Deformations of log canonical and F -pure singularities

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ABSTRACT

We introduce a lifting property for local cohomology which leads to a unified treatment of the dualizing complex for flat morphisms with semi-log-canonical, Du Bois or F -pure fibers. As a consequence, we obtain that in all three cases, the cohomology sheaves of the relative dualizing complex are flat and commute with base change. We also derive several consequences for deformations of semi-log-canonical, Du Bois and F -pure singularities.

1. Introduction

One of the difficulties of higher-dimensional birational geometry and moduli theory is that the occurring singularities are frequently not Cohen–Macaulay.

On a proper Cohen–Macaulay scheme, we have a dualizing *sheaf* ω_X and Serre duality. By contrast, on an arbitrary proper scheme, we have a dualizing *complex* $\omega_X^\bullet$, and the isomorphism of Serre duality is replaced by a spectral sequence of Grothendieck duality.

The “most important” cohomology sheaf of the dualizing complex $\omega_X^\bullet$ is

$$h^{-\dim X}(\omega_X^\bullet) \simeq \omega_X,$$

and X is Cohen–Macaulay if and only if the other cohomology sheaves $h^{-i}(\omega_X^\bullet)$ are all zero; cf. [Con00, Theorem 3.5.1]. Thus these $h^{-i}(\omega_X^\bullet)$ measure “how far” X is from being Cohen–Macaulay; see Proposition 8.1 for a more precise claim. Our main result implies that in flat families $X \rightarrow B$ with log canonical or F -pure fibers, the cohomology sheaves $h^{-i}(\omega_{X/B}^\bullet)$ are flat over B and commute with base change. In particular, being Cohen–Macaulay is a deformation-invariant property for such singularities. Note that for flat families, Cohen–Macaulay is always an open condition but usually not a closed one.

One of the puzzles of higher-dimensional singularity theory is that while the singularities of the minimal model program (log terminal, log canonical, Du Bois, etc.) and of positive characteristic commutative algebra (F -pure, F -injective, etc.) are very closely related, the methods to study them are completely different. Here we isolate the following quite powerful common property for some of these classes.

Received 22 October 2018, accepted in final form 14 May 2020.

2010 *Mathematics Subject Classification* 14B07, 14D20, 14D99.

Keywords: log canonical, Du Bois, F -pure.

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János Kollár was supported in part by NSF Grants DMS-1362960 and DMS-1901855. Sándor Kovács was supported in part by NSF Grant DMS-1565352 and the Craig McKibben and Sarah Merner Endowed Professorship in Mathematics.

DEFINITION 1.1. Let A be a noetherian ring and $(T, \mathfrak{n})$ a noetherian local A -algebra. We say that T has *liftable local cohomology over A* if for any noetherian local A -algebra $(R, \mathfrak{m})$ and nilpotent ideal $I \subset R$ such that $R/I \simeq T$, the natural morphism on local cohomology

$$H_{\mathfrak{m}}^i(R) \longrightarrow H_{\mathfrak{n}}^i(T)$$

is surjective for all i .

We say that T has *liftable local cohomology* if it has liftable local cohomology over $\mathbb{Z}$.

Remark 1.2. Notice that using the above notation, if $\phi: A' \rightarrow A$ is a ring homomorphism from another noetherian ring A' , then if T has liftable local cohomology over A' , then it also has liftable local cohomology over A . In particular, if T has liftable local cohomology over $\mathbb{Z}$, then it has liftable local cohomology over any noetherian ring A , justifying the above terminology.

Furthermore, if $A = k$ is a field of characteristic zero, then the notions of having liftable local cohomology over k and over $\mathbb{Z}$ are equivalent. This follows in one direction by the above and in the other by the Cohen structure theorem [Sta20, Tag 032A].

Remark 1.3. The results of this article are used in [CV18] to settle a conjecture of Herzog on ideals with square-free initial ideals. A closely related notion, a ring being *cohomologically full*, is defined in [DDM18].

