The Donaldson–Thomas partition function of the banana manifold

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With an appendix coauthored with Stephen Pietromonaco

ABSTRACT

A banana manifold is a compact Calabi–Yau threefold, fibered by Abelian surfaces, whose singular fibers have a singular locus given by a “banana configuration of curves.” A basic example is given by $X_{\text{ban}} := \text{Bl}_{\Delta}(S \times_{\mathbb{P}^1} S)$, the blowup along the diagonal of the fibered product of a generic rational elliptic surface $S \rightarrow \mathbb{P}^1$ with itself.

In this paper, we give a closed formula for the Donaldson–Thomas partition function of the banana manifold X_{ban} restricted to the 3-dimensional lattice Γ of curve classes supported in the fibers of $X_{\text{ban}} \rightarrow \mathbb{P}^1$. It is given by

$$Z_{\Gamma}(X_{\text{ban}}) = \prod_{d_1, d_2, d_3 \geq 0} \prod_k (1 - p^k Q_1^{d_1} Q_2^{d_2} Q_3^{d_3})^{-12c(\|\mathbf{d}\|, k)},$$

where $\|\mathbf{d}\| = 2d_1d_2 + 2d_2d_3 + 2d_3d_1 - d_1^2 - d_2^2 - d_3^2$ and the coefficients $c(a, k)$ have a generating function given by an explicit ratio of theta functions. This formula has interesting properties and is closely related to the equivariant elliptic genera of $\text{Hilb}(\mathbb{C}^2)$. In an appendix with S. Pietromonaco, it is shown that the corresponding genus g Gromov–Witten potential F_g is a genus 2 Siegel modular form of weight $2g - 2$ for $g \geq 2$; namely, it is the Skoruppa–Maass lift of a multiple of an Eisenstein series, specifically

$$\frac{12|B_{2g}|}{2g(2g-2)!} E_{2g}(\tau).$$

1. Introduction

1.1 Donaldson–Thomas invariants

The Donaldson–Thomas invariants of a Calabi–Yau threefold X encode subtle information about the enumerative geometry of X . They are a mathematical incarnation of counts of BPS states in B-model topological string theory compactified on X .

Let X be a Calabi–Yau threefold, that is, a smooth complex threefold with trivial canonical

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class. Let $\beta \in H_2(X, \mathbb{Z})$ be a curve class, let $n \in \mathbb{Z}$, and let

$$\text{Hilb}^{\beta, n}(X) = \{Z \subset X : [Z] = \beta, \chi(\mathcal{O}_Z) = n\}$$

be the Hilbert scheme parameterizing dimension 1 subschemes in the class β and having holomorphic Euler characteristic n . The Donaldson–Thomas invariant $\text{DT}_{\beta, n}(X)$ can be defined [Beh09] as a weighted Euler characteristic of the Hilbert scheme:

$$\text{DT}_{\beta, n}(X) = e(\text{Hilb}^{\beta, n}(X), \nu) := \sum_{k \in \mathbb{Z}} k \cdot e(\nu^{-1}(k)),$$

where $e(\cdot)$ is topological Euler characteristic and

$$\nu: \text{Hilb}^{\beta, n}(X) \rightarrow \mathbb{Z}$$

is Behrend’s constructible function. One can regard $\text{DT}_{\beta, n}(X)$ as a virtual count of the number of curves in the class β with Euler characteristic n .

The Donaldson–Thomas partition function is a generating function for the invariants which we define¹ as

$$Z(X) = \sum_{\beta \in H_2(X, \mathbb{Z})} \sum_{n \in \mathbb{Z}} \text{DT}_{\beta, n}(X) Q^\beta (-p)^n.$$

Here $Q^\beta = Q_1^{d_1} \cdots Q_r^{d_r}$, where $\beta = d_1 C_1 + \cdots + d_r C_r$ and $\{C_1, \dots, C_r\}$ is a basis for $H_2(X, \mathbb{Z})$ chosen such that $d_i \geq 0$ for all effective curve classes. The partition function $Z(X)$ is then a formal power series in $Q_1, \dots, Q_r$ whose coefficients are formal Laurent series in p .

The Donaldson–Thomas partition function is very hard to compute.

Remark 1.1. Currently, there is not a single compact Calabi–Yau threefold X with trivial first Betti number, for which $Z(X)$ is completely known, not even conjecturally.²

Given some sublattice $\Gamma \subset H_2(X, \mathbb{Z})$, we can define a restricted partition function:

$$Z_\Gamma(X) = \sum_{\beta \in \Gamma} \sum_{n \in \mathbb{Z}} \text{DT}_{\beta, n}(X) Q^\beta (-p)^n.$$

Even for restricted partition functions, there are very few results for compact X . For elliptic or $K3$ fibrations $\pi: X \rightarrow B$, the restricted partition functions $Z_\Gamma(X)$ have been computed for Γ given by the lattice of fiber classes, by Toda [Tod12, Theorem 6.9] in the case of elliptic fibrations, and by Maulik, Pandharipande, and Thomas [MP13, KMPS10] in the case of $K3$ fibrations. In the case of $K3$ fibrations, the partition functions exhibit modularity properties.

In this paper, we compute $Z_\Gamma(X)$ when X is a certain kind of Calabi–Yau threefold (a banana manifold) and Γ is a rank 3 lattice. We give an explicit product formula for $Z_\Gamma(X)$, we derive a generating function for the corresponding Gopakumar–Vafa invariants, and we show (assuming the Gromov–Witten/Donaldson–Thomas (GW/DT) correspondence) that the associated Gromov–Witten potentials are Siegel modular forms.

¹Our insertion of a minus sign is somewhat non-standard.

²In the case of positive first Betti number, the ordinary Donaldson–Thomas invariants are zero except for trivial cases. However, there is a modification of the usual invariants (the *reduced* Donaldson–Thomas invariants) which leads to an interesting theory. In the cases of Abelian threefolds, products of a $K3$ surface and an elliptic curve, and quotients of such a product by a cyclic group acting K -trivially on both factors, we do have complete conjectural answers for the partition functions of the modified invariants [OP16, BOPY18, BO18]. The case of a product of a $K3$ surface and an elliptic curve has now been proven [OP18].

1.2 Banana manifolds and their partition functions

We study a class of compact Calabi–Yau threefolds which we call banana manifolds. The basic example, which we denote by X_{ban} and refer to as the basic banana manifold (or “the” banana manifold for short), is defined as follows.³

DEFINITION 1.2. Let $S \rightarrow \mathbb{P}^1$ be a generic rational elliptic surface. Let

$$X_{\text{ban}} = \text{Bl}_{\Delta}(S \times_{\mathbb{P}^1} S)$$

be the fibered product of S with itself blown up along the diagonal Δ ; see Figure 1.

FIGURE 1. The banana manifold $X_{\text{ban}} = \text{Bl}_{\Delta}(S \times_{\mathbb{P}^1} S)$

The map $S \rightarrow \mathbb{P}^1$ is singular at 12 points which gives rise to 12 conifold singularities in the fibered product which all lie on the divisor Δ . Consequently, X_{ban} is a conifold resolution of $S \times_{\mathbb{P}^1} S$. It is a simply connected compact Calabi–Yau threefold with Hodge numbers $h^{1,1}(X_{\text{ban}}) = 20$ and $h^{1,2}(X_{\text{ban}}) = 8$ (see Section 3).

The generic fiber of $\pi: X_{\text{ban}} \rightarrow \mathbb{P}^1$ is $E \times E$, the product of an elliptic curve with itself, and π has 12 singular fibers which are non-normal toric surfaces F_{ban} , each of which is a compactification of $\mathbb{C}^* \times \mathbb{C}^*$ by a *banana configuration* of rational curves.

DEFINITION 1.3. A *banana configuration*⁴ in a Calabi–Yau threefold X is a union $C = C_1 \cup C_2 \cup C_3$ of three curves $C_i \cong \mathbb{P}^1$ with $N_{C_i/X} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ such that $C_1 \cap C_2 = C_2 \cap C_3 = C_3 \cap C_1 = \{p, q\}$, where $p, q \in X$ are distinct points. Moreover, there exist coordinates on formal neighborhoods of p and q such that the curves C_i are given by the coordinate axes in those coordinates. See Figure 2; the meaning of the picture on the right is discussed in Section 4.5.

The banana curves C_1, C_2, C_3 of any of the singular fibers generate

$$\Gamma \subset H_2(X_{\text{ban}}, \mathbb{Z}),$$

the lattice of the fiber classes (Lemma 3.3). Let

$$\beta_{\underline{d}} = d_1 C_1 + d_2 C_2 + d_3 C_3,$$

³Essentially, this construction of the banana manifold is mentioned briefly in [KL19, end of § 5.2]; we learned about this from Georg Oberdieck.

⁴The Strominger–Yau–Zaslow (SYZ) mirror of this configuration was studied by Abouzaid, Auroux, and Katzarkov in [AAK16]. The banana configuration also appears in the Landau–Ginzburg mirror of a genus 2 curve as studied by Gross, Katzarkov, and Ruddat in [GKR17] and Ruddat in [Rud17].

FIGURE 2. The banana configuration of rational curves

and define

$$\|\underline{\mathbf{d}}\| = 2d_1d_2 + 2d_2d_3 + 2d_3d_1 - d_1^2 - d_2^2 - d_3^2.$$

This quadratic form is twice the intersection form of a smooth fiber, by which we mean that if $\beta_{\underline{\mathbf{d}}}$ is represented by a cycle C supported on a smooth fiber, then $\|\underline{\mathbf{d}}\| = 2C \cdot C$, where $C \cdot C$ is the self-intersection of C in the smooth fiber.

We define banana manifolds in general as follows.

DEFINITION 1.4. We say that a compact Calabi–Yau threefold X is a *banana manifold* with N banana fibers if there is an Abelian surface fibration $\pi: X \rightarrow \mathbb{P}^1$, where the singular locus of π consists of N disjoint banana configurations and the smooth locus of π is an Abelian group scheme over $\mathbb{P}^1$ whose natural action on itself extends to an action on X .

Our main theorem is the following.

THEOREM 1.5. Let X_{ban} be the basic banana manifold, and let $\Gamma \cong \mathbb{Z}^3$ be the lattice of classes supported on the fibers of $\pi: X_{\text{ban}} \rightarrow \mathbb{P}^1$. Then the Donaldson–Thomas partition function of X_{ban} restricted to Γ is given by

$$Z_{\Gamma}(X_{\text{ban}}) = \prod_{d_1, d_2, d_3 \geq 0} \prod_k (1 - p^k Q_1^{d_1} Q_2^{d_2} Q_3^{d_3})^{-12c(\|\underline{\mathbf{d}}\|, k)},$$

where the second product is over all $k \in \mathbb{Z}$ unless $(d_1, d_2, d_3) = (0, 0, 0)$, in which case $k > 0$. The $c(\|\underline{\mathbf{d}}\|, k)$ are positive integers given by

$$\sum_{a=-1}^{\infty} \sum_{k \in \mathbb{Z}} c(a, k) Q^a y^k = \frac{\sum_{k \in \mathbb{Z}} Q^{k^2} (-y)^k}{\left(\sum_{k \in \mathbb{Z} + \frac{1}{2}} Q^{2k^2} (-y)^k \right)^2} = \frac{\theta_4(2\tau, z)}{\theta_1(4\tau, z)^2},$$

where θ_4 and θ_1 are the usual theta functions with $Q = e^{2\pi i \tau}$ and $y = e^{2\pi i z}$.

Remark 1.6. While we have formulated the theorem for the basic banana manifold, our proof will work for banana manifolds in general, where the 12 in the exponent on the right-hand side will be replaced by N , the number of banana configurations. This also applies to Corollary 1.9, Corollary 1.10, and Theorem 1.11 stated below with the number 12 replaced by N in each case.

Remark 1.7. The coefficients $c(a, k)$ are also the coefficients of the equivariant elliptic genus of the plane $\mathbb{C}^2$; see Section 2.

Remark 1.8. The study of the partition function of X_{ban} for the rank 4 lattice spanned by the fiber classes and one section class has been initiated by Leigh [Lei19], who has obtained the partition functions corresponding to various affine planes in the rank 4 lattice.

COROLLARY 1.9 (Pietromonaco). *Assuming that the GW/DT correspondence holds for X_{ban} , the genus g Gromov–Witten potential function $F_g(Q_1, Q_2, Q_3)$ is a meromorphic Siegel modular form of weight $2g - 2$ for all $g \geq 2$, where $Q_1 = e^{2\pi iz}$, $Q_2 = e^{2\pi i(\tau - z)}$, $Q_3 = e^{2\pi i(\sigma - z)}$, and where $\begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix} \in \mathbb{H}_2$ is in the genus 2 Siegel upper half plane. Namely, F_g is given by the Skoruppa–Maass lift of $a_g E_{2g}(\tau)$, the $(2g)$ th Eisenstein series multiplied by the constant*

$$a_g = \frac{12|B_{2g}|}{2g(2g-2)!}.$$

See Appendix 2 for full definitions of the terms used in the corollary. See Remark A.4 for a natural interpretation of the corollary in terms of mirror symmetry.

Our Theorem 1.5 also completely determines the Gopakumar–Vafa invariants of the banana manifold for the fiber classes $\beta_{\underline{\mathbf{d}}}$. One corollary is the following.

COROLLARY 1.10. *The Gopakumar–Vafa invariants of X_{ban} in the class $\beta_{\underline{\mathbf{d}}}$ depend only on $\|\underline{\mathbf{d}}\|$. We thus streamline the notation by writing*

$$n_{\beta_{\underline{\mathbf{d}}}}^g(X_{\text{ban}}) = n_a^g(X_{\text{ban}}), \quad \text{where } a = \|\underline{\mathbf{d}}\|.$$

The property that $n_{\beta_{\underline{\mathbf{d}}}}^g(X_{\text{ban}})$ depends only on $\|\underline{\mathbf{d}}\|$ and so, in particular, the invariants are independent of the divisibility of $\beta_{\underline{\mathbf{d}}}$, is an unusual property of X_{ban} which is also shared by the local $K3$ surface by a deep result of Pandharipande–Thomas [PT16].

We can reformulate our main result in terms of the Gopakumar–Vafa invariants. After some manipulation of generating functions (see Sections 2 and A.5), we can deduce the following result.

THEOREM 1.11. *The Gopakumar–Vafa invariants $n_{\beta_{\underline{\mathbf{d}}}}^g(X_{\text{ban}}) = n_a^g(X_{\text{ban}})$ in the classes $\beta_{\underline{\mathbf{d}}}$ with $\|\underline{\mathbf{d}}\| = a$ are given by*

$$\sum_{a=-1}^{\infty} \sum_{g \geq 0} n_a^g(X_{\text{ban}}) \left(y^{\frac{1}{2}} + y^{-\frac{1}{2}} \right)^{2g} Q^{a+1} = 12 \prod_{n=1}^{\infty} \frac{(1 + yQ^{2n-1})(1 + y^{-1}Q^{2n-1})(1 - Q^{2n})}{(1 + yQ^{4n})^2(1 + y^{-1}Q^{4n})^2(1 - Q^{4n})^2}.$$

It is interesting to compare the above formula to the analogous result for the local $K3$ surface, namely the Katz–Klemm–Vafa formula.

THEOREM 1.12 (Pandharipande–Thomas [PT16]). *The Gopakumar–Vafa invariants $n_{\beta}^g(K3) = n_a^g(K3)$ in the class β with $\beta^2/2 = a$ are given by*

$$\sum_{a=-1}^{\infty} \sum_{g \geq 0} n_a^g(K3) \left(y^{\frac{1}{2}} + y^{-\frac{1}{2}} \right)^{2g} Q^{a+1} = \prod_{n=1}^{\infty} \frac{1}{(1 + yQ^n)^2(1 + y^{-1}Q^n)^2(1 - Q^n)^{20}}.$$

Note that after the substitution $Q = -yq$, the right-hand side of the above equation is equal to $\sum_n \chi_y(\text{Hilb}^n(K3))q^n$, the generating function for the χ_y -genus of the Hilbert schemes of points on $K3$. It would be interesting to find an analogous interpretation of the right-hand side of the equation in Theorem 1.11.

See Section A.6 for a table of $n_a^g(X_{\text{ban}})$ for small values of a and g .

2. The coefficients $c(a, k)$ and the elliptic genera of $\text{Hilb}(\mathbb{C}^2)$

Let M be a compact complex manifold of dimension d , and let $x_1, \dots, x_d$ be the Chern roots of TM . Then the elliptic genus is defined by

$$\text{Ell}_{q,y}(M) = \int_M \prod_{j=1}^d x_j y^{-\frac{1}{2}} \prod_{n=1}^{\infty} \frac{(1 - ye^{-x_j} q^{n-1})(1 - y^{-1} e^{x_j} q^n)}{(1 - e^{-x_j} q^{n-1})(1 - e^{x_j} q^n)}.$$

If M has a $\mathbb{C}^*$ -action with isolated fixed points, then by Atiyah–Bott localization, we get

$$\text{Ell}_{q,y}(M) = \sum_{p \in M^{\mathbb{C}^*}} \prod_{j=1}^d y^{-\frac{1}{2}} \prod_{n=1}^{\infty} \frac{(1 - yt^{-k_j(p)} q^{n-1})(1 - y^{-1} t^{k_j(p)} q^n)}{(1 - t^{-k_j(p)} q^{n-1})(1 - t^{k_j(p)} q^n)}, \quad (2.1)$$

where $k_1(p), \dots, k_d(p) \in \mathbb{Z}$ are the weights of the $\mathbb{C}^*$ -action on $T_p M$.

