f_\#$ is the T -stabilization of $f_\#: \mathcal{H}_\bullet(Y) \rightarrow \mathcal{H}_\bullet(X)$ and is left adjoint to f^* . Similarly, if $i: Y \rightarrow X$ is a closed immersion, the maps $i_* = i_!: \mathrm{SH}^G(Y) \rightarrow \mathrm{SH}^G(X)$ are the T -stabilizations of the unstable i_* .

To give the reader some intuition about the suspension functors, we mention that for $V \rightarrow X$ a vector bundle with 0-section s , the suspension $\Sigma^V(1_X)$ is the stabilization of the *Thom space* $\mathrm{Th}_X(V) := p_\# s_*(1_V) \in \mathcal{H}_\bullet^G(X)$ and that $\mathrm{Th}_X(V)$ is canonically isomorphic to the cofiber of $V \setminus s(X) \rightarrow V$ in $\mathcal{H}_\bullet^G(X)$; that is, $\mathrm{Th}_X(V) \cong V/(V \setminus s(X))$. We will not be needing these facts here.

There are *exchange morphisms* associated to a cartesian diagram

$$\begin{array}{ccc} Z & \xrightarrow{q} & Y \\ g \downarrow & \Delta & \downarrow f \\ W & \xrightarrow{p} & X \end{array}$$

as follows:

1. We have $\mathrm{Ex}(\Delta_*^*): p^* f_* \rightarrow g_* q^*$ defined as the composition

$$p^* f_* \xrightarrow{u_g} g_* g^* p^* f_* = g_* q^* f^* f_* \xrightarrow{e_f} g_* q^*.$$

This is an isomorphism if p is smooth or if f is proper.

2. Suppose that p is smooth. The isomorphism $\mathrm{Ex}(\Delta_\#^*): q_\# g^* \rightarrow f^* p_\#$ is defined as the composition

$$q_\# g^* \xrightarrow{u_p} q_\# g^* p^* p_\# = q_\# q^* f^* p_\# \xrightarrow{e_q} f^* p_\#.$$

3. Suppose that p is smooth. We have $\mathrm{Ex}(\Delta_{\#*}): p_\# g_* \rightarrow f_* q_\#$ defined as the composition

$$p_\# g_* \xrightarrow{u_f} f_* f^* p_\# g_* \xrightarrow{\mathrm{Ex}(\Delta_\#^*)^{-1}} f_* q_\# g^* g_* \xrightarrow{e_g} f_* q_\#.$$

4. Suppose that p is smooth. We have the isomorphism $\mathrm{Ex}(\Delta^!): g^* p^! \rightarrow q^! f^*$ defined as the composition

$$g^* p^! \cong g^* \Sigma^{T_{W/X}} p^* \cong \Sigma^{T_{Z/Y}} g^* p^* \cong \Sigma^{T_{Z/Y}} q^* f^* \cong q^! f^*.$$

5. Suppose that p is smooth. We have $\mathrm{Ex}(\Delta_{!*}): p_! g_* \rightarrow f_* q_!$ defined as the composition

$$p_! g_* \cong p_\# \Sigma^{-T_{W/X}} g_* \cong p_\# g_* \Sigma^{-T_{Z/Y}} \xrightarrow{\mathrm{Ex}(\Delta_{\#*})} f_* q_\# \Sigma^{-T_{Z/Y}} \cong f_* q_!.$$

6. For arbitrary p , we have the base-change isomorphism $\mathrm{Ex}(\Delta_!^*): p^* f_! \rightarrow g_! q^*$ (see [Hoy17, Theorem 6.12]) satisfying $\eta_{!*}^g \circ \mathrm{Ex}(\Delta_!^*) = \mathrm{Ex}(\Delta_*^*) \circ \eta_{!*}^f$. If f is smooth, combining $\mathrm{Ex}(\Delta_!^*)$ with the naturality of Σ^- gives the base-change isomorphism $\mathrm{Ex}(\Delta_\#^*): p^* f_\# \rightarrow g_\# q^*$.

Suppose that we have a closed immersion $i: Z \rightarrow X$ in $\mathbf{Sch}^G/B$, with open complement $j: U \rightarrow X$. This yields the *localization distinguished triangle* in $\mathrm{SH}^G(X)$

$$j_! j^! \xrightarrow{e_j} \mathrm{Id}_{\mathrm{SH}^G(X)} \xrightarrow{u_i} i_* i^* \rightarrow j_! j^! [1]. \quad (2.1)$$

Note that $j_! = j_\#$, $j^! = j^*$ and $i_* = i_!$.

DEFINITION 2.1. Let $\pi_Z: Z \rightarrow B$ be in $\mathbf{Sch}^G/B$. The *Borel–Moore motive* of Z over B is the object $Z_{\text{B.M.}} := \pi_{Z!}(1_Z)$ in $\text{SH}^G(B)$. For $v := \mathbb{V}(E_\bullet)$, we have the twisted Borel–Moore motive $Z(v)_{\text{B.M.}} := \pi_{Z!}(\Sigma^v(1_Z))$. If we need to mention the base scheme B , we write these as $Z/B_{\text{B.M.}}$ and $Z/B(v)_{\text{B.M.}}$, respectively.

Let $p: Z \rightarrow W$ be a morphism in $\mathbf{Sch}^G/B$. For p proper, we have the natural transformation (*proper pull-back*)

$$p^*: \pi_{W!} \rightarrow \pi_{Z!} \circ p^* \quad (2.2)$$

defined as the composition

$$\pi_{W!} \xrightarrow{u_p} \pi_{W!} p_* p^* \xrightarrow{(\eta_{!*}^p)^{-1}} \pi_{W!} p_! p^* \cong \pi_{Z!} \circ p^* .$$

Applying p^* to $\Sigma^{\mathbb{V}(E_\bullet)}(1_W)$ gives the morphism

$$p^*: W(v)_{\text{B.M.}} \rightarrow Z(p^*v)_{\text{B.M.}}$$

in $\text{SH}^G(B)$. One checks easily that $(pq)^* = q^*p^*$ for composable proper morphisms p and q .

Let $f: Z \rightarrow W$ be a smooth morphism in $\mathbf{Sch}^G/B$. We have the natural transformation (*smooth push-forward*)

$$f_*: \pi_{Z!} \Sigma^{T_f} f^* \rightarrow \pi_{W!} \quad (2.3)$$

defined by the composition

$$\pi_{Z!} \Sigma^{T_f} f^* \cong \pi_{W!} f_! f^! \xrightarrow{e_f} \pi_{W!} .$$

One checks that $f \mapsto f_*$ is functorial: for smooth morphisms $f: Z \rightarrow W$, $g: Y \rightarrow Z$, we have

$$(fg)_* \circ \theta_{Y/Z/W} = f_* \circ [g_* \circ (\Sigma^{T_{fg}} \circ f^*)] ,$$

where

$$\theta_{Y/Z/W}: (\pi_{Y!} \Sigma^{T_g} g^*) \circ (\Sigma^{T_f} \circ f^*) \rightarrow \pi_{Y!} \Sigma^{T_{fg}} \circ (fg)^*$$

is the isomorphism induced by the exact sequence $0 \rightarrow T_g \rightarrow T_{fg} \rightarrow g^*T_f \rightarrow 0$. Applying f_* to $\Sigma^{\mathbb{V}(E_\bullet)}(1_W)$ gives the smooth push-forward map

$$f_*: Z(f^*v + T_f)_{\text{B.M.}} \rightarrow W(v)_{\text{B.M.}} .$$

Suppose that a smooth morphism $f: Z \rightarrow W$ admits a section $s: W \rightarrow Z$. We have the canonical isomorphism

$$f_! s_! \cong (fs)_! , \quad (a)$$

which induces an isomorphism on the adjoints

$$s^! f^! \cong (fs)^! . \quad (b)$$

Let $e_s: s_! s^! \rightarrow \text{Id}$, $e_f: f_! f^! \rightarrow \text{Id}$, $e_{fs}: (fs)_!(fs)^! \rightarrow \text{Id}$ be the co-units of adjunction. This gives us the commutative diagram

$$\begin{array}{ccccc} \text{Id} & \cong & f_! s_! s^! f^! & \xrightarrow{f_! e_s f^!} & f_! f^! & \xrightarrow{e_f} & \text{Id} \\ \parallel & & \downarrow (a) \circ (b) & & & & \parallel \\ \text{Id} & \cong & (fs)_! (fs)^! & \xrightarrow{e_{fs}} & & & \text{Id} ; \\ & & & \searrow & & & \\ & & & \text{Id} & & & \end{array}$$

in other words, $f_! e_s f^!$ is a right inverse to e_f .³ Define the natural transformation

$$s_! : \pi_{W!} \rightarrow \pi_{Z!} \circ \Sigma^{T_f} \circ f^* \quad (2.4)$$

as the composition

$$\pi_{W!} \xrightarrow{\pi_{W!} \circ f_! e_s f^!} \pi_{W!} \circ f_! \circ f^! \cong \pi_{W!} \circ f_! \circ \Sigma^{T_f} \circ f^* \cong \pi_{Z!} \circ \Sigma^{T_f} \circ f^*.$$

We call $s_!$ the *Gysin push-forward*.

LEMMA 2.2. (1) *Let $f : Z \rightarrow W$ be a smooth morphism in $\mathbf{Sch}^G/B$ with a section $s : W \rightarrow Z$. Then the composition*

$$\pi_{W!} \xrightarrow{s_!} \pi_{Z!} \circ \Sigma^{T_f} \circ f^* \xrightarrow{f_*} \pi_{W!}$$

is the identity.

(2) *If $f : Z \rightarrow W$ is a vector bundle over W , then $f_* : \pi_{Z!} \circ \Sigma^{T_f} \circ f^* \rightarrow \pi_{W!}$ is an isomorphism, with inverse $s_!$.*

(3) *If $f : Z \rightarrow W$ is an affine space bundle, then $f_* : \pi_{Z!} \circ \Sigma^{T_f} \circ f^* \rightarrow \pi_{W!}$ is an isomorphism.*

Proof. The fact that $e_f \circ (f_! e_s f^!) = \text{Id}$ implies assertion (1).

For assertion (2), it suffices by assertion (1) to show that f_* is an isomorphism. Since $Z \rightarrow W$ is a vector bundle, it follows by $\mathbb{A}^1$ -homotopy invariance that $e_f : f_! f^! = f_{\#} f^* \rightarrow \text{Id}$ is an isomorphism; applying $\pi_{W!} \circ -$ yields assertion (2).

As an affine space bundle is locally (in the Zariski topology) a vector bundle, assertion (3) follows from assertion (2) and the Mayer–Vietoris sequence. $\square$

LEMMA 2.3. *Suppose that we have a cartesian diagram in $\mathbf{Sch}^G/B$*

$$\begin{array}{ccc} Z & \xrightarrow{q} & Y \\ g \downarrow & \Delta & \downarrow f \\ W & \xrightarrow[p]{} & X \end{array}$$

with p proper and f smooth. Then the diagram

$$\begin{array}{ccc} \pi_{Y!} \circ \Sigma^{T_f} \circ f^* & \xrightarrow{q^* \circ (\Sigma^{T_f} \circ f^*)} & \pi_{Z!} \circ q^* \circ \Sigma^{T_f} \circ f^* = \pi_{Z!} \circ \Sigma^{T_g} \circ g^* \circ p^* \\ f_* \downarrow & & \downarrow g_* \circ p^* \\ \pi_{X!} & \xrightarrow[p^*]{} & \pi_{W!} \circ p^* \end{array}$$

commutes. In other words, proper pull-back commutes with smooth push-forward.

Proof. In what follows, we simply write $\simeq$ for isomorphisms that follow from functoriality, such as $\pi_{X!} f_! \cong \pi_{Y!}$, or that follow from the isomorphisms $\Sigma^{T_f} f^* \cong f^!$ or $\Sigma^{T_g} g^* \cong g^!$.

³I am grateful to Frédéric Déglise and Denis-Charles Cisinski for this argument.

We fit a number of diagrams together:

$$\begin{array}{ccc}
 \pi_{Y!}\Sigma^{T_f}f^* & \xrightarrow{\pi_{Y!}\circ u_q} & \pi_{Y!}q_*q^*\Sigma^{T_f}f^* \\
 \downarrow \wr & & \downarrow \wr \\
 \pi_{X!}f!f^! & \xrightarrow{\pi_{X!}f!u_qf^!} & \pi_{X!}f!q_*q^*f^! \\
 \downarrow \wr & & \downarrow \wr \\
 \pi_{X!}p_*g!q^*f^! & & \pi_{X!}p_*g!q^*f^! \\
 \downarrow \wr & & \downarrow \wr \\
 \pi_{X!}p_*g!g^!p^* & & \pi_{X!}p_*g!g^!p^* \\
 \downarrow \wr & & \downarrow \wr \\
 \pi_{X!}p_*e_gp^* & & \pi_{X!}p_*e_gp^* \\
 \downarrow \wr & & \downarrow \wr \\
 \pi_{X!} & \xrightarrow{\pi_{X!}u_p} & \pi_{X!}p_*p^*
 \end{array} \tag{a}$$

$$\begin{array}{ccc}
 \pi_{Y!}q_*q^*\Sigma^{T_f}f^* & \xleftarrow[\sim]{\pi_{Y!}\eta_{!*}^q\Sigma^{T_f}f^*} & \pi_{Y!}q!q^*\Sigma^{T_f}f^* \\
 \downarrow \wr & & \downarrow \wr \\
 \pi_{Y!}q_*q^*f^! & \xleftarrow[\sim]{\pi_{Y!}\eta_{!*}^qf^!} & \pi_{Y!}q!q^*f^! \\
 \downarrow \wr & & \downarrow \wr \\
 \pi_{X!}f!q_*q^*f^! & & \pi_{X!}p!g!q^*f^! \\
 \downarrow \wr & & \downarrow \wr \\
 \pi_{X!}p_*g!q^*f^! & & \pi_{X!}p!g!q^*f^! \\
 \downarrow \wr & & \downarrow \wr \\
 \pi_{X!}p_*g!g^!p^* & & \pi_{X!}p!g!g^!p^* \\
 \downarrow \wr & & \downarrow \wr \\
 \pi_{X!}p_*e_gp^* & & \pi_{X!}p!e_gp^* \\
 \downarrow \wr & & \downarrow \wr \\
 \pi_{X!}p_*p^* & \xleftarrow[\sim]{\pi_{X!}\eta_{!*}^p p^*} & \pi_{X!}p!p^*
 \end{array}$$

$$\begin{array}{ccc}
 \pi_{Y!}q!q^*\Sigma^{T_f}f^* & \xrightarrow{\sim} & \pi_{Z!}\Sigma^{T_g}g^*p^* \\
 \downarrow \wr & & \downarrow \wr \\
 \pi_{Y!}q!q^*f^! & & \pi_{Z!}g^!p^* \\
 \downarrow \wr & & \downarrow \wr \\
 \pi_{X!}p!g!q^*f^! & \xrightarrow[\sim]{\pi_{W!}g!Ex(\Delta^{*!})} & \pi_{W!}g!g^!p^* \\
 \downarrow \wr & & \downarrow \wr \\
 \pi_{X!}p!g!g^!p^* & \xleftarrow[\sim]{\sim \circ Id_{g!g^!p^*}} & \pi_{W!}g!g^!p^* \\
 \downarrow \wr & & \downarrow \wr \\
 \pi_{X!}p!e_gp^* & & \pi_{W!}e_gp^* \\
 \downarrow \wr & & \downarrow \wr \\
 \pi_{X!}p!p^* & \xrightarrow{\sim} & \pi_{W!}p^*
 \end{array} \tag{b1,b2}$$

$$\begin{array}{ccc}
 \pi_{Y!}q!q^*\Sigma^{T_f}f^* & \xrightarrow{\sim} & \pi_{Z!}\Sigma^{T_g}g^*p^* \\
 \downarrow \wr & & \downarrow \wr \\
 \pi_{Y!}q!q^*f^! & & \pi_{Z!}g^!p^* \\
 \downarrow \wr & & \downarrow \wr \\
 \pi_{X!}p!g!q^*f^! & \xrightarrow[\sim]{\pi_{W!}g!Ex(\Delta^{*!})} & \pi_{W!}g!g^!p^* \\
 \downarrow \wr & & \downarrow \wr \\
 \pi_{X!}p!g!g^!p^* & \xleftarrow[\sim]{\sim \circ Id_{g!g^!p^*}} & \pi_{W!}g!g^!p^* \\
 \downarrow \wr & & \downarrow \wr \\
 \pi_{X!}p!e_gp^* & & \pi_{W!}e_gp^* \\
 \downarrow \wr & & \downarrow \wr \\
 \pi_{X!}p!p^* & \xrightarrow{\sim} & \pi_{W!}p^*
 \end{array} \tag{c}$$

These fit together as

$$\begin{array}{ccc}
 \pi_{Y!} \circ \Sigma^{T_f} \circ f^* & \xrightarrow{q^* \circ (\Sigma^{T_f} \circ f^*)} & \pi_{Z!} \circ \Sigma^{T_{Z/W}} \circ g^* \circ p^* \\
 \downarrow f_* & & \downarrow g_* \circ p^* \\
 \pi_{X!} & \xrightarrow{p^*} & \pi_{W!} \circ p^* .
 \end{array}
 \quad \begin{array}{c}
 \text{(b1)} \\
 \text{(a)} \quad \text{(c)} \\
 \text{(b2)}
 \end{array}$$

The four diagrams (a), (b1), (b2) and (c) all commute; this follows from the commutativity of transformations acting on separate parts of a composition of functors, or the naturality of the unit and co-unit of an adjunction, or the fact that the exchange isomorphisms $\text{Ex}(\Delta^{*!})$ and $\text{Ex}(\Delta_{*!})$ are derived from the functoriality of composition for $(-)^*$ and $(-)_*$, combined with units and co-units of various adjunctions. For instance, the commutativity of the lower square in (a) is equivalent to the commutativity of the square

$$\begin{array}{ccc}
 f_{\#} f^* & \xrightarrow{u_q} & f_{\#} q_* q^* f^* \\
 \downarrow e_f & & \downarrow \text{Ex}(\Delta_{\#*}) \\
 & & p_* g_{\#} q^* f^* \\
 & & \downarrow \wr \\
 & & p_* g_{\#} g^* p^* \\
 & & \downarrow e_g \\
 \text{Id} & \xrightarrow{u_p} & p_* p^* .
 \end{array}$$

We fill this in as follows:

$$\begin{array}{ccccc}
 f_{\#} f^* & \xrightarrow{u_q} & f_{\#} q_* q^* f^* & & \\
 \downarrow e_f & \searrow u_p & & \swarrow u_p & \downarrow \text{Ex}(\Delta_{\#*}) \\
 & p_* p^* f_{\#} f^* & \xrightarrow{u_q} & p_* p^* f_{\#} q_* q^* f^* & \\
 & \uparrow \text{Ex}(\Delta_{\#}^*) & & \uparrow \text{Ex}(\Delta_{\#}^*) & \\
 & p_* g_{\#} q^* f^* & \xrightarrow{u_q} & p_* g_{\#} q^* q^* f^* & \xrightarrow{e_q} p_* g_{\#} q^* f^* \\
 & \downarrow \wr & & \downarrow \wr & \\
 & p_* g_{\#} g^* p^* & \xrightarrow{\text{Id}} & p_* g_{\#} g^* p^* & \downarrow e_g \\
 & & & \searrow e_g & \\
 \text{Id} & \xrightarrow{u_p} & p_* p^* & &
 \end{array}
 \quad \begin{array}{c}
 \text{(i)} \\
 \text{(ii)} \\
 \text{(iii)} \\
 \text{(iv)} \\
 \text{(v)} \\
 \text{(vi)}
 \end{array}$$

The commutativity of (i), (iii) and (vi) is obvious, that of (ii) is the definition of $\text{Ex}(\Delta_{\#*})$, and that of (iv) is the standard identity $(e_q \circ q^*) \circ (q^* \circ u_q) = \text{Id}$ for the unit and co-unit of an

adjunction. The commutativity of (v) reduces to that of

$$\begin{array}{ccccc}
 f_{\#}f^* & \xrightarrow{u_p} & p_*p^*f_{\#}f^* & \xleftarrow[\sim]{\text{Ex}(\Delta_{\#}^*)} & p_*g_{\#}q^*f^* \\
 e_f \downarrow & & e_f \downarrow & & \downarrow \wr \\
 \text{Id} & \xrightarrow{u_p} & p_*p^* & \xleftarrow{e_g} & p_*g_{\#}g^*p^* .
 \end{array}$$

The commutativity of the left side is obvious and, using the definition of $\text{Ex}(\Delta_{\#}^*)$, that of the right side reduces to the commutativity of

$$\begin{array}{ccccc}
 g_{\#}q^*f^* & \xleftarrow{\sim} & g_{\#}g_*p^* & \xrightarrow{e_g} & p^* \\
 u_f \downarrow & & & & \uparrow e_f \\
 g_{\#}q^*f^*f_{\#}f^* & \xrightarrow{\sim} & g_{\#}g^*p^*f_{\#}f^* & \xrightarrow{e_g} & p^*f_{\#}f^* .
 \end{array}$$

Filling this in as

$$\begin{array}{ccccc}
 & & g_{\#}g_*p^* & \xrightarrow{e_g} & p^* \\
 & \swarrow \sim & \uparrow \wr & & \uparrow e_f \\
 g_{\#}q^*f^* & \xlongequal{\quad} & g_{\#}q^*f^* & & \\
 u_f \downarrow & \nearrow e_f & & & \\
 g_{\#}q^*f^*f_{\#}f^* & \xrightarrow{\sim} & g_{\#}g^*p^*f_{\#}f^* & \xrightarrow{e_g} & p^*f_{\#}f^* ,
 \end{array}$$

we see that the commutativity follows from the identity $(e_f \circ f^*) \circ (f^* \circ u_f) = \text{Id}$.

The commutativity of the remaining diagrams is much easier to verify, and we leave the details to the reader. $\square$

Remark 2.4. Proper pull-back, smooth push-forward and Gysin push-forward are all compatible with base change in the following sense. Fix a morphism $g: B' \rightarrow B$.