We prove in Proposition 6.1 that Du Bois singularities have liftable local cohomology. On the Frobenius side, the right concept seems to be F -anti-nilpotent singularities, a notion introduced in [EH08] that lies between F -pure and F -injective [Ma14, MQ18]. We are very grateful to L. Ma and K. Schwede for pointing out that, by a result of Ma–Schwede–Shimomoto [MSS17], F -anti-nilpotent singularities also have liftable local cohomology over their ground field. We discuss this in Proposition 7.2.

With this definition (cf. Definition 4.2), our main technical theorem is the following.

THEOREM 1.4 (Theorem 5.13). *Let $f: X \rightarrow B$ be a flat morphism of schemes which is essentially of finite type, and let $b \in B$ such that X_b has liftable local cohomology over B . Then there exists an open neighborhood $X_b \subset U \subset X$ such that $\mathbf{h}^{-i}(\omega_{U/B}^\bullet)$ is flat over B and commutes with base change for each $i \in \mathbb{Z}$.*

For applications, the following consequences are especially important.

COROLLARY 1.5. *Let $f: X \rightarrow B$ be a flat morphism of schemes, essentially of finite type over a field k . Let $b \in B$ be a point. Assume that*

- (i) *either $\text{char } k = 0$ and X_b is Du Bois, for example semi-log-canonical,*
- (ii) *or $\text{char } k > 0$ and X_b is F -anti-nilpotent, for example F -pure.*

Then there exists an open neighborhood $X_b \subset U \subset X$ such that $\mathbf{h}^{-i}(\omega_{U/B}^\bullet)$ is flat over B and commutes with base change for each $i \in \mathbb{Z}$.

Corollary 1.5(i) can be viewed as a generalization of the following Corollary 1.6(i), proved in [KK10] for projective morphisms and in [MSS17] in general (cf. [KS16b]).

COROLLARY 1.6. *Let (X, x) be a local scheme, essentially of finite type over a field, and assume that*

- (i) *either $\text{char } k = 0$ and X is Du Bois, for example, semi-log-canonical,*

(ii) or $\text{char } k > 0$ and X is F -anti-nilpotent, for example F -pure.

If (X, x) admits a flat deformation whose generic fiber is Cohen–Macaulay, then (X, x) is also Cohen–Macaulay.

In many cases, this is quite sharp; see Example 2.1 and Theorem 8.5 for some stronger versions. This also gives the following immediate corollary.

COROLLARY 1.7 (cf. Corollary 8.7). *Let X be an abelian variety of dimension at least 2 defined over a field k . If $\text{char } k > 0$, assume that X is ordinary. Then the cone over an arbitrary projective embedding of X is not smoothable.*

Note that a special case of Corollary 1.7 over $\mathbb{C}$, the non-smoothability of the projective cone over an abelian variety of dimension at least 2, was proved in [Som79]. See Corollary 8.7 for a stronger version.

1.8. THE ORGANIZATION OF THIS PAPER. In Section 2, we give examples and show some applications of the main results. In Sections 3 and 4, we study infinitesimal deformations of schemes with liftable local cohomology and prove the main result for families over Artinian bases. In Section 5, we prove a rather general flatness and base change criterion, Theorem 5.11, which may be of independent interest, and derive Theorem 1.4 as a relatively easy consequence of this and of the results of Section 4. In Section 6, we prove that a Du Bois local scheme has *liftable local cohomology*. In Section 7, we recall a few basic notions about singularities defined by the behavior of the Frobenius morphism in positive characteristic and recall that F -anti-nilpotent singularities have liftable local cohomology over their ground field. In Section 8, we prove a criterion for S_n singularities in terms of the dualizing complex, cf. Proposition 8.1, and use this and Theorem 1.4 to prove Corollary 1.6.

1.9. THE DUALIZING COMPLEX AND ITS RELATIVES. The (*normalized*) *dualizing complex* of X is denoted by $\omega_X^\bullet$, and if X is of pure dimension n , the *canonical sheaf* of X is defined as $\omega_X := h^{-n}(\omega_X^\bullet)$. Note that if X is not normal, then this is not necessarily the push-forward of the canonical sheaf from the non-singular locus.