If M is non-compact, we take equation (2.1) to be the definition of the elliptic genus $\text{Ell}_{q,y}(M, t)$, which then may depend on t , the equivariant parameter. (The parameter $t \in H_{\mathbb{C}^*}^*(pt)$ is the Chern character of the universal line bundle, so $t = e^{c_1}$, where $c_1 \in H_{\mathbb{C}^*}^2(pt)$ is an integral generator.)

It is convenient to rewrite this expression in terms of the Fourier expansion of the theta function θ_1 . Let $q = \exp(2\pi i \tau)$ and $y = \exp(2\pi i z)$; then $\theta_1(q, y)$ is given by

$$\theta_1(q, y) = - \sum_{k \in \mathbb{Z} + \frac{1}{2}} q^{k^2/2} (-y)^k = -iq^{\frac{1}{8}} y^{-\frac{1}{2}} \prod_{n=1}^{\infty} (1 - q^n)(1 - yq^{n-1})(1 - y^{-1} q^n).$$

Then equation (2.1) becomes

$$\text{Ell}_{q,y}(M, t) = \sum_{p \in M^{\mathbb{C}^*}} \prod_{j=1}^d \frac{\theta_1(q, yt^{-k_j(p)})}{\theta_1(q, t^{-k_j(p)})}.$$

For example, if we let $\mathbb{C}^*$ act on $\mathbb{C}^2$ with weights ± 1 , then the induced action of $\mathbb{C}^*$ on $\text{Hilb}^m(\mathbb{C}^2)$ has isolated fixed points corresponding to monomial ideals which are in bijective correspondence with integer partitions of m . The tangent weights associated with a partition R are given by $\{\pm h_{j,k}\}$, where $h_{j,k} = h_{j,k}(R)$ is the hook length of the box in position (j, k) in the diagram of R (see Section 5.2 for this notation). Thus

$$\sum_{m=0}^{\infty} \text{Ell}_{q,y}(\text{Hilb}^m(\mathbb{C}^2), t) Q^m = \sum_R Q^{|R|} \prod_{(j,k) \in R} \frac{\theta_1(q, yt^{-h_{j,k}}) \theta_1(q, yt^{h_{j,k}})}{\theta_1(q, t^{-h_{j,k}}) \theta_1(q, t^{h_{j,k}})}. \quad (2.2)$$

Dijkgraaf–Moore–Verlinde–Verlinde (DMVV) conjectured [DMVV97] that

$$\sum_{m=0}^{\infty} \text{Ell}_{q,y}(\text{Hilb}^m(\mathbb{C}^2), t) Q^m = \sum_{m=0}^{\infty} \text{Ell}_{q,y}^{\text{orb}}(\text{Sym}^m(\mathbb{C}^2), t) Q^m,$$

where $\text{Ell}_{q,y}^{\text{orb}}(\text{Sym}^m(\mathbb{C}^2), t)$ is the orbifold elliptic genus of $\text{Sym}^m(\mathbb{C}^2)$. This is a special case of the crepant resolution conjecture for equivariant elliptic genera which was proven by Waelder [Wae08, Theorem 12] based on the non-equivariant case proven by Borisov–Libgobner [BL05]. The right-hand side of the DMVV conjecture is easy to compute; consequently, Waelder’s result leads to the following formula.

THEOREM 2.1 (Waelder). *We have*

$$\sum_{m=0}^{\infty} \text{Ell}_{q,y}(\text{Hilb}^m(\mathbb{C}^2), t) Q^m = \prod_{m=1}^{\infty} \prod_{n=0}^{\infty} \prod_{l,k \in \mathbb{Z}} (1 - t^k q^n y^l Q^m)^{-c(nm,l,k)},$$

where the coefficients $c(n, l, k)$ are defined by

$$\text{Ell}_{q,y}(\mathbb{C}^2, t) = \frac{\theta_1(q, yt) \theta_1(q, yt^{-1})}{\theta_1(q, t) \theta_1(q, t^{-1})} = \sum_{n=0}^{\infty} \sum_{k,l \in \mathbb{Z}} c(n, l, k) q^n y^l t^k.$$

We prove the following result about the coefficients $c(n, l, k)$.

PROPOSITION 2.2. *Let*

$$\text{Ell}_{q,y}(\mathbb{C}^2, t) = \sum_{n=0}^{\infty} \sum_{l,k \in \mathbb{Z}} c(n, l, k) q^n y^l t^k.$$

Then $c(n, l, k)$ depends only on the pair $(4n - l^2, k)$. Writing

$$c(n, l, k) = c(4n - l^2, k),$$

we have $c(a, k) = 0$ if $a < -1$ and

$$\sum_{a=-1}^{\infty} \sum_{k \in \mathbb{Z}} c(a, k) Q^a t^k = \frac{\sum_{k \in \mathbb{Z}} Q^{k^2} (-t)^k}{\left(\sum_{k \in \mathbb{Z} + \frac{1}{2}} Q^{2k^2} (-t)^k \right)^2} = \frac{\theta_4(Q^2, t)}{\theta_1(Q^4, t)^2}.$$

Examining the first two terms in the Q expansion, we get the following easy corollary.

COROLLARY 2.3. *The coefficients $c(a, k)$ for $a = -1, 0$ are given by*

$$c(-1, k) = \begin{cases} 0 & \text{if } -k \leq 0, \\ -k & \text{if } k > 0 \end{cases} \quad \text{and} \quad c(0, k) = \begin{cases} 0 & \text{if } k < 0, \\ 1 & \text{if } k = 0, \\ 2k & \text{if } k > 0. \end{cases}$$

Applying the Jacobi triple product identity, we also get the following corollary.

COROLLARY 2.4. *We have*

$$\sum_{a=-1}^{\infty} \sum_{k \in \mathbb{Z}} c(a, k) Q^{a+1} t^k = \frac{-t}{(1-t)^2} \prod_{n=1}^{\infty} \frac{(1 - Q^{2n})(1 - tQ^{2n-1})(1 - t^{-1}Q^{2n-1})}{(1 - Q^{4n})^2 (1 - tQ^{4n})^2 (1 - t^{-1}Q^{4n})^2}.$$

2.1 Proof of Proposition 2.2

From the product formula for $\theta_1(q, t)$ and the fact that $\theta_1(q, t^{-1}) = -\theta_1(q, t)$, we see that we may write

$$\theta_1(q, t)^{-1} \theta_1(q, t^{-1})^{-1} = -\theta_1(q, t)^{-2} = q^{-\frac{1}{4}} \frac{t}{(1-t)^2} \sum_{i=0}^{\infty} \delta_i(t) q^i,$$

where $\delta_0 = 1$ and $\delta_i(t) \in \mathbb{Z}[t, t^{-1}]$. Then

$$\text{Ell}_{q,y}(\mathbb{C}^2, t) = q^{-\frac{1}{4}} \frac{t}{(1-t)^2} \sum_{i=0}^{\infty} \delta_i(t) q^i \sum_{n,m \in \mathbb{Z}} q^{\frac{1}{2}(n+\frac{1}{2})^2 + \frac{1}{2}(m+\frac{1}{2})^2} (-yt)^{n+\frac{1}{2}} (-yt^{-1})^{m+\frac{1}{2}}.$$

If we let $l = m + n + 1$ and $b = n - m$ and we note that then $l \equiv b + 1 \pmod{2}$ and $\frac{1}{2}(n + \frac{1}{2})^2 + \frac{1}{2}(m + \frac{1}{2})^2 = \frac{1}{4}(l^2 + b^2)$, we get

$$\text{Ell}_{q,y}(\mathbb{C}^2, t) = \frac{-t}{(1-t)^2} \sum_{i=0}^{\infty} \delta_i(t) q^i \sum_{\substack{l, b \in \mathbb{Z} \\ l \equiv b+1 \pmod{2}}} q^{\frac{1}{4}(l^2 + b^2 - 1)} y^l (-t)^b.$$

The terms with $q^n y^l$ in the above sum occur when $n = i + \frac{1}{4}(l^2 + b^2 - 1)$ and thus when $4n - l^2 = 4i + b^2 - 1 \geq -1$. Therefore, the $q^n y^l$ coefficient of $\text{Ell}_{(q,y)}(\mathbb{C}^2, t)$ depends only on $a = 4n - l^2$ and is zero if $a < -1$:

$$\text{Coef}_{q^n y^l} [\text{Ell}_{(q,y)}(\mathbb{C}^2, t)] = \sum_{k \in \mathbb{Z}} c(n, l, k) t^k = \sum_{k \in \mathbb{Z}} c(4n - l^2, k) t^k,$$

where

$$\begin{aligned} \sum_{a=-1}^{\infty} \sum_{k \in \mathbb{Z}} c(a, k) Q^a t^k &= \frac{-t}{(1-t)^2} \sum_{i=0}^{\infty} \sum_{b \in \mathbb{Z}} Q^{4i+b^2-1} (-t)^b \delta_i(t) \\ &= \frac{-tQ^{-1}}{(1-t)^2} \sum_{i=0}^{\infty} \delta_i(t) Q^{4i} \sum_{b \in \mathbb{Z}} Q^{b^2} (-t)^b \\ &= \theta_1(Q^4, t)^{-2} \sum_{b \in \mathbb{Z}} Q^{b^2} (-t)^b = \frac{\sum_{k \in \mathbb{Z}} Q^{k^2} (-t)^k}{\left(\sum_{k \in \mathbb{Z} + \frac{1}{2}} Q^{2k^2} (-t)^k \right)^2}. \quad \square \end{aligned}$$

3. Geometry of the banana manifold

In this section, we compute the Hodge numbers of the banana manifold X_{ban} , and we show that the fiber classes are spanned by the banana curves C_1, C_2, C_3 .

Let $p: S \rightarrow \mathbb{P}^1$ be a rational elliptic surface with 12 singular fibers, each having one node. Let $p^\#: S^\# \rightarrow \mathbb{P}^1$ be an isomorphic copy of S , and consider the fibered product

$$X_{\text{sing}} = S \times_{\mathbb{P}^1} S^\#.$$

The space X_{sing} is singular at the 12 points where both p and $p^\#$ are not smooth, namely at the product of the nodes. To see that these points are conifold singularities, note that for each node $n \in S$ and corresponding node $n^\# \in S^\#$, there exist formal local coordinates $(x, y), (x^\#, y^\#), t$ about $n, n^\#$, and $p(n) = p^\#(n^\#) \in \mathbb{P}^1$, respectively, such that the maps p and $p^\#$ are given by $xy = t$ and $x^\#y^\# = t$. Consequently, $xy = x^\#y^\#$ is the local equation of X_{sing} at $(n, n^\#)$.

Let $\Delta \subset X_{\text{sing}}$ be the divisor given by the diagonal in the fibered product. From the above local description of the singularities, all of which lie on Δ , we see that

$$X_{\text{ban}} = \text{Bl}_\Delta(X_{\text{sing}}),$$

the blowup of X_{sing} along the diagonal is smooth, and

$$X_{\text{ban}} \longrightarrow X_{\text{sing}}$$

is a conifold resolution.

3.1 Relation to the Schoen threefold and the Hodge numbers

LEMMA 3.1. *We have $e(X_{\text{ban}}) = 24$.*

Proof. Since the map $X_{\text{ban}} \rightarrow X_{\text{sing}}$ contracts 12 rational curves to 12 singular points, we can write the following equation in the Grothendieck group of varieties:

$$[X_{\text{ban}}] = [X_{\text{sing}}^o] + 12[\mathbb{P}^1],$$

where $X_{\text{sing}}^o \subset X_{\text{sing}}$ is the non-singular locus. Since $e(-)$ is a homomorphism from the Grothendieck group to the integers, we see that $e(X_{\text{ban}}) = e(X_{\text{sing}}^o) + 24$. We claim that the Euler characteristic of each fiber of $X_{\text{sing}}^o \rightarrow \mathbb{P}^1$ is zero. The smooth fibers are $E \times E$, a product of smooth elliptic curves, and hence have zero Euler characteristic. The singular fibers have a $(\mathbb{C}^* \times \mathbb{C}^*)$ -action constructed in Section 4.5. The fixed points of the action on the fibers of $X_{\text{sing}} \rightarrow \mathbb{P}^1$ are exactly the conifold points, and so the action on the fibers of $X_{\text{sing}}^o \rightarrow \mathbb{P}^1$ is free. Consequently, the Euler characteristics of these fibers are zero. The lemma follows. $\square$

The banana manifold is related to the Schoen Calabi–Yau threefold [Sch88] by a conifold transition. This will allow us to compute the Hodge numbers of the banana manifold in terms of the (well-known) Hodge numbers of the Schoen threefold. The Schoen threefold can be defined as

$$X_{\text{Sch}} \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1,$$

the intersection of two generic hypersurfaces of multi-degree $(3, 0, 1)$ and $(0, 3, 1)$. It is a simply connected Calabi–Yau threefold with $h^{1,1}(X_{\text{Sch}}) = 19$ and $h^{2,1}(X_{\text{Sch}}) = 19$.

PROPOSITION 3.2. *The space X_{ban} is a simply connected Calabi–Yau manifold with $h^{2,1}(X_{\text{ban}}) = 8$ and $h^{1,1}(X_{\text{ban}}) = 20$.*

Proof. We first show that there is a conifold transition from X_{Sch} to X_{ban} . The generic hypersurface in $\mathbb{P}^2 \times \mathbb{P}^1$ of degree $(3, 1)$ is a rational elliptic surface $S \rightarrow \mathbb{P}^1$, and consequently the projections of $X_{\text{Sch}} \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1$ onto various subfactors realize X_{Sch} as a fibered product

$$X_{\text{Sch}} = S_0 \times_{\mathbb{P}^1} S_1,$$

where the $S_i \rightarrow \mathbb{P}^1$ are distinct generic rational surfaces. We may choose a 1-parameter family of elliptic surfaces S_t which interpolates between S_0 and S_1 . Then the family of threefolds

$$X_t = S_0 \times_{\mathbb{P}^1} S_t$$

are all Schoen threefolds for $t \neq 0$ and $X_0 = X_{\text{sing}}$, and since $X_{\text{ban}} \rightarrow X_{\text{sing}}$ is a conifold resolution, we get a conifold transition $X_{\text{Sch}} \rightsquigarrow X_{\text{ban}}$. Since conifold transitions preserve simple connectivity and the Calabi–Yau condition, we see that X_{ban} is a simply connected Calabi–Yau threefold. The change in $h^{1,1}$ through a conifold transition is determined by the codimension of the family of singular threefolds inside the full deformation space of the threefold (see [Mor99, § 3.1]). The family of Schoen threefolds is 19-dimensional: there are the two 8-dimensional families of rational elliptic surfaces $S_i \rightarrow \mathbb{P}^1$ and the 3-dimensional family of isomorphisms between the bases of $S_i \rightarrow \mathbb{P}^1$ required to form the fibered product $S_0 \times_{\mathbb{P}^1} S_1$. The locus of such fibered products with $\delta = 12$ conifold singularities is 8-dimensional: the 12 nodes of S_0 and S_1 must occur over the same fibers, which implies that $S_0 \cong S_1$, and the isomorphism of the base is the identity. The codimension σ is thus $\sigma = 19 - 8 = 11$. Following [Mor99, § 3.1], we compute

$$h^{1,1}(X_{\text{ban}}) = h^{1,1}(X_{\text{Sch}}) + \delta - \sigma = 19 + 12 - 11 = 20.$$

Then since $e(X_{\text{ban}}) = 24 = 2(h^{1,1}(X_{\text{ban}}) - h^{2,1}(X_{\text{ban}}))$, we see that $h^{2,1}(X_{\text{ban}}) = 8$ (in particular, the 8-dimensional space of banana manifolds constructed above is the whole deformation space). $\square$

LEMMA 3.3. *Let C_1 , C_2 , and C_3 be the banana curves in a singular fiber F_{ban} . Let $\Gamma \subset H_2(X_{\text{ban}}, \mathbb{Z})$ be the lattice spanned by cycles supported on the fibers of π . Then Γ is spanned by the classes of C_1 , C_2 , and C_3 .*

Proof. By the previous discussion of the conifold transition, we have that

$$H_2(X_{\text{ban}}, \mathbb{Z}) \cong H_2(X_{\text{Sch}}, \mathbb{Z}) \oplus \mathbb{Z},$$

where the $\mathbb{Z}$ -summand is spanned by the exceptional curves of the conifold resolution; in particular, the exceptional curves are all homologous. Let

$$p: X_{\text{ban}} \longrightarrow S, \quad p^\#: X_{\text{ban}} \longrightarrow S^\#$$

be the projections onto the first and second factors of the fibered product. Let $x_1, \dots, x_{12} \in \mathbb{P}^1$ be points corresponding to singular fibers; we label the banana curves $C_1(i)$, $C_2(i)$, $C_3(i)$ in the singular fiber over x_i as follows. Let $C_3(i)$ be an exceptional curve for the conifold resolution, and let $C_1(i)$ and $C_2(i)$ be such that

$$p^{-1}(n_i) = C_1(i) \cup C_3(i), \quad (p^\#)^{-1}(n_i^\#) = C_2(i) \cup C_3(i),$$

where $n_i \in S$ and $n_i^\# \in S^\#$ are the nodal points in the fibers over x_i .