Let $p: Z \rightarrow W$ in $\mathbf{Sch}^G/B$ be proper. Using the base-change isomorphisms $\text{Ex}(\Delta_!^*)$, we see that the proper pull-back morphism $p^*: \pi_W! \rightarrow \pi_Z! \circ p^*$ is natural with respect to pull-back by $g: B' \rightarrow B$. In detail, we have the cartersian square

$$\begin{array}{ccc}
 Z_{B'} & \xrightarrow{g_Z} & Z \\
 p_{B'} \downarrow & & \downarrow p \\
 W_{B'} & \xrightarrow{g_W} & W
 \end{array}$$

with $Z_{B'} = Z \times_B B'$ and $W_{B'} = W \times_B B'$, both considered as objects in $\mathbf{Sch}^G/B'$. The base-change isomorphisms $\text{Ex}(\Delta_!^*)$ gives the commutative diagram

$$\begin{array}{ccc}
 \pi_{W_{B'}!} \circ g_W^* & \xrightarrow{\sim} & g^* \circ \pi_W! \\
 p_{B'}^* \downarrow & & \downarrow p^* \\
 \pi_{Z_{B'}!} \circ p_{B'}^* \circ g_W^* & \xrightarrow{\sim} & g^* \circ \pi_Z! .
 \end{array}$$

For $f: Z \rightarrow W$ smooth in $\mathbf{Sch}^G/B$, inducing $f_{B'}: Z_{B'} \rightarrow W_{B'}$, the base-change isomorphisms $\mathrm{Ex}(\Delta_!^*)$ and $\mathrm{Ex}(\Delta^{*!})$ and the naturality of Σ^- give us the commutative diagram

$$\begin{array}{ccc} \pi_{Z_{B'}!} \circ \Sigma^{T_{f_{B'}}} \circ f_{B'}^* \circ g_W^* & \xrightarrow{\sim} & g^* \circ \pi_{Z!} \circ \Sigma^{T_f} \circ f^* \\ f_{B*} \downarrow & & \downarrow f_* \\ \pi_{W_{B'}!} \circ g_W^* & \xrightarrow{\sim} & g^* \circ \pi_{W!} . \end{array}$$

If $s: W \rightarrow Z$ is a section to a smooth $f: Z \rightarrow W$, we have the induced section $s_{B'}: W_{B'} \rightarrow Z_{B'}$ to $f_{B'}$, and the base-change isomorphisms $\mathrm{Ex}(\Delta_!^*)$ and $\mathrm{Ex}(\Delta^{*!})$ and the naturality of Σ^- give us the commutative diagram

$$\begin{array}{ccc} \pi_{Z_{B'}!} \circ \Sigma^{T_{f_{B'}}} \circ f_{B'}^* \circ g_W^* & \xrightarrow{\sim} & g^* \circ \pi_{Z!} \circ \Sigma^{T_f} \circ f^* \\ s_{B!} \uparrow & & \uparrow s_! \\ \pi_{W_{B'}!} \circ g_W^* & \xrightarrow{\sim} & g^* \circ \pi_{W!} . \end{array}$$

These are all diagrams of functors from $\mathrm{SH}^G(W)$ to $\mathrm{SH}^G(B')$, and all the arrows marked with a “ $\sim$ ” are isomorphisms.

In the next definition, we adapt the notation for a bivariate theory introduced in [DJK21].

DEFINITION 2.5. For $\mathcal{E} \in \mathrm{SH}^G(B)$, $\pi_Z: Z \rightarrow B$ in $\mathbf{Sch}^G/B$, $E_\bullet \in \mathcal{V}^G(Z)$ and $v = \mathbb{V}(E_\bullet)$, define the *twisted Borel–Moore homology* with values in $\mathcal{E}$ as

$$\mathcal{E}_{a,b}^{\mathrm{B.M.}}(Z, v) := \mathrm{Hom}_{\mathrm{SH}^G(B)}(\Sigma^{a,b} Z(v)_{\mathrm{B.M.}}, \mathcal{E}) .$$

To simplify the notation, we write $\mathcal{E}^{\mathrm{B.M.}}(Z, v)$ for $\mathcal{E}_{0,0}^{\mathrm{B.M.}}(Z, v)$.

For an object $\mathcal{F} \in \mathrm{SH}^G(B)$, we have the $\mathcal{E}$ -cohomology

$$\mathcal{E}^{a,b}(\mathcal{F}) := \mathrm{Hom}_{\mathrm{SH}^G(B)}(\mathcal{F}, \Sigma^{a,b} \mathcal{E}) ;$$

for $\mathcal{X} \in \mathcal{H}_\bullet^G(B)$, define $\mathcal{E}^{a,b}(\mathcal{X}) := \mathcal{E}^{a,b}(\Sigma_T^\infty \mathcal{X})$, and for $\mathcal{Y} \in \mathcal{H}^G(B)$, define $\mathcal{E}^{a,b}(\mathcal{Y}) := \mathcal{E}^{a,b}(\mathcal{Y}_+)$. Finally, for $X \in \mathbf{Sm}^G/B$, $E_\bullet \in \mathcal{V}^G(X)$ and $v = \mathbb{V}(E_\bullet)$, define the twisted $\mathcal{E}$ -cohomology

$$\mathcal{E}^{a,b}(X, v) := \mathcal{E}^{a,b}(\pi_{X\#}(\Sigma^{-v}(1_X))) = \mathrm{Hom}_{\mathrm{SH}^G(B)}(\Sigma_T^\infty X_+, \Sigma^{a,b} \Sigma^v \mathcal{E}) .$$

Remark 2.6. Let

$$\begin{array}{ccc} Z' & \xrightarrow{q} & Z \\ g \downarrow & & \downarrow f \\ W' & \xrightarrow{p} & W \end{array}$$

be a cartesian diagram in $\mathbf{Sch}^G/B$ with p and q smooth and with $f: Z \rightarrow W$ a vector bundle. Let $s: W \rightarrow Z$ be a section, and let $t: W' \rightarrow Z'$ be the induced section. Then the diagram

$$\begin{array}{ccc} (Z, f^*v + T_f)_{\mathrm{B.M.}} & \xleftarrow{s_!} & (W, v)_{\mathrm{B.M.}} \\ q^* \uparrow & & \uparrow p_* \\ (Z', q^*(f^*v + T_f) + T_q)_{\mathrm{B.M.}} & \xlongequal{\quad} & (Z', g^*(p^*v + T_p) + T_g)_{\mathrm{B.M.}} \xleftarrow{t_!} (W', p^*v + T_p)_{\mathrm{B.M.}} \end{array}$$

commutes. To see this, we may replace the maps $s_!$ and $t_!$ with their respective inverses f_* and g_* (Lemma 2.2), and then the commutativity follows from the functoriality of smooth push-forward: $p_*g_* = f_*q_*$.

In fact, the Gysin push-forward commutes with smooth push-forward in cartesian squares, without assuming that the smooth maps are vector bundles, but as we do not need this result, we omit the proof.

Remark 2.7. The usual operations of proper push-forward, smooth pull-back and intersection with a section on Borel–Moore homology all follow by applying a proper pull-back p^* , smooth push-forward f_* or Gysin push-forward $s_!$ to morphisms $\Sigma^{**}(-)/B^{\text{B.M.}}(-) \rightarrow \mathcal{E}$. Explicitly, a proper map $p: Z \rightarrow W$ in $\mathbf{Sch}^G/B$ induces the functorial proper push-forward

$$p_*: \mathcal{E}_{a,b}^{\text{B.M.}}(Z, p^*v) \rightarrow \mathcal{E}_{a,b}^{\text{B.M.}}(W, v)$$

defined by $p_* := (p^*)^*$, a smooth map $f: Z \rightarrow W$ in $\mathbf{Sch}^G/B$ induces the functorial smooth pull-back

$$f^*: \mathcal{E}_{a,b}^{\text{B.M.}}(W, v) \rightarrow \mathcal{E}_{a,b}^{\text{B.M.}}(Z, f^*v + T_f)$$

defined by $f^* := (f_*)^*$ and for $f: Z \rightarrow W$ a smooth morphism in $\mathbf{Sch}^G/B$ with a section $s: W \rightarrow Z$, we have the Gysin map

$$s^!: \mathcal{E}_{a,b}^{\text{B.M.}}(Z, f^*v + T_f) \rightarrow \mathcal{E}_{a,b}^{\text{B.M.}}(W, v)$$

defined by $s^! := (s_!)^*$ and satisfying $s^! \circ f^* = \text{Id}$. We hope that the context will enable the reader to distinguish the *maps* p_* , f^* and $s^!$ from the *functors* p_* , f^* and $s^!$.

With this translation, Lemma 2.3 says that on twisted Borel–Moore homology, proper push-forward commutes with smooth pull-back in a cartesian square, and Remark 2.6 says that the Gysin map commutes with smooth pull-back in a cartesian square of vector bundles.

For $\pi_X: X \rightarrow B$ in $\mathbf{Sm}^G/B$, the purity isomorphism $\pi_{X!} \cong \pi_{X\#} \circ \Sigma^{-T_X}$ gives the isomorphism

$$\mathcal{E}_{a,b}^{\text{B.M.}}(X, v) \cong \mathcal{E}^{-a,-b}(X, T_X - v).$$

Finally, for $f: Z \rightarrow X$ a morphism in $\mathbf{Sch}^G/B$ with $X \in \mathbf{Sm}^G/B$ and $\mathcal{E} \in \text{SH}^G(B)$ a commutative ring spectrum (that is, commutative monoid object in $\text{SH}^G(B)$), we have the cap product

$$- \cap f^*(-): \mathcal{E}_{a,b}^{\text{B.M.}}(Z, v) \times \mathcal{E}^{p,q}(W, w) \rightarrow \mathcal{E}_{a-p,b-q}^{\text{B.M.}}(Z, v - f^*w)$$

defined via the monoidal structure

$$\begin{aligned} \text{Hom}_{\text{SH}^G(B)}(\Sigma^{a,b}Z(v)_{\text{B.M.}}, \mathcal{E}) \times \text{Hom}_{\text{SH}^G(B)}(\Sigma_T^\infty W_+, \Sigma^{p,q}\Sigma^w \mathcal{E}) \\ \rightarrow \text{Hom}_{\text{SH}^G(B)}(\Sigma^{a-p,b-q}Z(v)_{\text{B.M.}} \wedge \Sigma^{-w}\Sigma_T^\infty W_+, \mathcal{E} \wedge \mathcal{E}) \end{aligned}$$

followed by the multiplication map $\mathcal{E} \wedge \mathcal{E} \rightarrow \mathcal{E}$, the purity isomorphism

$$Z(v)_{\text{B.M.}} \wedge \Sigma^{-w}\Sigma_T^\infty W_+ \cong Z \times_B W(p_1^*v - p_2^*w + p_2^*T_W)_{\text{B.M.}}$$

and the Gysin map for the graph morphism $\gamma_f: Z \rightarrow Z \times_B W$

$$\gamma_f^!: \mathcal{E}_{a-p,b-q}^{\text{B.M.}}(Z \times_B W, p_1^*v - p_2^*w + p_2^*T_W) \rightarrow \mathcal{E}_{a-p,b-q}^{\text{B.M.}}(Z, v - f^*w).$$

3. The intrinsic stable normal cone

Take $Z \in \mathbf{Sch}^G/B$. Since Z is G -quasi-projective over B , it admits a closed immersion $i: Z \rightarrow M$ in $\mathbf{Sch}^G/B$ with $M \in \mathbf{Sm}^G/B$. As in [BF97], we have the *normal cone* $\mathfrak{C}_i$:

$$\mathfrak{C}_i := \mathrm{Spec}_{\mathcal{O}_X} \left(\bigoplus_{n \geq 0} \mathcal{I}_Z^n / \mathcal{I}_Z^{n+1} \right),$$

where $\mathcal{I}_Z$ is the ideal sheaf of $Z \subset M$. Let $p_i: \mathfrak{C}_i \rightarrow Z$ be the projection and $\sigma_i: \mathfrak{C}_i \rightarrow M$ the composition $i \circ p_i$. As before, we denote the structure morphism for $Y \in \mathbf{Sch}^G/B$ by $\pi_Y: Y \rightarrow B$. For $f: Y \rightarrow Z$ a smooth morphism in $\mathbf{Sch}^G/B$, we often denote the relative tangent bundle T_f by $T_{Y/Z}$ and in case $f = \pi_Y: Y \rightarrow B$ is the (smooth) structure morphism, we may write T_Y instead of $T_{Y/B}$.

LEMMA 3.1. *Suppose that we have closed immersions $i: Z \rightarrow M$ and $i': Z \rightarrow M'$ in $\mathbf{Sch}_B^G$ with $M, M' \in \mathbf{Sm}^G/B$. Then there is a canonical isomorphism*

$$\psi_{i,i'}: \mathfrak{C}_{i'}(\sigma_{i'}^* T_{M'})_{\mathrm{B.M.}} \xrightarrow{\sim} \mathfrak{C}_i(\sigma_i^* T_M)_{\mathrm{B.M.}}.$$

If we have a third closed immersion $i'': Z \rightarrow M''$, then $\psi_{i,i'} \circ \psi_{i',i''} = \psi_{i,i''}$.

Proof. Suppose that we have defined the required isomorphism $\psi_g := \psi_{i,i'}$ for each commutative diagram in $\mathbf{Sch}_B^G$

$$\begin{array}{ccc} Z & & \\ i' \downarrow & \searrow i & \\ M' & \xrightarrow{g} & M \end{array} \quad (3.1)$$

with g a smooth morphism in $\mathbf{Sm}^G/B$ and i, i' closed immersions, satisfying $\psi_{gg'} = \psi_g \circ \psi_{g'}$ for each commutative diagram

$$\begin{array}{ccccc} & & Z & & \\ & i'' \swarrow & \downarrow i' & \searrow i & \\ M'' & \xrightarrow{g'} & M' & \xrightarrow{g} & M. \end{array}$$

For an arbitrary pair $i: Z \rightarrow M, i': Z \rightarrow M'$, we have the closed immersion $(i, i'): Z \rightarrow M \times_B M'$. We set $\psi_{i,i'} := \psi_{p_1} \circ \psi_{p_2}^{-1}$, which solves our problem.

We consider a commutative diagram (3.1). The smooth morphism g induces a smooth morphism $\mathfrak{C}(g): \mathfrak{C}_{i'} \rightarrow \mathfrak{C}_i$, giving the commutative diagram

$$\begin{array}{ccc} \mathfrak{C}_{i'} & \xrightarrow{\mathfrak{C}(g)} & \mathfrak{C}_i \\ \sigma_{i'} \downarrow & & \downarrow \sigma_i \\ M' & \xrightarrow{g} & M. \end{array}$$

The projection $\mathfrak{C}(g)$ makes $\mathfrak{C}_{i'}$ into a torsor over $\mathfrak{C}_i$ for the vector bundle $p_i^* i'^* T_{M'/M}$, giving a canonical identification $T_{\mathfrak{C}_{i'}/\mathfrak{C}_i} \cong \sigma_{i'}^* T_{M'/M}$.

The exact sequence $0 \rightarrow T_{M'/M} \rightarrow T_{M'} \rightarrow g^* T_M \rightarrow 0$ gives the canonical isomorphism

$$\Sigma^{\sigma_{i'}^* T_{M'}} \xrightarrow{\theta_g} \Sigma^{\sigma_{i'}^* T_{M'/M}} \circ \Sigma^{\mathfrak{C}(g)^* \sigma_i^* T_M}.$$

By Lemma 2.2, the smooth push-forward

$$\mathfrak{C}(g)_*: \pi_{\mathfrak{C}_{i'}}! \circ \Sigma^{\sigma_{i'}^* T_{M'/M}} \circ \mathfrak{C}(g)^* \rightarrow \pi_{\mathfrak{C}_i}!$$

is an isomorphism.

Composing $\mathfrak{C}(g)_*$ with the isomorphism Θ_g defined as the composition

$$\begin{aligned} \pi_{\mathfrak{C}_{i'}}! \circ \Sigma^{\sigma_{i'}^* T_{M'}} \circ \mathfrak{C}(g)^* &\xrightarrow[\sim]{\theta_g} \pi_{\mathfrak{C}_{i'}}! \circ \Sigma^{\sigma_{i'}^* T_{M'}/M} \circ \Sigma^{\mathfrak{C}(g)^* \sigma_i^* T_M} \circ \mathfrak{C}(g)^* \\ &\cong \pi_{\mathfrak{C}_{i'}}! \circ \Sigma^{\sigma_{i'}^* T_{M'}/M} \circ \mathfrak{C}(g)^* \circ \Sigma^{\sigma_i^* T_M} \end{aligned}$$

gives us the isomorphism

$$\mathfrak{C}(g)_* \circ \Theta_g : \pi_{\mathfrak{C}_{i'}}! \circ \Sigma^{\sigma_{i'}^* T_{M'}/B} \circ \mathfrak{C}(g)^* \rightarrow \pi_{\mathfrak{C}_i}! \circ \Sigma^{\sigma_i^* T_M}.$$

Evaluating at $1_{\mathfrak{C}_i}$ gives the isomorphism

$$\psi_g : \mathfrak{C}_{i'}(\sigma_{i'}^* T_{M'})_{\text{B.M.}} \rightarrow \mathfrak{C}_i(\sigma_i^* T_M)_{\text{B.M.}}.$$

Suppose that we have another smooth morphism $g' : M'' \rightarrow M'$ and a closed immersion $i'' : Z \rightarrow M''$ with $g' \circ i'' = i'$. Let $\Xi_{g,g'}$ be the isomorphism

$$\begin{aligned} \pi_{\mathfrak{C}_{i''}}! \circ \Sigma^{\sigma_{i''}^* T_{M''}/M} \circ \mathfrak{C}(gg')^* \circ \Sigma^{\sigma_i^* T_M} \\ \xrightarrow{\Xi_{g,g'}} \pi_{\mathfrak{C}_{i''}}! \circ \Sigma^{\sigma_{i''}^* T_{M''}/M'} \circ \mathfrak{C}(g')^* \circ \Sigma^{\sigma_{i'}^* T_{M'}/M} \circ \mathfrak{C}(g)^* \circ \Sigma^{\sigma_i^* T_M} \end{aligned}$$

defined as the composition

$$\begin{aligned} \pi_{\mathfrak{C}_{i''}}! \circ \Sigma^{\sigma_{i''}^* T_{M''}/M} \circ \mathfrak{C}(gg')^* \circ \Sigma^{\sigma_i^* T_M} \\ \xrightarrow{\theta_{g'}} \pi_{\mathfrak{C}_{i''}}! \circ \Sigma^{\sigma_{i''}^* T_{M''}/M'} \circ \Sigma^{\mathfrak{C}(g')^* \sigma_{i'}^* T_{M'}/M} \circ \mathfrak{C}(g')^* \circ \mathfrak{C}(g)^* \circ \Sigma^{\sigma_i^* T_M} \\ \cong \pi_{\mathfrak{C}_{i''}}! \circ \Sigma^{\sigma_{i''}^* T_{M''}/M'} \circ \mathfrak{C}(g')^* \circ \Sigma^{\sigma_{i'}^* T_{M'}/M} \circ \mathfrak{C}(g)^* \circ \Sigma^{\sigma_i^* T_M}. \end{aligned}$$

The functoriality of smooth push-forward gives the identity

$$\mathfrak{C}(gg')_* = \mathfrak{C}(g)_* \circ \mathfrak{C}(g')_* \circ \Xi_{g,g'} \quad (3.2)$$

as maps from $\pi_{\mathfrak{C}_{i''}}! \circ \Sigma^{\sigma_{i''}^* T_{M''}} \circ \mathfrak{C}(gg')^* \circ \Sigma^{\sigma_i^* T_M}$ to $\pi_{\mathfrak{C}_i}! \circ \Sigma^{\sigma_i^* T_M}$.

We then have

$$\begin{aligned} \psi_g \circ \psi_{g'} &= \mathfrak{C}(g)_* \circ \Theta_g \circ \mathfrak{C}(g')_* \circ \Theta_{g'}(1_{\mathfrak{C}_i}) \\ &= \mathfrak{C}(g)_* \circ \mathfrak{C}(g')_* \circ \Theta'_g \circ \Theta_{g'}(1_{\mathfrak{C}_i}) \\ &= \mathfrak{C}(g)_* \circ \mathfrak{C}(g')_* \circ \Xi_{g,g'} \circ \Theta_{gg'}(1_{\mathfrak{C}_i}) \\ &= \mathfrak{C}(gg')_* \circ \Theta_{gg'}(1_{\mathfrak{C}_i}) \\ &= \psi_{gg'}, \end{aligned}$$

where Θ'_g is the isomorphism

$$\begin{aligned} \pi_{\mathfrak{C}_{i''}}! \circ \Sigma^{\sigma_{i''}^* T_{M''}/M'} \circ \mathfrak{C}(g')^* \circ \Sigma^{\sigma_{i'}^* T_{M'}} \circ \mathfrak{C}(g)^* \\ \rightarrow \pi_{\mathfrak{C}_{i''}}! \circ \Sigma^{\sigma_{i''}^* T_{M''}/M'} \circ \mathfrak{C}(g')^* \circ \Sigma^{\sigma_{i'}^* T_{M'}/M} \circ \mathfrak{C}(g)^* \circ \Sigma^{\sigma_i^* T_M} \end{aligned}$$

constructed similarly to Θ_g . All the above identities are easy consequences of the naturality of the functor $\Sigma^- : D_{G,\text{iso}}^{\text{perf}}(-) \rightarrow \text{AutSH}^G(-)$ and (3.2). $\square$

Relying on Lemma 3.1, we can prove the main result of this section: the construction of an analog of the Behrend–Fantechi intrinsic normal cone in the setting of the stable motivic homotopy category.

THEOREM 3.2. For each $Z \in \mathbf{Sch}^G/B$, there is an object $\mathfrak{C}_Z^{\text{st}} \in \text{SH}^G(B)$ satisfying the following:

- (1) Let $i: Z \rightarrow M$ be a closed immersion in $\mathbf{Sch}^G/B$ with M smooth over B . Then there is a canonical isomorphism

$$\alpha_i: \mathfrak{C}_Z^{\text{st}} \xrightarrow{\sim} \mathfrak{C}_i(\sigma_i^* T_M)_{\text{B.M.}} .$$

- (2) For $i': Z \rightarrow M$ another closed immersion in $\mathbf{Sch}^G/B$, we have

$$\alpha_i = \psi_{i,i'} \circ \alpha_{i'} ,$$

where $\psi_{i,i'}: \mathfrak{C}_{i'}(\sigma_{i'}^* T_{M'})_{\text{B.M.}} \xrightarrow{\sim} \mathfrak{C}_i(\sigma_i^* T_M)_{\text{B.M.}}$ is the isomorphism defined in Lemma 3.1.

Proof. Choose a closed immersion $i_0: Z \rightarrow M_0$ in $\mathbf{Sch}^G/B$ with $M_0 \in \mathbf{Sm}^G/B$, which exists by the definition of the category $\mathbf{Sch}^G/B$, and define

$$\mathfrak{C}_Z^{\text{st}} := \mathfrak{C}_{i_0}(\sigma_{i_0}^* T_{M_0})_{\text{B.M.}} .$$

For $i: Z \rightarrow M$ a closed immersion $\mathbf{Sch}^G/B$ with $M \in \mathbf{Sm}^G/B$, Lemma 3.1 gives us the isomorphism $\alpha_i := \psi_{i,i_0}: \mathfrak{C}_Z^{\text{st}} \rightarrow \mathfrak{C}_i(\sigma_i^* T_M)_{\text{B.M.}}$ and shows that if $i': Z \rightarrow M'$ is another closed immersion, the diagram

$$\begin{array}{ccc} \mathfrak{C}_Z^{\text{st}} & \xrightarrow{\alpha_{i'}} & \mathfrak{C}_{i'}(\sigma_{i'}^* T_{M'})_{\text{B.M.}} \\ & \searrow \alpha_i & \downarrow \psi_{i,i'} \\ & & \mathfrak{C}_i(\sigma_i^* T_M)_{\text{B.M.}} \end{array}$$

commutes. □

DEFINITION 3.3. For Z in $\mathbf{Sch}^G/B$, we call $\mathfrak{C}_Z^{\text{st}} \in \text{SH}^G(B)$ the *intrinsic stable normal cone* of Z .

Example 3.4. Suppose that $\pi_Z: Z \rightarrow B$ is smooth over B . Then we may take the identity for $i: Z \rightarrow M$, giving $\mathfrak{C}_i = Z$ and $\mathfrak{C}_Z^{\text{st}} = \pi_{Z!} \circ \Sigma^{T_Z}(1_Z) = \pi_{Z\#}(1_Z) = \Sigma_T^\infty Z_+$ in $\text{SH}^G(B)$.

Remark 3.5. Here is an intuitive justification for our definition of $\mathfrak{C}_Z^{\text{st}}$. Let $V \rightarrow Z$ be a rank r vector bundle on a B -scheme Z , and let $p: C \rightarrow Z$ be a morphism in $\mathbf{Sch}^G/B$. We consider V as a group scheme over Z via fiberwise vector addition.