We will work with *three* closely related, but generally different, *objects*:

- the dualizing complex $\omega_X^\bullet$,
- the canonical sheaf $\omega_X = h^{-n}(\omega_X^\bullet)$ and
- the object defined by $\underline{\omega}_X := \mathcal{R}\mathcal{H}om_X(\underline{\Omega}_X^0, \omega_X^\bullet)$. (See Section 6 for a description of $\underline{\Omega}_X^0$.)

Note that one has a natural morphism $\underline{\omega}_X \rightarrow \omega_X^\bullet$ dual to $\eta: \mathcal{O}_X \rightarrow \underline{\Omega}_X^0$.

For a morphism $f: X \rightarrow B$, the (*normalized*) *relative dualizing complex* of f will be denoted by $\omega_{X/B}^\bullet$, and if f has equidimensional fibers of dimension n , then the *relative canonical sheaf* of f is $\omega_{X/B} := h^{-n}(\omega_{X/B}^\bullet)$. If B consists of a single (closed) point, then these notions reduce to the ones discussed above. For more details on relative dualizing complexes, see [Sta20, Tag 0E2S].

2. Examples

2.A. Characteristic zero

In this section, we review rational, Cohen–Macaulay and Du Bois singularities of cones in characteristic zero and demonstrate some consequences of the main results.

Example 2.1 (Deformations of cones). Let X be a projective variety and $\mathcal{L}$ an ample line bundle on X . Assume for simplicity that X has rational singularities. Let

$$C_a(X, \mathcal{L}) := \operatorname{Spec}_k \bigoplus_{r=0}^{\infty} H^0(X, \mathcal{L}^r)$$

be the affine cone over X with conormal bundle $\mathcal{L}$ and vertex v . Then the singularity $v \in C_a(X, \mathcal{L})$ is

$$(2.1.1) \quad \text{rational} \Leftrightarrow H^i(X, \mathcal{L}^r) = 0 \text{ for every } i > 0, r \geq 0,$$

$$(2.1.2) \quad \text{Cohen–Macaulay} \Leftrightarrow H^i(X, \mathcal{L}^r) = 0 \text{ for every } \dim X > i > 0, r \geq 0 \text{ and}$$

$$(2.1.3) \quad \text{Du Bois} \Leftrightarrow H^i(X, \mathcal{L}^r) = 0 \text{ for every } i > 0, r > 0;$$

see [Kol13, Corollary 3.11, Proposition 3.13] and [GK14, Theorem 2.5] for proofs.

Let $D \subset X$ be an effective divisor with rational singularities such that $\mathcal{L} \simeq \mathcal{O}_X(D)$. Set $\mathcal{L}_D := \mathcal{L}|_D$. There is a natural morphism $C_a(D, \mathcal{L}_D) \rightarrow C_a(X, \mathcal{L})$ which is an embedding if and only if $H^0(X, \mathcal{L}^r) \twoheadrightarrow H^0(D, \mathcal{L}_D^r)$ is surjective for every $r \geq 0$ or, equivalently, if and only if $H^1(X, \mathcal{L}^r) \hookrightarrow H^1(X, \mathcal{L}^{r+1})$ is injective for every $r \geq 0$. As in [Kol13, Proposition 3.10], using Serre vanishing, we get that

$$(2.1.4) \quad C_a(D, \mathcal{L}_D) \text{ is a Cartier divisor of } C_a(X, \mathcal{L}) \Leftrightarrow H^1(X, \mathcal{L}^r) = 0 \text{ for every } r \geq 0.$$

In this case, if $C_a(D, \mathcal{L}_D)$ is defined by $(g = 0)$ near the vertex, then $g: C_a(X, \mathcal{L}) \rightarrow \mathbb{A}^1$ is a flat deformation of a neighborhood of the vertex in $C_a(D, \mathcal{L}_D)$. In particular, $C_a(D, \mathcal{L}_D)$ is smoothable if $X \setminus D$ is smooth.