The fibers of $p: X_{\text{ban}} \rightarrow S$ and $p^\#: X_{\text{ban}} \rightarrow S^\#$ are all irreducible except for the fibers $C_1(i) \cup C_3(i)$ and $C_2(i) \cup C_3(i)$ for $i = 1, \dots, 12$. Since the exceptional curves are all homologous, we have a single class $C_3 = C_3(i)$ for all i , and since the fibers of p are all homologous and the fibers of $p^\#$ are all homologous, we get

$$C_1(i) + C_3 = C_1(i') + C_3 \quad \text{and} \quad C_2(i) + C_3 = C_2(i') + C_3.$$

Hence all $C_1(i)$ are homologous to a single class C_1 , and all $C_2(i)$ are homologous to a single class C_2 .

Moreover, the three classes C_1 , C_2 , and C_3 are independent: The classes $p_*(C_1)$ and $p_*(C_3)$ are independent in $H_2(S, \mathbb{Z})$, and, likewise, the classes $p^\#_*(C_2)$ and $p^\#_*(C_3)$ are independent in $H_2(S^\#, \mathbb{Z})$. Then pushing forward any relation $a_1 C_1 + a_2 C_2 + a_3 C_3 = 0$ by p_* and $p^\#_*$ shows $a_1 = a_2 = a_3 = 0$.

It remains to show that C_1 , C_2 , and C_3 generate Γ , which is by definition spanned by classes represented by cycles supported on fibers. As we will prove in Proposition 4.10, the singular fibers F_{ban} are non-normal toric surfaces whose boundary divisor is the union of the banana curves which thus generate all the classes in $H_2(F_{\text{ban}}, \mathbb{Z})$. Then we need only prove that all classes represented by classes supported on some smooth fiber $E_y \times E_y^\#$ are in the span of C_1 , C_2 , and C_3 . By the Künneth theorem, all classes in $H_2(E_y \times E_y^\#, \mathbb{Z})$ are integral combinations of $E_y \times \{\text{pt}\}$, $\{\text{pt}\} \times E_y^\#$, and classes given by a product of 1-cycles in each factor. The latter classes are 0 as classes in $H_2(X_{\text{ban}}, \mathbb{Z})$ since 1-cycles in E_y and $E_y^\#$ are generated by vanishing cycles in the elliptic surfaces S and $S^\#$. Finally, as argued above, the classes $E_y \times \{\text{pt}\}$ and $\{\text{pt}\} \times E_y^\#$ are homologous to $C_1 + C_3$ and $C_2 + C_3$, respectively. $\square$

4. Computing the partition function

4.1 Overview

Our basic strategy for computing the partition function $Z_\Gamma(X_{\text{ban}})$ is to stratify the Hilbert scheme, where each stratum parameterizes subschemes supported on a prescribed set of fibers of the map $\pi: X \rightarrow \mathbb{P}^1$. Each stratum admits an action by the Mordell–Weil group of sections on

an infinitesimal neighborhood of a fiber.

We use these group actions to reduce the ν -weighted Euler characteristic computation to the fixed-point set of the group actions. The fixed points correspond to subschemes which are supported on an infinitesimal neighborhood of the banana configurations and which are formally locally given by monomial ideals. We can count these fixed points (weighted by ν) using a technique from [MNOP06] adapted to our setting. The outcome is an expression for the partition function in terms of the topological vertex.

The vertex expression we get has essentially been computed in the physics literature. The bulk of the computation is done by Hollowood, Iqbal, and Vafa [HIV08] using geometric engineering. Their derivation requires a certain geometrically motivated combinatorial conjecture. The conjecture was a special case of the equivariant crepant resolution conjecture for elliptic genera (also called the equivariant DMVV conjecture after Dijkgraaf–Moore–Verlinde–Verlinde [DMVV97]), which was subsequently proven by Borisov–Libgobner and Waelder [BL05, Wae08] using motivic integration. We present a mathematically self-contained version of the Hollowood–Iqbal–Vafa derivation in Section 5.

4.2 Preliminaries on notation and Euler characteristics

For any separated scheme of finite type Y over $\mathbb{C}$, let $e(Y)$ be the topological Euler characteristic of Y in the complex analytic topology. Note that the Euler characteristic is independent of any nilpotent structures; that is, $e(Y) = e(Y_{\text{red}})$.

For any constructible function $\mu: Y \rightarrow \mathbb{Z}$, let

$$e(Y, \mu) = \sum_{k \in \mathbb{Z}} k \cdot e(\mu^{-1}(k))$$

be the μ -weighted Euler characteristic of Y .

We will need the following standard facts about Euler characteristics:

- The Euler characteristic defines a ring homomorphism $e: K_0(\text{Var}_{\mathbb{C}}) \rightarrow \mathbb{Z}$; that is, it is additive under the decomposition of a scheme into an open set and its complement, and it is multiplicative on Cartesian products.
- For any constructible morphism⁵ $f: Y \rightarrow Z$, we have (see [Mac74])

$$e(Y, \mu) = e(Z, f_*\mu), \tag{4.1}$$

where $f_*\mu$ is the constructible function given by

$$(f_*\mu)(x) = e(f^{-1}(x), \mu).$$

- If $\mathbb{C}^*$ acts on a scheme Y with fixed point locus $Y^{\mathbb{C}^*} \subset Y$, then (see [Bia73])

$$e(Y) = e(Y^{\mathbb{C}^*}).$$

We will also use the following.

LEMMA 4.1. *Let G be an algebraic group acting on V , a separated scheme of finite type over $\mathbb{C}$. Let μ be a G -invariant constructible function on V . Suppose that each G -orbit has zero Euler characteristic; that is, $e(O_x) = 0$ for all $x \in V$. Then $e(V, \mu) = 0$.*

⁵A constructible morphism is a map which is regular on each piece of a decomposition of its domain into locally closed subsets.

Proof. Since we are considering Euler characteristics, we may replace V with V_{red} and thus assume without loss of generality that V is a variety.

By a theorem of Rosenlicht [Ros56, Theorem 2], [Ros63], we have that for the action of any algebraic group G on any variety V , there exist a G -invariant dense open set $U \subset V$ and a morphism $\tau: U \rightarrow W$ to a variety W such that for all $x \in U$, the fiber $\tau^{-1}(x)$ is a G -orbit. By iteratively applying the same theorem to $V \setminus U$, we obtain a locally closed G -equivariant stratification $V = \cup_{\alpha} U_{\alpha}$ such that the G -action on each stratum has a geometric quotient

$$\tau_{\alpha}: U_{\alpha} \longrightarrow W_{\alpha}.$$

Suppose that every G -orbit has Euler characteristic zero, $e(O_x) = 0$. Let μ be a G -invariant constructible function on V ; then

$$e(V, \mu) = \sum_{\alpha} e(U_{\alpha}, \mu) = \sum_{\alpha} e(W_{\alpha}, (\tau_{\alpha})_*(\mu)) = \sum_{\alpha} e(W_{\alpha}, 0) = 0. \quad \square$$

We will use the notation

$$\text{Hilb}^{\bullet}(X_{\text{ban}}) = \sum_{\underline{d}, n} \text{Hilb}^{\beta_{\underline{d}}, n}(X_{\text{ban}}) Q_1^{d_1} Q_2^{d_2} Q_3^{d_3} (-p)^n,$$

where we regard the right-hand side as a formal series in Q_1, Q_2, Q_3 , Laurent in p , having coefficients in $K_0(\text{Var}_{\mathbb{C}})$, the Grothendieck group of varieties.⁶ We extend the operations in the Grothendieck group (addition, multiplication, and Euler characteristic) to the series in the obvious way. So, for example, in this notation, the partition function is given by

$$Z_{\Gamma}(X_{\text{ban}}) = e(\text{Hilb}^{\bullet}(X_{\text{ban}}), \nu).$$

We will apply the above notation more generally. Whenever we have any collection $\{A^{\underline{d}, n}\}$ of elements of a set (for example $\mathbb{Z}$ or $K_0(\text{Var}_{\mathbb{C}})$) indexed by $(\underline{d}, n)$, we will write

$$A^{\bullet} = \sum_{\underline{d}, n} A^{\underline{d}, n} Q_1^{d_1} Q_2^{d_2} Q_3^{d_3} (-p)^n.$$

4.3 Pushing forward the ν -weighted Euler characteristic measure

Let

$$\text{Conf}^k \mathbb{P}^1 = \text{Sym}^k \mathbb{P}^1 \setminus \Delta$$

be the configuration space of k unordered distinct points in $\mathbb{P}^1$, that is, the k th symmetric product of $\mathbb{P}^1$ with the big diagonal deleted. Let

$$\text{Conf} \mathbb{P}^1 = \bigcup_k \text{Conf}^k \mathbb{P}^1.$$

We define a constructible morphism by

$$\begin{aligned} \rho^{\underline{d}, n}: \text{Hilb}^{\beta_{\underline{d}}, n}(X_{\text{ban}}) &\longrightarrow \text{Conf} \mathbb{P}^1, \\ Z &\longmapsto \text{Supp}(\pi_* \mathcal{O}_Z). \end{aligned}$$

Informally, $\rho^{\underline{d}, n}(Z)$ records the points in the base of $\pi: X_{\text{ban}} \rightarrow \mathbb{P}^1$ over which the subscheme $Z \subset X_{\text{ban}}$ is supported. Then using equation (4.1) and our bullet notation, we have

$$Z_{\Gamma}(X_{\text{ban}}) = e(\text{Hilb}^{\bullet}(X_{\text{ban}}), \nu) = e(\text{Conf} \mathbb{P}^1, \rho_{\bullet}^{\bullet} \nu).$$

⁶We note that $\text{Hilb}^{0,0}(X_{\text{ban}})$ is a single point corresponding to the empty subscheme. Thus the constant term of the series $\text{Hilb}^{\bullet}(X_{\text{ban}})$ is 1.

Note that the constructible function $\rho_*^\bullet \nu$ takes values in $\mathbb{Z}[[Q_1, Q_2, Q_3]]((p))$.

4.4 Subschemes supported in an infinitesimal neighborhood of a fiber

From here on out, we will write X for X_{ban} for the sake of brevity when there is no ambiguity. We will also suppress the superscripts β and n when they are understood and/or unimportant.

DEFINITION 4.2. Let F_y be the fiber of $\pi: X \rightarrow \mathbb{P}^1$ over $y \in \mathbb{P}^1$. Let $F_y^{(k)}$ be the k -order infinitesimal neighborhood of F_y in X , and let $\widehat{F}_y$ be the formal neighborhood of F_y in X . We define

$$\text{Hilb}(\widehat{F}_y) \subset \text{Hilb}(X)$$

to be the locally closed subscheme of $\text{Hilb}(X)$ parameterizing subschemes $Z \subset X$ supported on $\widehat{F}_y$, that is, subschemes which are set-theoretically, but not necessarily scheme-theoretically, supported in F_y . Finally, we define

$$\widehat{\text{Hilb}}(\widehat{F}_y)$$

to be the formal neighborhood of $\text{Hilb}(\widehat{F}_y)$ in $\text{Hilb}(X)$.

The value of the Behrend function $\nu: \text{Hilb}(X) \rightarrow \mathbb{Z}$ at a point depends only on the formal neighborhood of that point inside the Hilbert scheme [Jia17]. Therefore, ν restricted to $\text{Hilb}(\widehat{F}_y)$ is completely determined by the formal scheme $\widehat{\text{Hilb}}(\widehat{F}_y)$.

Remark 4.3. The closed points of $\text{Hilb}(\widehat{F}_y)$ and $\widehat{\text{Hilb}}(\widehat{F}_y)$ are of course the same; they correspond to subschemes $Z \subset X$ such that $Z_{\text{red}} \subset F_y$. However, $\text{Hilb}(\widehat{F}_y)$ and $\widehat{\text{Hilb}}(\widehat{F}_y)$ classify different families of subschemes. For example, the infinitesimal deformations of Z parameterized by $\text{Hilb}(\widehat{F}_y)$ must preserve the closed condition $Z_{\text{red}} \subset F_y$, whereas $\widehat{\text{Hilb}}(\widehat{F}_y)$ includes *all* infinitesimal deformations of Z , even those which (infinitesimally) violate the condition $Z_{\text{red}} \subset F_y$.

4.5 Mordell–Weil groups and actions on $\text{Hilb}(\widehat{F}_y)$

Let

$$X^\circ = \{x \in X \text{ such that } \pi: X \rightarrow \mathbb{P}^1 \text{ is smooth at } x\}.$$

In other words, X° is X with the 12 banana configurations removed. Then, once we fix a section $s_0: \mathbb{P}^1 \rightarrow X^\circ$,

$$\pi^\circ: X^\circ \longrightarrow \mathbb{P}^1$$

has the structure of an Abelian group scheme over $\mathbb{P}^1$. Let $F_y^\circ = X^\circ \cap F_y$ be the group of the fiber over y . Let

$$\{x_1, \dots, x_{12}\} \subset \mathbb{P}^1$$

be the points with singular fiber. Then for $y \notin \{x_1, \dots, x_{12}\}$, we have $F_y^\circ \cong E_y \times E_y$, where E_y is an elliptic curve, and for $y \in \{x_1, \dots, x_{12}\}$, we have $F_y^\circ \cong \mathbb{C}^* \times \mathbb{C}^*$.

DEFINITION 4.4. Let $\pi^\circ: X_B^\circ \rightarrow B$ and $\pi: X_B \rightarrow B$ denote the B -schemes obtained from X° and X by some base change $B \rightarrow \mathbb{P}^1$. We define $\text{MW}(B)$ to be the Mordell–Weil group of sections of π° over B .

If $B \rightarrow \text{Spec } \mathbb{C}$ is finite, then $\text{MW}(B)$ is the Weil restriction of $X_B^\circ \rightarrow B$ with respect to $B \rightarrow \text{Spec } \mathbb{C}$ and is thus an algebraic group over $\mathbb{C}$; see [BLR90, § 7, Theorem 4].

We get an action of $\mathrm{MW}(B)$ on X_B defined as follows (cf. [BLR90, § 7, Theorem 6]). The group scheme structure morphism $X_B^\circ \times_B X_B^\circ \rightarrow X_B^\circ$ extends (see [DR73]) to a morphism

$$X_B^\circ \times_B X_B \xrightarrow{+} X_B.$$

We define the $\mathrm{MW}(B)$ -action on X_B by the composition

$$\mathrm{MW}(B) \times X_B \xrightarrow{\phi} X_B^\circ \times_B X_B \xrightarrow{+} X_B,$$

where $\phi = (\mathrm{ev} \circ (\mathrm{Id} \times \pi), \mathrm{pr}_{X_B})$ with $\mathrm{ev}: \mathrm{MW}(B) \times B \rightarrow X_B$ the tautological evaluation map $(s, y) \mapsto s(y)$ and

$$\mathrm{pr}_{X_B}: \mathrm{MW}(B) \times X_B \rightarrow X_B$$

the projection. The map ϕ is well defined because $\pi \circ \mathrm{ev} \circ (\mathrm{Id} \times \pi) = \pi \circ \mathrm{pr}_{X_B}$.

Concretely, if $x \in X_B$ is a point, then the action of s on x is given by translation: $(s, x) \mapsto s(\pi(x)) + x$.