Suppose that we have a (smooth) morphism $q: C \rightarrow \bar{C}$ over Z that makes C a principal V -bundle over $\bar{C}$; let $\bar{p}: \bar{C} \rightarrow Z$ be the structure morphism, and let $\pi_C: C \rightarrow B$ and $\pi_{\bar{C}}: \bar{C} \rightarrow B$ be the structure morphisms to B . We have a canonical isomorphism $T_{C/\bar{C}} \cong p^*V$, the purity isomorphism $q_! \circ \Sigma^{p^*V} \cong q_\#$ and the homotopy invariance isomorphism $1_{\bar{C}} \cong q_\#(1_C)$. This gives the isomorphism in $\text{SH}^G(B)$

$$\bar{C}_{\text{B.M.}} := \pi_{\bar{C}!}(1_{\bar{C}}) \cong \pi_{\bar{C}!}(q_\#(1_C)) \cong \pi_{\bar{C}!} \circ q_! \circ \Sigma^{p^*V}(1_C) \cong \pi_{C!}(\Sigma^{p^*V}(1_C)) = C(p^*V)_{\text{B.M.}} .$$

Now suppose that we just have an action of V on C over Z , giving us the quotient stack $[V \backslash C]$ with projection $\pi: C \rightarrow [V \backslash C]$. In the category of Artin stacks, π is a principal V -bundle, so we could reasonably *define* the Borel–Moore motive of $[V \backslash C]$ by

$$[V \backslash C]_{\text{B.M.}} := C(p^*V)_{\text{B.M.}} .$$

The cone $p: \mathfrak{C}_i \rightarrow Z$ for a closed immersion $i: Z \rightarrow M$ has the canonical action by i^*T_M , giving the Behrend–Fantechi intrinsic normal cone $\mathfrak{C}_Z := [i^*T_M \backslash \mathfrak{C}_i]$. We should thus define the Borel–Moore motive of $\mathfrak{C}_Z$ by

$$(\mathfrak{C}_Z)_{\text{B.M.}} = [i^*T_M \backslash \mathfrak{C}_i]_{\text{B.M.}} := \mathfrak{C}_i(\sigma_i^* T_M)_{\text{B.M.}} ,$$

which is exactly our definition of $\mathfrak{C}_Z^{\text{st}}$.

LEMMA 3.6. Suppose that we have a closed immersion $i: Z \rightarrow M$ with $M \in \mathbf{Sm}^G/B$ and an affine space bundle $q: V \rightarrow M$, giving the cartesian diagram

$$\begin{array}{ccc} Z' & \xrightarrow{i'} & V \\ q_Z \downarrow & & \downarrow q \\ Z & \xrightarrow{i} & M. \end{array}$$

Let $\mathfrak{C}(q): \mathfrak{C}_{i'} \rightarrow \mathfrak{C}_i$ be the morphism induced by (q_Z, q) , and let

$$\mathfrak{C}^{\text{st}}(q): \mathfrak{C}_{Z'}^{\text{st}} \rightarrow \mathfrak{C}_Z^{\text{st}} \quad (3.3)$$

be defined as the composition

$$\begin{aligned} \mathfrak{C}_{Z'}^{\text{st}} &\xrightarrow{\alpha_{i'}} \pi_{\mathfrak{C}_{i'}!} \circ \Sigma^{\sigma_{i'}^* T_V} (1_{\mathfrak{C}_{i'}}) \\ &\xrightarrow{\sim} \pi_{\mathfrak{C}_{i'}!} \circ \Sigma^{\sigma_{i'}^* q^* V} \circ \Sigma^{\sigma_{i'}^* q^* T_M} \circ \mathfrak{C}(q)^* (1_{\mathfrak{C}_i}) \\ &\xrightarrow{\sim} \pi_{\mathfrak{C}_{i'}!} \circ \Sigma^{\sigma_{i'}^* q^* V} \circ \Sigma^{\mathfrak{C}(q)^* \sigma_i^* T_M} \circ \mathfrak{C}(q)^* (1_{\mathfrak{C}_i}) \\ &\xrightarrow{\sim} \pi_{\mathfrak{C}_{i'}!} \circ \Sigma^{\sigma_{i'}^* q^* V} \circ \mathfrak{C}(q)^* \circ \Sigma^{\sigma_i^* T_M} (1_{\mathfrak{C}_i}) \\ &\xrightarrow{\mathfrak{C}(q)_*} \pi_{\mathfrak{C}_i!} \circ \Sigma^{\sigma_i^* T_M} (1_{\mathfrak{C}_i}) \\ &\xrightarrow{\alpha_i^{-1}} \mathfrak{C}_Z^{\text{st}}, \end{aligned}$$

where the isomorphism θ is induced by the exact sequence

$$0 \rightarrow q^* V \rightarrow T_V \rightarrow q^* T_M \rightarrow 0.$$

Then $\mathfrak{C}^{\text{st}}(q)$ is an isomorphism. Moreover, if we have an extension of our diagram to a cartesian diagram in $\mathbf{Sch}^G/B$

$$\begin{array}{ccc} Z'' & \xrightarrow{i''} & W \\ q'_Z \downarrow & & \downarrow q' \\ Z' & \xrightarrow{i'} & V \\ q_Z \downarrow & & \downarrow q \\ Z & \xrightarrow{i} & M \end{array}$$

with $q': W \rightarrow V$ an affine space bundle, then

$$\mathfrak{C}^{\text{st}}(q \circ q') = \mathfrak{C}^{\text{st}}(q) \circ \mathfrak{C}^{\text{st}}(q').$$

Proof. We use the closed immersion $i': Z' \rightarrow V$ to compute $\mathfrak{C}_{Z'}^{\text{st}}$. The morphism $\mathfrak{C}(q): \mathfrak{C}_{i'} \rightarrow \mathfrak{C}_i$ identifies $\mathfrak{C}_{i'}$ with $\mathfrak{C}_i \times_M V$, and hence by Lemma 2.2, the map $\mathfrak{C}(q)_*$ is an isomorphism, which implies that $\mathfrak{C}^{\text{st}}(q)$ is an isomorphism as well.

The functoriality $\mathfrak{C}^{\text{st}}(q \circ q') = \mathfrak{C}^{\text{st}}(q) \circ \mathfrak{C}^{\text{st}}(q')$ follows the functoriality of smooth push-forward, the naturality of the isomorphisms θ_- and the naturality of $\Sigma^-: \mathcal{V}^G(-) \rightarrow \text{Aut}(\text{SH}^G(-))$, as in the proof of Lemma 3.1. $\square$

4. The fundamental class

The next step in the construction is to define a fundamental class in the co-homotopy of our intrinsic stable normal cone: $[\mathfrak{C}_Z^{\text{st}}] \in \mathbb{S}_B^{0,0}(\mathfrak{C}_Z^{\text{st}})$. We do this by the method of specialization to the normal cone, suitably interpreted.

Choose, as before, a closed immersion $i: Z \rightarrow M$ in $\mathbf{Sch}^G/B$ with $M \in \mathbf{Sm}^G/B$. Let

$$\tilde{\pi}: \widetilde{M \times \mathbb{A}^1} \rightarrow M \times \mathbb{A}^1$$

be the blow-up of $M \times \mathbb{A}^1$ along $Z \times 0$, that is,

$$\widetilde{M \times \mathbb{A}^1} = \text{Proj}_{M \times \mathbb{A}^1} \oplus_{n \geq 0} \mathcal{I}_{Z \times 0}^n.$$

Writing $M \times \mathbb{A}^1 = \text{Spec } \mathcal{O}_M[t]$, the element $t \in \mathcal{I}_{Z \times 0}$, considered as an element of $\oplus_{n \geq 0} \mathcal{I}_{Z \times 0}^n$ of degree 1, gives a G -invariant section of $\mathcal{O}(1)$, which we denote by T . We define

$$\text{Def}(i) := \widetilde{M \times \mathbb{A}^1} \setminus (T = 0) \in \mathbf{Sch}^G/B,$$

so

$$\text{Def}(i) = \text{Spec}_{M \times \mathbb{A}^1} (\oplus_{n \geq 0} \mathcal{I}_{Z \times 0}^n [T^{-1}])_0,$$

where the subscript 0 denotes the subsheaf of homogeneous sections of degree 0. The projection $p: \text{Def}(i) \rightarrow M \times \mathbb{A}^1$ is flat, $p^{-1}(M \times (\mathbb{A}^1 \setminus \{0\}))$ is isomorphic via p to $M \times (\mathbb{A}^1 \setminus \{0\})$, and $p^{-1}(M \times 0) = \mathfrak{C}_i$. Thus $\mathfrak{C}_i$ is a principal effective Cartier divisor on $\text{Def}(i)$, with ideal sheaf $(t)\mathcal{O}_{\text{Def}(i)}$. Let $i_{\mathfrak{C}}: \mathfrak{C}_i \rightarrow \text{Def}(i)$ be the inclusion, and let $j: M \times (\mathbb{A}^1 \setminus \{0\}) \rightarrow \text{Def}(i)$ be the open complement.

The localization triangle (2.1), twisted by $\Sigma^{p^*p_1^*T_{M/B}}$ and pushed forward by $\pi_{\text{Def}(i)!}$, gives us the distinguished triangle in $\text{SH}^G(B)$

$$\pi_{M \times (\mathbb{A}^1 \setminus \{0\})!} \circ \Sigma^{p_1^*T_M} (1_{M \times (\mathbb{A}^1 \setminus \{0\})}) \rightarrow \pi_{\text{Def}(i)!} \circ \Sigma^{p^*p_1^*T_M} (1_{\text{Def}(i)}) \rightarrow \mathfrak{C}_i(\sigma_i^*T_M)_{\text{B.M.}} \xrightarrow{+1}.$$

The isomorphism

$$T_{M \times (\mathbb{A}^1 \setminus \{0\})} \cong p_1^*T_M \oplus p_2^*T_{\mathbb{A}^1 \setminus \{0\}}$$

and the canonical isomorphism $T_{\mathbb{A}^1 \setminus \{0\}} \cong \mathcal{O}_{\mathbb{A}^1 \setminus \{0\}}$ give the isomorphisms

$$\begin{aligned} \pi_{M \times (\mathbb{A}^1 \setminus \{0\})!} \circ \Sigma^{p_1^*T_M} (1_{M \times (\mathbb{A}^1 \setminus \{0\})}) &\cong \pi_{M\#} (1_M) \wedge_B \pi_{\mathbb{A}^1 \setminus \{0\}\#} (\Sigma_T^{-1} 1_{\mathbb{A}^1 \setminus \{0\}}) \\ &\cong \Sigma_T^{-1} M \times (\mathbb{A}^1 \setminus \{0\})_+ \\ &\cong \Sigma_{S^1}^{-1} \Sigma_{\mathbb{G}_m}^{-1} M \times (\mathbb{A}^1 \setminus \{0\})_+. \end{aligned}$$

Our distinguished triangle thus gives us the map in $\text{SH}^G(B)$

$$\partial: \mathfrak{C}_i(\sigma_i^*T_M)_{\text{B.M.}} \rightarrow \Sigma_{\mathbb{G}_m}^{-1} M \times (\mathbb{A}^1 \setminus \{0\})_+.$$

Let

$$\bar{p}_2^M: M \times (\mathbb{A}^1 \setminus \{0\})_+ \rightarrow \mathbb{G}_m$$

be the projection $p_2: M \times (\mathbb{A}^1 \setminus \{0\})_+ \rightarrow (\mathbb{A}^1 \setminus \{0\})_+$ followed by the quotient map $(\mathbb{A}^1 \setminus \{0\})_+ \rightarrow (\mathbb{A}^1 \setminus \{0\}, \{1\}) = \mathbb{G}_m$. This gives us the map

$$\Sigma_{\mathbb{G}_m}^{-1} \bar{p}_2^M \circ \partial: \mathfrak{C}_i(\sigma_i^*T_M)_{\text{B.M.}} \rightarrow \mathbb{S}_B.$$

DEFINITION 4.1. The fundamental class $[\mathfrak{C}_i] \in \mathbb{S}_B^{\text{B.M.}}(\mathfrak{C}_i, \sigma_i^*T_M)$ is the element represented by $\Sigma_{\mathbb{G}_m}^{-1} \bar{p}_2^M \circ \partial$.

If $\mathcal{E}$ is a commutative monoid in $\mathrm{SH}^G(B)$ with unit $\epsilon_{\mathcal{E}}: \mathbb{S}_B \rightarrow \mathcal{E}$, we define the fundamental class $[\mathfrak{C}_i]_{\mathcal{E}} \in \mathcal{E}^{\mathrm{B.M.}}(\mathfrak{C}_i, \sigma_i^* T_M)$ by composing $[\mathfrak{C}_i]$ with $\epsilon_{\mathcal{E}}$.

We have the canonical isomorphism $\alpha_i: \mathfrak{C}_Z^{\mathrm{st}} \rightarrow \mathfrak{C}_i(\sigma_i^* T_M)_{\mathrm{B.M.}}$, giving the map in $\mathrm{SH}^G(B)$

$$[\mathcal{C}_i] \circ \alpha_i: \mathfrak{C}_Z^{\mathrm{st}} \rightarrow \mathbb{S}_B. \quad (4.1)$$

LEMMA 4.2. *The map $[\mathcal{C}_i] \circ \alpha_i: \mathfrak{C}_Z^{\mathrm{st}} \rightarrow \mathbb{S}_B$ is independent of the choice of closed immersion $i: Z \rightarrow M$.*

Proof. We reduce as in the proof of Lemma 3.1 to the case in which we have a commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{i'} & M' \\ & \searrow i & \downarrow g \\ & & M \end{array}$$

with g smooth. The map g induces a smooth morphism $\mathrm{Def}(g): \mathrm{Def}(i') \rightarrow \mathrm{Def}(i)$, giving us the commutative diagram

$$\begin{array}{ccc} \mathrm{Def}(i') & \xrightarrow{p_{M'}} & M' \times \mathbb{A}^1 \\ \mathrm{Def}(g) \downarrow & & \downarrow g \times \mathrm{Id} \\ \mathrm{Def}(i) & \xrightarrow{p_M} & M \times \mathbb{A}^1 \end{array}$$

The restriction of $\mathrm{Def}(g)$ to $\mathfrak{C}_{i'}$ is the map $\mathfrak{C}(g): \mathfrak{C}_{i'} \rightarrow \mathfrak{C}_i$ induced by g .

This gives us the map of distinguished triangles (we suppress the isomorphisms on the suspension operations induced by various exact sequences)

$$\begin{array}{ccc} \pi_{M' \times (\mathbb{A}^1 \setminus \{0\})!} \circ \Sigma^{p_1^* T_{M'}}(1_{M' \times (\mathbb{A}^1 \setminus \{0\})}) & \xrightarrow{(g \times \mathrm{Id})_* \circ \Theta_g} & \pi_{M \times (\mathbb{A}^1 \setminus \{0\})!} \circ \Sigma^{p_1^* T_M}(1_{M \times (\mathbb{A}^1 \setminus \{0\})}) \\ \downarrow & & \downarrow \\ \pi_{\mathrm{Def}(i')!} \circ \Sigma^{p_{M'}^* p_1^* T_{M'}}(1_{\mathrm{Def}(i')}) & \xrightarrow{\mathrm{Def}(g)_* \circ \Theta_g} & \pi_{\mathrm{Def}(i)!} \circ \Sigma^{p_M^* p_1^* T_M}(1_{\mathrm{Def}(i)}) \\ \downarrow & & \downarrow \\ \mathfrak{C}_{i'}(\sigma_{i'}^* T_{M'})_{\mathrm{B.M.}} & \xrightarrow{\psi(g) = \mathfrak{C}(g)_* \circ \Theta_g} & \mathfrak{C}_i(\sigma_i^* T_M)_{\mathrm{B.M.}} \end{array}$$

which in turn gives the commutative diagram

$$\begin{array}{ccc}
 \mathfrak{C}_Z^{\text{st}} & \xrightarrow{\alpha_i} & \mathfrak{C}_i(\sigma_i^* T_M)_{\text{B.M.}} \\
 \alpha_{i'} \downarrow & \searrow \psi_g & \downarrow \partial \\
 \mathfrak{C}_{i'}(\sigma_{i'}^* T_{M'})_{\text{B.M.}} & \xrightarrow{\psi_g} & \mathfrak{C}_i(\sigma_i^* T_M)_{\text{B.M.}} \\
 \partial \downarrow & \searrow \Sigma_{\mathbb{G}_m}^{-1} g \times \text{Id} & \downarrow \partial \\
 \Sigma_{\mathbb{G}_m}^{-1} M' \times (\mathbb{A}^1 \setminus \{0\})_+ & \xrightarrow{\Sigma_{\mathbb{G}_m}^{-1} g \times \text{Id}} & \Sigma_{\mathbb{G}_m}^{-1} M \times (\mathbb{A}^1 \setminus \{0\})_+ \\
 & \searrow \bar{p}_2^{M'} & \downarrow \bar{p}_2^M \\
 & & \mathbb{S}_B,
 \end{array}$$

completing the proof. $\square$

Lemma 4.2 allows us to make the following definition.

DEFINITION 4.3. Let Z be in $\mathbf{Sch}^G/B$. The *fundamental class* $[\mathfrak{C}_Z^{\text{st}}] \in \mathbb{S}_B^{0,0}(\mathfrak{C}_Z^{\text{st}})$ is the element represented by the map $[\mathfrak{C}_i] \circ \alpha_i: \mathfrak{C}_Z^{\text{st}} \rightarrow \mathbb{S}_B$ for any choice of closed immersion $i: Z \rightarrow M$ in $\mathbf{Sch}^G/B$ with M smooth over B .

If $\mathcal{E}$ is a commutative monoid in $\text{SH}^G(B)$ with unit $\epsilon_{\mathcal{E}}: \mathbb{S}_B \rightarrow \mathcal{E}$, we define the fundamental class $[\mathfrak{C}_Z^{\text{st}}]_{\mathcal{E}} \in \mathcal{E}^{0,0}(\mathfrak{C}_Z^{\text{st}})$ by composing $[\mathfrak{C}_Z^{\text{st}}]$ with $\epsilon_{\mathcal{E}}$.

5. The virtual fundamental class for a reduced normalized representative

Let $\mathbb{L}_{Z/B}$ be the relative cotangent complex on $Z \in \mathbf{Sch}^G/B$, and let $[\phi]: E_{\bullet} \rightarrow \mathbb{L}_{Z/B}$ be a perfect obstruction theory on Z . Recall that we use the homological notation for complexes.

If we choose a closed immersion $i: Z \rightarrow M$ in $\mathbf{Sch}^G/B$ with $M \in \mathbf{Sm}^G/B$, we have the explicit model $(\mathcal{I}_Z/\mathcal{I}_Z^2 \xrightarrow{d} i^* \Omega_{M/B})$ for $\tau_{\leq 1} \mathbb{L}_{Z/B}$. Since $E_{\bullet}$ is by definition supported in $[0, 1]$ and Z satisfies the G -resolution property, we have a global resolution, that is, we have a two-term complex $F_{\bullet} := (F_1 \rightarrow F_0)$ of locally free sheaves in Coh_Z^G , and a map of complexes

$$\phi: (F_1 \xrightarrow{d_F} F_0) \rightarrow (\mathcal{I}_Z/\mathcal{I}_Z^2 \xrightarrow{d} i^* \Omega_{M/B})$$

which induces an isomorphism on h_0 and a surjection on h_1 , representing $\tau_{\leq 1}$ applied to $[\phi]: E_{\bullet} \rightarrow \mathbb{L}_{Z/B}$.

DEFINITION 5.1. We call a representative $(F_{\bullet}, \phi)$ of $[\phi]$ as above a *normalized* representative if the maps ϕ_0 and ϕ_1 are surjective. If in addition $F_0 = i^* \Omega_{M/B}$ and ϕ_0 is the identity, we call ϕ *reduced*.

Let $i: Z \rightarrow M$ be a closed immersion in $\mathbf{Sch}^G/B$ with M smooth over B , and let

$$\phi: (F_1 \rightarrow F_0 = i^* \Omega_{M/B}) \rightarrow (\mathcal{I}_Z/\mathcal{I}_Z^2 \rightarrow i^* \Omega_{M/B})$$

be a reduced normalized representative of a perfect obstruction theory $[\phi]$ on Z . The surjection $\phi_1: F_1 \rightarrow \mathcal{I}_Z/\mathcal{I}_Z^2$ induces a surjection of graded algebras $\text{Sym}^* F_1 \rightarrow \oplus_n \mathcal{I}_Z^n/\mathcal{I}_Z^{n+1}$, and thereby a closed immersion of cones over Z

$$i_{\phi}: \mathfrak{C}_i \rightarrow F^1 := \mathbb{V}(F_1).$$

Let $p_{F^1}: F^1 \rightarrow Z$ be the projection, and let $0_{F^1}: Z \rightarrow F^1$ be the 0-section.

The map i_ϕ induces the proper push-forward

$$i_{\phi*}: \mathbb{S}_B^{\text{B.M.}}(\mathfrak{C}_i, \sigma_i^* T_M) \rightarrow \mathbb{S}_B^{\text{B.M.}}(F^1, p_{F^1}^* i_Z^* T_M).$$

As $T_{F^1/Z} = p_{F^1}^*(F^1)$, the 0-section 0_{F^1} gives us the Gysin map

$$0_{F^1}^!: \mathbb{S}_B^{\text{B.M.}}(F^1, p_{F^1}^* i_Z^* T_M) \rightarrow \mathbb{S}_B^{\text{B.M.}}(Z, i^* T_M - F^1).$$

We have the fundamental class $[\mathfrak{C}_i] \in \mathbb{S}_B^{\text{B.M.}}(\mathfrak{C}_i, \sigma_i^* T_M)$ (see Definition 4.1).

DEFINITION 5.2. The *virtual fundamental class* for a reduced normalized representative ϕ of a perfect obstruction theory $[\phi]: E_\bullet \rightarrow \mathbb{L}_{Z/B}$ with respect to a closed immersion i is defined as

$$[Z, \phi, i]^{\text{vir}} := 0_{F^1}^!(i_{\phi*}([\mathfrak{C}_i])) \in \mathbb{S}_B^{\text{B.M.}}(Z, i^* T_M - F^1) = \mathbb{S}_B^{\text{B.M.}}(Z, \mathbb{V}(E_\bullet)).$$

The identity $\mathbb{S}_B^{\text{B.M.}}(Z, i^* T_M - F^1) = \mathbb{S}_B^{\text{B.M.}}(Z, \mathbb{V}(E_\bullet))$ is really the isomorphism induced by the given isomorphism $(F_1 \rightarrow F_0) \cong E_\bullet$ in $D_G^{\text{perf}}(Z)$.

6. Reduced normalized representatives of a perfect obstruction theory

We show how a given perfect obstruction theory $[\phi]: E_\bullet \rightarrow \mathbb{L}_{Z/B}$ on some $Z \in \mathbf{Sch}^G/B$ admits a reduced normalized representative “up to $\mathbb{A}^1$ -homotopy equivalence,” in a way which will be clarified in this section. If Z is affine, $[\phi]$ already admits a reduced normalized representative. In general, we replace Z with a Jouanolou cover $p_Z: \tilde{Z} \rightarrow Z$, and we construct an “induced perfect obstruction theory” $p_Z^![\phi]: p_Z^! E_\bullet \rightarrow \mathbb{L}_{\tilde{Z}/B}$ on $\tilde{Z}$.