By looking at the cohomology of the sequences

$$0 \rightarrow \mathcal{L}^{r-1} \rightarrow \mathcal{L}^r \rightarrow \mathcal{L}_D^r \rightarrow 0, \quad (2.1.5)$$

we see that

$$(2.1.6) \quad C_a(D, \mathcal{L}_D) \text{ is Du Bois} \Leftrightarrow H^1(X, \mathcal{L}^r) \twoheadrightarrow H^1(X, \mathcal{L}^{r+1}) \text{ is surjective for every } r \geq 0 \text{ and } H^i(X, \mathcal{L}^r) = 0 \text{ for every } i > 1, r \geq 0.$$

Putting all these together, we see that if $C_a(D, \mathcal{L}_D)$ is Du Bois and has a flat deformation to $X \setminus D$, then $v \in C_a(X, \mathcal{L})$ is a rational singularity. In particular, $C_a(D, \mathcal{L}_D)$ is Cohen–Macaulay.

This is actually stronger than Corollary 1.6, but here we also assumed that $X \setminus D$ has rational singularities. See also [KS16b] for closely related results.

By Corollary 1.6, if a local Du Bois scheme (X, x) is smoothable, then it is Cohen–Macaulay. The next example shows that this is close to being optimal for some cones.

Example 2.2. Let (S, H) be a polarized K3 surface, and set $X := S \times \mathbb{P}^2$. Fix $a, b \geq 1$, set $\mathcal{L}(a, b) := \pi_1^* \mathcal{O}_S(aH) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^2}(b)$, and let $D(a, b) \subset X$ be a smooth member of the associated linear system.

The affine cone $C_a(D(a, b), \mathcal{L}(a, b)|_{D(a, b)})$ is a hyperplane section of the cone $C_a(X, \mathcal{L}(a, b))$, hence smoothable. It is not Cohen–Macaulay since $H^2(D(a, b), \mathcal{O}_{D(a, b)}) = 1$ and also not Du Bois since $H^1(D(a, b), \mathcal{L}(a, b)|_{D(a, b)}) = 1$.

However, for any $a' > a, b' > b$, the cone $C_a(D(a, b), \mathcal{L}(a', b')|_{D(a, b)})$ is Du Bois but still not Cohen–Macaulay. Thus the cones $C_a(D(a, b), \mathcal{L}(a', b')|_{D(a, b)})$ are not smoothable.

More generally, one gets similar examples starting with any smooth variety X for which $H^1(X, \mathcal{O}_X) = 0$ but $H^i(X, \mathcal{O}_X) \neq 0$ for some $2 \leq i \leq \dim X - 2$.

Example 2.3 (Singularities of cones I). Let X be a smooth projective variety such that $K_X \equiv 0$. Kodaira vanishing and (2.1.2–3) show that $C_a(X, \mathcal{L})$ is Du Bois. It is Cohen–Macaulay if and only if $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim X$. This and Corollary 1.6 imply that if $C_a(X, \mathcal{L})$ is smoothable, then $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim X$. In particular, if X is an abelian variety, then $C_a(X, \mathcal{L})$ is not smoothable. In Corollary 8.7, we prove that if X is an abelian variety, then $C_a(X, \mathcal{L})$ cannot be deformed even to an S_3 scheme.

2.B. All characteristics

Example 2.4 (Singularities of cones II). Let us use the notation introduced in Example 2.1 and first note that (2.1.1) and (2.1.2) remain true in all characteristics:

$$(2.4.1) \quad C_a(X, \mathcal{L}) \text{ is rational} \Leftrightarrow H^i(X, \mathcal{L}^r) = 0 \text{ for every } i > 0, r \geq 0, \text{ and}$$

$$(2.4.2) \quad C_a(X, \mathcal{L}) \text{ is Cohen–Macaulay} \Leftrightarrow H^i(X, \mathcal{L}^r) = 0 \text{ for every } \dim X > i > 0, r \geq 0.$$

For future reference, we add a more sophisticated version of (2.1.2):

$$(2.4.3) \quad C_a(X, \mathcal{L}) \text{ is } S_n \text{ for some } n \in \mathbb{N} \Leftrightarrow H^i(X, \mathcal{L}^r) = 0 \text{ for every } n - 1 > i > 0, r \geq 0.$$

The reader may find a proof in, for instance, [Pat13, Lemma 4.3]; cf. [Koll13, Corollary 3.11].