Let $\Delta_y^{(k)} \subset \mathbb{P}^1$ be the k th order thickening of $y \in \mathbb{P}^1$. Since

$$\Delta_y^{(k)} \cong \mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^{k+1}) \rightarrow \mathrm{Spec} \mathbb{C}$$

is finite, $\mathrm{MW}(\Delta_y^{(k)})$ is an algebraic group. Restriction of a section to the closed fiber expresses $\mathrm{MW}(\Delta_y^{(k)})$ as an extension of F_y° by a vector group of some dimension $D = D(k)$:

$$0 \rightarrow \mathbb{C}^D \rightarrow \mathrm{MW}(\Delta_y^{(k)}) \xrightarrow{r} F_y^\circ \rightarrow 0. \quad (4.2)$$

We write $F_k^{(k)}$ for $X_{\Delta_y^{(k)}}$, that is, the k th order thickening of the fiber F_y . The action of $\mathrm{MW}(\Delta_y^{(k)})$ on $F_y^{(k)}$ is compatible with the restriction homomorphisms $\mathrm{MW}(\Delta_y^{(k+1)}) \rightarrow \mathrm{MW}(\Delta_y^{(k)})$ and the inclusions $F_y^{(k)} \subset F_y^{(k+1)}$. Let

$$\mathrm{MW}_y^{(\infty)} = \varprojlim_k \mathrm{MW}(\Delta_y^{(k)})$$

be the inverse limit group. Then by construction, $\mathrm{MW}_y^{(\infty)}$ acts on $\widehat{F}_y$, and this induces an action of $\mathrm{MW}_y^{(\infty)}$ on $\mathrm{Hilb}(\widehat{F}_y)$ and on $\widehat{\mathrm{Hilb}}(\widehat{F}_y)$. Note that

$$\mathrm{Hilb}^{\beta, n}(\widehat{F}_y) \subset \mathrm{Hilb}^{\beta, n}(F_y^{(N)})$$

for some large $N = N(\beta, n)$ which depends on β and n . Therefore, $\mathrm{Hilb}^{\beta, n}(\widehat{F}_y)$ is acted on by the algebraic group $\mathrm{MW}(\Delta_y^{(N)})$. However, $\mathrm{MW}(\Delta_y^{(N)})$ does not act on $\widehat{\mathrm{Hilb}}^{\beta, n}(\widehat{F}_y)$ since this includes all infinitesimal deformations of any $Z \subset \widehat{F}_y$, which can involve finite neighborhoods of arbitrarily big orders. Thus only the pro-algebraic group $\mathrm{MW}_y^{(\infty)}$ acts on $\widehat{\mathrm{Hilb}}^{\beta, n}(\widehat{F}_y)$.

Since the Behrend function $\nu: \mathrm{Hilb}(\widehat{F}_y) \rightarrow \mathbb{Z}$ is determined by $\widehat{\mathrm{Hilb}}(\widehat{F}_y)$, the action of $\mathrm{MW}_y^{(\infty)}$ must preserve ν . Consequently we have the following.

LEMMA 4.5. *The Behrend function ν is invariant under the action of $\mathrm{MW}(\Delta_y^{(N)})$ on $\mathrm{Hilb}^{\beta, n}(\widehat{F}_y)$.*

In general, we do not know if the sequence (4.2) splits. However, if $y = x_i$ so that F_{x_i} is a singular fiber and thus $F_{x_i}^\circ \cong \mathbb{C}^* \times \mathbb{C}^*$, then (4.2) must split:

$$0 \rightarrow \mathbb{C}^D \rightarrow \mathrm{MW}(\Delta_{x_i}^{(k)}) \xleftarrow[r]{} \mathbb{C}^* \times \mathbb{C}^* \rightarrow 1.$$

Indeed, any infinitesimal deformation of an algebraic group of multiplicative type is trivial by [BCO14, Corollary B.2.7], and thus

$$X_{\Delta_{x_i}^{(k)}}^{\circ} \cong \Delta_{x_i}^{(k)} \times \mathbb{C}^* \times \mathbb{C}^*,$$

which induces the splitting.

Consequently, we have the following.

LEMMA 4.6. *The group $\mathbb{C}^* \times \mathbb{C}^*$ acts on $\mathrm{Hilb}(\widehat{F}_{\mathrm{ban}})$, where $\widehat{F}_{\mathrm{ban}}$ is any singular fiber. Moreover, the action extends to $\widehat{\mathrm{Hilb}}(\widehat{F}_{\mathrm{ban}})$, and thus it preserves the Behrend function ν .*

4.6 Reduction of the computation to the singular fibers

LEMMA 4.7. *Let F_y be a smooth fiber; then $e(\mathrm{Hilb}(\widehat{F}_y), \nu) = 0$.*

Proof. By Lemma 4.5, the group $\mathrm{MW} = \mathrm{MW}(\Delta_y^{N(\beta, n)})$ acts on $\mathrm{Hilb}^{\beta_{\mathbf{d}}, n}(\widehat{F}_y)$ preserving ν . Moreover, the action is fixed-point free since the group MW acts transitively on $\widehat{F}_y$, and so any subscheme $Z \subset \widehat{F}_y$ not supported on all of $\widehat{F}_y$ must be moved by some element of the group. Let Stab_x be the MW -stabilizer of $x \in \mathrm{Hilb}(\widehat{F}_y)$, and let $Z_x \subset \widehat{F}_y$ be the subscheme corresponding to x . Then the image $r(\mathrm{Stab}_x)$ is a proper subgroup of F_y since there is some element in F_y which does not preserve $\mathrm{Supp}(Z_x)$. Then the orbit $O_x = \mathrm{MW}/\mathrm{Stab}_x$ of x is an Abelian group fitting into the exact sequence

$$0 \longrightarrow \frac{\mathbb{C}^D}{\mathrm{Stab}_x \cap \mathbb{C}^D} \longrightarrow O_x \longrightarrow \frac{F_y}{r(\mathrm{Stab}_x)} \longrightarrow 0.$$

Therefore, O_x is given as the total space of a smooth fibration with base $F_y/r(\mathrm{Stab}_x)$, a positive-dimensional Abelian variety. It follows that $e(O_x) = 0$. We then can apply Lemma 4.1 with $G = \mathrm{MW}$, $V = \mathrm{Hilb}(\widehat{F}_y)$, and $\mu = \nu$ to complete the proof. $\square$

Recall that $Z_{\Gamma}(X) = e(\mathrm{Conf} \mathbb{P}^1, \rho_{\bullet}^* \nu)$. To compute $\rho_{\bullet}^* \nu$, we note that the preimage satisfies

$$(\rho_{\bullet}^*)^{-1}(\{y_1, \dots, y_k\}) = \mathrm{Hilb}^{\bullet}(\widehat{F}_{y_1} \cup \dots \cup \widehat{F}_{y_k}) = \prod_{i=1}^k \mathrm{Hilb}^{\bullet}(\widehat{F}_{y_i}),$$

where the product takes place in the ring $K_0(\mathrm{Var}_{\mathbb{C}})[[Q_1, Q_2, Q_3]]((p))$.

Lemma 4.7 implies that

$$(\rho_{\bullet}^* \nu)(\{y_1, \dots, y_k\}) = 0 \quad \text{if} \quad \{y_1, \dots, y_k\} \not\subset \{x_1, \dots, x_{12}\},$$

where $F_{x_1}, \dots, F_{x_{12}}$ are the 12 singular fibers.

In other words, we have shown that the constructible function $\rho_{\bullet}^* \nu$ is supported on the subset $\mathrm{Conf}(\{x_1, \dots, x_{12}\}) \subset \mathrm{Conf} \mathbb{P}^1$. Therefore,

$$\begin{aligned} Z_{\Gamma}(X) &= e(\mathrm{Conf} \mathbb{P}^1, \nu) = e(\mathrm{Conf}(\{x_1, \dots, x_{12}\}), \rho_{\bullet}^*) \\ &= e(\mathrm{Hilb}^{\bullet}(\widehat{F}_{x_1} \cup \dots \cup \widehat{F}_{x_{12}}), \nu) = \prod_{i=1}^{12} e(\mathrm{Hilb}^{\bullet}(\widehat{F}_{x_i}), \nu). \end{aligned}$$

Since all the formal neighborhoods $\widehat{F}_{x_i}$ are isomorphic, we will write $\widehat{F}_{\mathrm{ban}}$ for this formal scheme, that is, the formal neighborhood of F_{ban} , a fiber containing a banana configuration. In conclusion, we have

$$Z_{\Gamma}(X_{\mathrm{ban}}) = e(\widehat{\mathrm{Hilb}}^{\bullet}(\widehat{F}_{\mathrm{ban}}), \nu)^{12}. \quad (4.3)$$

Note that here ν is the restriction of the Behrend function on $\text{Hilb}(X)$ to $\text{Hilb}(\widehat{F}_{\text{ban}})$, but since this is determined by $\widehat{\text{Hilb}}(\widehat{F}_{\text{ban}})$, we may regard ν as the Behrend function of the formal scheme $\widehat{\text{Hilb}}(\widehat{F}_{\text{ban}})$. We will write

$$Z(\widehat{F}_{\text{ban}}) = e(\widehat{\text{Hilb}}^\bullet(\widehat{F}_{\text{ban}}), \nu).$$

Remark 4.8. For more general banana manifolds, the same proof shows that equation (4.3) holds with the 12 replaced by N , the number of singular fibers (that is, the number of banana configurations).

4.7 Reduction to $(\mathbb{C}^* \times \mathbb{C}^*)$ -fixed subschemes

The $(\mathbb{C}^* \times \mathbb{C}^*)$ -action on $\widehat{F}_{\text{ban}}$ preserves the canonical class by construction. In particular, the action of $\mathbb{C}^* \times \mathbb{C}^*$ is compatible with the symmetric obstruction theory [BF08] of $\text{Hilb}(X)$, restricted to $\widehat{\text{Hilb}}(X)$. We note that in order to work in this formal setting, we must use the symmetric obstruction theory associated with the formal moduli space (defined by Jiang in [Jia17]). The result of Behrend and Fantechi [BF08] is that for an isolated fixed point $P \in \widehat{\text{Hilb}}(\widehat{F}_{\text{ban}})$, the value of the Behrend function is given by

$$\nu(P) = (-1)^{\dim T_P},$$

where $T_P = T_P \widehat{\text{Hilb}}(\widehat{F}_{\text{ban}})$ is the Zariski tangent space of $P \in \widehat{\text{Hilb}}(\widehat{F}_{\text{ban}})$. We note that $T_P = \text{Ext}_0^1(I_{Z_P}, I_{Z_P})$, where Z_P is the subscheme associated with P (see [MNOP06]).

We will see that the fixed points of the action are isolated. It then follows that

$$\begin{aligned} Z(\widehat{F}_{\text{ban}}) &= e(\widehat{\text{Hilb}}^\bullet(\widehat{F}_{\text{ban}}), \nu) = e(\widehat{\text{Hilb}}^\bullet(\widehat{F}_{\text{ban}})^{\mathbb{C}^* \times \mathbb{C}^*}, \nu) \\ &= \sum_{P \in \widehat{\text{Hilb}}(\widehat{F}_{\text{ban}})^{\mathbb{C}^* \times \mathbb{C}^*}} (-1)^{\dim T_P} Q_1^{d_1(P)} Q_2^{d_2(P)} Q_3^{d_3(P)} (-p)^{\chi(\mathcal{O}_{Z_P})}, \end{aligned}$$

where $d_i(P)$ is defined by

$$[Z_P] = d_1(P)C_1 + d_2(P)C_2 + d_3(P)C_3.$$

PROPOSITION 4.9. *The $(\mathbb{C}^* \times \mathbb{C}^*)$ -fixed points $P \in \widehat{\text{Hilb}}(\widehat{F}_{\text{ban}})^{\mathbb{C}^* \times \mathbb{C}^*}$ are isolated, and the fixed-point set is in bijective correspondence with the set $\{R_1, R_2, R_3, \pi_1, \pi_2\}$, where R_1, R_2, R_3 is a triple of 2D partitions and π_1, π_2 is a pair of 3D partitions asymptotic to (R_1, R_2, R_3) and the conjugate partitions (R'_1, R'_2, R'_3) , respectively. Moreover, the discrete invariants of the corresponding subscheme are given by*

$$\begin{aligned} d_i(P) &= |R_i|, \\ \chi(\mathcal{O}_{Z_P}) &= |\pi_1| + |\pi_2| + \frac{1}{2} \sum_{i=1}^3 \|R_i\|^2 + \|R'_i\|^2, \\ (-1)^{\dim T_P} &= (-1)^{\chi(\mathcal{O}_{Z_P}) + |R_1| + |R_2| + |R_3|}, \end{aligned}$$

where $|R_i|$ is the size of a partition, $|\pi_i|$ is the normalized volume [BKY18, ORV06], and $\|R_i\|^2$ denotes the sum of the squares of the parts (cf. Section 5).

See, for example, [BKY18, Definition 1] for a definition of 3D partitions asymptotic to a triple of 2D partitions. We use A' to denote the partition conjugate to A ; see Section 5.

This proposition will be proved in the next section, but we first use it to finish the computation

of $Z(\widehat{F}_{\text{ban}})$. First, recall that the topological vertex is defined to be the generating function

$$\mathbf{V}_{R_1 R_2 R_3}(p) = \sum_{\pi} p^{|\pi|},$$

where the sum is over all 3D-partitions asymptotic to (R_1, R_2, R_3) ; see [BKY18, Definition 2], [ORV06]. Then by the proposition, we obtain

$$\begin{aligned} Z(\widehat{F}_{\text{ban}}) &= \sum_{R_1 R_2 R_3} (-Q_1)^{|R_1|} (-Q_2)^{|R_2|} (-Q_3)^{|R_3|} p^{\frac{1}{2} \sum_{i=1}^3 \|R_i\|^2 + \|R'_i\|^2} \mathbf{V}_{R_1 R_2 R_3}(p) \mathbf{V}_{R'_1 R'_2 R'_3}(p) \\ &= M(p)^2 \sum_{R_1 R_2 R_3} (-Q_1)^{|R_1|} (-Q_2)^{|R_2|} (-Q_3)^{|R_3|} \overline{\mathbf{V}}_{R_1 R_2 R_3}(p) \overline{\mathbf{V}}_{R'_1 R'_2 R'_3}(p), \end{aligned}$$

where

$$\overline{\mathbf{V}}_{R_1 R_2 R_3} = M(p)^{-1} p^{\frac{1}{2}(\|R_1\|^2 + \|R'_1\|^2 + \|R_3\|^2)} \mathbf{V}_{R_1 R_2 R_3}$$

is the normalized vertex. The main result of Section 5 (Theorem 5.1) then asserts

$$Z(\widehat{F}_{\text{ban}}) = \prod_{a_i \geq 0} \prod_{\substack{k \in \mathbb{Z}, \\ k > 0 \text{ if } \underline{a} = \underline{0}}} (1 - Q_1^{a_1} Q_2^{a_2} Q_3^{a_3} p^k)^{-c(\|\underline{a}\|, k)}.$$

This formula, along with equation (4.3), then proves Theorem 1.5.

4.8 Analysis of $(\mathbb{C}^* \times \mathbb{C}^*)$ -fixed subschemes

The goal of this section is to prove Proposition 4.9. To analyze the action of $\mathbb{C}^* \times \mathbb{C}^*$ on $\widehat{F}_{\text{ban}}$ and determine the $(\mathbb{C}^* \times \mathbb{C}^*)$ -invariant subschemes, we obtain an explicit toric description of $\widehat{F}_{\text{ban}}$ using Proposition 4.10 below. Informally, the proposition says that

- F_{ban} is obtained from $\text{Bl}(\mathbb{P}^1 \times \mathbb{P}^1)$, the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at the points $(0, \infty)$, and $(\infty, 0)$, by gluing the three pairs of disjoint boundary divisors to each other in normal crossings;
- $\widehat{F}_{\text{ban}}$ is obtained from $\widehat{\text{Bl}(\mathbb{P}^1 \times \mathbb{P}^1)}$, the formal neighborhood of $\text{Bl}(\mathbb{P}^1 \times \mathbb{P}^1)$ inside the total space of its canonical bundle, by an étale gluing (which restricts to the normal crossing gluing of $\text{Bl}(\mathbb{P}^1 \times \mathbb{P}^1)$);
- the gluing is $(\mathbb{C}^* \times \mathbb{C}^*)$ -equivariant.

Let $F_{\text{sing}} \subset X_{\text{sing}}$ be the image of F_{ban} under the blowup $X_{\text{ban}} \rightarrow X_{\text{sing}}$. Let

$$\sigma: F_{\text{sing}}^{\text{norm}} \longrightarrow F_{\text{sing}}, \quad \tau: F_{\text{ban}}^{\text{norm}} \longrightarrow F_{\text{ban}}$$

be the normalizations. Define $\widehat{F}_{\text{ban}}^{\text{norm}}$ to be the formal scheme given by the ringed space $(F_{\text{ban}}^{\text{norm}}, \tau^* \widehat{\mathcal{O}}_{\widehat{F}_{\text{ban}}})$.

PROPOSITION 4.10. *The following hold:*

- (i) $F_{\text{sing}}^{\text{norm}} \cong \mathbb{P}^1 \times \mathbb{P}^1$.
- (ii) We have that $F_{\text{ban}}^{\text{norm}} \cong \text{Bl}(\mathbb{P}^1 \times \mathbb{P}^1)$, the blowup of $F_{\text{sing}}^{\text{norm}}$ at the two points $(0, \infty)$ and $(\infty, 0)$ (cf. [AAK16, Figure 4]).
- (iii) We have that $\widehat{F}_{\text{ban}}^{\text{norm}} \cong \widehat{\text{Bl}(\mathbb{P}^1 \times \mathbb{P}^1)}$, the formal neighborhood of $\text{Bl}(\mathbb{P}^1 \times \mathbb{P}^1)$ viewed as the zero section in $\text{Tot}(K_{\text{Bl}(\mathbb{P}^1 \times \mathbb{P}^1)})$, the total space of its canonical bundle.
- (iv) The induced map $\widehat{\tau}: \widehat{F}_{\text{ban}}^{\text{norm}} \rightarrow \widehat{F}_{\text{ban}}$ is étale.
- (v) All of the above maps are $(\mathbb{C}^* \times \mathbb{C}^*)$ -equivariant.