As in the previous section, let $\mathbb{L}_{Z/B}$ be the relative cotangent complex on Z , and let $[\phi]: E_\bullet \rightarrow \mathbb{L}_{Z/B}$ be a perfect obstruction theory on Z . Choose a closed immersion $i: Z \rightarrow M$ in $\mathbf{Sch}^G/B$ with $M \in \mathbf{Sm}^G/B$, a two-term complex $F_\bullet := (F_1 \rightarrow F_0)$ of locally free sheaves in Coh_Z^G and a map of complexes

$$\phi: (F_1 \xrightarrow{d_F} F_0) \rightarrow (\mathcal{I}_Z/\mathcal{I}_Z^2 \xrightarrow{d} i^* \Omega_{M/B})$$

which induces an isomorphism on h_0 and a surjection on h_1 , representing $\tau_{\leq 1}$ applied to $[\phi]: E_\bullet \rightarrow \mathbb{L}_{Z/B}$.

By the G -resolution property for Z , there is a locally free sheaf $\mathcal{F}$ on Z with a surjection $p: \mathcal{F} \rightarrow \mathcal{I}_Z/\mathcal{I}_Z^2$. We may then replace $(F_\bullet, \phi)$ with

$$F_1 \oplus \mathcal{F} \xrightarrow{d_F \oplus \text{Id}_{\mathcal{F}}} F_0 \oplus \mathcal{F}$$

and map the copy of $\mathcal{F}$ in degree 0 to $i^* \Omega_{M/B}$ by $d \circ p$, giving the map

$$\phi': (F_1 \oplus \mathcal{F} \xrightarrow{d_F \oplus \text{Id}_{\mathcal{F}}} F_0 \oplus \mathcal{F}) \rightarrow (\mathcal{I}_Z/\mathcal{I}_Z^2 \xrightarrow{d} i^* \Omega_{M/B}).$$

The map $\phi'_1 := \phi + p$ is surjective by construction, and the assumption that $h_0(\phi)$ is an isomorphism implies that ϕ'_0 is surjective as well. Thus, each $[\phi]$ admits a normalized representative.

For a normalized representative $\phi: F_\bullet \rightarrow (\mathcal{I}_Z/\mathcal{I}_Z^2 \xrightarrow{d} i^* \Omega_{M/B})$, we let $K_i \subset F_i$ be the kernel of ϕ_i . We let $F^i \rightarrow Z$ be the vector bundle $\mathbb{V}(F_i) := \text{Spec}_{\mathcal{O}_Z} \text{Sym}^* F_i$ and similarly define $K^i := \mathbb{V}(K_i)$.

LEMMA 6.1. Let $\phi: F_\bullet \rightarrow (\mathcal{I}_Z/\mathcal{I}_Z^2 \xrightarrow{d} i^* \Omega_{M/B})$ be a normalized representative of a perfect obstruction theory $[\phi]$. Let $K(h_1(F_\bullet))$ be the kernel of the surjection $h_1(\phi): h_1(F_\bullet) \rightarrow h_1(\mathbb{L}_{Z/B})$.

Then K_0 is locally free, and in the commutative diagram

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & K(h_1(F_\bullet)) & \longrightarrow & K_1 & \longrightarrow & K_0 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & h_1(F_\bullet) & \longrightarrow & F_1 & \xrightarrow{d_F} & F_0 \longrightarrow h_0(F_\bullet) \longrightarrow 0 \\
 & & \downarrow h_1(\phi) & & \downarrow \phi_1 & & \downarrow \phi_0 & & \downarrow h_0(\phi) \\
 0 & \longrightarrow & h_1(\mathbb{L}_{Z/B}) & \longrightarrow & \mathcal{I}_Z/\mathcal{I}_Z^2 & \xrightarrow{d} & i^*\Omega_{M/B} \longrightarrow h_0(\mathbb{L}_{Z/B}) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0 & & 0,
 \end{array}$$

all the rows and columns are exact.

Proof. Our assumption that ϕ is a normalized representative is just that ϕ_0 , ϕ_1 and $h_1(\phi)$ are surjective and $h_0(\phi)$ is an isomorphism. The rest follows by the snake lemma. $\square$

LEMMA 6.2. Suppose that Z is affine, choose a closed immersion $i: Z \rightarrow M$ in $\mathbf{Sch}^G/B$ with $M \in \mathbf{Sm}^G/B$, and let $[\phi]$ be a perfect obstruction theory on Z . Then $[\phi]$ admits a reduced normalized representative.

Proof. We have already seen that $[\phi]$ admits a normalized representative

$$\phi: (F_1 \rightarrow F_0) \rightarrow (\mathcal{I}_Z/\mathcal{I}_Z^2 \rightarrow i^*\Omega_{M/B}).$$

We use the notation of Lemma 6.1. Since Z is affine, each locally free $\mathcal{F}$ in QCoh_Z^G is a projective object in QCoh_Z^G . Thus, we may choose a splitting $s_K: K_0 \rightarrow K_1$ to the surjection $K_1 \rightarrow K_0$. This gives us the commutative diagram

$$\begin{array}{ccccccc}
 & & i_1 & & & & \\
 & \searrow & \curvearrowright & \searrow & & & \\
 K_0 & \xrightarrow{s_K} & K_1 & \longrightarrow & F_1 & \xrightarrow{\phi_1} & \mathcal{I}_Z/\mathcal{I}_Z^2 \\
 \parallel & & \downarrow & & \downarrow d_F & & \downarrow d \\
 K_0 & \xrightarrow{=} & K_0 & \longrightarrow & F_0 & \xrightarrow{\phi_0} & i^*\Omega_{M/B}.
 \end{array}$$

Replacing F_1 with $F'_1 := F_1/i_1(K_0)$ and F_0 with $i^*\Omega_{M/B} \cong F_0/K_0$, we have the reduced normalized representative

$$(F'_1 \xrightarrow{d_{F'_1}} i^*\Omega_{M/B}) \xrightarrow{(\phi'_1, \mathrm{Id})} (\mathcal{I}_Z/\mathcal{I}_Z^2 \rightarrow i^*\Omega_{M/B})$$

for $[\phi]$. $\square$

LEMMA 6.3. Suppose that Z is affine, and choose a closed immersion $i: Z \rightarrow M$ in $\mathbf{Sch}^G/B$ with $M \in \mathbf{Sm}^G/B$. If $\phi: F_\bullet \rightarrow (\mathcal{I}_Z/\mathcal{I}_Z^2 \rightarrow i^*\Omega_{M/B})$ and $\phi': F'_\bullet \rightarrow (\mathcal{I}_Z/\mathcal{I}_Z^2 \rightarrow i^*\Omega_{M/B})$ are two reduced normalized representatives of a given perfect obstruction theory $[\phi]: E_\bullet \rightarrow \mathbb{L}_{Z/B}$ on Z , then the induced isomorphism $F_\bullet \cong E_\bullet \cong F'_\bullet$ in $D_G^{\mathrm{perf}}(Z)$ arises from an isomorphism of

complexes $\rho_\bullet: F_\bullet \rightarrow F'_\bullet$ making the diagram

$$\begin{array}{ccc} F_\bullet & \xrightarrow{\rho_\bullet} & F'_\bullet \\ & \searrow & \downarrow \\ & & (\mathcal{I}_Z/\mathcal{I}_Z^2 \rightarrow i^*\Omega_{M/B}) \end{array}$$

commute. Moreover, $\rho_\bullet$ is unique up to a chain homotopy of the form $h: i^*\Omega_{M/B} \rightarrow h_1(F'_\bullet) \subset F'_1$ satisfying $d_{F'_\bullet} h = 0 = \phi'_1 h d_{F_\bullet}$.

Proof. We denote the homology of the complex $(\mathcal{I}_Z/\mathcal{I}_Z^2 \rightarrow i^*\Omega_{M/B})$ by h_1, h_0 . As we are given an isomorphism $F_\bullet \cong F'_\bullet$ in $D_G^{\text{perf}}(Z)$, we have isomorphisms $\xi_i: h_i(F_\bullet) \cong h_i(F'_\bullet)$ in Coh_Z^G such that the induced map on the Ext groups

$$\text{Ext}_{\text{Coh}_Z^G}^2(h_0(F_\bullet), h_1(F_\bullet)) \rightarrow \text{Ext}_{\text{Coh}_Z^G}^2(h_0(F'_\bullet), h_1(F'_\bullet))$$

sends the class of $0 \rightarrow h_1(F_\bullet) \rightarrow F_1 \rightarrow F_0 \rightarrow h_0(F_\bullet) \rightarrow 0$ to the class of $0 \rightarrow h_1(F'_\bullet) \rightarrow F'_1 \rightarrow F'_0 \rightarrow h_0(F'_\bullet) \rightarrow 0$.

Let $M_0 \subset i^*\Omega_{M/B}$ be the image subsheaf $d(\mathcal{I}_Z/\mathcal{I}_Z^2)$. The maps ϕ and ϕ' give rise to exact sequences

$$0 \rightarrow h_1(F_\bullet) \rightarrow F_1 \xrightarrow{d_F} M_0 \rightarrow 0, \quad (6.1)$$

$$0 \rightarrow h_1(F'_\bullet) \rightarrow F'_1 \xrightarrow{d_{F'}} M_0 \rightarrow 0 \quad (6.2)$$

and

$$0 \rightarrow M_0 \rightarrow i^*\Omega_{M/B} \rightarrow h_0 \rightarrow 0. \quad (6.3)$$

Thus the Ext^2 classes considered above arise from the Ext^1 -classes of (6.1) and (6.2), respectively, by taking the boundary in the long exact Ext sequence associated to (6.3). Since Z is affine, $\text{Ext}_{\text{Coh}_Z^G}^1(i^*\Omega_{M/B}, -) = 0$, so the isomorphism $h_1(F_\bullet) \cong h_1(F'_\bullet)$ sends the Ext^1 -class of (6.1) to that of (6.2), giving us the isomorphism $\rho_1: F_1 \rightarrow F'_1$ making the diagram

$$\begin{array}{ccccc} h_1(F_\bullet) & \longrightarrow & F_1 & \longrightarrow & M_0 \\ \xi_1 \downarrow & & \rho_1 \downarrow & & \parallel \\ h_1(F'_\bullet) & \longrightarrow & F'_1 & \longrightarrow & M_0 \end{array}$$

commute. This gives us the isomorphism $\rho := (\rho_1, \text{Id}): F_\bullet \rightarrow F'_\bullet$ of perfect complexes.

Since Z is affine and G is tame, Coh_Z^G has enough projectives, so $D^-(\text{Coh}_Z^G)$ is equivalent to the bounded-above homotopy category of the full subcategory of projective objects in Coh_Z^G ; the projective objects are the $\mathcal{F} \in \text{Coh}_Z^G$ which are locally free as $\mathcal{O}_Z$ -modules.

Consider the two morphisms $\phi, \phi' \circ \rho: F_\bullet \rightarrow (\mathcal{I}_Z/\mathcal{I}_Z^2 \rightarrow i^*\Omega_{M/B})$. We have $[\phi] = [\phi' \circ \rho]$ as maps in $D^b(\text{Coh}_Z^G)$, and as $F_\bullet$ is a perfect complex, the maps $\phi, \phi' \circ \rho$ are chain homotopic. Since ϕ'_1 is surjective, for $h: F_0 \rightarrow \mathcal{I}_Z/\mathcal{I}_Z^2$ a chain homotopy, there is an $\tilde{h}: F_0 \rightarrow F'_1$ with $\phi'_1 \circ \tilde{h} = h$. Since $\phi_0 = \text{Id}_{i^*\Omega_{M/B}}$, we have

$$0 = d \circ h = d \circ \phi'_1 \circ \tilde{h} = \text{Id}_{i^*\Omega_{M/B}} \circ d_{F'_\bullet} \circ \tilde{h},$$

so we may consider $\tilde{h}$ as a map $\tilde{h}: i^*\Omega_{M/B} \rightarrow h_1(F'_\bullet)$. Replacing ρ with $\rho' := \rho - d\tilde{h}$, we see that $\rho': F_\bullet \rightarrow F'_\bullet$ satisfies $\phi' \circ \rho' = \phi$ as maps of complexes. Changing notation, we may assume that ρ is an isomorphism satisfying $\phi' \circ \rho = \phi$.

For two isomorphisms $\rho_1, \rho_2: F_\bullet \rightarrow F'_\bullet$ over $\mathbb{L}_{Z/B}$ both representing the same morphism in $D_G^{\text{perf}}(Z)$, the maps $\tau_{\leq 1}\rho_1$ and $\tau_{\leq 1}\rho_2$ are chain homotopic by a map $h: F_0 = i^*\Omega_{M/B} \rightarrow F'_1$. The fact that both $\tau_{\leq 1}\rho_i$ are isomorphisms over $(\mathcal{I}_Z/\mathcal{I}_Z^2 \rightarrow i^*\Omega_{M/B})$ implies the identities $d_{F'_\bullet}h = 0 = \phi'_1 h d_{F_\bullet}$, the first of which shows that $h(F_0) \subset h_1(F'_\bullet)$. $\square$

We return to the case of arbitrary $Z \in \mathbf{Sch}^G/B$ with a closed immersion $i_Z: Z \rightarrow M$ in $\mathbf{Sch}^G/B$ with $M \in \mathbf{Sm}^G/B$. Suppose that we have a smooth morphism $p_M: \tilde{M} \rightarrow M$. Form the cartesian square

$$\begin{array}{ccc} \tilde{Z} & \xrightarrow{i_{\tilde{Z}}} & \tilde{M} \\ p_Z \downarrow & & \downarrow p_M \\ Z & \xrightarrow{i_Z} & M \end{array}$$

Since p_Z is smooth, we have the distinguished triangle in $D_G^{\text{perf}}(\tilde{Z})$

$$p_Z^* \mathbb{L}_{Z/B} \xrightarrow{\mathbb{L}(p_Z)} \mathbb{L}_{\tilde{Z}/B} \rightarrow \Omega_{\tilde{Z}/Z} \rightarrow p_Z^* \mathbb{L}_{Z/B}[1] \quad (6.4)$$

and the isomorphism $\Omega_{\tilde{Z}/Z} \cong i_{\tilde{Z}}^* \Omega_{\tilde{M}/M}$ of locally free sheaves on $\tilde{Z}$. Applying the truncation functor h_0 gives us the exact sequence of sheaves

$$0 \rightarrow p_Z^* \Omega_{Z/B} \rightarrow \Omega_{\tilde{Z}/B} \xrightarrow{\pi} \Omega_{\tilde{Z}/Z} \rightarrow 0.$$

Given a perfect obstruction theory $[\phi]: E_\bullet \rightarrow \mathbb{L}_{Z/B}$, we say that a perfect obstruction theory $[\tilde{\phi}]: \tilde{E}_\bullet \rightarrow \mathbb{L}_{\tilde{Z}/B}$ is *induced from* $[\phi]$ if

1. $\tilde{E}_\bullet = p_Z^* E_\bullet \oplus \Omega_{\tilde{Z}/Z}$,
2. the perfect obstruction theory $[\tilde{\phi}]$ fits into a commutative diagram

$$\begin{array}{ccccc} p_Z^* E_\bullet & \xrightarrow{i} & \tilde{E}_\bullet & \xrightarrow{p} & \Omega_{\tilde{Z}/Z} \\ p_Z^* [\phi] \downarrow & & \downarrow [\tilde{\phi}] & & \parallel \\ p_Z^* \mathbb{L}_{Z/B} & \xrightarrow{\mathbb{L}(p_Z)} & \mathbb{L}_{\tilde{Z}/B} & \longrightarrow & \Omega_{\tilde{Z}/Z} \end{array}$$

with i and p the canonical inclusion and projection.

Since $h_0([\phi]): h_0(E_\bullet) \rightarrow \Omega_{Z/B}$ and $h_0([\tilde{\phi}]): h_0(\tilde{E}_\bullet) \rightarrow \Omega_{\tilde{Z}/B}$ are both isomorphisms, a necessary condition for an induced obstruction theory to exist is that the surjection $\Omega_{\tilde{Z}/B} \rightarrow \Omega_{\tilde{Z}/Z}$ should split.

LEMMA 6.4. *Suppose that $\tilde{Z}$ is affine. Then for each perfect obstruction theory $[\phi]$ on Z , there exists an induced obstruction theory $[\tilde{\phi}]: \tilde{E}_\bullet \rightarrow \mathbb{L}_{\tilde{Z}/B}$ on $\tilde{Z}$.*

Moreover, $[\tilde{\phi}]$ is unique up to a canonical isomorphism $\alpha: (\tilde{E}, [\tilde{\phi}]) \simeq (\tilde{E}, [\tilde{\phi}'])$ of obstruction theories.

Proof. Since $\tilde{Z} = \text{Spec } A$ is affine and G is tame, $\text{Hom}_{D_G^{\text{perf}}(\tilde{Z})}(\Omega_{\tilde{Z}/Z}, p_Z^* \mathbb{L}_{Z/B}[1]) = 0$. This gives us a splitting $\tilde{s}$ of the distinguished triangle

$$p_Z^* \mathbb{L}_{Z/B} \rightarrow \mathbb{L}_{\tilde{Z}/B} \xrightarrow{\pi} \Omega_{\tilde{Z}/Z} \rightarrow p_Z^* \mathbb{L}_{Z/B}[1]$$

and thereby an isomorphism $\mathbb{L}_{\tilde{Z}/B} \cong p_Z^* \mathbb{L}_{Z/B} \oplus \Omega_{\tilde{Z}/Z}$ in $D_G^{\text{perf}}(\tilde{Z})$. From this, we see that each perfect obstruction theory $[\phi]: E_\bullet \rightarrow \mathbb{L}_{Z/B}$ admits an induced perfect obstruction theory of the form

$$[\tilde{\phi}_{\tilde{s}}] := [\phi] \oplus \text{Id}_{\Omega_{\tilde{Z}/Z}}: p_Z^* E_\bullet \oplus \Omega_{\tilde{Z}/Z} \rightarrow p_Z^* \mathbb{L}_{Z/B} \oplus \Omega_{\tilde{Z}/Z} \cong \mathbb{L}_{\tilde{Z}/B}$$

with the isomorphism depending on the choice of a splitting $\tilde{s}$ of the distinguished triangle.

We have the distinguished triangle

$$\tau_{\geq 1} \mathbb{L}_{\tilde{Z}/B} \rightarrow \mathbb{L}_{\tilde{Z}/B} \rightarrow h_0 \mathbb{L}_{\tilde{Z}/B} \rightarrow (\tau_{\geq 1} \mathbb{L}_{\tilde{Z}/B})[1].$$

Since $\tilde{Z}$ is affine, $\Omega_{\tilde{Z}/Z}$ is locally free and $h_n \tau_{\geq 1} \mathbb{L}_{\tilde{Z}/B} = 0$ for $n \leq 0$ and for $n \gg 0$, we have $\text{Hom}_{D_G^{\text{perf}}(\tilde{Z})}(\Omega_{\tilde{Z}/Z}, \tau_{\geq 1} \mathbb{L}_{\tilde{Z}/B}[i]) = 0$ for all $i \geq 0$, so each map of sheaves

$$s: \Omega_{\tilde{Z}/Z} \rightarrow h_0 \mathbb{L}_{\tilde{Z}/B} = \Omega_{\tilde{Z}/B}$$

lifts uniquely to a map $\tilde{s}: \Omega_{\tilde{Z}/Z} \rightarrow \mathbb{L}_{\tilde{Z}/B}$ in $D_G^{\text{perf}}(\tilde{Z})$. This shows that splittings of the map π are in bijection with splittings of the surjection $\Omega_{\tilde{Z}/B} \rightarrow \Omega_{\tilde{Z}/Z}$.

Since $[\phi]$ is a perfect obstruction theory, we also have $h_0(p_Z^* E_\bullet) = p_Z^* \Omega_{Z/B}$, and the same argument as above shows that a second splitting s' of $\Omega_{\tilde{Z}/B} \rightarrow \Omega_{\tilde{Z}/Z}$ determines a map $\lambda: \Omega_{\tilde{Z}/Z} \rightarrow p_Z^* E_\bullet$. The map $\alpha: p_Z^* E_\bullet \oplus \Omega_{\tilde{Z}/Z} \rightarrow p_Z^* E_\bullet \oplus \Omega_{\tilde{Z}/Z}$ with matrix

$$\alpha = \begin{pmatrix} \text{Id} & \lambda \\ 0 & \text{Id} \end{pmatrix}$$

thus defines an automorphism of $p_Z^* E_\bullet \oplus \Omega_{\tilde{Z}/Z}$ intertwining $[\tilde{\phi}_{\tilde{s}}]$ and $[\tilde{\phi}_{s'}]$. $\square$

Assuming that $\tilde{Z}$ is affine as above, we write $p_Z^![\phi]$ for the (unique up to canonical isomorphism) perfect obstruction theory induced by a given perfect obstruction theory $[\phi]$.

Remark 6.5 (Jouanolou covers). Hoyois [Hoy17, Proposition 2.20] has shown that the Jouanolou trick extends to the equivariant case: for each $M \in \mathbf{Sch}^G/B$, there is an affine space bundle $\tilde{M} \rightarrow M$ in $\mathbf{Sch}^G/B$ such that $\pi_{\tilde{M}}: \tilde{M} \rightarrow B$ is an affine morphism. We call such a map $\tilde{M} \rightarrow M$ a *Jouanolou cover* of M . For affine B , Lemma 3.6 will thus enable us to reduce various constructions to the case of affine $Z \in \mathbf{Sch}^G/B$.

We now assume that B is affine. By Remark 6.5, for each $Z \in \mathbf{Sch}^G/B$ and each closed immersion $Z \rightarrow M$ with $M \in \mathbf{Sch}^G/B$, there is an affine space bundle $\tilde{M} \rightarrow M$ with $\tilde{M}$ affine, and thus $\tilde{Z} := Z \times_M \tilde{M}$ is affine as well.

LEMMA 6.6. *Suppose that B is affine and that we have a closed immersion $i_Z: Z \rightarrow M$ with M in $\mathbf{Sch}^G/B$. Let $\tilde{M} \rightarrow M$ be a Jouanolou cover of M , and form the cartesian square*

$$\begin{array}{ccc} \tilde{Z} & \xrightarrow{i_{\tilde{Z}}} & \tilde{M} \\ p_Z \downarrow & & \downarrow p_M \\ Z & \xrightarrow{i_Z} & M \end{array}$$

Let $(E_\bullet, [\phi])$ be a perfect obstruction theory on Z . Then an induced obstruction theory $p_Z^![\phi]$ on $\tilde{Z}$ exists, is unique up to canonical isomorphism and admits a reduced normalized representative $\tilde{\phi}$ for $p_Z^![\phi]$.

Proof. Since $\tilde{Z}$ is affine, this follows from Lemmas 6.2 and 6.4. $\square$

7. The virtual fundamental class: The general case

In this section, we assume that B is affine.

We use our construction in Section 5 of the virtual fundamental class for a reduced normalized representative together with the results of Section 6 to give a well-defined virtual fundamental class for a perfect obstruction theory on a general $Z \in \mathbf{Sch}^G/B$.