2.C. Positive characteristic

For the definition of F -singularities appearing in this section, please refer to Section 7.

Example 2.5. Let k be a field of characteristic $p > 0$, and let Y be the curve consisting of the three coordinate axes in $\mathbb{A}_k^3$; that is, let $Y = \operatorname{Spec} k[x, y, z]/(xy, xz, yz)$. Then Y is F -pure by [HR76, Proposition 5.38], and hence it is also F -anti-nilpotent by [Ma14].

The following two conditions frequently play important roles in showing that a class of singularities is invariant under small deformation:

- (i) The class of singularities in question satisfies an *inversion of adjunction type* property; that is, if the singularities of a Cartier divisor $Y \subseteq X$ belong to this class, then there is an open neighborhood $Y \subset X^\circ \subset X$ such that the singularities of X° belong to this class.
- (ii) The class of singularities in question satisfies a *Bertini type* property; that is, if X belongs to this class and $Y \subseteq X$ is a general member of a very ample linear system, then Y also belongs to this class.

Various flavors of F -singularities satisfy one or both of these properties: Property (i) holds for Gorenstein F -pure singularities by [Fed83, Theorem 3.4(2)], and property (ii), at least over an algebraically closed field, holds by [SZ13]. Property (i) is conjectured to hold for $\mathbb{Q}$ -Gorenstein F -pure singularities, but this seems to be known only when the index of the canonical class is not divisible by the characteristic [Sch09a, Proposition 7.2]. Without the $\mathbb{Q}$ -Gorenstein assumption, F -pure singularities do not satisfy property (i) by [Fed83, Sin99].

By contrast, F -anti-nilpotent singularities satisfy property (i) by [MQ18], but it is not known at the moment whether they satisfy property (ii). Similarly, F -injective singularities with liftable local cohomology satisfy property (i) by [HMS14].

The fact that F -anti-nilpotent singularities satisfy property (i), but F -pure singularities in general do not, lead to simple examples of F -anti-nilpotent singularities that are not F -pure:

Example 2.6 ([Fed83, Sin99, QS17, MQ18]). Let $X = \operatorname{Spec} k[x, y, z, t]/(xy, xz, y(z - t^2))$ and $Y = (t = 0) \subseteq X$. Then $Y \simeq \operatorname{Spec} k[x, y, z]/(xy, xz, yz)$, and hence it is F -pure by Example 2.5.

Furthermore, X is then also F -anti-nilpotent by [MQ18, Theorem 4.2], but it is not F -pure by [Sin99, Example 3.2].

Example 2.7 (Singularities of cones III). We have seen in Example 2.3 that in characteristic zero, a cone over an abelian variety has Du Bois singularities. We have a similar statement in positive characteristic: Let X be an ordinary abelian variety over a field of positive characteristic and $\mathcal{L}$ an ample line bundle on X . Then $C_a(X, \mathcal{L})$ has F -pure singularities by [MS87, Lemma 1.1] and hence F -anti-nilpotent singularities by [Ma14].

3. Filtrations on modules over Artinian local rings

We will use the following notation throughout.

3.1. MAXIMAL FILTRATIONS. Let $(S, \mathfrak{m}, k)$ be an Artinian local ring and N a finite S -module with a filtration $N = N_0 \supsetneq N_1 \supsetneq \cdots \supsetneq N_q \supsetneq N_{q+1} = 0$ such that $N_j/N_{j+1} \simeq k$ as S -modules for each $j = 0, \dots, q$. Further, let $f: (X, x) \rightarrow (\text{Spec } S, \mathfrak{m})$ be a flat local morphism, and denote the fiber of f over $\mathfrak{m}$ by $X_{\mathfrak{m}}$. It follows that then for each $j = 0, \dots, q$,

$$f^* \left(N_j / N_{j+1} \right) \simeq \mathcal{O}_{X_{\mathfrak{m}}}. \quad (3.1.1)$$