These results are summarized in the following diagram:

$$\begin{array}{ccccccc}
 & & \widehat{F}_{\text{ban}} & \xleftarrow{\widehat{\tau} \text{ étale}} & \widehat{F}_{\text{ban}}^{\text{norm}} & \cong & \text{Bl}(\widehat{\mathbb{P}^1 \times \mathbb{P}^1}) \\
 \text{formal neighborhood} \nearrow & & \uparrow & & \uparrow & & \nwarrow \text{formal neighborhood} \\
 X_{\text{ban}} & \longleftrightarrow & F_{\text{ban}} & \xleftarrow{\tau \text{ normalization}} & F_{\text{ban}}^{\text{norm}} & \cong & \text{Bl}(\mathbb{P}^1 \times \mathbb{P}^1) \longrightarrow \text{Tot}(K_{\text{Bl}(\mathbb{P}^1 \times \mathbb{P}^1)}) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \text{Blow up } (\infty, 0) \text{ and } (0, \infty) \\
 X_{\text{sing}} & \longleftrightarrow & F_{\text{sing}} & \xleftarrow{\sigma \text{ normalization}} & F_{\text{sing}}^{\text{norm}} & \cong & \mathbb{P}^1 \times \mathbb{P}^1
 \end{array}$$

Proof. The proof of this proposition is primarily based on computations in formal local coordinates. Let $C_{\text{sing}} \subset S$ be a singular fiber of $S \rightarrow \mathbb{P}^1$, and let $C_{\text{sing}}^{\#} \subset S^{\#}$ be an isomorphic copy, so that

$$X_{\text{sing}} = S \times_{\mathbb{P}^1} S^{\#} \quad \text{and} \quad F_{\text{sing}} = C_{\text{sing}} \times C_{\text{sing}}^{\#}.$$

The curve C_{sing} is a nodal rational curve whose normalization $\mathbb{P}^1 \rightarrow C_{\text{sing}}$ identifies the points $0, \infty \in \mathbb{P}^1$ to the nodal point $n \in C_{\text{sing}}$. Thus $F_{\text{sing}}^{\text{norm}}$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and the map $F_{\text{sing}}^{\text{norm}} \rightarrow F_{\text{sing}}$ identifies $0 \times \mathbb{P}^1$ and $\infty \times \mathbb{P}^1$ to $n \times C_{\text{sing}}^{\#}$ and identifies $\mathbb{P}^1 \times 0$ and $\mathbb{P}^1 \times \infty$ to $C_{\text{sing}} \times n^{\#}$. Note that this map is $(\mathbb{C}^* \times \mathbb{C}^*)$ -equivariant with respect to the usual toric action on $\mathbb{P}^1 \times \mathbb{P}^1$ and our constructed action on F_{sing} .

The formal neighborhood $\widehat{F}_{\text{ban}}$ is obtained from $\widehat{F}_{\text{sing}}$ by blowing up the diagonal $\Delta \subset \widehat{F}_{\text{sing}}$. The blowdown $\widehat{F}_{\text{ban}} \rightarrow \widehat{F}_{\text{sing}}$ contracts the exceptional $\mathbb{P}^1$ to the conifold point $n \times n^{\#}$ and is an isomorphism elsewhere. We study this blowup and the normalizations in formal local coordinates about $n \times n^{\#}$.

Let (x, y) and $(x^{\#}, y^{\#})$ be formal local coordinates on S and $S^{\#}$ about the points n and $n^{\#}$ such that the maps $S \rightarrow \mathbb{P}^1$ and $S^{\#} \rightarrow \mathbb{P}^1$ are given by $t = xy$ and $t = x^{\#}y^{\#}$, where t is a formal local coordinate on $\mathbb{P}^1$. Moreover, we may choose the coordinates such that the action of $(\lambda, \lambda^{\#}) \in \mathbb{C}^* \times \mathbb{C}^*$ on $\widehat{F}_{\text{sing}}$ is given by

$$(x, y, x^{\#}, y^{\#}) \mapsto (\lambda x, \lambda^{-1} y, \lambda^{\#} x^{\#}, (\lambda^{\#})^{-1} y^{\#}).$$

Then the formal neighborhood of $n \times n^{\#} \in \widehat{F}_{\text{sing}}$ is isomorphic to

$$\{xy = x^{\#}y^{\#}\} \subset \text{Spec } \mathbb{C}[[x, y, x^{\#}, y^{\#}]],$$

where the closed fiber $F_{\text{sing}} \subset \widehat{F}_{\text{sing}}$ is given by $\{xy = x^{\#}y^{\#} = 0\}$.

The blowup of $\widehat{F}_{\text{sing}}$ along the diagonal $\Delta = \{x = x^{\#}, y = y^{\#}\}$ is canonically isomorphic to the blowup along any of the planes⁷

$$\{ax = a^{\#}x^{\#}, a^{\#}y = ay^{\#}\}, \quad (a : a^{\#}) \in \mathbb{P}^1.$$

Choosing $(a : a^{\#}) = (1 : 0)$, we get two affine toric charts for the blowup with coordinate rings given by

$$\begin{aligned}
 \mathbb{C}[[x, x^{\#}, y, y^{\#}]] [u] / (x - uy^{\#}, x^{\#} - uy) &\cong \mathbb{C}[[y, y^{\#}]] [u], \\
 \mathbb{C}[[x, x^{\#}, y, y^{\#}]] [v] / (y - vx^{\#}, y^{\#} - vx) &\cong \mathbb{C}[[x, x^{\#}]] [v].
 \end{aligned}$$

⁷There are canonically two small resolutions of the conifold singularity $xy = x^{\#}y^{\#}$. These two are given by blowing up any plane given by an affine cone over a line in one of the two rulings of the quadric surface $\{xy = x^{\#}y^{\#}\} \subset \mathbb{P}^3$.

The coordinate change between the charts is given by $u = v^{-1}$, $y = vx^\#$, $y^\# = vx$, and the induced $(\lambda, \lambda^\#)$ -action is given by

$$(\lambda x, \lambda^\# x^\#, (\lambda \lambda^\#)^{-1} v), \quad (\lambda^{-1} y, (\lambda^\#)^{-1} y^\#, \lambda \lambda^\# u).$$

The coordinates on the blowup of the formal neighborhood of $n \times n^\# \in \widehat{F}_{\text{sing}}$ and the corresponding blowdown are encoded in the momentum “polytopes” in Figure 3, where the coordinate lines are labeled by their corresponding variables.

FIGURE 3. Momentum “polytopes”

The x - and y -coordinates about $n \in S$ (which correspond to the two branches of the node in C_{sing}) become local coordinates near 0 and ∞ in $\mathbb{P}^1$, the normalization of C_{sing} .

Thus $F_{\text{sing}}^{\text{norm}}$ and $F_{\text{ban}}^{\text{norm}}$, when endowed with local coordinate rings given by $\sigma^* \widehat{\mathcal{O}}_{\widehat{F}_{\text{sing}}}$ and $\tau^* \widehat{\mathcal{O}}_{\widehat{F}_{\text{ban}}}$, respectively, have the momentum polytopes given in Figure 4.

FIGURE 4. Momentum “polytopes” of $F_{\text{sing}}^{\text{norm}}$ and $F_{\text{ban}}^{\text{norm}}$

We see that

$$F_{\text{ban}}^{\text{norm}} \cong \text{Bl}_{(\infty, 0)(0, \infty)}(\mathbb{P}^1 \times \mathbb{P}^1), \quad \widehat{F}_{\text{ban}}^{\text{norm}} \cong \widehat{\text{Bl}(\mathbb{P}^1 \times \mathbb{P}^1)},$$

as asserted. □

We can now finish the proof of Proposition 4.9. Using the equivariant étale morphism

$$\hat{\tau}: \hat{F}_{\text{ban}}^{\text{norm}} \longrightarrow \hat{F}_{\text{ban}}$$

provided by the lemma, we may study $(\mathbb{C}^* \times \mathbb{C}^*)$ -invariant subschemes of $\hat{F}_{\text{ban}}$ by studying $(\mathbb{C}^* \times \mathbb{C}^*)$ -invariant subschemes of $\hat{F}_{\text{ban}}^{\text{norm}}$ satisfying the descent condition. Since the only proper invariant subschemes of

$$K_{\text{Bl}} := \text{Tot}(K_{\text{Bl}}(\mathbb{P}^1 \times \mathbb{P}^1))$$

are supported on $\hat{F}_{\text{ban}}^{\text{norm}} \subset K_{\text{Bl}}$, we may consider torus-invariant subschemes of the toric threefold K_{Bl} and use the method of Maulik–Nekrasov–Okounkov–Pandharipande (MNOP) [MNOP06] to count such subschemes—subject to the condition that such subschemes descend under the étale relation given by $\hat{\tau}$.

A torus-invariant subscheme of a toric Calabi–Yau threefold is determined by combinatorial data attached to its web diagram, a trivalent planar graph (which includes non-compact edges). Namely, each edge is labeled by a 2D partition and each vertex is labeled by a 3D partition which is asymptotic to the three 2D partitions given by the incident edges (see [MNOP06] or also [BCY12, §3 and Appendix B]). The web diagram of K_{Bl} is a hexagon with six additional non-compact edges (the graph on the right in Figure 3). The étale relation identifies the six vertices to two distinct vertices corresponding to the points $\{p, q\}$ in the banana configuration. The relation also identifies the edges to three distinct edges corresponding to the curves C_1, C_2, C_3 in the banana configuration. Each $(\mathbb{C}^* \times \mathbb{C}^*)$ -invariant subscheme is thus determined by three 2D partitions R_1, R_2, R_3 and two 3D partitions π_1, π_2 . The quantities $d_i(P)$, $\chi(\mathcal{O}_{Z_P})$, and $(-1)^{\dim T_P}$ are computed by MNOP in terms of the combinatorial data associated with the fixed point P for toric Calabi–Yau threefolds. Their computation applies in our setting (of a formal Calabi–Yau threefold) as well. Their proofs are based on computing with the Čech open cover given by the $\mathbb{C}^3$ coordinate charts, but we can equally well use the Čech cover obtained by intersecting the toric open cover on K_{Bl} with the formal neighborhood of the banana configuration and then identifying via the étale relation. The proof that fixed points are isolated [MNOP06, Lemma 6], the computation of the degree and Euler characteristic [MNOP06, §4.4], and the computation of the parity of the dimension of the tangent space [MNOP06, Theorem 2] all work with this more general Čech cover.

Applying the formulas of MNOP is then straightforward. The formulas in our Proposition 4.9 follow directly from the formulas in [MNOP06, §4.4] (in particular Lemma 5), [MNOP06, Theorem 2], and the fact that

$$\sum_{(i,j) \in R} (i + j + 1) = \frac{1}{2} (\|R\|^2 + \|R'\|^2).$$

This completes the proof of Proposition 4.9.

5. The vertex calculation

Let

$$\bar{V}_{R_1 R_2 R_3}(p) = s_{R'_3}(p^{-\rho}) \sum_A s_{R'_1/A}(p^{-R_3-\rho}) s_{R_2/A}(p^{-R'_3-\rho})$$

be the normalized vertex, and let (see Section 4.7)

$$Z(\widehat{F}_{\text{ban}}) = M(p)^2 \sum_{R_1, R_2, R_3} (-Q_1)^{R_1} (-Q_2)^{R_2} (-Q_3)^{R_3} \overline{V}_{R_1 R_2 R_3}(p) \overline{V}_{R'_1 R'_2 R'_3}(p),$$

where $M(p) = \prod_{m=1}^{\infty} (1 - p^m)^{-m}$.

The purpose of this section is to prove the following.

THEOREM 5.1.

$$Z(\widehat{F}_{\text{ban}}) = \prod_{a_i \geq 0} \prod_{\substack{k \in \mathbb{Z}, \\ k > 0 \text{ if } \underline{a} = \underline{0}}} (1 - Q_1^{a_1} Q_2^{a_2} Q_3^{a_3} p^k)^{-c(\|\underline{a}\|, k)},$$

where $c(a, k)$ is given by Proposition 2.2.

Most of this computation has previously appeared in the physics literature under the guise of geometric engineering (in this case, a duality between a certain 6-dimensional $U(1)$ gauge theory and a certain topological string theory). The main reference is Hollowood–Iqbal–Vafa [HIV08]. This calculation was also studied in [LLZ06]. These computations assumed an equality between the generating function for the equivariant elliptic genus of $\text{Hilb}^n(\mathbb{C}^2)$ and the orbifold equivariant elliptic genus of $\text{Sym}^n(\mathbb{C}^2)$, which they call the DMVV conjecture (after [DMVV97]). This is an instance of the crepant resolution conjecture for elliptic genera, proven in the compact case by Borisov–Libgober [BL05] and in the equivariant case by Waelder [Wae08].

We give the derivation here in full detail. We have filled in some details that are missing from the previous accounts (in particular, the proof of Lemma 5.2 seems to be missing from the literature) and have collected all the needed results in one place.

5.1 Overview of the computation

After collecting some standard Schur function identities and proving a few new ones in the next subsection, we proceed to the main computation. The basic structure is as follows.

(1) Writing

$$Z'_{\text{ban}} = M(p)^{-2} Z(\widehat{F}_{\text{ban}}) = \sum_R (-Q_3)^{|R|} Z_R(Q_1, Q_2, p),$$

we use a series of Schur function identities to simplify Z_R and write it as a product

$$Z_R = Z_{\text{prod}} \cdot Z_{\text{hook}, R},$$

where Z_{prod} is a product of terms which do not depend on R and $Z_{\text{hook}, R}$ is also a product of terms which depend on the hook-lengths of R .

(2) We observe that after the change of variables

$$Q_1 = y, \quad Q_2 = y^{-1}q, \quad Q_3 = y^{-1}Q, \quad p = t, \quad (5.1)$$

the product $Z_{\text{hook}, R}$ is exactly the contribution of a $\mathbb{C}^*$ -fixed point to the computation of

$$\sum_{n=0}^{\infty} \text{Ell}_{q, y}(\text{Hilb}(\mathbb{C}^2), t) Q^n,$$

the elliptic genera of the Hilbert schemes, via Atiyah–Bott localization. Here $\lambda \in \mathbb{C}^*$ acts on the factors of $\mathbb{C}^2$ with opposite weights, and t is the equivariant parameter.

- (3) A product formula for $\sum_{n=0}^{\infty} \text{Ell}_{q,y}(\text{Hilb}(\mathbb{C}^2), t) Q^n$ was conjectured by Dijkgraaf–Moore–Verlinde–Verlinde [DMVV97] and proven by Borisov–Libgobner and Waelder [BL05, Wae08]. Using that formula, substituting back to the variables Q_1, Q_2, Q_3, p , and performing a few easy manipulations, we arrive at Theorem 1.5.

5.2 Notation and Schur function identities

We will use capital letters R, A, B, C , etc. to denote partitions. Via its diagram, we regard a partition A as a finite subset of $\mathbb{N} \times \mathbb{N}$, where if $(i, j) \in A$, then $(i-1, j) \in A$ and $(i, j-1) \in A$. The *rows* or *parts* of A are the integers $A_j = \max\{i \mid (i, j) \in A\}$. We use $'$ to denote the conjugate partition

$$A' = \{(i, j) : (j, i) \in A\},$$

and we write

$$|A| = \sum_j A_j, \quad \|A\|^2 = \sum_j A_j^2.$$

For each $(i, j) \in A$, we define the *hook length* by

$$h_{ij}(A) = A_i + A'_j - i - j + 1.$$

We write $\square$ for the unique partition of size 1.

We also use the notation

$$M(u, p) = \prod_{m=1}^{\infty} (1 - up^m)^{-m}$$

and the shorthand $M(p) = M(1, p)$.

For a collection of variables $x = (x_1, x_2, \dots)$ and two partitions A and B , let $s_{A/B}(x) = s_{A/B}(x_1, x_2, \dots)$ denote the skew Schur function (see, for example, [Mac95, § I.5]).