Let $[\phi]: E \rightarrow \mathbb{L}_{Z/B}$ be a perfect obstruction theory on some $Z \in \mathbf{Sch}^G/B$. Choose a closed immersion $i: Z \rightarrow M$ with $M \in \mathbf{Sm}^G/B$, and take a Jouanolou cover

$$\begin{array}{ccc} \tilde{Z} & \xrightarrow{i_{\tilde{Z}}} & \tilde{M} \\ p_Z \downarrow & & \downarrow p_M \\ Z & \xrightarrow{i_Z} & M. \end{array}$$

By Lemma 6.6, we have an induced perfect obstruction theory $(\tilde{E}_\bullet, [\tilde{\phi}]) := (p_Z^! E_\bullet, p_Z^! [\phi])$ on $\tilde{Z}$ and a reduced normalized representative $\tilde{\phi}: \tilde{F}_\bullet \rightarrow \tau_{\leq 1} \mathbb{L}_{\tilde{Z}/B}$ for $[\tilde{\phi}]$. In particular, we have the identity

$$\tilde{E}_\bullet = p_Z^* E_\bullet \oplus \Omega_{\tilde{Z}/Z}$$

inducing an isomorphism of suspension operators

$$\Sigma^{p_Z^* \mathbb{V}(E_\bullet) + T_{\tilde{Z}/Z}} \cong \Sigma^{\mathbb{V}(\tilde{E}_\bullet)}.$$

The quasi-isomorphism $\tilde{F}_\bullet \rightarrow \tilde{E}_\bullet$ and the fact that $\tilde{F}_\bullet$ is reduced and normalized give us a canonical isomorphism

$$\Sigma^{i_Z^* T_{\tilde{M}} - \tilde{F}^1} \cong \Sigma^{\mathbb{V}(\tilde{E}_\bullet)}.$$

Since $p_Z: \tilde{Z} \rightarrow Z$ is an affine space bundle, homotopy invariance implies that the smooth push-forward map

$$p_{Z*}: \pi_{\tilde{Z}!} \circ \Sigma^{p_Z^* \mathbb{V}(E_\bullet) + T_{\tilde{Z}/Z}} \circ p_Z^* \rightarrow \pi_{Z!} \circ \Sigma^{\mathbb{V}(E_\bullet)}$$

is an isomorphism. Applying this to 1_Z , using the isomorphisms

$$\Sigma^{p_Z^* \mathbb{V}(E_\bullet) + T_{\tilde{Z}/Z}} \cong \Sigma^{\mathbb{V}(\tilde{E}_\bullet)} \cong \Sigma^{i_Z^* T_{\tilde{M}} - \tilde{F}^1}$$

described above and taking the inverse gives us the isomorphism

$$\vartheta_{i_Z, p_Z, \tilde{F}_\bullet}^*: Z(\mathbb{V}(E_\bullet))_{\text{B.M.}} \rightarrow \tilde{Z}(i_Z^* T_{\tilde{M}} - \tilde{F}^1)_{\text{B.M.}}. \quad (7.1)$$

We have the virtual fundamental class $[\tilde{Z}, \tilde{\phi}, i_{\tilde{Z}}]^{\text{vir}} \in \mathbb{S}_B^{\text{B.M.}}(\tilde{Z}, i_Z^* T_{\tilde{M}} - \tilde{F}^1)$, see Definition 5.2, giving us the class

$$\vartheta_{i_Z, p_Z, \tilde{F}_\bullet}^*([\tilde{Z}, \tilde{\phi}, i_{\tilde{Z}}]^{\text{vir}}) \in \mathbb{S}_B^{\text{B.M.}}(Z, \mathbb{V}(E_\bullet)).$$

The main point is that this class is independent of the various choices made. This is proven with the help of the following two results.

LEMMA 7.1. Let $p_M: \tilde{M} \rightarrow M$ be a Jouanolou cover, and let $q_M: \hat{M} \rightarrow \tilde{M}$ be a vector bundle. Form the cartesian diagram

$$\begin{array}{ccc}
 \hat{Z} & \xrightarrow{i_{\hat{Z}}} & \hat{M} \\
 q_Z \downarrow & & \downarrow q_M \\
 \tilde{Z} & \xrightarrow{i_{\tilde{Z}}} & \tilde{M} \\
 p_Z \downarrow & & \downarrow p_M \\
 Z & \xrightarrow{i_Z} & M
 \end{array}
 \quad \begin{array}{c} \hat{p}_Z \quad \hat{p}_M \end{array}
 \quad (7.2)$$

Let $[\phi]: E_\bullet \rightarrow \mathbb{L}_{Z/B}$ be a perfect obstruction theory, form the induced obstruction theories $(p_Z^! E_\bullet, p_Z^! [\phi])$ and $(\hat{p}_Z^! E_\bullet, \hat{p}_Z^! [\phi])$, and let

$$\tilde{\phi}: \tilde{F}_\bullet \rightarrow (\mathcal{I}_{\tilde{Z}}/\mathcal{I}_{\tilde{Z}}^2 \xrightarrow{d} i_{\tilde{Z}}^* \Omega_{\tilde{M}/B}), \quad \hat{\phi}: \hat{F}_\bullet \rightarrow (\mathcal{I}_{\hat{Z}}/\mathcal{I}_{\hat{Z}}^2 \xrightarrow{d} i_{\hat{Z}}^* \Omega_{\hat{M}/B})$$

be reduced normalized representatives of $p_Z^! [\phi]$ and $\hat{p}_Z^! [\phi]$, respectively. We have the isomorphisms of Lemma 3.6,

$$\mathfrak{C}^{\text{st}}(p_M): \mathfrak{C}_{\tilde{Z}}^{\text{st}} \rightarrow \mathfrak{C}_Z^{\text{st}}, \quad \mathfrak{C}^{\text{st}}(\hat{p}_M): \mathfrak{C}_{\hat{Z}}^{\text{st}} \rightarrow \mathfrak{C}_Z^{\text{st}}.$$

The respective 0-sections $0_{\tilde{F}^1}: \tilde{Z} \rightarrow \tilde{F}^1$ and $0_{\hat{F}^1}: \hat{Z} \rightarrow \hat{F}^1$ induce the Gysin push-forward maps $0_{\tilde{F}^1!}$ and $0_{\hat{F}^1!}$, and the respective closed immersions $i_{\tilde{\phi}}: \mathfrak{C}_{i_{\tilde{Z}}} \rightarrow \tilde{F}^1$ and $i_{\hat{\phi}}: \mathfrak{C}_{i_{\hat{Z}}} \rightarrow \hat{F}^1$ induce the proper pull-back maps $i_{\tilde{\phi}}^*$ and $i_{\hat{\phi}}^*$. We have as well the isomorphisms $\alpha_{i_{\tilde{Z}}}: \mathfrak{C}_{\tilde{Z}}^{\text{st}} \rightarrow \mathfrak{C}_{i_{\tilde{Z}}}(\sigma_i^* T_{\tilde{M}})_{\text{B.M.}}$ and $\alpha_{i_{\hat{Z}}}: \mathfrak{C}_{\hat{Z}}^{\text{st}} \rightarrow \mathfrak{C}_{i_{\hat{Z}}}(\sigma_i^* T_{\hat{M}})_{\text{B.M.}}$.

Define maps $s_{\tilde{\phi}}: \tilde{Z}(i_{\tilde{Z}}^* T_{\tilde{M}} - \tilde{F}^1)_{\text{B.M.}} \rightarrow \mathfrak{C}_{\tilde{Z}}^{\text{st}}$ and $s_{\hat{\phi}}: \hat{Z}(i_{\hat{Z}}^* T_{\hat{M}} - \hat{F}^1)_{\text{B.M.}} \rightarrow \mathfrak{C}_{\hat{Z}}^{\text{st}}$ by

$$s_{\tilde{\phi}} := \alpha_{i_{\tilde{Z}}}^{-1} \circ i_{\tilde{\phi}}^* \circ 0_{\tilde{F}^1!} \quad \text{and} \quad s_{\hat{\phi}} := \alpha_{i_{\hat{Z}}}^{-1} \circ i_{\hat{\phi}}^* \circ 0_{\hat{F}^1!}. \quad (7.3)$$

Suppose that $\hat{F}_1 = q_Z^* \tilde{F}_1$. Then

$$\mathfrak{C}^{\text{st}}(p_M) \circ s_{i_{\tilde{\phi}}} \circ \vartheta_{i_Z, p_Z, \tilde{F}_\bullet} = \mathfrak{C}^{\text{st}}(\hat{p}_M) \circ s_{i_{\hat{\phi}}} \circ \vartheta_{i_Z, \hat{p}_Z, \hat{F}_\bullet}.$$

Proof. The identity $\hat{F}_1 = q_Z^* \tilde{F}_1$ gives us the map of vector bundles over q_Z , $q_F: \hat{F}^1 \rightarrow \tilde{F}^1$. This identifies $\hat{F}^1$ with the vector bundle $\hat{Z} \times_{\tilde{Z}} \tilde{F}^1 \rightarrow \tilde{F}^1$, which we denote by $q_F: V \rightarrow \tilde{F}^1$.

Saying that $(\tilde{F}_\bullet, \tilde{\phi})$ represents $p_Z^! [\phi]$ gives us an isomorphism $\tilde{F}_\bullet \cong p_Z^* E_\bullet \oplus \Omega_{\tilde{Z}/Z}$; similarly, we are given an isomorphism $\hat{F}_\bullet \cong \hat{p}_Z^* E_\bullet \oplus \Omega_{\hat{Z}/Z}$. These give us isomorphisms

$$\begin{aligned}
 \tilde{\alpha}: \tilde{Z}(i_{\tilde{Z}}^* T_{\tilde{M}} - \tilde{F}^1)_{\text{B.M.}} &\xrightarrow{\sim} \tilde{Z}(p_Z^* \mathbb{V}(E_\bullet) + T_{\tilde{Z}/Z})_{\text{B.M.}}, \\
 \hat{\alpha}: \hat{Z}(i_{\hat{Z}}^* T_{\hat{M}} - \hat{F}^1)_{\text{B.M.}} &\xrightarrow{\sim} \hat{Z}(\hat{p}_Z^* \mathbb{V}(E_\bullet) + T_{\hat{Z}/Z})_{\text{B.M.}}.
 \end{aligned}$$

The exact sequence

$$0 \rightarrow T_{\hat{Z}/\tilde{Z}} \rightarrow T_{\hat{Z}/Z} \rightarrow q_Z^* T_{\tilde{Z}/Z} \rightarrow 0$$

gives the isomorphism

$$\alpha: \hat{Z}(\hat{p}_Z^* \mathbb{V}(E_\bullet) + T_{\hat{Z}/Z})_{\text{B.M.}} \rightarrow \hat{Z}(q_Z^*(p_Z^* \mathbb{V}(E_\bullet) + T_{\tilde{Z}/Z}) + T_{\hat{Z}/\tilde{Z}})_{\text{B.M.}}.$$

Since $q_Z: \hat{Z} \rightarrow \tilde{Z}$ is a vector bundle over $\tilde{Z}$, the smooth push-forward map

$$q_{Z*}: \hat{Z}(q_Z^*(p_Z^* \mathbb{V}(E_\bullet) + T_{\tilde{Z}/Z}) + T_{\hat{Z}/\tilde{Z}})_{\text{B.M.}} \rightarrow \tilde{Z}(p_Z^* \mathbb{V}(E_\bullet) + T_{\tilde{Z}/Z})_{\text{B.M.}}$$

is an isomorphism. Putting these together gives us the isomorphism

$$\beta: \hat{Z}(i_{\hat{Z}}^* T_{\hat{M}} - \hat{F}^1)_{\text{B.M.}} \xrightarrow{\sim} \tilde{Z}(i_{\tilde{Z}}^* T_{\tilde{M}} - \tilde{F}^1)_{\text{B.M.}}$$

defined as $\beta := (\tilde{\alpha})^{-1} \circ q_{Z*} \circ \alpha \circ \hat{\alpha}$.

The functoriality of smooth push-forward for the composition $\hat{p}_Z = p_Z \circ q_Z$ yields the identity

$$\beta \circ \vartheta_{i_{\hat{Z}}, \hat{p}_Z, \hat{F}^\bullet} = \vartheta_{i_{\tilde{Z}}, p_Z, \tilde{F}^\bullet}. \quad (\text{A})$$

Let $p_{\hat{F}^1}: \hat{F}^1 \rightarrow \hat{Z}$ and $p_{\tilde{F}^1}: \tilde{F}^1 \rightarrow \tilde{Z}$ be the projections. The smooth push-forward map for the smooth map q_F combined with the exact sequence of vector bundles on $\hat{F}^1$

$$0 \rightarrow q_F^* V \rightarrow p_{\hat{F}^1}^* i_{\hat{Z}}^* T_{\hat{M}} \rightarrow q_F^* p_{\tilde{F}^1}^* i_{\tilde{Z}}^* T_{\tilde{M}} \rightarrow 0 \quad (7.4)$$

induces the morphism

$$q_{F*}: \hat{F}^1(p_{\hat{F}^1}^* i_{\hat{Z}}^* T_{\hat{M}})_{\text{B.M.}} \rightarrow \tilde{F}^1(p_{\tilde{F}^1}^* i_{\tilde{Z}}^* T_{\tilde{M}})_{\text{B.M.}},$$

giving us the diagram

$$\begin{array}{ccc} \hat{Z}(i_{\hat{Z}}^* T_{\hat{M}} - \hat{F}^1)_{\text{B.M.}} & \xrightarrow{\beta} & \tilde{Z}(i_{\tilde{Z}}^* T_{\tilde{M}} - \tilde{F}^1)_{\text{B.M.}} \\ \downarrow 0_{\hat{F}^1!} & & \downarrow 0_{\tilde{F}^1!} \\ \hat{F}^1(p_{\hat{F}^1}^* i_{\hat{Z}}^* T_{\hat{M}})_{\text{B.M.}} & \xrightarrow{q_{F*}} & \tilde{F}^1(p_{\tilde{F}^1}^* i_{\tilde{Z}}^* T_{\tilde{M}})_{\text{B.M.}} \end{array} \quad (\text{B})$$

This commutes by Remark 2.6: the Gysin push-forward commutes with smooth push-forward in a cartersian square of vector bundles.

The cartesian diagram (7.2) gives rise to the cartesian diagram

$$\begin{array}{ccc} \mathfrak{C}_{i_{\hat{Z}}} & \xrightarrow{i_{\hat{\phi}}} & \hat{F}^1 \\ \mathfrak{C}(q_M) \downarrow & & \downarrow q_F \\ \mathfrak{C}_{i_{\tilde{Z}}} & \xrightarrow{i_{\tilde{\phi}}} & \tilde{F}^1 \end{array}$$

Using the exact sequence (7.4) again, we see that smooth push-forward for the smooth morphism $\mathfrak{C}(q_M)$ gives the map

$$\mathfrak{C}(q_M)_*: \mathfrak{C}_{i_{\hat{Z}}}(\sigma_{i_{\hat{Z}}}^* T_{\hat{M}})_{\text{B.M.}} \rightarrow \mathfrak{C}_{i_{\tilde{Z}}}(\sigma_{i_{\tilde{Z}}}^* T_{\tilde{M}})_{\text{B.M.}}.$$

The compatibility of smooth push-forward and proper pull-back in cartesian squares (see Lemma 2.3) implies that the diagram

$$\begin{array}{ccc} \hat{F}^1(p_{\hat{F}^1}^* i_{\hat{Z}}^* T_{\hat{M}})_{\text{B.M.}} & \xrightarrow{q_{F*}} & \tilde{F}^1(p_{\tilde{F}^1}^* i_{\tilde{Z}}^* T_{\tilde{M}})_{\text{B.M.}} \\ \downarrow i_{\hat{\phi}}^* & & \downarrow i_{\tilde{\phi}}^* \\ \mathfrak{C}_{i_{\hat{Z}}}(\sigma_{i_{\hat{Z}}}^* T_{\hat{M}})_{\text{B.M.}} & \xrightarrow{\mathfrak{C}(q_M)_*} & \mathfrak{C}_{i_{\tilde{Z}}}(\sigma_{i_{\tilde{Z}}}^* T_{\tilde{M}})_{\text{B.M.}} \end{array} \quad (\text{C})$$

commutes.

Finally, it follows directly from the definitions of the various morphisms involved that the

diagram

$$\begin{array}{ccc}
 \mathfrak{C}_{\tilde{Z}}^{\text{st}} & \xrightarrow{\mathfrak{C}^{\text{st}}(q_M)} & \mathfrak{C}_{\tilde{Z}}^{\text{st}} \\
 \alpha_{i_{\tilde{Z}}} \downarrow & & \downarrow \alpha_{i_{\tilde{Z}}} \\
 \mathfrak{C}_{i_{\tilde{Z}}}(\sigma_{i_{\tilde{Z}}}^* T_{\tilde{M}})_{\text{B.M.}} & \xrightarrow{\mathfrak{C}_{(q_M)_*}} & \mathfrak{C}_{i_{\tilde{Z}}}(\sigma_{i_{\tilde{Z}}}^* T_{\tilde{M}})_{\text{B.M.}}
 \end{array} \tag{D}$$

commutes. Putting together the identity (A) with the commutativity of the diagrams (B), (C) and (D) gives the identity

$$s_{i_{\tilde{\phi}}} \circ \vartheta_{i_Z, p_Z, \tilde{F}_{\bullet}} = \mathfrak{C}^{\text{st}}(q_M) \circ s_{i_{\hat{\phi}}} \circ \vartheta_{i_Z, \hat{p}_Z, \hat{F}_{\bullet}};$$

composing on the left with $\mathfrak{C}^{\text{st}}(p_M)$ and using the functoriality $\mathfrak{C}^{\text{st}}(p_M) \circ \mathfrak{C}^{\text{st}}(q_M) = \mathfrak{C}^{\text{st}}(\hat{p}_M)$ completes the proof. $\square$

PROPOSITION 7.2. *Suppose that B is affine, take $Z \in \mathbf{Sch}^G/B$, and let $[\phi]: E_{\bullet} \rightarrow \mathbb{L}_{Z/B}$ be a perfect obstruction theory on Z . Choose a closed immersion $i_Z: Z \rightarrow M$ with $M \in \mathbf{Sm}^G/B$. Choose a Jouanolou cover $p_M: \tilde{M} \rightarrow M$, and let $p_Z: \tilde{Z} \rightarrow Z$ be the pull-back $\tilde{M} \times_M Z$. Choose a reduced normalized obstruction theory $\tilde{\phi}: \tilde{F}_{\bullet} \rightarrow (\mathcal{I}_{\tilde{Z}}/\mathcal{I}_{\tilde{Z}}^2 \rightarrow i^* \Omega_{\tilde{M}/B})$ representing $p_Z^![\phi]$. This gives us the isomorphisms $\mathfrak{C}^{\text{st}}(p_M)$ (3.3) and $\vartheta_{i_Z, p_Z, \tilde{F}_{\bullet}}$ (7.1) and the map $s_{\tilde{\phi}}$ (7.3). Composing these morphisms gives*

$$\Phi_{E_{\bullet}, [\phi]} := \mathfrak{C}^{\text{st}}(p_M) \circ s_{\tilde{\phi}} \circ \vartheta_{i_Z, p_Z, \tilde{F}_{\bullet}}: Z(\mathbb{V}(E_{\bullet}))_{\text{B.M.}} \rightarrow \mathfrak{C}_Z^{\text{st}}.$$

Then the morphism $\Phi_{E_{\bullet}, [\phi]}$ depends only on the perfect obstruction theory $(E_{\bullet}, [\phi])$; that is, it is independent of the choice of closed immersion i_Z , the choice of Jouanolou cover p_M , the choice of induced obstruction theory $p_Z^![\phi]$ and the choice of reduced normalized obstruction theory $\tilde{\phi}$ representing $p_Z^![\phi]$.

Moreover, we have the identity

$$\vartheta_{i_Z, p_Z, \tilde{F}_{\bullet}}^*([\tilde{Z}, \tilde{\phi}, i_{\tilde{Z}}]^{\text{vir}}) = [\mathfrak{C}^{\text{st}}] \circ \Phi_{E_{\bullet}, [\phi]}.$$

Relying on Proposition 7.2, we may make the following definition.

DEFINITION 7.3. Suppose that B is affine, take $Z \in \mathbf{Sch}^G/B$, and let $[\phi]: E_{\bullet} \rightarrow \mathbb{L}_{Z/B}$ be a perfect obstruction theory on Z . We have the fundamental class $[\mathfrak{C}^{\text{st}}] \in \mathbb{S}_B^{0,0}(\mathfrak{C}^{\text{st}})$. Define the *virtual fundamental class*

$$[Z, [\phi]]^{\text{vir}} \in \mathbb{S}_B^{\text{B.M.}}(Z, \mathbb{V}(E_{\bullet}))$$

by

$$[Z, [\phi]]^{\text{vir}} = [\mathfrak{C}^{\text{st}}] \circ \Phi_{E_{\bullet}, [\phi]}.$$

If we have a motivic ring spectrum $\mathcal{E}$ in $\text{SH}^G(B)$ with unit $\epsilon_{\mathcal{E}}: \mathbb{S}_B \rightarrow \mathcal{E}$, we define

$$[Z, [\phi]]_{\mathcal{E}}^{\text{vir}} := \epsilon_{\mathcal{E}}([Z, [\phi]]^{\text{vir}}) \in \mathcal{E}^{\text{B.M.}}(Z, \mathbb{V}(E_{\bullet})).$$

Remark 7.4. It follows from the last assertion in Proposition 7.2 that given a closed immersion $i_Z: Z \rightarrow M$ with $M \in \mathbf{Sm}^G/B$, a Jouanolou cover $\tilde{M} \rightarrow M$ inducing the Jouanolou cover $p_Z: \tilde{Z} \rightarrow Z$ and a reduced normalized representative $\tilde{\phi}: \tilde{F}_{\bullet} \rightarrow L_{\tilde{Z}/B}$ for $p_Z^![\phi]$, we have

$$[Z, [\phi]]^{\text{vir}} = \vartheta_{i_Z, p_Z, \tilde{F}_{\bullet}}^*([\tilde{Z}, \tilde{\phi}, i_{\tilde{Z}}]^{\text{vir}}).$$

Proof of Proposition 7.2. Suppose that we have fixed a closed immersion $i_Z: Z \rightarrow M$ and a Jouanolou cover $\tilde{M} \rightarrow M$. By Lemma 6.6, there exists an induced obstruction theory $p_Z^![\phi]$ on $\tilde{Z}$, which is unique up to canonical isomorphism.

Suppose that we have two reduced normalized obstruction theories $(\tilde{F}_\bullet, \phi)$ and $(\tilde{F}'_\bullet, \phi')$ representing $p_Z^![\phi]$. By Lemma 6.3, there is an isomorphism $\rho_\bullet: \tilde{F}_\bullet \rightarrow \tilde{F}'_\bullet$ of perfect obstruction theories. In particular, the map $\rho^1: F'^1 \rightarrow F^1$ satisfies $\rho^1 \circ i_{\phi'} = i_\phi$. This gives us the commutative diagram

$$\begin{array}{ccc} Z(\mathbb{V}(E_\bullet)_{\text{B.M.}}) & \xrightarrow{\vartheta_{i_Z, p_Z, \tilde{F}_\bullet}} & \tilde{Z}(i_Z^* T_{\tilde{M}} - \tilde{F}^1)_{\text{B.M.}} \\ \vartheta_{i_Z, p_Z, \tilde{F}'_\bullet} \downarrow & \nearrow \gamma(\rho^1) & \downarrow s_{i_\phi} \\ \tilde{Z}(i_Z^* T_{\tilde{M}} - \tilde{F}'^1)_{\text{B.M.}} & \xrightarrow{s_{i_{\phi'}}} & \mathfrak{C}_Z^{\text{st}}, \end{array}$$

where $\gamma(\rho^1)$ is the isomorphism induced by ρ^1 . This yields the independence of $\Phi_{E_\bullet, [\phi]}$ of the choice of reduced normalized obstruction theory representing $p_Z^![\phi]$.

To show the independence of the choice of Jouanolou cover $p_M: \tilde{M} \rightarrow M$ over a fixed closed immersion $i: Z \rightarrow M$, we may assume that we are comparing one cover $p_M: \tilde{M} \rightarrow M$ with a second cover $\hat{p}_M: \hat{M} \rightarrow M$ which factors as

$$\hat{M} \xrightarrow{q_M} \tilde{M} \xrightarrow{p_M} M$$

with $q_M: \hat{M} \rightarrow \tilde{M}$ a vector bundle over $\tilde{M}$. The independence here follows from Lemma 7.1 and what we have already shown.