Let

$$\rho = \left(-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots\right)$$

so that, for example, $p^{-R-\rho}$ is notation for the variable list

$$\left(p^{-R_1+\frac{1}{2}}, p^{-R_2+\frac{3}{2}}, \dots\right).$$

Okounkov–Reshetikhin–Vafa derived a formula for the topological vertex in terms of skew Schur functions. Translating their formulas [ORV06, (3.20) and (3.21)] into our notation, we get

$$\mathbf{V}_{R_1 R_2 R_3}(p) = M(p) p^{-\frac{1}{2}(\|R_1\|^2 + \|R_2'\|^2 + \|R_3\|^2)} \bar{\mathbf{V}}_{R_1 R_2 R_3}(p), \quad (5.2)$$

where we have defined

$$\bar{\mathbf{V}}_{R_1 R_2 R_3}(p) = s_{R_3'}(p^{-\rho}) \sum_A s_{R_1'/A}(p^{-R_3-\rho}) s_{R_2/A}(p^{-R_3'-\rho}). \quad (5.3)$$

We will need the following Schur function identities. We remark that the Schur functions which appear are all Laurent expansions in p or in p^{-1} of rational functions in p , and many of the identities should be understood as equalities of rational functions.

From [ORV06, (3.10)], we have⁸

$$s_{A/B}(p^{C+\rho}) = (-1)^{|A|-|B|} s_{A'/B'}(p^{-C'-\rho}). \quad (5.4)$$

From [Mac95, § I.3, Example 2, p. 45]:

$$s_R(1, p, p^2, \dots) = p^{n(R)} \prod_{(i,j) \in R} \frac{1}{1 - p^{h_{ij}(R)}},$$

where $n(R) = \frac{1}{2} \|R'\|^2 - \frac{1}{2} |R|$ (see [Mac95, § I.1, (1.6), p. 3]) and so using the homogeneity of s_R , we see that

$$s_R(p^{-\rho}) = p^{\frac{1}{2} \|R'\|^2} \prod_{i,j \in R} \frac{1}{(1 - p^{h_{ij}(R)})},$$

and so we arrive at

$$s_R(p^{-\rho}) s_{R'}(p^{-\rho}) = (-1)^{|R|} \prod_{i,j \in R} \frac{1}{(1 - p^{h_{ij}(R)})(1 - p^{-h_{ij}(R)})}. \quad (5.5)$$

We will also need the following identities [Mac95, § I.5, Example 26(2), p. 93 and (5.10), p. 72]

$$\sum_R s_{R/B}(x) s_{R'/A}(y) = \prod_{i,j=1}^{\infty} (1 + x_i y_j) \sum_C s_{A'/C}(x) s_{B'/C}(y), \quad (5.6)$$

$$\sum_B s_{A/B}(x) s_{B/C}(y) = s_{A/C}(x, y). \quad (5.7)$$

Finally, we will need the following lemma. The proof of this lemma appears to be absent in the physics literature.

LEMMA 5.2. *For a fixed partition R , the following equalities hold as formal power series in u whose coefficients are rational functions of p :*

$$\prod_{j,k=1}^{\infty} (1 - up^{R_j+R'_k-j-k+1}) = M(u, p)^{-1} \prod_{j,k \in R} (1 - up^{h_{jk}(R)})(1 - up^{-h_{jk}(R)}), \quad (5.8)$$

$$\prod_{j,k=1}^{\infty} (1 - up^{-R_j+R_k+j-k}) = M(u, p) \prod_{j,k \in R} (1 - up^{h_{jk}(R)})^{-1} (1 - up^{-h_{jk}(R)})^{-1}. \quad (5.9)$$

Note that the right-hand sides of the equations are invariant under $p \leftrightarrow p^{-1}$, and so we get two more identities by replacing p with p^{-1} on the left-hand sides of the above equations.

Proof. Moving the $M(u, p)$ to the left-hand side of the first equation and then taking the log, we get

$$\begin{aligned} & \log \left(M(u, p) \prod_{j,k=1}^{\infty} (1 - up^{R_j+R'_k-j-k+1}) \right) \\ &= \log \left(\prod_{j,k=1}^{\infty} (1 - up^{j+k-1})^{-1} \right) + \log \left(\prod_{j,k=1}^{\infty} (1 - up^{R_j+R'_k-j-k+1}) \right) \\ &= \sum_{n \geq 1} \frac{u^n}{n} \sum_{j,k \geq 1} (p^{n(j+k-1)} - p^{n(R_j+R'_k-j-k+1)}) = \sum_{n \geq 1} -\frac{u^n}{n} P_R(p^n), \end{aligned}$$

⁸There is a typo in equation (3.10) in [ORV06]—the exponent on the right-hand side should be $-\nu' - \rho$.

where we have defined

$$\begin{aligned} P_R(x) &= \sum_{j,k \geq 1} (x^{R_j+R'_k-j-k+1} - x^{j+k-1}) = \left(\sum_{j \geq 1} x^{R_j-j+\frac{1}{2}} \sum_{k \geq 1} x^{R_k-k+\frac{1}{2}} \right) - \frac{x}{(1-x)^2} \\ &= s_{\square}(x^{R+\rho}) s_{\square}(x^{R'+\rho}) - \frac{x}{(1-x)^2}. \end{aligned}$$

Since by equation (5.5), we have

$$s_{\square}(x^{R+\rho}) = -s_{\square}(x^{-R'-\rho}),$$

we see from the above equation that $P_R(x)$ is a sum of two rational functions in x , each invariant under $x \leftrightarrow x^{-1}$. Moreover, all but a finite number of terms in the sum

$$\sum_{j,k \geq 1} (x^{R_j+R'_k-j-k+1} - x^{-j-i+1}) = P_R(x)$$

cancel, and so we deduce that $P_R(x)$ is a Laurent *polynomial* which is invariant under $x \leftrightarrow 1/x$. Consequently, $P_R(x)$ is uniquely determined by its terms with positive exponents. Since

$$R_j + R'_k - j - k + 1 = \begin{cases} h_{jk}(R) & \text{if } (j, k) \in R, \\ \text{negative} & \text{if } (j, k) \notin R, \end{cases}$$

we see that the terms with positive exponent in the above expression for P_R are precisely the $x^{h_{jk}(R)}$ where $(j, k) \in R$. Therefore,

$$P_R(x) = \sum_{(j,k) \in R} (x^{h_{jk}(R)} + x^{-h_{jk}(R)}).$$

Substituting back, we find

$$\begin{aligned} \log \left(M(u, p) \prod_{j,k=1}^{\infty} (1 - up^{R_j+R'_k-j-k+1}) \right) &= \sum_{n \geq 1} -\frac{u^n}{n} \sum_{(j,k) \in R} (x^{h_{jk}(R)} + x^{-h_{jk}(R)}) \\ &= \sum_{j,k \in R} \log(1 - up^{h_{jk}(R)}) + \log(1 - up^{-h_{jk}(R)}) = \log \prod_{(j,k) \in R} (1 - up^{h_{jk}(R)}) (1 - up^{-h_{jk}(R)}), \end{aligned}$$

which proves equation (5.8). To prove equation (5.9), we observe that since by equation (5.5)

$$\sum_{k \geq 1} p^{R'_k-k+\frac{1}{2}} = - \sum_{k \geq 1} p^{-R_k+k-\frac{1}{2}},$$

we have

$$- \sum_{j,k \geq 1} (x^{-R_j+R_k+j-k} + x^{j+k-1}) = \sum_{(j,k) \in R} (x^{h_{jk}(R)} + x^{-h_{jk}(R)}).$$

Equation (5.9) then follows from a logarithm argument similar to the one we used for equation (5.8). $\square$

5.3 The main derivation

Recall that $Z'_{\text{ban}} = M(p)^{-2} Z(\widehat{F}_{\text{ban}})$, so that

$$Z'_{\text{ban}} = \sum_{R_1 R_2 R_3} (-Q_1)^{R_1} (-Q_2)^{R_2} (-Q_3)^{R_3} \bar{V}_{R_1 R_2 R_3} \bar{V}_{R'_1 R'_2 R'_3}.$$

We write

$$Z'_{\text{ban}} = \sum_R (-Q_3)^R Z_R,$$

where

$$\begin{aligned} Z_R &= \sum_{R_1, R_2} (-Q_1)^{R_1} (-Q_2)^{R_2} \bar{V}_{R_1 R_2 R} \bar{V}_{R'_1 R'_2 R} \\ &= s_R(p^{-\rho}) s_{R'}(p^{-\rho}) \sum_{A, B, R_1, R_2} (-Q_1)^{R_1} (-Q_2)^{R_2} \left(s_{R'_1/A}(p^{-R-\rho}) s_{R_2/A}(p^{-R'-\rho}) \right. \\ &\quad \left. \cdot s_{R_1/B}(p^{-R'-\rho}) s_{R'_2/B}(p^{-R-\rho}) \right) \\ &= s_R(p^{-\rho}) s_{R'}(p^{-\rho}) \sum_{A, B, R_2} \left((-Q_2)^{R_2} (-Q_1)^B s_{R_2/A}(p^{-R'-\rho}) s_{R'_2/B}(p^{-R-\rho}) \right. \\ &\quad \left. \cdot \sum_{R_1} s_{R_1/B}(-Q_1 p^{-R'-\rho}) s_{R'_1/A}(p^{-R-\rho}) \right). \end{aligned}$$

We apply equation (5.6) to the last sum in the above and equation (5.5) to the first two factors in the above. The result is

$$\begin{aligned} Z_R &= (-1)^R \prod_{i,j \in R} (1 - p^{h_{ij}(R)})^{-1} (1 - p^{-h_{ij}(R)})^{-1} \prod_{i,j=1}^{\infty} (1 - Q_1 p^{i+j-1-R'_i-R_j}) \\ &\quad \cdot \sum_{A B R_2 C} (-Q_2)^{R_2} (-Q_1)^B \left(s_{R_2/A}(p^{-R'-\rho}) s_{R'_2/B}(p^{-R-\rho}) \right. \\ &\quad \left. \cdot s_{A'/C}(-Q_1 p^{-R'-\rho}) s_{B'/C'}(p^{-R-\rho}) \right). \end{aligned}$$

Using equation (5.8) from Lemma 5.2, we get

$$\begin{aligned} Z_R &= H_R \sum_{C R_2} (-Q_2)^{R_2} (-Q_1)^C \cdot \sum_A s_{R_2/A}(p^{-R'-\rho}) s_{A'/C}(-Q_1 p^{-R'-\rho}) \\ &\quad \cdot \sum_B s_{R'_2/B}(p^{-R-\rho}) s_{B'/C'}(-Q_1 p^{-R-\rho}), \end{aligned} \tag{5.10}$$

where

$$H_R = (-1)^R M(Q_1, p)^{-1} \prod_{i,j \in R} \frac{(1 - Q_1 p^{h_{ij}(R)})(1 - Q_1 p^{-h_{ij}(R)})}{(1 - p^{h_{ij}(R)})(1 - p^{-h_{ij}(R)})}.$$

Using equations (5.7) and (5.4), we can rewrite the second and third sums in equation (5.10) as

$$\begin{aligned} \sum_A s_{R_2/A}(p^{-R'-\rho}) \cdot s_{A'/C}(-Q_1 p^{-R'-\rho}) &= s_{R_2/C'}(p^{-R'-\rho}, Q_1 p^{R+\rho}), \\ \sum_B s_{R'_2/B}(p^{-R-\rho}) \cdot s_{B'/C'}(-Q_1 p^{-R-\rho}) &= s_{R'_2/C}(p^{-R-\rho}, Q_1 p^{R+\rho}). \end{aligned}$$

Substituting back into equation (5.10), we get

$$\begin{aligned} Z_R &= H_R \sum_{C, R_2} (-Q_2)^{R_2} (-Q_1)^C \cdot s_{R_2/C'}(p^{-R'-\rho}, Q_1 p^{R+\rho}) \cdot s_{R'_2/C}(p^{-R-\rho} Q_1 p^{R'+\rho}) \\ &= H_R \sum_{C, R_2} (Q_1 Q_2)^{R_2} \cdot s_{R_2/C'}(\mathbf{y}, \mathbf{y}') \cdot s_{R'_2/C}(\mathbf{x}, \mathbf{x}'), \end{aligned}$$

where

$$\mathbf{y} = \{-Q_1^{-1} p^{-R'-\rho}\}, \quad \mathbf{y}' = \{-p^{R+\rho}\}, \quad \mathbf{x} = \{p^{-R-\rho}\}, \quad \mathbf{x}' = \{Q_1 p^{R'+\rho}\}.$$

We now use [Mac95, § I.5, Example 28(b), p. 94] to obtain

$$Z_R = H_R \prod_{i=1}^{\infty} (1 - Q_1 Q_2)^{-1} \prod_{j,k} (1 + Q_1^i Q_2^i x_j y_k) (1 + Q_1^i Q_2^i x'_j y_k) (1 + Q_1^i Q_2^i x_j y'_k) (1 + Q_1^i Q_2^i x'_j y'_k).$$

We deal with each of four factors in the product over j and k using Lemma 5.2:

$$\begin{aligned} \prod_{j,k} (1 + Q_1^i Q_2^i x_j y_k) &= \prod_{j,k} (1 - Q_1^{i-1} Q_2^i p^{j+k-1-R'_k-R_j}) \\ &= M(Q_1^{i-1} Q_2^i, p)^{-1} \prod_{j,k \in R} (1 - Q_1^{i-1} Q_2^i p^{h_{jk}(R)}) (1 - Q_1^{i-1} Q_2^i p^{-h_{jk}(R)}), \\ \prod_{j,k} (1 + Q_1^i Q_2^i x'_j y_k) &= \prod_{j,k} (1 - Q_1^i Q_2^i p^{-j+k-R'_k+R'_j}) \\ &= M(Q_1^i Q_2^i, p) \prod_{j,k \in R} (1 - Q_1^i Q_2^i p^{h_{jk}(R)})^{-1} (1 - Q_1^i Q_2^i p^{-h_{jk}(R)})^{-1}, \\ \prod_{j,k} (1 + Q_1^i Q_2^i x_j y'_k) &= \prod_{j,k} (1 - Q_1^i Q_2^i p^{j-k-1+R_k-R_j}) \\ &= M(Q_1^i Q_2^i, p) \prod_{j,k \in R} (1 - Q_1^i Q_2^i p^{h_{jk}(R)})^{-1} (1 - Q_1^i Q_2^i p^{-h_{jk}(R)})^{-1}, \\ \prod_{j,k} (1 + Q_1^i Q_2^i x'_j y'_k) &= \prod_{j,k} (1 - Q_1^{i+1} Q_2^i p^{-j-k+1+R_k+R'_j}) \\ &= M(Q_1^{i+1} Q_2^i, p)^{-1} \prod_{j,k \in R} (1 - Q_1^{i+1} Q_2^i p^{h_{jk}(R)}) (1 - Q_1^{i+1} Q_2^i p^{-h_{jk}(R)}). \end{aligned}$$

Substituting back, we see that Z_R equals

$$\begin{aligned} H_R \prod_{i=1}^{\infty} &\left(\frac{M(Q_1^i Q_2^i, p)^2}{(1 - Q_1^i Q_2^i) M(Q_1^{i-1} Q_2^i, p) M(Q_1^{i+1} Q_2^i, p)} \right. \\ &\cdot \prod_{j,k \in R} \frac{(1 - Q_1^{i-1} Q_2^i p^{h_{jk}(R)}) (1 - Q_1^{i-1} Q_2^i p^{-h_{jk}(R)}) (1 - Q_1^{i+1} Q_2^i p^{h_{jk}(R)}) (1 - Q_1^{i+1} Q_2^i p^{-h_{jk}(R)})}{(1 - Q_1^i Q_2^i p^{h_{jk}(R)})^2 (1 - Q_1^i Q_2^i p^{-h_{jk}(R)})^2} \Big). \end{aligned}$$

Putting H_R back in we get that Z_R equals

$$\begin{aligned}
 Z_{\text{prod}} \cdot (-1)^R & \prod_{j,k \in R} \left(\frac{(1 - Q_1 p^{h_{jk}(R)})(1 - Q_1 p^{-h_{jk}(R)})}{(1 - p^{h_{jk}(R)})(1 - p^{-h_{jk}(R)})} \right) \\
 & \cdot \prod_{i=1}^{\infty} \frac{(1 - Q_1^{i+1} Q_2^i p^{h_{jk}(R)})(1 - Q_1^{i+1} Q_2^i p^{-h_{jk}(R)})(1 - Q_1^{i-1} Q_2^i p^{h_{jk}(R)})(1 - Q_1^{i-1} Q_2^i p^{-h_{jk}(R)})}{(1 - Q_1^i Q_2^i p^{h_{jk}(R)})^2 (1 - Q_1^i Q_2^i p^{-h_{jk}(R)})^2} \\
 & = Z_{\text{prod}} \cdot (-1)^R \\
 & \cdot \prod_{j,k \in R} \prod_{i=1}^{\infty} \frac{(1 - Q_1^{i-1} Q_2^i p^{h_{jk}(R)})(1 - Q_1^{i-1} Q_2^i p^{-h_{jk}(R)})(1 - Q_1^{i-1} Q_2^i p^{h_{jk}(R)})(1 - Q_1^{i-1} Q_2^i p^{-h_{jk}(R)})}{(1 - Q_1^{i-1} Q_2^{i-1} p^{h_{jk}(R)})(1 - Q_1^{i-1} Q_2^{i-1} p^{-h_{jk}(R)})(1 - Q_1^i Q_2^i p^{h_{jk}(R)})(1 - Q_1^i Q_2^i p^{-h_{jk}(R)})},
 \end{aligned}$$

where we have defined

$$\begin{aligned}
 Z_{\text{prod}} & = M(Q_1, p)^{-1} \prod_{i=1}^{\infty} \frac{M(Q_1^i Q_2^i, p)^2}{(1 - Q_1^i Q_2^i) M(Q_1^{i-1} Q_2^i, p) M(Q_1^{i+1} Q_2^i, p)} \\
 & = \prod_{i=1}^{\infty} \frac{M(Q_1^i Q_2^i, p)^2}{(1 - Q_1^i Q_2^i) M(Q_1^{i-1} Q_2^i, p) M(Q_1^i Q_2^{i-1}, p)}.
 \end{aligned}$$

We now make the variable change given by equation (5.1), sum over the remaining partition, and write the result in terms of elliptic genera:

$$\begin{aligned}
 Z'_{\text{ban}} & = Z_{\text{prod}} \cdot \sum_R (y^{-1} Q)^R \cdot \prod_{i=1}^{\infty} \prod_{j,k \in R} \frac{(1 - t^{h_{jk}} y q^{i-1})(1 - t^{-h_{jk}} y q^{i-1})(1 - t^{h_{jk}} y^{-1} q^i)(1 - t^{-h_{jk}} y^{-1} q^i)}{(1 - t^{h_{jk}} q^{i-1})(1 - t^{-h_{jk}} q^{i-1})(1 - t^{h_{jk}} q^i)(1 - t^{-h_{jk}} q^i)} \\
 & = Z_{\text{prod}} \cdot \sum_{k=0}^{\infty} Q^k \text{Ell}_{q,y}((\mathbb{C}^2)^{[k]}, t) \\
 & = Z_{\text{prod}} \cdot \prod_{n=0}^{\infty} \prod_{m=1}^{\infty} \prod_{l,k \in \mathbb{Z}} (1 - t^k q^n y^l Q^m)^{-c(nm, l, k)} \\
 & = Z_{\text{prod}} \cdot \prod_{n=0}^{\infty} \prod_{m=1}^{\infty} \prod_{l,k \in \mathbb{Z}} (1 - t^k q^n y^l Q^m)^{-c(4nm - l^2, k)},
 \end{aligned}$$

where the last three equalities come from equation (2.1), Theorem 2.1, and Proposition 2.2, respectively.