Finally, suppose that we have a smooth morphism $q: N \rightarrow M$ and a closed immersion $i'_Z: Z \rightarrow N$ with $q \circ i'_Z = i_Z$. By what we have already shown, we may suppose that M is affine; if we take a Jouanolou cover $\tilde{N} \rightarrow N$, then as Z is affine, the cover admits a section over Z , so we may replace N with $\tilde{N}$, change notation and assume that N is also affine. With what we have already proven, we may choose a reduced normalized representative $\phi: F_\bullet \rightarrow (\mathcal{I}_{i(Z)}/\mathcal{I}_{i(Z)}^2 \rightarrow i_Z^* \Omega_{M/B})$ for $[\phi]$.

We have the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & q_Z^*(\mathcal{I}_{i(Z)}/\mathcal{I}_{i(Z)}^2) & \longrightarrow & \mathcal{I}_{i'(Z)}/\mathcal{I}_{i'(Z)}^2 & \longrightarrow & i'^* \Omega_{N/M} \longrightarrow 0 \\ & & \downarrow d & & \downarrow d & & \parallel \\ 0 & \longrightarrow & i'^* q^* \Omega_{M/B} & \longrightarrow & i'^* \Omega_{N/B} & \longrightarrow & i'^* \Omega_{N/M} \longrightarrow 0. \end{array}$$

Choosing a splitting $\rho: i'^* \Omega_{N/M} \rightarrow \mathcal{I}_{i'(Z)}/\mathcal{I}_{i'(Z)}^2$ gives us the splitting $\sigma := d \circ \rho: i'^* \Omega_{N/M} \rightarrow i'^* \Omega_{N/B}$. This gives the reduced normalized representative for $[\phi]$ (with respect to i'_Z)

$$F'_\bullet := (F_1 \oplus i'^* \Omega_{N/M} \xrightarrow{q^* d_F \oplus \sigma} i'^* \Omega_{N/B}) \xrightarrow{(q^* \phi_1 + \rho, \text{Id})} (\mathcal{I}_{i'(Z)}/\mathcal{I}_{i'(Z)}^2 \xrightarrow{d} i'^* \Omega_{N/B}).$$

We have the cartesian diagram

$$\begin{array}{ccc} \mathfrak{C}_{i'_Z} & \xrightarrow{i_{\phi'}} & F^1 \oplus i'^* T_{N/M} \\ \mathfrak{C}(q) \downarrow & & \downarrow p_1 \\ \mathfrak{C}_{i_Z} & \xrightarrow{i_\phi} & F^1 \end{array}$$

identifying $\mathfrak{C}_{i'_Z}$ with the bundle $\sigma_{i'_Z}^* T_{N/M} = \mathfrak{C}_{i_Z} \times_Z i'^* T_{N/M}$ over $\mathfrak{C}_{i_Z}$ and $i_{\phi'}$ with $i_{\phi} \times \text{Id}$.

We have the isomorphism (Lemma 3.1)

$$\psi_q := \psi_{i'_Z, i_Z} : \mathfrak{C}_{i'_Z}(\sigma_{i'_Z}^* T_N)_{\text{B.M.}} \rightarrow \mathfrak{C}_{i_Z}(\sigma_{i_Z}^* T_M)_{\text{B.M.}}.$$

By Theorem 3.2, the diagram

$$\begin{array}{ccc} \mathfrak{C}_Z^{\text{st}} & \xrightarrow[\sim]{\alpha_i} & \mathfrak{C}_{i_Z}(\sigma_{i_Z}^* T_M)_{\text{B.M.}} \\ \alpha_{i'} \downarrow \wr & \nearrow \psi_q & \\ \mathfrak{C}_{i'_Z}(\sigma_{i'_Z}^* T_N)_{\text{B.M.}} & & \end{array} \quad (7.5)$$

commutes.

We have the isomorphism θ_q

$$\Sigma^{\sigma_{i'_Z}^* T_N} \cong \Sigma^{\sigma_{i'_Z}^* T_{N/M}} \circ \Sigma^{\sigma_{i'_Z}^* q^* T_M}$$

induced by the exact sequence

$$0 \rightarrow T_{N/M} \rightarrow T_N \rightarrow q^* T_M \rightarrow 0 \quad (7.6)$$

and the isomorphism nat

$$\Sigma^{\sigma_{i'_Z}^* q^* T_M} \circ \mathfrak{C}(q)^* = \Sigma^{\mathfrak{C}(q)^* \sigma_{i'_Z}^* T_M} \circ \mathfrak{C}(q)^* \cong \mathfrak{C}(q)^* \circ \Sigma^{\sigma_{i_Z}^* T_M},$$

giving the composition

$$\begin{aligned} \pi_{\mathfrak{C}_{i'_Z}!} \circ \Sigma^{\sigma_{i'_Z}^* T_N} \circ \mathfrak{C}(q)^* &\xrightarrow{\theta_q} \pi_{\mathfrak{C}_{i'_Z}!} \circ \Sigma^{\sigma_{i'_Z}^* T_{N/M}} \circ \Sigma^{\sigma_{i'_Z}^* q^* T_M} \circ \mathfrak{C}(q)^* \\ &\xrightarrow{\text{nat}} \pi_{\mathfrak{C}_{i'_Z}!} \circ \Sigma^{\sigma_{i'_Z}^* T_{N/M}} \circ \mathfrak{C}(q)^* \circ \Sigma^{\sigma_{i_Z}^* T_M} \xrightarrow{\mathfrak{C}(q)_*} \pi_{\mathfrak{C}_{i_Z}!} \Sigma^{\sigma_{i_Z}^* T_M}. \end{aligned}$$

Let $\Theta_q := \text{nat} \circ \theta_q$. By the definition of ψ_q , we have $\psi_q = (\mathfrak{C}(q)_* \circ \Theta_q)(1_{\mathfrak{C}_{i_Z}})$.

We have the isomorphism

$$\theta'_q : \Sigma^{-F'^1} \circ \Sigma^{i'^* T_N} \xrightarrow{\sim} \Sigma^{-F^1} \circ \Sigma^{i^* T_M}$$

induced by the exact sequence (7.6) and the identity $F'^1 \cong F^1 \oplus i'^* T_{N/M}$. Let $p_1 : F'^1 \rightarrow F^1$ be the projection.

Consider the diagram

$$\begin{array}{ccc} \pi_{Z!} \circ \Sigma^{-F'^1} \circ \Sigma^{i'^* T_N} & \xrightarrow[\sim]{\theta'_q} & \pi_{Z!} \circ \Sigma^{-F^1} \circ \Sigma^{i^* T_M} \\ \downarrow 0_{F'^1!} & & \downarrow 0_{F^1!} \\ \pi_{F'^1!} \circ \Sigma^{p_{F'^1}^* i'^* T_N} \circ p_{F'^1}^* & & \pi_{F^1!} \circ \Sigma^{p_{F^1}^* i^* T_M} \circ p_{F^1}^* \\ & \searrow \Theta_q \quad \nearrow p_{1*} & \\ & \pi_{F'^1!} \circ \Sigma^{p_{F'^1}^* i'^* T_{N/M}} \circ p_1^* \circ \Sigma^{p_{F^1}^* i^* T_M} \circ p_{F^1}^* & \end{array} \quad (7.7)$$

As in the proof of Remark 2.6, one shows that this diagram commutes by using the functoriality of smooth push-forward, replacing $0_{F^1!}$ and $0_{F'^1!}$ with their respective inverses p_{F^1*} and $p_{F'^1*}$.

The commutativity of the left-hand side of the diagram

$$\begin{array}{ccc}
 \pi_{F'^1!} \circ \Sigma^{p_{F'^1}^* i_Z'^* T_N} \circ p_{F'^1}^* & & \pi_{F^1!} \circ \Sigma^{p_{F^1}^* i_Z'^* T_M} \circ p_{F^1}^* \\
 \downarrow i_{\phi'}^* & \searrow \Theta_q \sim & \nearrow p_{1*} \sim \\
 \pi_{F'^1!} \circ \Sigma^{p_{F'^1}^* i_Z'^* T_{N/M}} \circ p_1^* \circ \Sigma^{p_{F^1}^* i_Z'^* T_M} \circ p_{F^1}^* & & \\
 \downarrow i_{\phi'}^* & & \downarrow i_{\phi}^* \\
 \pi_{\mathfrak{C}_{i_Z}!} \circ \Sigma^{\sigma_{i_Z}^* T_N} \circ p_{i_Z}^* & & \pi_{\mathfrak{C}_{i_Z}!} \circ \Sigma^{\sigma_{i_Z}^* T_M} \circ p_{i_Z}^* \\
 \searrow \Theta_q \sim & & \nearrow \mathfrak{C}(q)_* \sim \\
 \pi_{\mathfrak{C}_{i_Z}!} \circ \Sigma^{\sigma_{i_Z}^* T_{N/M}} \circ \mathfrak{C}(q)^* \circ \Sigma^{\sigma_{i_Z}^* T_M} \circ p_{i_Z}^* & &
 \end{array} \tag{7.8}$$

follows from the naturality of $\Sigma^{(-)}: D_{G,\text{iso}}^{\text{perf}}(?) \rightarrow \text{Aut}(\text{SH}^G(?))$ and that of the right-hand side from the commutativity of smooth push-forward with proper pull-back, Lemma 2.3.

Putting these two diagrams together and evaluating at 1_Z gives us the commutative diagram

$$\begin{array}{ccc}
 Z(i'^* T_N - F'^1)_{\text{B.M.}} & \xrightarrow[\sim]{\theta'_q(1_Z)} & Z(i^* T_M - F^1)_{\text{B.M.}} \\
 \downarrow i_{\phi'}^* \circ 0_{F'^1!} & & \downarrow i_{\phi}^* \circ 0_{F^1!} \\
 \mathfrak{C}_{i_Z}(\sigma_{i_Z}^* T_N)_{\text{B.M.}} & \xrightarrow[\sim]{\psi_q} & \mathfrak{C}_{i_Z}(\sigma_{i_Z}^* T_M)_{\text{B.M.}} .
 \end{array} \tag{7.9}$$

The functoriality of $\Sigma^{(-)}: D_{G,\text{iso}}^{\text{perf}}(Z) \rightarrow \text{Aut}(\text{SH}^G(Z))$ gives us the commutative diagram of isomorphisms

$$\begin{array}{ccc}
 & Z(\mathbb{V}(E_{\bullet})_{\text{B.M.}}) & \\
 \vartheta_{i_Z, \text{Id}_M, F_{\bullet}} \swarrow & & \searrow \vartheta_{i'_Z, \text{Id}_N, F'_{\bullet}} \\
 Z(i'^* T_N - F'^1)_{\text{B.M.}} & \xrightarrow[\sim]{\theta'_q(1_Z)} & Z(i^* T_M - F^1)_{\text{B.M.}} .
 \end{array} \tag{7.10}$$

Putting the diagrams (7.5), (7.9) and (7.10) together, we see that the definition of s_{ϕ} and $s_{\phi'}$ gives the identity

$$\alpha_{i_Z} \circ s_{\phi'} \circ \vartheta_{i'_Z, \text{Id}_N, F'_{\bullet}} = \psi_q \circ \alpha_{i'_Z} \circ s_{\phi'} \circ \vartheta_{i'_Z, \text{Id}_N, F'_{\bullet}} = \alpha_{i_Z} \circ s_{\phi} \circ \vartheta_{i_Z, \text{Id}_M, F_{\bullet}} .$$

or

$$s_{\phi'} \circ \vartheta_{i'_Z, \text{Id}_N, F'_{\bullet}} = s_{\phi} \circ \vartheta_{i_Z, \text{Id}_M, F_{\bullet}} .$$

As we are taking the trivial Jouanolou covers, we have $p_M = \text{Id}_M$, $p_N = \text{Id}_N$, completing the proof. $\square$

We conclude with a result on the compatibility of the fundamental class and virtual fundamental class with respect to base change.

PROPOSITION 7.5. Let $\pi_Z: Z \rightarrow B$ be in $\mathbf{Sch}^G/B$, let $f: B' \rightarrow B$ be a morphism in $\mathbf{Sch}/B$, and let $\pi_{Z'}: Z' = Z \times_B B' \rightarrow B'$ be the pull-back. Suppose that the cartesian square

$$\begin{array}{ccc} Z' & \xrightarrow{p_1} & Z \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

is Tor-independent: $\mathrm{Tor}_{\mathcal{O}_B}^i(\mathcal{O}_{B'}, \mathcal{O}_Z) = 0$ for $i > 0$.

- (1) Suppose that we have a closed immersion $i: Z \rightarrow M$ in $\mathbf{Sch}^G/B$ with M in $\mathbf{Sm}^G/B$, let $M' = M \times_B B'$, and let $i_{Z'}: Z' \rightarrow M'$ be the induced closed immersion. Then we have natural isomorphisms $\mathfrak{C}_{i_{Z'}} \cong \mathfrak{C}_{i_Z} \times_B B'$ and $\mathrm{Def}(i_{Z'}) \cong \mathrm{Def}(i) \times_B B'$ over $M' \times \mathbb{A}^1$.
- (2) The isomorphism $\mathfrak{C}_{i_{Z'}} \cong \mathfrak{C}_{i_Z} \times_B B'$ of assertion (1) induces an isomorphism $\mathfrak{C}_{Z'}^{\mathrm{st}} \cong f^* \mathfrak{C}_Z^{\mathrm{st}}$ sending the fundamental class $[\mathfrak{C}_{Z'}^{\mathrm{st}}] \in \mathbb{S}_{B'}^{0,0}(\mathfrak{C}_{Z'}^{\mathrm{st}})$ to $f^*[\mathfrak{C}_Z^{\mathrm{st}}]$.
- (3) Let $[\phi]: E_\bullet \rightarrow \mathbb{L}_{Z/B}$ be a perfect obstruction theory on Z . Then the map $[\phi']: p_1^* E_\bullet \rightarrow \mathbb{L}_{Z'/B'}$ defined as the composition

$$p_1^* E_\bullet \xrightarrow{p_1^*[\phi]} p_1^* \mathbb{L}_{Z/B} \xrightarrow{\mathrm{can}} \mathbb{L}_{Z'/B'}$$

is a perfect obstruction theory on Z' , we have a canonical isomorphism $f^*(Z(\mathbb{V}(E_\bullet))_{\mathrm{B.M.}}) \cong Z'(\mathbb{V}(p_1^* E_\bullet))_{\mathrm{B.M.}}$, and, via this isomorphism, we have $p_1^*([Z, [\phi]]^{\mathrm{vir}}) = [Z'[\phi']]^{\mathrm{vir}}$.

Proof. The Tor-independence implies that $p_1^* \mathbb{L}_{Z/B} \cong \mathbb{L}_{Z'/B'}$ and that $p_1^*[\phi]: p_1^* E_\bullet \rightarrow \mathbb{L}_{Z'/B'}$ is a perfect obstruction theory on Z' .

Let $p_M: M' := M \times_B B' \rightarrow M$ be the projection; the Tor-independence implies that the canonical map $p_M^* \mathcal{I}_Z \rightarrow \mathcal{I}_{Z'}$ is an isomorphism. This readily implies assertion (1) and gives us the isomorphism of the diagram

$$\begin{array}{ccccc} \mathfrak{C}_{i_{Z'}} & \longrightarrow & \mathrm{Def}(i_{Z'}) & \longleftarrow & M' \times (\mathbb{A}^1 \setminus \{0\}) \\ \downarrow & & \downarrow & & \parallel \\ Z' & \longrightarrow & M' \times \mathbb{A}^1 & \longleftarrow & M' \times (\mathbb{A}^1 \setminus \{0\}) \end{array}$$

with

$$\left[\begin{array}{ccccc} \mathfrak{C}_{i_Z} & \longrightarrow & \mathrm{Def}(i_Z) & \longleftarrow & M \times (\mathbb{A}^1 \setminus \{0\}) \\ \downarrow & & \downarrow & & \parallel \\ Z & \longrightarrow & M \times \mathbb{A}^1 & \longleftarrow & M \times (\mathbb{A}^1 \setminus \{0\}) \end{array} \right] \times_B B'.$$

Assertion (2) follows from this, the naturality of $\Sigma^-: \mathcal{V}(?) \rightarrow \mathrm{SH}^G(?)$ and the localization distinguished triangle with respect to f^* , and the base-change isomorphism $\mathrm{Ex}(\Delta_!^*)$. This also gives us the isomorphism $f^*(Z(\mathbb{V}(E_\bullet))_{\mathrm{B.M.}}) \cong Z'(\mathbb{V}(p_1^* E_\bullet))_{\mathrm{B.M.}}$. By the definition of the virtual fundamental class, assertion (3) follows from assertions (1) and (2) together with the compatibility of proper pull-back, smooth push-forward and Gysin push-forward with base change by f^* ; see Remark 2.4. $\square$

8. Comparisons and examples

We relate our constructions to the constructions of [BF97] in the case of motivic cohomology/Chow groups or, more generally, the case of an oriented theory, for example, K -theory or algebraic cobordism. For simplicity, we take $G = \{\text{Id}\}$, and we remind the reader that we are assuming that B is affine.

8.1 Oriented theories

Let $\mathcal{E}$ be an oriented commutative ring spectrum in $\text{SH}(B)$. For $Z \in \mathbf{Sch}^G/B$ and $v \in D_G^{\text{perf}}(Z)$ of virtual rank r , we have the *Thom isomorphism*

$$\vartheta_v : \Sigma^v \pi_Z^! \mathcal{E} \cong \Sigma^{2r, r} \pi_Z^! \mathcal{E}$$

giving the isomorphism on $\mathcal{E}$ -Borel–Moore homology

$$\vartheta_v^* : \mathcal{E}_{a,b}^{\text{B.M.}}(Z, v) \xrightarrow{\sim} \mathcal{E}_{2r+a, r+b}^{\text{B.M.}}(Z).$$

If $p_Z : Z \rightarrow B$ is smooth of relative dimension d_Z , the purity isomorphism $p_{Z\#} \circ \Sigma^{-T_Z} \cong p_Z!$ gives the purity isomorphism

$$\mathcal{E}_{a,b}^{\text{B.M.}}(Z, v) \cong (\Sigma^{T_X - v} \mathcal{E})^{-a, -b}(Z) \cong \mathcal{E}^{2d_Z - 2r - a, d_Z - r - b}(Z).$$

For $i : Z \rightarrow M$ a closed immersion in $\mathbf{Sch}^G/B$, in a smooth dimension d_M B -scheme M , we thus have

$$\mathcal{E}^{a,b}(\mathfrak{C}_Z^{\text{st}}) \xrightarrow[\sim]{(\alpha_i^{-1})^*} \mathcal{E}^{a,b}(\mathfrak{C}_i(\sigma_i^* T_M)_{\text{B.M.}}) \cong \mathcal{E}_{2d_M - a, d_M - b}^{\text{B.M.}}(\mathfrak{C}_i).$$

As $\mathfrak{C}_i$ has pure dimension $d_{\mathfrak{C}} = d_M$ over B , the fundamental class $[\mathfrak{C}_Z^{\text{st}}]_{\mathcal{E}}$ is thus an element of $\mathcal{E}^{0,0}(\mathfrak{C}_Z^{\text{st}}) \cong \mathcal{E}_{2d_{\mathfrak{C}}, d_{\mathfrak{C}}}^{\text{B.M.}}(\mathfrak{C}_i)$, and the virtual fundamental class $[Z, [\phi]]_{\mathcal{E}}^{\text{vir}}$ associated to a perfect obstruction theory $(E_{\bullet}, [\phi])$ of virtual rank r lives in $\mathcal{E}^{\text{B.M.}}(Z, \mathbb{V}(E_{\bullet})) \cong \mathcal{E}_{2r, r}^{\text{B.M.}}(Z)$.

8.2 Fundamental classes

Let $\mathcal{E}$ be an oriented theory. Suppose that we have an integral B -scheme D and a principal effective Cartier divisor $\mathfrak{C}$ on D such that $D \setminus \mathfrak{C}$ is smooth over B . Suppose that D has pure dimension $d_{\mathfrak{C}} + 1$ over B , so $\mathfrak{C}$ has pure dimension $d_{\mathfrak{C}}$ over B . Let $t \in \Gamma(D, \mathcal{O}_D)$ be a generator for $\mathcal{I}_{\mathfrak{C}}$. The map $t : D \setminus \mathfrak{C} \rightarrow \mathbb{G}_m$ determines an element

$$[t] \in \mathcal{E}^{1,1}(D \setminus \mathfrak{C}) \cong \mathcal{E}_{2d_{\mathfrak{C}}+1, d_{\mathfrak{C}}}^{\text{B.M.}}(D \setminus \mathfrak{C}).$$

We have the localization sequence

$$\cdots \rightarrow \mathcal{E}_{2d_{\mathfrak{C}}+1, d_{\mathfrak{C}}}^{\text{B.M.}}(D \setminus \mathfrak{C}) \xrightarrow{\partial} \mathcal{E}_{2d_{\mathfrak{C}}, d_{\mathfrak{C}}}^{\text{B.M.}}(\mathfrak{C}) \xrightarrow{i_{\mathfrak{C}*}} \mathcal{E}_{2d_{\mathfrak{C}}, d_{\mathfrak{C}}}^{\text{B.M.}}(D) \rightarrow \cdots,$$

and we have the fundamental class $[\mathfrak{C}]_{\mathcal{E}} \in \mathcal{E}_{2d_{\mathfrak{C}}, d_{\mathfrak{C}}}^{\text{B.M.}}(\mathfrak{C})$ defined by $[\mathfrak{C}]_{\mathcal{E}} := \partial[t]$. This class is independent of the choice of defining equation t . If $D = \text{Def}(i)$ for our closed immersion $i : Z \rightarrow M$ and $t : \text{Def}(i) \rightarrow \mathbb{A}^1$ is the structure morphism $\text{Def}(i) \xrightarrow{p_1} M \times \mathbb{A}^1 \xrightarrow{p_2} \mathbb{A}^1$, we have $\mathfrak{C} = \mathfrak{C}_i$ and $[\mathfrak{C}]$ agrees with the fundamental class $[\mathfrak{C}_Z^{\text{st}}]_{\mathcal{E}} \in \mathcal{E}^{0,0}(\mathfrak{C}_Z^{\text{st}})$ (see Section 4) after identifying $\mathcal{E}_{2d_{\mathfrak{C}}, d_{\mathfrak{C}}}^{\text{B.M.}}(\mathfrak{C})$ with $\mathcal{E}^{0,0}(\mathfrak{C}_Z^{\text{st}})$ as above. The construction of the fundamental class of $\mathfrak{C}_i$ by this method in the case of algebraic cobordism was described to us by Parker Lowrey and appears in his paper with Timo Schürg [LS16].

Example 8.1. Take $B = \text{Spec } k$ and $Z \in \mathbf{Sch}/B$. For $\mathcal{E} = H\mathbb{Z}$, the ring spectrum representing motivic cohomology, $H\mathbb{Z}_{2d, d}^{\text{B.M.}}(Z)$ is the classical Chow group $\text{CH}_d(Z)$. For $\mathcal{E} = \text{KGL}$, the

ring spectrum representing Quillen K -theory, $\mathrm{KGL}_{2d,d}^{\mathrm{B.M.}}(Z)$ is $G_0(Z)$, the Grothendieck group of coherent sheaves on Z . For k a field of characteristic 0 and $\mathcal{E} = \mathrm{MGL}$, the ring spectrum representing Voevodsky's algebraic cobordism [Voe98], $\mathrm{MGL}_{2d,d}^{\mathrm{B.M.}}$ is $\Omega_d(Z)$, the algebraic cobordism of [LM07].