Returning to the DT-variables via

$$t = p, \quad q = Q_1 Q_2, \quad y = Q_1, \quad Q = Q_1 Q_3,$$

reindexing by

$$d_1 = n + l + m, \quad d_2 = n, \quad d_3 = m,$$

and observing that

$$\|d\| = 2d_1 d_2 + 2d_2 d_3 + 2d_3 d_1 - d_1^2 - d_2^2 - d_3^2 = 4mn - l^2,$$

we find

$$Z'_{\text{ban}} = Z_{\text{prod}} \prod_{d_3=1}^{\infty} \prod_{d_2=0}^{\infty} \prod_{d_1, k \in \mathbb{Z}} (1 - p^k Q_1^{d_1} Q_2^{d_2} Q_3^{d_3})^{-c(\|d\|, k)}.$$

Observing further that when $d_1 < 0$ and $d_3 > 0$, we have $\|d\| = 4d_3d_1 - (d_2 - d_1 - d_3)^2 < -1$, and so $c(\|d\|, k) = 0$, we get

$$Z(\widehat{F}_{\text{ban}}) = M(p)^2 \cdot Z'_{\text{ban}} = M(p)^2 \cdot Z_{\text{prod}} \prod_{d_3=1}^{\infty} \prod_{d_2, d_3=0}^{\infty} \prod_{k \in \mathbb{Z}} (1 - p^k Q_1^{d_1} Q_2^{d_2} Q_3^{d_3})^{-c(\|d\|, k)}.$$

Finally, we claim that

$$M(p)^2 \cdot Z_{\text{prod}} = \prod_{(*)} (1 - p^k Q_1^{d_1} Q_2^{d_2} Q_3^{d_3})^{-c(\|d\|, k)},$$

where the product is over

$$(*) \quad d_3 = 0, \quad d_1, d_2 \geq 0, \quad \text{and} \quad \begin{cases} k \in \mathbb{Z} & \text{if } (d_1, d_2) \neq (0, 0), \\ k > 0 & \text{if } (d_1, d_2) = (0, 0). \end{cases}$$

Indeed, if $d_3 = 0$, then $\|d\| = -(d_2 - d_1)^2$ and by Corollary 2.3, the product over $(*)$ reduces to the terms where

$$k \in \mathbb{Z}, \quad (d_1, d_2) = (d, d-1), (d-1, d), (d, d), \quad d > 1$$

and the special case $d_1 = d_2 = 0, k > 0$. Applying Corollary 2.3, we easily deduce the above claim.

Substituting the claimed equation back into the previous equation for $Z(\widehat{F}_{\text{ban}})$ then finishes the proof of Theorem 1.5. $\square$

Appendix. The Gromov–Witten potentials are Siegel modular forms

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A.1 Overview

Let $F_g(Q_1, Q_2, Q_3)$ be the genus $g \geq 2$ Gromov–Witten potential for fiber classes in X_{ban} . Namely, let

$$F_g(Q_1, Q_2, Q_3) = \sum_{d_1, d_2, d_3 \geq 0} GW_{\underline{d}}^g(X_{\text{ban}}) Q_1^{d_1} Q_2^{d_2} Q_3^{d_3},$$

where $GW_{\underline{d}}^g(X_{\text{ban}})$ is the genus g Gromov–Witten invariant of X_{ban} in the class $\beta_{\underline{d}} = d_1 C_1 + d_2 C_2 + d_3 C_3$.

Assuming that the Gromov–Witten/Donaldson–Thomas (GW/DT) correspondence conjectured in [MNOP06] holds, we may compute $F_g(Q_1, Q_2, Q_3)$ using the formula for the Donaldson–Thomas partition function derived in the main text. The main result of this appendix is that F_g is an explicit genus 2 Siegel modular form of weight $2g - 2$.

DEFINITION A.1. Let $\Omega = \begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix} \in \mathbb{H}_2$ be the standard coordinates on the genus 2 Siegel upper half plane. A holomorphic (respectively, meromorphic) genus 2 Siegel modular form of weight k is a holomorphic (respectively, meromorphic) function $F(\Omega)$ satisfying

$$F((A\Omega + B)(C\Omega + D)^{-1}) = \det((C\Omega + D)^k) F(\Omega)$$

for all $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_4(\mathbb{Z})$ (cf. [EZ85]). We denote the space of meromorphic genus 2 Siegel forms of weight k by Siegel_k .

As we will detail in Section A.2, a standard way to construct a genus 2 Siegel modular form of weight $2g - 2$ is to take what is called the Maass lift of a Jacobi form of weight $2g - 2$ and

index 1. Moreover, such Jacobi forms are easily obtained by taking a modular form of weight $2g$ and multiplying it by $\phi_{-2,1}$, the unique weak Jacobi form of weight -2 and index 1. Some authors call this the Skoruppa lift of the modular form. Schematically, we have

$$\mathrm{Mod}_{2g} \xrightarrow{\text{Skoruppa}} \mathrm{Jac}_{2g-2,1} \xrightarrow{\text{Maass}} \mathrm{Siegel}_{2g-2}.$$

We call the composition the Skoruppa–Maass lift. It takes weight $2g$ modular forms to genus 2, weight $2g-2$ Siegel modular forms. (This lift is different from the famous Saito–Kurokawa lift which also takes Mod_{2g} to Siegel_{2g-2} . While both use the Maass lift, the Saito–Kurokawa lift uses a combination of the Shimura correspondence and a lift studied by Eichler–Zagier to go from Mod_{2g} to $\mathrm{Jac}_{2g-2,1}$; see [EZ85].) Our main result is the following.

THEOREM A.2. *Assume that the GW/DT correspondence holds for X_{ban} . Then for $g \geq 2$, the genus g Gromov–Witten potentials $F_g(Q_1, Q_2, Q_3)$ of X_{ban} are meromorphic genus 2 Siegel modular forms of weight $2g-2$, where*

$$Q_1 = e^{2\pi i z}, \quad Q_2 = e^{2\pi i(\tau-z)}, \quad Q_3 = e^{2\pi i(\sigma-z)}.$$

Specifically, F_g is the Skoruppa–Maass lift of $a_g E_{2g}(\tau)$, the $(2g)$ th Eisenstein series times the constant

$$a_g = 12|B_{2g}|/(2g(2g-2)!),$$

where B_{2g} is the $(2g)$ th Bernoulli number.

The ring of holomorphic, even weight, genus 2 Siegel modular forms is a polynomial ring generated by the Igusa cusp forms χ_{10} and χ_{12} of weight 10 and 12, respectively, and the Siegel Eisenstein series $\mathcal{E}_4$ and $\mathcal{E}_6$ of weight 4 and 6, respectively [Igu62, Igu67]. Although the F_g are meromorphic, the denominators can be determined explicitly.

COROLLARY A.3. *For $g \geq 2$, the product $\chi_{10}^{g-1} \cdot F_g$ is a holomorphic Siegel form of weight $12g-12$.*

This corollary follows from a proof given by Aoki in an upcoming paper in Pure Appl. Math. Q. In terms of the generators χ_{10} , χ_{12} , $\mathcal{E}_4$, $\mathcal{E}_6$, the first few potentials are given explicitly as

$$\begin{aligned} F_2 &= \frac{1}{240} \left(\frac{\chi_{12}}{\chi_{10}} \right), \\ F_3 &= \frac{-1}{60480} \left(6\mathcal{E}_4 - 5 \left(\frac{\chi_{12}}{\chi_{10}} \right)^2 \right), \\ F_4 &= \frac{1}{3628800} \left(\frac{35}{2} \left(\frac{\chi_{12}}{\chi_{10}} \right)^3 - \frac{63}{2} \left(\frac{\chi_{12}}{\chi_{10}} \right) \mathcal{E}_4 + 15\mathcal{E}_6 \right), \\ F_5 &= \frac{-1}{106444800} \left(\frac{-175}{3} \left(\frac{\chi_{12}}{\chi_{10}} \right)^4 + 140 \left(\frac{\chi_{12}}{\chi_{10}} \right)^2 \mathcal{E}_4 - \frac{200}{3} \left(\frac{\chi_{12}}{\chi_{10}} \right) \mathcal{E}_6 - 14\mathcal{E}_4^2 \right). \end{aligned}$$

Note that the prefactor is given by

$$F_g(0, 0, 0) = \frac{12B_{2g-2}|B_{2g}|}{2g(2g-2)(2g-2)!},$$

the degree 0 genus g Gromov–Witten invariant of X_{ban} . See [Pie18] for details of this computation.

Remark A.4. Theorem A.2 has a nice interpretation in terms of mirror symmetry. The Gromov–Witten potentials are functions of local coordinates on the Kähler moduli space. Under mirror

symmetry, these become coordinates on the complex moduli space of the mirror. Since the arguments of a genus 2 Siegel modular form are coordinates on the moduli space of genus 2 curves (or Abelian surfaces), we expect the complex moduli space of $\check{X}_{\text{ban}}$, the mirror of the banana manifold, to contain a subspace isomorphic to the moduli space of genus 2 curves. Indeed, it has already been observed that the mirror of a *local* banana configuration should be a genus 2 curve [AAK16, GKR17, Rud17].

Remark A.5. We also determine the genus 0 and genus 1 potentials. Up to degree 0 terms (which are unstable for $g = 0$ or $g = 1$), F_0 is the Skoruppa–Maass lift of the constant 12 (viewed as a weight 0 modular form), and F_1 is the Maass lift of $12\wp \cdot \phi_{-2,1}$, where $\wp$ is the Weierstrass $\wp$ -function.⁹

A.2 Modular forms and lifts

DEFINITION A.6. A weight k *modular form* is a holomorphic function $f(\tau)$ on $\mathbb{H} = \{\tau \in \mathbb{C} \text{ with } \text{Im } \tau > 0\}$ satisfying the following conditions:

- For all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$,

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau).$$

- The function $f(\tau)$ admits a Fourier series of the form

$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n, \quad q = e^{2\pi i \tau}.$$

We denote the space of weight k modular forms by Mod_k . We sometimes abuse notation by writing $f(q)$ for the Fourier expansion of $f(\tau)$.

The $(2g)$ th Eisenstein series is given by

$$E_{2g}(q) = 1 - \frac{4g}{B_{2g}} \sum_{n=1}^{\infty} \sum_{d|n} d^{2g-1} q^n,$$

where B_{2g} is the $(2g)$ th Bernoulli number. The Eisenstein series E_{2g} is a modular form of weight $2g$ for all $g \geq 2$.

DEFINITION A.7. A *weak Jacobi form* of weight k and index m is a holomorphic function $\phi(\tau, z)$ on $\mathbb{H} \times \mathbb{C}$ satisfying the following conditions:

- For all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$,

$$\phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^k \exp\left(\frac{2\pi i c m z^2}{c\tau + d}\right) \phi(\tau, z).$$

- For all $u, v \in \mathbb{Z}$,

$$\phi(\tau, z + u\tau + v) = e^{-2\pi i m(\tau u^2 + 2zu)} \phi(\tau, z).$$

- The function ϕ admits a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{n=0}^{\infty} \sum_{l \in \mathbb{Z}} c_{\phi}(n, l) q^n y^l,$$

where $q = e^{2\pi i \tau}$ and $y = e^{2\pi i z}$.

⁹Interestingly, the Jacobi form $\phi_{0,1} = 12\wp \cdot \phi_{-2,1}$ is also equal to $\frac{1}{2} \text{Ell}(K3)$.

In the case of index $m = 1$, the Fourier coefficients $c_\phi(n, l)$ depend only on $4n - l^2$ and so, in this case, we will sometimes write

$$c_\phi(4n - l^2) = c_\phi(n, l).$$

We denote the space of weak Jacobi forms of weight k and index m by $\text{Jac}_{k,m}$. We sometimes abuse notation by writing $\phi(q, y)$ for the Fourier expansion of $\phi(\tau, z)$. A basic example is given by $\phi_{-2,1}$, whose Fourier expansion is given by

$$\phi_{-2,1}(q, y) = y^{-1}(1-y)^2 \prod_{n=1}^{\infty} \frac{(1-yq^n)^2(1-y^{-1}q^n)^2}{(1-q^n)^4}.$$

Up to a multiplicative constant, $\phi_{-2,1}$ is the unique weak Jacobi form of weight -2 and index 1 . We also will use the Weierstrass $\wp$ -function:

$$\wp(q, y) = \frac{1}{12} + \frac{y}{(1-y)^2} + \sum_{n=1}^{\infty} \sum_{d|n} d(y^d + y^{-d} - 2)q^n,$$

and we note that up to a multiplicative constant,

$$\phi_{0,1} = 12 \cdot \phi_{-2,1} \cdot \wp$$

is the unique weak Jacobi form of weight 0 and index 1 .

The product of a weak Jacobi form of weight k and index m with a modular form of weight n is a weak Jacobi form of $k+n$ and index m . In particular, multiplication by $\phi_{-2,1}$ defines a map which we call the *Skoruppa lift*:

$$\text{Mod}_{2g} \xrightarrow{\text{Skoruppa}} \text{Jac}_{2g-2,1}.$$

DEFINITION A.8. Let k be even. For a non-negative integer m , the m th Hecke operator

$$V_m : \text{Jac}_{k,1} \rightarrow \text{Jac}_{k,m}$$

is given by taking

$$\phi(q, y) = \sum_{n=0}^{\infty} \sum_{l \in \mathbb{Z}} c_\phi(n, l) q^n y^l$$

to

$$\begin{aligned} (\phi|V_m) &= \sum_{n=0}^{\infty} \sum_{r \in \mathbb{Z}} \sum_{d|(n,r,m)} d^{k-1} c_\phi\left(\frac{nm}{d^2}, \frac{r}{d}\right) q^n y^r & \text{for } m > 0, \\ (\phi|V_0) &= c_\phi(0, 0) \frac{-B_k}{2k} + \sum_{n=0}^{\infty} \sum_{\substack{r \in \mathbb{Z} \\ r > 0 \text{ if } n=0}} \sum_{d|(n,r)} d^{k-1} c_\phi\left(0, \frac{r}{d}\right) q^n y^r & \text{for } m = 0. \end{aligned}$$

DEFINITION A.9. Let $\phi \in \text{Jac}_{k,1}$ with k even. The *Maass lift* of ϕ is given by

$$ML(\phi) = \sum_{m=0}^{\infty} (\phi|V_m) Q^m.$$

The following is due to Eichler–Zagier [EZ85] in the case of holomorphic Jacobi forms and to Borcherds [Bor95, Theorem 9.3] and Aoki in the case of weak Jacobi forms (upcoming paper).¹⁰

¹⁰We thank H. Aoki and G. Oberdieck for a discussion on this point.