For $\mathcal{E} = H\mathbb{Z}$, the fundamental class $[\mathfrak{C}]_{H\mathbb{Z}} \in \mathcal{E}_{2d_{\mathfrak{C}},d_{\mathfrak{C}}}^{\mathrm{B.M.}}(\mathfrak{C})$ is the cycle class of the scheme $\mathfrak{C}$. For $\mathcal{E} = \mathrm{KGL}$, the fundamental class $[\mathfrak{C}]_{\mathrm{KGL}}$ is the class in $G_0(\mathfrak{C})$ of the structure sheaf $\mathcal{O}_{\mathfrak{C}}$. For $\mathcal{E} = \mathrm{MGL}$, the fundamental class $[\mathfrak{C}]_{\mathrm{MGL}}$ is the class of the pseudo-divisor $\mathrm{div}[t]$ on D , applied to any resolution of singularities $f: \tilde{D} \rightarrow D$; that is, $[\mathfrak{C}]_{\mathrm{MGL}} = f_*(\mathrm{div}_{\tilde{D}}(f^*t))$.

8.3 Push-forward and intersection with the 0-section

Let $p: Y \rightarrow X$ be a projective map in $\mathbf{Sch}/B$, and let $\mathcal{E}$ be an oriented theory. We have the push-forward map $p_*: \mathcal{E}_{a,b}^{\mathrm{B.M.}}(Y) \rightarrow \mathcal{E}_{a,b}^{\mathrm{B.M.}}(X)$. Given a rank r vector bundle $f: V \rightarrow Z$ with a section $s: Z \rightarrow V$, we have the maps

$$\begin{aligned} f^*: \mathcal{E}_{a,b}^{\mathrm{B.M.}}(Z) &\rightarrow \mathcal{E}_{a,b}^{\mathrm{B.M.}}(V, T_{V/Z}) \cong \mathcal{E}_{a+2r,b+r}^{\mathrm{B.M.}}(V), \\ s^!: \mathcal{E}_{a+2r,b+r}^{\mathrm{B.M.}}(V) &\cong \mathcal{E}_{a,b}^{\mathrm{B.M.}}(V, T_{V/Z}) \rightarrow \mathcal{E}_{a,b}^{\mathrm{B.M.}}(Z) \end{aligned}$$

with $s^! = (f^*)^{-1}$, by Lemma 2.2, the map f^* is the usual pull-back for the smooth morphism f , and $s^!$ is the classical “intersection with the 0-section” defined as the inverse of f^* .

Remark 8.2. Suppose that we have a closed immersion $i: Z \hookrightarrow M$ with M smooth of dimension d_M over B . The intrinsic normal cone $\mathfrak{C}_Z$ as defined by Behrend–Fantechi [BF97] is the quotient stack $[\mathfrak{C}_i/i^*T_{M/B}]$. They also define the normal sheaf $\mathcal{N}_i := \mathrm{Spec}_{\mathcal{O}_Z} \mathrm{Sym}^* \mathcal{I}_Z / \mathcal{I}_Z^2$; the surjection $\mathrm{Sym}^* \mathcal{I}_Z / \mathcal{I}_Z^2 \rightarrow \oplus_n \mathcal{I}_Z^n / \mathcal{I}_Z^{n+1}$ defines the closed immersion $\mathfrak{C}_i \hookrightarrow \mathcal{N}_i$. This induces the closed immersion of quotient stacks $\mathfrak{C}_Z \hookrightarrow \mathcal{N}_Z := [\mathcal{N}_i/i^*T_{M/B}]$.

Suppose that we have a perfect obstruction theory $[\phi]$ of virtual rank r on $i: Z \hookrightarrow M$, with global resolution $(F_1 \rightarrow F_0) \xrightarrow{\phi_\bullet} (\mathcal{I}_Z / \mathcal{I}_Z^2 \rightarrow i^* \Omega_{M/B})$. We may assume that $(F_\bullet, \phi)$ is normalized. The assumption that ϕ is a perfect obstruction theory implies that ϕ induces closed immersions

$$\mathfrak{C}_Z \hookrightarrow \mathcal{N}_Z \hookrightarrow [F^1 / F^0].$$

Let $\mathfrak{C}(F_\bullet) \subset \mathcal{N}(F_\bullet) \subset F^1$ be the pull-back of this sequence of closed immersions by the quotient map $F^1 \rightarrow [F^1 / F^0]$.

One can describe $\mathcal{N}(F_\bullet)$ explicitly as follows: we have the commutative diagram

$$\begin{array}{ccc} F_1 & \xrightarrow{d_F} & F_0 \\ \phi_1 \downarrow & & \downarrow \phi_0 \\ \mathcal{I}_Z / \mathcal{I}_Z^2 & \xrightarrow{d} & i^* \Omega_{M/B}. \end{array}$$

Let $F := \mathcal{I}_Z / \mathcal{I}_Z^2 \times_{i^* \Omega_{M/B}} F_0$. The map (ϕ_1, d_F) gives a surjection $\phi_{\mathcal{N}}: F_1 \rightarrow F$; the scheme $\mathcal{N}(F_\bullet)$ is the closed subscheme $\mathrm{Spec}_{\mathcal{O}_Z} \mathrm{Sym}^* F$ of $F^1 = \mathrm{Spec}_{\mathcal{O}_Z} \mathrm{Sym}^* F_1$.

The virtual fundamental class $[Z, [\phi]]_{\mathrm{BF}}^{\mathrm{vir}}$ as defined by Behrend–Fantechi [BF97] is the element of $\mathrm{CH}_r(Z)$ given by

$$[Z, [\phi]]_{\mathrm{BF}}^{\mathrm{vir}} := 0_{F^1}^!([\mathfrak{C}(F_\bullet)]).$$

Suppose that we have a Jouanolou cover $p_M: \tilde{M} \rightarrow M$, with pull-back $p_Z: \tilde{Z} \rightarrow Z$. An induced perfect obstruction theory $p_Z^! [F_\bullet] \rightarrow (\mathcal{I}_{\tilde{Z}} / \mathcal{I}_{\tilde{Z}}^2 \rightarrow i_{\tilde{Z}}^* \Omega_{\tilde{M}/B})$ is defined (see Lemma 6.6).

Writing $(\tilde{F}_1 \rightarrow \tilde{F}_0) := p_Z^! [F_\bullet]$, we have $\tilde{F}_1 = p_Z^* F_1$ and $\tilde{F}_0 = p_Z^* F_0 \oplus \Omega_{\tilde{Z}/Z}$. Thus, we have the isomorphism of quotients of $\tilde{F}_1$

$$\tilde{F} := \mathcal{I}_{\tilde{Z}} / \mathcal{I}_{\tilde{Z}}^2 \times_{i_Z^* \Omega_{\tilde{M}/B}} \tilde{F}_0 \cong p_Z^* F,$$

which shows that $\mathcal{N}(\tilde{F}_\bullet) \subset \tilde{F}^1$ is equal to $p_Z^* \mathcal{N}(F_\bullet)$. Thus $\mathfrak{C}(\tilde{F}_\bullet) = p_Z^* \mathfrak{C}(F_\bullet) \subset p_Z^* F^1 = \tilde{F}^1$, which implies that

$$p_Z^* ([Z, [\phi]]_{\text{BF}}^{\text{vir}}) = [\tilde{Z}, p_Z^! [\phi]]_{\text{BF}}^{\text{vir}}$$

in $\text{CH}_{r+d}(\tilde{Z})$, where d is the rank of $\Omega_{\tilde{Z}/Z}$.

Since $p_Z: \tilde{Z} \rightarrow Z$ induces an isomorphism $p_Z^*: \text{CH}_r(Z) \rightarrow \text{CH}_{r+d}(\tilde{Z})$, we may assume that Z is affine for the purpose of comparing our construction of virtual fundamental classes with that of Behrend–Fantechi.

Assuming then that Z is affine, with a closed immersion $i: Z \rightarrow M$ into a smooth affine B -scheme M , we may take a reduced normalized representative $(F_1 \rightarrow F_0) \rightarrow (\mathcal{I}_Z / \mathcal{I}_Z^2 \rightarrow i^* \Omega_{M/B})$ of a given rank r perfect obstruction theory $[\phi]$; let r_i be $\text{rank}(F_i)$. In this case, we have $\mathfrak{C}(F_\bullet) = \mathfrak{C}_i \subset F^1$ and $d_{\mathfrak{C}} = r_0$. We have already identified our construction of the fundamental class $[\mathfrak{C}_i]_{H\mathbb{Z}}$ with the cycle class $[\mathfrak{C}_i] \in \text{CH}_{r_0}(\mathfrak{C}_i)$; we have also identified the push-forward map $i_{\mathfrak{C}*}: \text{CH}_{r_0}(\mathfrak{C}_i) \rightarrow \text{CH}_{r_0}(F^1)$ and the intersection with the 0-section $0_{F^1}^!: \text{CH}_{r_0}(F^1) \rightarrow \text{CH}_{r_0-r_1}(Z)$, as defined here, with the classical ones. This gives the identity of virtual fundamental classes in $\text{CH}_r(Z)$

$$[Z, [\phi]]_{\text{BF}}^{\text{vir}} = [Z, [\phi]]^{\text{vir}}.$$

Of course, the Behrend–Fantechi theory is defined for perfect obstruction theories on Deligne–Mumford stacks, whereas the theory presented here is limited to quasi-projective B -schemes for affine B .

8.4 Gromov–Witten invariants for oriented theories

If $[\phi]$ is a virtual rank 0 perfect obstruction theory on some $Z \in \mathbf{Sch}/B$ and $\mathcal{E}$ is an oriented ring spectrum in $\text{SH}(B)$, the virtual fundamental class lives in $\mathcal{E}_{0,0}^{\text{B.M.}}(Z) = \mathcal{E}^{0,0}(\pi_{Z!}(1_Z))$. If $\pi_Z: Z \rightarrow B$ is proper, we have the push-forward map in $\mathcal{E}$ -cohomology

$$\pi_{Z*}: \mathcal{E}_{a,b}^{\text{B.M.}}(Z) \rightarrow \mathcal{E}_{a,b}^{\text{B.M.}}(B) = \mathcal{E}^{-a,-b}(B)$$

induced by the map $\pi_Z^*: 1_B \rightarrow \pi_{Z!}(1_Z)$, giving the GW-invariant

$$\deg_{\mathcal{E}}([Z, [\phi]]^{\text{vir}}) := \pi_{Z*}([Z, [\phi]]^{\text{vir}}) \in \mathcal{E}^{0,0}(B).$$

This is the classical “degree of the virtual fundamental class” in case $\mathcal{E} = H\mathbb{Z}$. For more general theories, we may have non-zero invariants for perfect obstruction theories of non-zero ranks which give rise to non-zero degrees

$$\deg_{\mathcal{E}}([Z, [\phi]]^{\text{vir}}) := \pi_{Z*}([Z, [\phi]]^{\text{vir}}) \in \mathcal{E}_{2r,r}^{\text{B.M.}}(B) = \mathcal{E}^{-2r,-r}(B)$$

for $[\phi]$ of virtual rank r .

If we have a morphism $f: Z \rightarrow W$, one can twist $[Z, [\phi]]^{\text{vir}}$ by classes coming from W ; if Z is proper over B , pushing forward gives the descendant classes in $\mathcal{E}_{*,*}^{\text{B.M.}}(B)$.

8.5 Gromov–Witten invariants for SL-oriented theories

We now consider theories $\mathcal{E}$ which are not oriented in the sense of the previous section but are rather SL-oriented. This means that given a perfect complex $E_\bullet$ on some $Z \in \mathbf{Sch}/B$ of virtual

rank r and virtual determinant $\det E_\bullet$, there is a canonical isomorphism

$$\lambda_{E_\bullet} : \Sigma^{\mathbb{V}(E_\bullet) - \mathbb{V}(\mathcal{O}_Z^r)} \pi_Z^* \mathcal{E} \cong \Sigma^{\mathbb{V}(\det E_\bullet) - \mathbb{V}(\mathcal{O}_Z)} \pi_Z^* \mathcal{E}.$$

Moreover, for L a line bundle on Z , we have a canonical isomorphism

$$\Sigma^{\mathbb{V}(L^{\otimes 2}) - \mathbb{V}(\mathcal{O}_Z)} \pi_Z^* \mathcal{E} \cong \pi_Z^* \mathcal{E}.$$

This gives us the isomorphisms

$$\mathcal{E}_{a,b}^{\text{B.M.}}(Z, \mathbb{V}(E_\bullet)) \cong \mathcal{E}_{a+2r-2, b+r-1}^{\text{B.M.}}(Z, \mathbb{V}(\det(E_\bullet))) \quad \text{and} \quad \mathcal{E}_{a,b}^{\text{B.M.}}(Z) \cong \mathcal{E}_{a-2, b-1}^{\text{B.M.}}(Z, L^{\otimes 2}).$$

This also implies that if $\beta: V \rightarrow V$ is an automorphism of a vector bundle $V \rightarrow Z$, the induced map

$$\Sigma^\beta : \mathcal{E}_{a,b}^{\text{B.M.}}(Z, V) \rightarrow \mathcal{E}_{a,b}^{\text{B.M.}}(Z, V)$$

is given by multiplication by the automorphism $\langle \det \beta \rangle \in \text{End}_{\text{SH}(Z)}(1_Z)$.

Suppose that we have a rank r perfect obstruction theory $([\phi], E_\bullet)$ on some $Z \in \mathbf{Sch}/B$, with $\pi_Z: Z \rightarrow B$ projective over B . The virtual fundamental class $[Z, [\phi]]^{\text{vir}}$ lives in $\mathcal{E}^{\text{B.M.}}(Z, \mathbb{V}(E_\bullet)) \cong \mathcal{E}_{2r,r}^{\text{B.M.}}(Z, \mathbb{V}(\det E_\bullet) - \mathbb{V}(\mathcal{O}_Z))$. Given an isomorphism $\alpha: \det E_\bullet \rightarrow L^{\otimes 2}$ for some line bundle L on Z , we have the isomorphism

$$\lambda_\alpha : \mathcal{E}_{2r,r}^{\text{B.M.}}(Z, \mathbb{V}(\det E_\bullet) - \mathbb{V}(\mathcal{O}_Z)) \xrightarrow{\sim} \mathcal{E}_{2r,r}^{\text{B.M.}}(Z),$$

so we may push $\lambda_\alpha([Z, [\phi]]^{\text{vir}})$ forward by π_Z to give

$$\deg_{\mathcal{E}}([Z, [\phi]]^{\text{vir}}, \alpha) := \pi_{Z*}(\lambda_\alpha([Z, [\phi]]^{\text{vir}})) \in \mathcal{E}_{2r,r}^{\text{B.M.}}(B).$$

Example 8.3. We take $B = \text{Spec } k$, k a perfect field, $G = \{\text{Id}\}$ and $\mathcal{E} = H_0(\mathbb{S}_k)$. We have the sheaf of graded rings on $\mathbf{Sm}_k$ given by *Milnor–Witt K -theory*, $\mathcal{K}_*^{\text{MW}}$ (see [Mor12] for details). The 0th homology $H_0(\mathbb{S}_k)$ with respect to the homotopy t -structure on $\text{SH}(k)$ represents the theory of Milnor–Witt K -theory: for $X \in \mathbf{Sm}/k$, there is a canonical isomorphism

$$H_0(\mathbb{S}_k)^{a+b,b}(X) \cong H_{\text{Nis}}^a(X, \mathcal{K}_b^{\text{MW}}).$$

The homology $H_0(\mathbb{S}_k)$ is SL-oriented, giving the canonical isomorphism

$$H_0(\mathbb{S}_k)_{a,b}^{\text{B.M.}}(Z, v - \mathbb{V}(\mathcal{O}_Z^r)) \cong H_0(\mathbb{S}_k)_{a,b}^{\text{B.M.}}(Z, \det v - \mathbb{V}(\mathcal{O}_Z))$$

for v of virtual rank r on $Z \in \mathbf{Sch}/B$. This gives the isomorphism

$$H_0(\mathbb{S}_k)_{2n,n}^{\text{B.M.}}(Z, \mathbb{V}(E_\bullet)) \cong \widetilde{\text{CH}}_n(Z, \det E_\bullet)$$

for $Z \in \mathbf{Sch}/k$ and $E_\bullet \in D^{\text{perf}}(Z)$, where $\widetilde{\text{CH}}_*$ is the Chow–Witt theory of Barge–Morel [BM00] and Fasel [Fas08] (also called the oriented Chow groups). For example, $\widetilde{\text{CH}}_0(\text{Spec } k)$ is $\text{GW}(k)$, the Grothendieck–Witt ring of non-degenerate symmetric bilinear forms over k .

For $i: Z \hookrightarrow M$ with M smooth of dimension d_M over k and $(\phi, E_\bullet)$ a rank r perfect obstruction theory, this gives us the fundamental class and virtual fundamental class

$$[\mathfrak{C}_Z^{\text{st}}] \in \widetilde{\text{CH}}_{d_M}(\mathfrak{C}_i, \sigma_i^* \omega_{M/k}),$$

$$[Z, [\phi]]^{\text{vir}} \in H_0(\mathbb{S}_k)_{2r,r}^{\text{B.M.}}(Z, \mathbb{V}(\det E_\bullet) - \mathbb{V}(\mathcal{O}_Z)) = \widetilde{\text{CH}}_r(Z, \det E_\bullet).$$

Thus, if we have a rank 0 perfect obstruction theory $(E_\bullet, [\phi])$ on some $Z \in \mathbf{Sch}/k$, with Z projective over k , and an isomorphism $\alpha: \det E_\bullet \xrightarrow{\sim} L^{\otimes 2}$ for some line bundle L on Z , we have

$$\deg_{H_0(\mathbb{S}_k)}([Z, [\phi]]^{\text{vir}}, \alpha) := \pi_{Z*}(\lambda_\alpha([Z, [\phi]]^{\text{vir}})) \in H_0(\mathbb{S}_k)^{\text{B.M.}}(\text{Spec } k) = K_0^{\text{MW}}(k) = \text{GW}(k).$$

More generally, if $E_\bullet$ has virtual rank r , and we have a morphism $f: Z \rightarrow W$ with $W \in \mathbf{Sm}/k$, a line bundle L' on W , a line bundle L on Z and an isomorphism $\alpha: \det E_\bullet \otimes f^*L' \rightarrow L^{\otimes 2}$, then a class

$$\beta \in H^r(W, \mathcal{K}_{r+s}^{\text{MW}}(L')) = H_0(\mathbb{S}_k)^{2r+s, r+s}(W, \mathbb{V}(\mathcal{O}_W) - \mathbb{V}(L'))$$

gives via the cap product

$$\begin{aligned} H_0(\mathbb{S}_k)_{2r, r}^{\text{B.M.}}(Z, \mathbb{V}(\det E_\bullet) - \mathbb{V}(\mathcal{O}_Z)) \times H_0(\mathbb{S}_k)^{2r+s, r+s}(W, \mathbb{V}(\mathcal{O}_W) - \mathbb{V}(L')) \\ \xrightarrow{-\cap f^*(-)} H_0(\mathbb{S}_k)_{-s, -s}^{\text{B.M.}}(Z, \mathbb{V}(\det E_\bullet \otimes f^*L') - \mathbb{V}(\mathcal{O}_Z)) \end{aligned}$$

the element

$$\lambda_\alpha([Z, [\phi]^{\text{vir}}] \cap f^*\beta) \in H_0(\mathbb{S}_k)_{-s, -s}^{\text{B.M.}}(Z),$$

and we can define the descendant class

$$\begin{aligned} \deg_{H_0(\mathbb{S}_k)}^s([Z, [\phi]^{\text{vir}}] \cap f^*\beta, \alpha) &:= \pi_{Z*}(\lambda_\alpha([Z, [\phi]^{\text{vir}}] \cap f^*\beta)) \\ &\in H_0(\mathbb{S}_k)_{-s, -s}^{\text{B.M.}}(\text{Spec } k) = H_0(\mathbb{S}_k)^{s, s}(\text{Spec } k) = K_s^{\text{MW}}(k). \end{aligned}$$

There is a universal SL-oriented theory, MSL, with MSL_n the Thom space of the universal bundle $\tilde{E}_n \rightarrow \text{BSL}_n$. The group $\text{MSL}_n^{-2r, -r}(k)$ is non-zero for all $r \geq 2$ (and for $r = 0$), so we have a non-trivial target for the degree map for perfect obstruction theories of all non-negative ranks different from 1, assuming that the associated determinant bundle is a square.

9. Generating ideals, $\mathbb{A}^1$ -local degree and critical loci

The most elementary type of obstruction theory is one that is already normalized and reduced. Fix a scheme B . For $X \rightarrow B$ a B -scheme, we let $\pi_X: X \rightarrow B$ denote the structure morphism. Let $M \rightarrow B$ be a smooth B -scheme and $i: Z \hookrightarrow M$ a closed subscheme. This gives us the cone $\mathfrak{C}_i$ and the fundamental class $[\mathfrak{C}_i] \in \mathbb{S}_B^{\text{B.M.}}(\mathfrak{C}_i, \sigma_i^*T_M)$.

We suppose that we have a locally free sheaf $\mathcal{V}$ on M and an $\mathcal{O}_Z$ -linear surjective map $F: \mathcal{V}^\vee \otimes \mathcal{O}_Z \rightarrow \mathcal{I}_Z/\mathcal{I}_Z^2$; if M is quasi-projective, this always exists. We let $\partial_\phi: \mathcal{V}^\vee \otimes \mathcal{O}_Z \rightarrow \Omega_{M/B} \otimes \mathcal{O}_Z$ be the map $d \circ \phi_1$, with $d: \mathcal{I}_Z/\mathcal{I}_Z^2 \rightarrow \Omega_{M/B} \otimes \mathcal{O}_Z$ induced by the canonical derivation $d: \mathcal{O}_M \rightarrow \Omega_{M/B}$. This gives us the perfect obstruction theory

$$\phi = (F, \text{Id}): (\mathcal{V}^\vee \otimes \mathcal{O}_Z \xrightarrow{\partial_\phi} \Omega_{M/B} \otimes \mathcal{O}_Z) \rightarrow (\mathcal{I}_Z/\mathcal{I}_Z^2 \xrightarrow{d} \Omega_{M/B} \otimes \mathcal{O}_Z)$$

on Z , which is already reduced and normalized.

The map F induces the closed immersion of Z -schemes $i_\phi: \mathfrak{C}_i \hookrightarrow i^*V$, where $V \rightarrow M$ is the vector bundle $\mathbb{V}(\mathcal{V}^\vee)$. The associated virtual fundamental class is then

$$[Z, [\phi]]^{\text{vir}} := 0_{i_*V}^! i_{\phi*}([\mathfrak{C}_i]) \in \mathbb{S}_B^{\text{B.M.}}(Z, i^*T_M - i^*V).$$

Since ϕ is determined by F , we write this as $[Z, F]^{\text{vir}}$.

A choice of an isomorphism $\psi: T_M \xrightarrow{\sim} V$ (if one exists) simplifies this to a virtual fundamental class $[Z, F]^{\text{vir}} \in \mathbb{S}_B^{\text{B.M.}}(Z)$. If we pass to an SL-oriented theory $\mathcal{E}$ and V and T_M have the same rank, then we only need an isomorphism $\rho: \det \mathcal{V} \otimes \omega_{M/B} \otimes \mathcal{O}_Z \rightarrow \mathcal{L}^{\otimes 2}$ for some invertible sheaf $\mathcal{L}$ on Z to reduce to $[Z, F]_{\mathcal{E}}^{\text{vir}} \in \mathcal{E}^{\text{B.M.}}(Z)$. For example, if $\mathcal{V} = \Omega_{M/B}$, we have the identity $\det \mathcal{V} \otimes \omega_{M/B} = \omega_{M/B}^{\otimes 2}$. If M has even dimension $2m$ and $\mathcal{V} = \Omega_{M/B} \otimes \mathcal{L}$ for some invertible sheaf $\mathcal{L}$, then we have the canonical isomorphism $\det \mathcal{V} \otimes \omega_{M/B} \cong (\omega_{M/B} \otimes \mathcal{L}^{\otimes m})^{\otimes 2}$.

Example 9.1. Let R be a commutative ring, let $B = \operatorname{Spec} R$, and let $Z \subset \mathbb{A}_B^n$ be a closed subscheme. Let $I \subset R[X_1, \dots, X_n]$ be the ideal of Z , and choose $F_1, \dots, F_n \in I$ that generate I/I^2 as $\mathcal{O}_Z$ -module.

We have an $\mathcal{O}_Z$ -linear surjection $F: \Omega_{\mathbb{A}_B^n/B}^\vee \otimes \mathcal{O}_Z \rightarrow I/I^2$ defined by sending the basis element $\partial/\partial X_i$ to the image of F_i in I/I^2 . Using the trivialization of $T_{\mathbb{A}_B^n/B}$ and $T_{\mathbb{A}_B^n/B}^\vee$ by the bases $\{\partial/\partial X_i\}_i$ and $\{dX_i\}_i$, we have the canonical isomorphism

$$\Sigma^{i^* T_{\mathbb{A}_B^n/B} - i^* T_{\mathbb{A}_B^n/B}^\vee} \cong \operatorname{Id},$$

giving us the virtual fundamental class $[Z, F]^{\operatorname{vir}} \in \mathbb{S}_B^{\operatorname{B.M.}}(Z) = \mathbb{S}_B^{0,0}(Z_{\operatorname{B.M.}})$. Note that $[Z, F]^{\operatorname{vir}}$ depends only on the image of the F_i modulo I^2 .

DEFINITION 9.2 ($\mathbb{A}^1$ -local degree [KW19, Definition 10]). Let $B = \operatorname{Spec} A$ be an affine scheme, and let $g = (g_1, \dots, g_n): \mathbb{A}_B^n \rightarrow \mathbb{A}_B^n$ be a polynomial map, where $g_i \in A[X_1, \dots, X_n]$. Suppose that $g^{-1}(0_B)_{\operatorname{red}}$ is a disjoint union, $g^{-1}(0_B) = x \amalg x'$, with $g|_x: x \rightarrow B$ finite. Choose $U \subset \mathbb{A}^n$ an open neighborhood of x with $U \cap g^{-1}(0_B) = x$. The $\mathbb{A}^1$ -local degree $\delta_{\mathbb{A}^1}(g, x)$ of g along x is the element of $\mathbb{S}_B^{0,0}(B) = \operatorname{End}_{\operatorname{SH}(B)}(\mathbb{S}_B)$ given by stabilizing the composition

$$S_B^{2n,n} = \mathbb{P}_B^n / \mathbb{P}_B^{n-1} \rightarrow \mathbb{P}_B^n / \mathbb{P}_B^n \setminus x \xleftarrow{\sim} U/U \setminus x \xrightarrow{g|_{(U,U \setminus x)}} \mathbb{P}_B^n / \mathbb{P}_B^n \setminus \{0_B\} \xleftarrow{\sim} \mathbb{P}_B^n / \mathbb{P}_B^{n-1} = S_B^{2n,n}.$$

The map $U/U \setminus x \rightarrow \mathbb{P}_B^n / \mathbb{P}_B^n \setminus x$ is an isomorphism by excision, and the map $\mathbb{P}_B^n / \mathbb{P}_B^{n-1} \rightarrow \mathbb{P}_B^n / \mathbb{P}_B^n \setminus \{0_B\}$ is an isomorphism by homotopy invariance. It is easy to see that this composition is independent of the choice of U .

Remarks 9.3. Let $g: \mathbb{A}_B^n \rightarrow \mathbb{A}_B^n$ and $x \subset g^{-1}(0_B) \subset \mathbb{A}_B^n$ be as in Definition 9.2.

1. Suppose that x is a disjoint union $x = x_1 \amalg x_2$. Then

$$\delta_{\mathbb{A}^1}(g, x) = \delta_{\mathbb{A}^1}(g, x_1) + \delta_{\mathbb{A}^1}(g, x_2).$$

This follows by considering the Nisnevich cover $U \setminus x_2 \amalg U \setminus x_1 \rightarrow U$, where $U \subset \mathbb{A}^n$ is an open subscheme with $g^{-1}(0_B) \cap U = x$

2. Letting $f: B' \rightarrow B$ be a morphism, we have $f^*(\delta_{\mathbb{A}^1}(g, x)) \in \mathbb{S}_B^{0,0}(B')$, we have the pull-back morphism $g': \mathbb{A}_{B'}^n \rightarrow \mathbb{A}_{B'}^n$ and $x' := f^{-1}(x) \subset g'^{-1}(0_{B'})$. Then

$$f^*(\delta_{\mathbb{A}^1}(g, x)) = \delta_{\mathbb{A}^1}(g', x').$$

Indeed, if we take $U \subset \mathbb{A}_B^n$ with $g^{-1}(0_B)_{\operatorname{red}} \cap U = x$, then letting $U' = U \times_B B'$, we have $g'^{-1}(0_{B'})_{\operatorname{red}} \cap U' = x'$, and the result follows by applying the base-change isomorphism $\operatorname{Ex}(\Delta_\#^*)$ to the sequence of maps defining $\delta_{\mathbb{A}^1}(g, x)$ and $\delta_{\mathbb{A}^1}(g', x')$.

3. In case $B = \operatorname{Spec} k$ with k a perfect field, Morel's theorem identifies $\mathbb{S}_k^{0,0}(\operatorname{Spec} k)$ with $\operatorname{GW}(k)$, so we have $\delta_{\mathbb{A}^1}(g, x) \in \operatorname{GW}(k)$.

Remark 9.4. Let K be a perfect field, $f: \mathbb{A}_K^n \rightarrow \mathbb{A}_K^n$ a linear automorphism and $p: \mathbb{A}_K^n \rightarrow \operatorname{Spec} K$ the projection. Then the map $f_*: p_!(1_{\mathbb{A}^n}) \rightarrow p_!(1_{\mathbb{A}^n})$ is multiplication by the rank 1 quadratic form $\langle \det f \rangle$. Indeed, we may use a matrix representation for f . Since $\operatorname{SL}_n(K)$ is generated by elementary matrices, f is $\mathbb{A}^1$ -homotopic to the map $(x_1, \dots, x_n) \mapsto (ux_1, x_2, \dots, x_n)$ with $u = \det f$.

Via the canonical isomorphism $p_! = p_\# \circ \Sigma^{-T_{\mathbb{A}^n/K}}$, we have $p_!(1_{\mathbb{A}^n}) \cong \Sigma_{\mathbb{P}^1}^n(1_{\operatorname{Spec} K})$, with the action of f going over to $\Sigma_{\mathbb{P}^1}^{n-1}(p_{\mathbb{A}^1!}(\det f))$ with $\det f: \mathbb{A}_K^1 \rightarrow \mathbb{A}_K^1$ the multiplication map. This reduces us to the case $n = 1$. In this case, via the isomorphism $p_!(1_{\mathbb{A}^1}) \cong \Sigma_{\mathbb{P}^1}(1_{\operatorname{Spec} K})$, multiplication by $u = \det f$ goes over to the map $\mathbb{P}^1 \rightarrow \mathbb{P}^1$ sending $[x_0 : x_1]$ to $[x_0 : ux_1]$. Morel's

isomorphism $\mathrm{GW}(K) \rightarrow \mathrm{End}_{\mathrm{SH}(K)}(1_{\mathrm{Spec} K})$ sends the quadratic form $\langle u \rangle$ to the stable version of this latter map.

LEMMA 9.5. *Let k be a perfect field, let $g: \mathbb{A}_k^n \rightarrow \mathbb{A}_k^n$ be a polynomial map, $g = (g_1, \dots, g_n)$, and suppose that we have closed points $x_1, \dots, x_r$ of $\mathbb{A}_k^n$ which are all isolated points of $g^{-1}(0)$. Let $Z = \{x_1, \dots, x_r\}$, and suppose that the map g is étale in a neighborhood of Z . Let $\pi_Z: Z \rightarrow \mathrm{Spec} k$ be the projection, giving the push-forward map $\pi_{Z*}: \mathbb{S}_k^{\mathrm{B.M.}}(Z) \rightarrow \mathbb{S}_k^{\mathrm{B.M.}}(\mathrm{Spec} k) = \mathbb{S}_k^{0,0}(\mathrm{Spec} k)$. Then*

$$\pi_{Z*}([Z, g]^{\mathrm{vir}}) = \delta_{\mathbb{A}^1}(g, Z).$$

Proof. We use the standard basis $dX_1, \dots, dX_n$ for $\Omega_{\mathbb{A}^n/k}$. This gives us the canonical isomorphisms $\mathfrak{C}_{Z \subset \mathbb{A}^n} \cong \mathbb{A}_Z^n$ and $T_{\mathbb{A}^n/k}^\vee \otimes \mathcal{O}_Z \cong \mathbb{A}_Z^n$ via which the isomorphism $i_g: \mathfrak{C}_{Z \subset \mathbb{A}^n} \rightarrow T_{\mathbb{A}^n/k}^\vee \otimes \mathcal{O}_Z$ becomes the $\mathcal{O}_Z$ -linear map $\mathbb{A}_Z^n \rightarrow \mathbb{A}_Z^n$ with matrix the Jacobian matrix $\mathcal{J}(g) = (\partial g_i / \partial X_j)$ restricted to Z . Letting $J(g) = \det \mathcal{J}(g)$, it follows from Remark 9.4 and the definition of $[Z, g]^{\mathrm{vir}}$ that $[Z, g]^{\mathrm{vir}}$ is the rank 1 quadratic form $\langle J(g)(Z) \rangle \in \mathrm{GW}(Z) = \mathbb{S}_k^{0,0}(Z)$; that is, at $x_i \in Z$, the form $[Z, g]^{\mathrm{vir}}$ takes the values $\langle J(g)(x_i) \rangle \in \mathrm{GW}(k(x_i)) = \mathbb{S}_k^{0,0}(x_i)$.

Since $Z \rightarrow \mathrm{Spec} k$ is étale, we have $Z_{\mathrm{B.M.}} = Z$, and the map $\pi_{Z*}: \mathbb{S}_k^{0,0}(Z) \rightarrow \mathbb{S}_k^{0,0}(\mathrm{Spec} k)$ is identified with the trace map

$$\mathrm{Tr}_{k[Z]/k} = \sum_i \mathrm{Tr}_{k(x_i)/k}: \prod_{i=1}^r \mathrm{GW}(k(x_i)) \rightarrow \mathrm{GW}(k)$$

(see [Hoy14, Lemma 5.3]). By [KW19, Proposition 14], we have $\delta_{\mathbb{A}^1}(g, Z) = \mathrm{Tr}_{k[Z]/k} \langle J(g)(Z) \rangle$, thus $\pi_{Z*}([Z, g]^{\mathrm{vir}}) = \delta_{\mathbb{A}^1}(g, Z)$, as claimed. $\square$

We can remove the condition that g is étale along Z by a deformation argument, taken from [KW19].

PROPOSITION 9.6. *Let k be perfect field of characteristic different from 2. Let $g = (g_1, \dots, g_n): \mathbb{A}_k^n \rightarrow \mathbb{A}_k^n$ be a polynomial map. Suppose that $g^{-1}(0)$ is a disjoint union $Z \amalg Z'$ of closed subschemes with Z of pure dimension 0, and write $Z_{\mathrm{red}} = z$. Then $\pi_{Z*}([Z, g]^{\mathrm{vir}}) \in \mathbb{S}_k^{0,0}(k) = \mathrm{GW}(k)$ is the $\mathbb{A}^1$ -local degree $\delta_{\mathbb{A}^1}(g, z)$.*

Proof. Since the map $\mathrm{GW}(k) \rightarrow \mathrm{GW}(k')$ is injective for $k \subset k'$ a field extension that is the union over a tower of finite extensions $k \subset k_\alpha$ of odd degree, we may assume that k is an infinite field.

Let $\mathfrak{m} \subset k[X_1, \dots, X_r]$ be the ideal of z . Then Z is a complete intersection component of the subscheme of $\mathbb{A}_k^n$ defined by $(g_1, \dots, g_n)$. Moreover, the cone $\mathfrak{C}_{Z \subset \mathbb{A}^n}$, the fundamental class $[\mathfrak{C}_{Z \subset \mathbb{A}^n}]$ and the map ϕ_g are unchanged if we replace the g_i with polynomials $g'_i \in k[X_1, \dots, X_n]$ such that $g'_i - g_i \in \mathfrak{m}^b$ for $b \gg 0$. By [KW19, Lemma 15(3)], the same holds for the $\mathbb{A}^1$ -local degree $\delta_{\mathbb{A}^1}(g, z)$.

Adding to each g_i a suitably general $h_i \in (X_1, \dots, X_n)^b$ for sufficiently high b , we may assume that the g_i all have the same degree d and that the map g extends to a morphism $\bar{g}: \mathbb{P}_k^n \rightarrow \mathbb{P}_k^n$ satisfying

- (1) $\bar{g}$ is finite, flat and of degree prime to the characteristic of k ;
 - (2) $\bar{g}$ is étale at each point of $F^{-1}(0) \setminus Z$;
 - (3) $\bar{g}^{-1}(\mathbb{A}^n) \subset \mathbb{A}^n$.
- (9.1)

This is proven in [KW19, Lemmas 19–21, Proposition 22] under the assumption that Z is supported at 0, but the same proof works in the more general case. We construct a morphism $\mathcal{G}: \mathbb{P}_{\mathbb{A}^1}^n \rightarrow \mathbb{P}_{\mathbb{A}^1}^n$ over $\mathbb{A}^1$ satisfying condition (1) of (9.1), and in addition

- (2') there is an open subset $V \subset \mathbb{A}^1$ containing 1 such that $\mathcal{G}$ is étale over a neighborhood of $\mathcal{G}^{-1}(0 \times V)$;
- (3') $\mathcal{G}^{-1}(\mathbb{A}^n \times \mathbb{A}^1) \subset \mathbb{A}^n \times \mathbb{A}^1$;
- (4') letting $\mathcal{G}_\lambda: \mathbb{P}_k^n \rightarrow \mathbb{P}_k^n$ be the pull-back of $\mathcal{G}$ over $\lambda \in \mathbb{A}^1(k)$, we have $\mathcal{G}_0 = \bar{g}$.

Indeed, since $\bar{g}$ is finite and of degree prime to the characteristic, $\bar{g}$ is étale over a dense open subset $U \subset \mathbb{P}_k^n$. Since k is infinite, there is a k -point $u \in U$, and thus $\bar{g}$ is étale over a neighborhood of $\bar{g}^{-1}(u)$. Let $\phi: \mathbb{A}^1 \rightarrow \text{Aut} \mathbb{P}_k^n$ be the morphism sending t to translation by tu (in the Euclidean subgroup of affine linear automorphisms of $\mathbb{A}^n$, embedded as a subgroup of $\text{Aut}(\mathbb{P}^n)$ in the usual way), and define $\mathcal{G}$ by $\mathcal{G}(x, t) = \phi(-t)(\bar{g}(x))$.

Let $Z \subset \mathbb{A}^n \times \mathbb{A}^1$ be the closed subscheme $\mathcal{G}^{-1}(0 \times \mathbb{A}^1)$. Then $\pi_Z: Z \rightarrow \mathbb{A}^1$ is finite and flat with fiber over 0 a disjoint union $Z \amalg Z'$ of two closed subschemes. Moreover, Z' is reduced, and the map g is étale on a neighborhood of Z' . Let $\tilde{\mathcal{G}} = \mathcal{G}|_{\mathbb{A}^n \times \mathbb{A}^1}: \mathbb{A}^n \times \mathbb{A}^1 \rightarrow \mathbb{A}^n \times \mathbb{A}^1$, and let $\tilde{\mathcal{G}}_\lambda$ be the restriction of $\tilde{\mathcal{G}}$ over $\lambda \in \mathbb{A}^1(k)$, so $\tilde{\mathcal{G}}_0 = g$. Similarly, let Z_λ be the fiber of Z over λ .

Replacing $\text{SH}(k)$ with $\text{SH}(\mathbb{A}_k^1)$, we have the virtual fundamental class

$$[Z, \tilde{\mathcal{G}}]^{\text{vir}} \in \text{Hom}_{\text{SH}(\mathbb{A}_k^1)}(\pi_{Z!}(1_Z), \mathbb{S}_{\mathbb{A}^1}),$$

its push-forward $\pi_{Z*}([Z, \tilde{\mathcal{G}}]^{\text{vir}}) \in \text{End}_{\text{SH}(\mathbb{A}_k^1)}(\mathbb{S}_{\mathbb{A}^1})$ and the $\mathbb{A}^1$ -local degree of $\tilde{\mathcal{G}}$, $\delta_{\mathbb{A}^1}(\tilde{\mathcal{G}}, Z_{\text{red}}) \in \text{End}_{\text{SH}(\mathbb{A}_k^1)}(\mathbb{S}_{\mathbb{A}^1})$. Since Z is flat over $\mathbb{A}^1$, we have for each $\lambda \in \mathbb{A}^1(k)$, with inclusion $i_\lambda: \text{Spec } k \rightarrow \mathbb{A}^1$, the identity

$$i_\lambda^*(\pi_{Z*}([Z, \tilde{\mathcal{G}}]^{\text{vir}})) = \pi_{Z_\lambda*}([Z, \tilde{\mathcal{G}}_\lambda]^{\text{vir}})$$

(see Proposition 7.5). Similarly,

$$i_\lambda^*(\delta_{\mathbb{A}^1}(\tilde{\mathcal{G}}, Z_{\text{red}})) = \delta_{\mathbb{A}^1}(\tilde{\mathcal{G}}_\lambda, Z_{\lambda \text{red}}).$$

On the other hand, for $p: \mathbb{A}_k^1 \rightarrow \text{Spec } k$ the projection, we have

$$\text{End}_{\text{SH}(\mathbb{A}_k^1)}(\mathbb{S}_{\mathbb{A}^1}) = \text{Hom}_{\text{SH}(\mathbb{A}^1)}(1_{\mathbb{A}^1}, p^*(1_k)) = \text{Hom}_{\text{SH}(k)}(p_{\#}1_{\mathbb{A}^1}, 1_k) = \mathbb{S}_k^{0,0}(\mathbb{A}_k^1),$$

so by homotopy invariance and Remark 9.3, we have

$$\begin{aligned} \pi_{Z_1*}([Z, \tilde{\mathcal{G}}_1]^{\text{vir}}) &= \pi_{Z_0*}([Z, \tilde{\mathcal{G}}_0]^{\text{vir}}) = \pi_{Z*}([Z, g]^{\text{vir}}) + \pi_{Z'*}([Z', g]^{\text{vir}}), \\ \delta_{\mathbb{A}^1}(\tilde{\mathcal{G}}_1, Z_{1\text{red}}) &= \delta_{\mathbb{A}^1}(\tilde{\mathcal{G}}_0, Z_{0\text{red}}) = \delta_{\mathbb{A}^1}(g_0, Z_{\text{red}}) + \delta_{\mathbb{A}^1}(g, Z'). \end{aligned}$$

By Lemma 9.5, we have

$$\pi_{Z_1*}([Z, \tilde{\mathcal{G}}_1]^{\text{vir}}) = \delta_{\mathbb{A}^1}(Z_1, \tilde{\mathcal{G}}_1) \quad \text{and} \quad \pi_{Z'*}([Z', g]^{\text{vir}}) = \delta_{\mathbb{A}^1}(g, Z'),$$

so $\pi_{Z*}([Z, g]^{\text{vir}}) = \delta_{\mathbb{A}^1}(g, Z_{\text{red}})$. $\square$

Remark 9.7. Proposition 9.6 deals with the trivial case of a virtual fundamental class, namely, the case of a local complete intersection. In the classical case with values in the Chow groups, the virtual fundamental class is just the cycle class of the local complete intersection, in other words, the result given by classical intersection theory. The refined version of this trivial case is still interesting, as it points out how the classical intersection multiplicity is replaced by the $\mathbb{A}^1$ -local degree.

As an example of the above construction, we have the virtual fundamental class of the critical locus of a function $f: M \rightarrow \mathbb{A}^1$ with M a smooth k -scheme. The critical locus Z of f is simply the 0-subscheme of the section df of $\Omega_{M/k}$.

Taking the Hessian matrix of f gives the globally defined morphism of sheaves

$$H: \mathcal{O}_Z \rightarrow \mathcal{H}om(\Omega_{M/k}^\vee \otimes \mathcal{O}_Z, \Omega_{M/k} \otimes \mathcal{O}_Z).$$

In a coordinate neighborhood with coordinates $x_1, \dots, x_n$, the map $H(1)$ sends $\partial/\partial x_i \otimes f$ to $\sum_j dx_j \otimes \partial^2 f / \partial x_i \partial x_j$. We have as well the commutative diagram of sheaves on Z

$$\begin{array}{ccc} \Omega_{M/k}^\vee \otimes \mathcal{O}_Z & \xrightarrow{H} & \Omega_{M/k} \otimes \mathcal{O}_Z \\ \text{ev}_{df} \downarrow & & \parallel \\ \mathcal{I}_Z / \mathcal{I}_Z^2 & \xrightarrow{d} & \Omega_{M/k} \otimes \mathcal{O}_Z, \end{array}$$

where ev_{df} is the map evaluating a vector field on df , giving us a perfect obstruction theory on Z .

Clearly $[\Omega_{M/k}^\vee \otimes \mathcal{O}_Z \xrightarrow{H} \Omega_{M/k} \otimes \mathcal{O}_Z]$ has virtual rank 0 and virtual determinant $\omega_{M/k}^{\otimes 2}$. Thus, this perfect obstruction theory gives us a virtual fundamental class

$$[Z, \partial_f]^{\text{vir}} \in \mathcal{E}_{0,0}^{\text{B.M.}}(Z)$$

for any cohomology theory $\mathcal{E} \in \text{SH}(k)$. If, in addition, Z is projective over k , and $\mathcal{E}$ is SL-oriented, then we have the push-forward map, giving

$$\deg_{\mathcal{E}}([Z, \partial_f]^{\text{vir}}) := \pi_{Z*}([Z, \partial_f]^{\text{vir}}) \in \mathcal{E}^{0,0}(\text{Spec } k).$$

For instance, we may take $\mathcal{E}$ to be hermitian K -theory or Chow–Witt theory; we then have $\deg_{\mathcal{E}}([Z, \partial_f]^{\text{vir}}) \in \text{GW}(k)$ with rank the degree of the usual virtual fundamental class.

One can generalize this construction slightly. Assuming that the critical subscheme of f is a disjoint union of components, $Z \amalg Z'$, we can restrict the whole construction to Z , giving the virtual fundamental class $[Z, \partial_f]^{\text{vir}} \in \mathcal{E}_{0,0}^{\text{B.M.}}(Z)$.

As a direct consequence of Proposition 9.6 we have the following description of the virtual fundamental class of a zero-dimensional component of the critical locus.

COROLLARY 9.8. *Let $f: \mathbb{A}^n \rightarrow \mathbb{A}^1$ be a function, and suppose that the critical subscheme of f is a disjoint of closed subschemes $Z \amalg Z'$ with Z of pure dimension 0. We have the polynomial map*

$$\partial f = (\partial f / \partial X_1, \dots, \partial f / \partial X_n): \mathbb{A}^n \rightarrow \mathbb{A}^n.$$

Then $\pi_{Z}([Z, \partial_f]^{\text{vir}}) = \delta_{\mathbb{A}^1}(\partial f, Z_{\text{red}})$ in $\text{GW}(k)$.*

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