THEOREM A.10. *The Maass lift of a weak Jacobi form of weight $k > 0$ and index 1 is a meromorphic genus 2 Siegel modular form of weight k . If the Jacobi form is holomorphic, then the Maass lift is also holomorphic. Here $Q = e^{2\pi i\sigma}$, $q = e^{2\pi i\tau}$, and $y = e^{2\pi iz}$, where $(\begin{smallmatrix} \tau & z \\ z & \sigma \end{smallmatrix}) \in \mathbb{H}_2$.*

We may reformulate the Maass lift in terms of polylogarithms. Let

$$\mathrm{Li}_a(x) = \sum_{n=1}^{\infty} n^{-a} x^n.$$

Then a straightforward computation yields the following.

LEMMA A.11. *Let $\phi = \sum_{n=0}^{\infty} \sum_{l \in \mathbb{Z}} c_{\phi}(4n - l^2) q^n y^l \in \mathrm{Jac}_{k,1}$ with k even. Then*

$$ML(\phi) = c_{\phi}(0) \frac{-B_k}{2k} + \sum_{n,m \geq 0} \sum_{\substack{l \in \mathbb{Z} \\ l > 0 \text{ if } (n,m)=(0,0)}} c_{\phi}(4nm - l^2) \mathrm{Li}_{1-k}(Q^m q^n y^l).$$

A.3 The λ -expansion of $\mathrm{Ell}_{q,y}(\mathbb{C}^2, t)$

Recall from Section 2 that the coefficients $c(d, k)$ are defined by the expansion of the equivariant elliptic genus of $\mathbb{C}^2$:

$$\mathrm{Ell}_{q,y}(\mathbb{C}^2, t) = \sum_{n=0}^{\infty} \sum_{l, k \in \mathbb{Z}} c(4n - l^2, k) q^n y^l t^k.$$

Let $t = e^{i\lambda}$. Theorem 4.4 in Zhou [Zho15] gives the expansion of $\mathrm{Ell}_{q,y}(\mathbb{C}^2, t)$ as a Laurent series in λ . His result is¹¹

$$\mathrm{Ell}_{q,y}(\mathbb{C}^2, e^{i\lambda}) = \lambda^{-2} \cdot \phi_{-2,1}(q, y) \cdot \left(1 + \wp(q, y) \lambda^2 + \sum_{g=2}^{\infty} \frac{|B_{2g}|}{2g(2g-2)!} E_{2g}(q) \lambda^{2g} \right).$$

Let

$$\psi_{2g-2}(q, y) = \mathrm{Coef}_{\lambda^{2g-2}} [\mathrm{Ell}_{q,y}(\mathbb{C}^2, e^{i\lambda})],$$

so that Zhou's result can be expressed as

$$\psi_{2g-2}(q, y) = \phi_{-2,1}(q, y) \cdot \begin{cases} 1 & \text{if } g = 0, \\ \wp(q, y) & \text{if } g = 1, \\ \frac{|B_{2g}|}{2g(2g-2)!} E_{2g}(q) & \text{if } g > 1. \end{cases}$$

We observe that ψ_{2g-2} is a weak Jacobi form of weight $2g - 2$ and index 1 and consequently has an expansion

$$\psi_{2g-2}(q, y) = \sum_{n=0}^{\infty} \sum_{l \in \mathbb{Z}} c_{2g-2}(4n - l^2) q^n y^l,$$

which defines the coefficients $c_{2g-2}(d)$.

Comparing coefficients in the λ and the t expansions of $\mathrm{Ell}_{q,y}(\mathbb{C}^2, t = e^{i\lambda})$, we get the following

¹¹We use the $r = 1$ case of Zhou's theorem. Our λ is $2\pi t$ in Zhou's notation (his t is not our t). There is a typo in his formula: the $\eta(\tau)$ should be $\eta(\tau)^3$.

fundamental relationship between the coefficients $c_{2g-2}(d)$ and $c(d, l)$:

$$\sum_{g=0}^{\infty} c_{2g-2}(d) \lambda^{2g-2} = \sum_{l \in \mathbb{Z}} c(d, l) e^{il\lambda}. \quad (\text{A.11})$$

A.4 Gromov–Witten potentials via the GW/DT correspondence

The GW/DT correspondence is a conjectural equivalence between the Gromov–Witten and the Donaldson–Thomas invariants of a Calabi–Yau threefold [MNOP06]. It has been proven for a broad class of Calabi–Yau threefolds including complete intersections in products of projective spaces [PP17], which unfortunately does not include X_{ban} .

However, if we assume that the GW/DT correspondence holds for X_{ban} , we may compute the genus g Gromov–Witten potentials from our formula for the Donaldson–Thomas partition function (Theorem 1.5). We define the reduced genus g Gromov–Witten potential (for banana curve classes) of X_{ban} by

$$F'_g(Q_1, Q_2, Q_3) = \sum_{\underline{d} > 0} GW_{\underline{d}}^g(X_{\text{ban}}) Q_1^{d_1} Q_2^{d_2} Q_3^{d_3}.$$

Here $GW_{\underline{d}}^g(X_{\text{ban}})$ denotes the genus g Gromov–Witten invariant in the class $\beta_{\underline{d}} = d_1 C_1 + d_2 C_2 + d_3 C_3$ and $\underline{d} > 0$ means $d_i \geq 0$ and $(d_1, d_2, d_3) \neq (0, 0, 0)$.

The GW/DT correspondence asserts that under the change of variables $p = e^{i\lambda}$,

$$\sum_{g=0}^{\infty} F'_g(Q_1, Q_2, Q_3) \lambda^{2g-2} = \log \left(\frac{Z_{\Gamma}(X_{\text{ban}})}{Z_{\Gamma}(X_{\text{ban}})|_{Q_i=0}} \right).$$

We now prove Theorem A.2. Applying Theorem 1.5 and using equation (A.11), we get

$$\begin{aligned} \sum_{g=0}^{\infty} F'_g(Q_1, Q_2, Q_3) \lambda^{2g-2} &= \log \left(\prod_{\underline{d} > 0} \prod_{k \in \mathbb{Z}} (1 - p^k Q_1^{d_1} Q_2^{d_2} Q_3^{d_3})^{-12c(\|\underline{d}\|, k)} \right) \\ &= \sum_{\underline{d} > 0} \sum_{k \in \mathbb{Z}} 12c(\|\underline{d}\|, k) \sum_{n=1}^{\infty} \frac{1}{n} p^{nk} Q_1^{nd_1} Q_2^{nd_2} Q_3^{nd_3} \\ &= 12 \sum_{\underline{d} > 0} \sum_{n=1}^{\infty} \frac{1}{n} Q_1^{nd_1} Q_2^{nd_2} Q_3^{nd_3} \sum_{g=0}^{\infty} c_{2g-2}(\|\underline{d}\|) n^{2g-2} \lambda^{2g-2}. \end{aligned}$$

Thus we find that

$$\begin{aligned} F'_g(Q_1, Q_2, Q_3) &= 12 \sum_{\underline{d} > 0} c_{2g-2}(\|\underline{d}\|) \sum_{n=1}^{\infty} n^{2g-3} Q_1^{nd_1} Q_2^{nd_2} Q_3^{nd_3} \\ &= 12 \sum_{\underline{d} > 0} c_{2g-2}(\|\underline{d}\|) \text{Li}_{3-2g}(Q_1^{d_1} Q_2^{d_2} Q_3^{d_3}). \end{aligned}$$

Substituting

$$Q_1 = e^{2\pi iz} = y, \quad Q_2 = e^{2\pi i(\tau-z)} = qy^{-1}, \quad Q_3 = e^{2\pi i(\sigma-z)} = Qy^{-1},$$

reindexing by

$$d_1 = l + n + m, \quad d_2 = n, \quad d_3 = m,$$

so that $\|\underline{\mathbf{d}}\| = 4nm - l^2$, and noting that $\underline{\mathbf{d}} > 0$ is equivalent to $n, m \geq 0, l \in \mathbb{Z}$ and $l > 0$ if $n = m = 0$, we find that

$$F'_g(q, y, Q) = 12 \sum_{n, m \geq 0} \sum_{\substack{l \in \mathbb{Z} \\ l > 0 \text{ if } (n, m) = (0, 0)}} c_{2g-2}(4nm - l^2) \text{Li}_{3-2g}(Q^m q^n y^l).$$

For $g > 1$, the full genus g Gromov–Witten potential is the reduced potential plus the constant term:

$$F_g = GW_{\underline{\mathbf{0}}}^g(X_{\text{ban}}) + F'_g.$$

Using for example the formulas in [MNOP06, § 2.1], we know

$$GW_{\underline{\mathbf{0}}}^g(X_{\text{ban}}) = (-1)^g \frac{1}{2} e(X_{\text{ban}}) \frac{|B_{2g}| |B_{2g-2}|}{2g(2g-2)(2g-2)!} = 12 \left(\frac{-B_{2g-2}}{4g-4} \right) \cdot \left(\frac{-|B_{2g}|}{g(2g-2)!} \right).$$

Examining the $y^0 q^0$ -term of

$$\psi_{2g-2}(q, y) = \phi_{-2,1} \cdot \frac{|B_{2g}|}{2g(2g-2)!} E_{2g}$$

for $g > 1$ yields

$$c_{2g-2}(0) = \frac{-|B_{2g}|}{g(2g-2)!},$$

and hence we find

$$F_g = 12 \left(c_{2g-2}(0) \frac{-B_{2g-2}}{4g-4} + \sum_{n, m \geq 0} \sum_{\substack{l \in \mathbb{Z} \\ l > 0 \text{ if } (n, m) = (0, 0)}} c_{2g-2}(4nm - l^2) \text{Li}_{3-2g}(Q^m q^n y^l) \right).$$

By Lemma A.11, the above is exactly the Maass lift of $12\psi_{2g-2}$, and hence we find that F_g is the Skoruppa–Maass lift of $a_g E_{2g}$, where

$$a_g = \frac{12 |B_{2g}|}{2g(2g-2)!}.$$

This completes the proof of Theorem A.2.

Remark A.12. Note that the proof of the theorem also shows that up to constant terms, the genus 0 and the genus 1 Gromov–Witten potentials are given by the Maass lifts of $12\phi_{-2,1}$ and $12\phi_{-2,1}\wp = \phi_{0,1}$, respectively. Although these are weak Jacobi forms of index 1, they are not of positive weight, and hence their Maass lifts are not guaranteed to be Siegel forms. See [Pie18] for a further discussion.

A.5 BPS invariants from a Donaldson–Thomas partition function in product form

The Gopakumar–Vafa invariants (also known as the BPS invariants) $n_\beta^g(X)$ can be defined in terms of Gromov–Witten or Donaldson–Thomas invariants and (conjecturally) have better finiteness properties. Unlike the Donaldson–Thomas invariants, it is expected that there are only a finite number of non-zero Gopakumar–Vafa invariants for each curve class β .

One definition of Gopakumar–Vafa invariants, which is equivalent to the usual one given in terms of Gromov–Witten invariants via the Gopakumar–Vafa formula, is the following.

DEFINITION–THEOREM A.13. *Let X be any Calabi–Yau threefold, and suppose that the Donald–*

son–Thomas partition function is given by

$$Z(X) = \prod_{\beta \in H_2(X)} \prod_{k \in \mathbb{Z}} (1 - p^k Q^\beta)^{-a(\beta, k)} \quad (\text{A.12})$$

for some integers $a(\beta, k) \in \mathbb{Z}$. Then for $\beta \neq 0 \in H_2(X)$ and $g \in \mathbb{Z}$, the Gopakumar–Vafa invariants $n_\beta^g(X)$ can be defined by the formula

$$\sum_g n_\beta^g(X) \left(y^{\frac{1}{2}} + y^{-\frac{1}{2}} \right)^{2g} = \left(y^{\frac{1}{2}} + y^{-\frac{1}{2}} \right)^2 \sum_{k \in \mathbb{Z}} a(\beta, k) (-y)^k.$$

This is equivalent, assuming the GW/DT correspondence, to defining $n_\beta^g(X)$ in terms of the Gromov–Witten invariants via the Gopakumar–Vafa formula.

Remark A.14. It is expected that $n_\beta^g(X)$ is zero unless $0 \leq g \leq g_{\max}$, where $g_{\max}$ is the maximal arithmetic genus of curves in the class β . Note that the left-hand side of the above formula is a palindromic polynomial in y (that is, invariant under $y \leftrightarrow y^{-1}$). One can see that the invariants $n_\beta^g(X)$ are well defined by the above definition as follows. Any power series $Z \in \mathbb{Z}((p))[[Q]]$ can be uniquely written in the form of equation (A.12) for some collection of integers $a(\beta, k)$. By a theorem of Bridgeland [Bri11], the right-hand side of equation (A.12) is a rational function of y , invariant under $y \leftrightarrow y^{-1}$, possibly having poles at $y = -1$. Such functions have a basis given by the functions

$$\left(y^{\frac{1}{2}} + y^{-\frac{1}{2}} \right)^{2g} = y^{-g} (1 + y)^{2g}, \quad g \in \mathbb{Z},$$

and so the left-hand side of (A.12) is well defined and uniquely determines $n_\beta^g(X)$. The conjecture that $n_\beta^g(X) = 0$ if $g < 0$ is equivalent to the right-hand side of (A.12) not having a pole.

Proof of Definition–Theorem A.13. The Gopakumar–Vafa formula [GV98] expresses the reduced Gromov–Witten potential function $F'(X)$ of a Calabi–Yau threefold X in terms of conjecturally integer invariants $n_\beta^g(X)$, which are commonly called Gopakumar–Vafa invariants or BPS invariants:

$$F'(X) = \sum_{g \geq 0} \sum_{\beta \neq 0} GW_\beta^g(X) \lambda^{2g-2} Q^\beta = \sum_{g \geq 0} \sum_{\beta \neq 0} \sum_{m > 0} n_\beta^g(X) \frac{1}{m} \left(2 \sin \frac{m\lambda}{2} \right)^{2g-2} Q^{m\beta}.$$

One can re-express this formula in terms of the partition function $Z'(X) = \exp(F'(X))$ as follows:¹²

$$Z'(X) = \prod_{\beta \neq 0} \prod_{m \in \mathbb{Z}} (1 - Q^\beta p^m)^{-\sum_{g \geq 0} n_\beta^g(X) \binom{2g-2}{g-1-m} (-1)^m},$$

where $p = \exp(i\lambda)$ and $\binom{n}{k} = n(n-1) \cdots (n-k+1)/k!$ is defined for all $k \geq 0$ and $n \in \mathbb{Z}$.

The Donaldson–Thomas partition function can always be written in the form

$$Z(X) = \prod_{\beta} \prod_{m \in \mathbb{Z}} (1 - Q^\beta p^m)^{-c(\beta, m)}$$

for some $c(\beta, m) \in \mathbb{Z}$. Then for $\beta \neq 0$,

$$c(\beta, m) = \sum_{g \geq 0} n_\beta^g(X) \binom{2g-2}{g-1-m} (-1)^m.$$

¹²This expression essentially appears in [Kat06, (18)], although our convention for $\binom{n}{k}$ allows us to write the formula more uniformly.

The definition–theorem is then an easy consequence of the binomial theorem. $\square$

A.6 Gopakumar–Vafa invariants

In this section, we give tables of values of the Gopakumar–Vafa invariants $n_a^g(X_{\text{ban}})$ for small values of g and a . Since all values are divisible by 12, we list $\frac{1}{12} n_a^g(X_{\text{ban}})$ (which can also be regarded as the Gopakumar–Vafa invariants of a local banana configuration). We note that the non-zero values have a congruent to 0 or -1 modulo 4, and so we organize the tables as such.

$\frac{1}{12} n_{4n-1}^g(X_{\text{ban}})$	$g = 0$	$g = 1$	$g = 2$	$g = 3$	$g = 4$	$g = 5$	$g = 6$
$n = 0$	1	0	0	0	0	0	0
$n = 1$	8	−6	1	0	0	0	0
$n = 2$	39	−46	17	−2	0	0	0
$n = 3$	152	−242	139	−34	3	0	0
$n = 4$	513	−1024	800	−304	56	−4	0
$n = 5$	1560	−3730	3683	−1912	548	−82	5

$\frac{1}{12} n_{4n}^g(X_{\text{ban}})$	$g = 0$	$g = 1$	$g = 2$	$g = 3$	$g = 4$	$g = 5$	$g = 6$
$n = 0$	−2	1	0	0	0	0	0
$n = 1$	−12	10	−2	0	0	0	0
$n = 2$	−56	72	−30	4	0	0	0
$n = 3$	−208	352	−220	60	−6	0	0
$n = 4$	−684	1434	−1194	492	−100	8	0
$n = 5$	−2032	5056	−5252	2908	−902	148	−10

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