3.2. FILTERING S . In particular, considering S as a module over itself, we choose a filtration of S by ideals $S = I_0 \supsetneq I_1 \supsetneq \cdots \supsetneq I_q \supsetneq I_{q+1} = 0$ such that $I_j/I_{j+1} \simeq k$ as S -modules for all $0 \leq j \leq q$. Observe that in this case, $I_1 = \mathfrak{m}$ and for every j , there exists a $t_j \in I_j$ such that the composition

$$S \xrightarrow{t_j} I_j \longrightarrow I_j / I_{j+1}$$

induces an isomorphism $S/\mathfrak{m} \simeq I_j/I_{j+1}$. In particular, $\text{ann}(I_j/I_{j+1}) = \mathfrak{m}$. Finally, let $S_j := S/I_j$. Note that $S_1 = S/\mathfrak{m}$ and $S_{q+1} = S$.

3.3. FILTERING ω_S . Applying Grothendieck duality to the closed embedding given by the surjection $S \twoheadrightarrow S_j$ implies that $\omega_{S_j} \simeq \text{Hom}_S(S_j, \omega_S)$, and we obtain injective S -module homomorphisms $\varsigma_j: \omega_{S_j} \hookrightarrow \omega_{S_{j+1}}$ induced by the natural surjection $S_{j+1} \twoheadrightarrow S_j$. Using the fact that ω_S is an injective S -module and applying the functor $\text{Hom}_S(_, \omega_S)$ to the short exact sequence of S -modules

$$0 \longrightarrow I_j / I_{j+1} \longrightarrow S_{j+1} \longrightarrow S_j \longrightarrow 0,$$

we obtain another short exact sequence of S -modules:

$$0 \longrightarrow \omega_{S_j} \xrightarrow{\varsigma_j} \omega_{S_{j+1}} \longrightarrow \text{Hom}_S(k, \omega_S) \simeq k \longrightarrow 0. \quad (3.3.1)$$

Therefore, we obtain a filtration of $N = \omega_S$ by the submodules $N_j := \omega_{S_{q+1-j}}$ as in Section 3.1, where $q+1 = \text{length}_S(S) = \text{length}_S(\omega_S)$. The composition of the embeddings in (3.3.1) will be denoted by $\varsigma := \varsigma_q \circ \cdots \circ \varsigma_1: \omega_{S_1} \hookrightarrow \omega_{S_{q+1}} = \omega_S$.

Recall that the *socle* of a module M over a local ring $(S, \mathfrak{m}, k)$ is

$$\text{Soc } M := (0 : \mathfrak{m})_M = \{x \in M \mid \mathfrak{m} \cdot x = 0\} \simeq \text{Hom}_S(k, M). \quad (3.3.2)$$

The socle is naturally a k -vector space, and $\dim_k \text{Soc } \omega_S = 1$ by the definition of the canonical module. In particular, $\text{Soc } \omega_S \simeq k$, and this is the only S -submodule of ω_S isomorphic to k .

LEMMA 3.4. Using the notation from Sections 3.2 and 3.3, we have that

$$\mathrm{im} \, \varsigma = \mathrm{Soc} \, \omega_S = I_q \omega_S. \quad (3.4.1)$$

Remark 3.4.2. Note that we are not simply stating that the modules in (3.4.1) are isomorphic, we are stating that they are equal as submodules of ω_S .

Proof of Lemma 3.4. Since $S_1 \simeq S/\mathfrak{m} \simeq k$ and hence $\omega_{S_1} \simeq k$, it follows that the image of the embedding $\varsigma: \omega_{S_1} \hookrightarrow \omega_S$ maps ω_{S_1} isomorphically onto $\mathrm{Soc} \, \omega_S$:

$$\mathrm{im} \, \varsigma = \mathrm{Soc} \, \omega_S. \quad (3.4.3)$$

As ω_S is a dualizing sheaf, $I_q \omega_S \neq 0$, and since $I_q \simeq S/\mathfrak{m}$, it follows that

$$0 \neq I_q \omega_S \subseteq (0 : \mathfrak{m})_{\omega_S} = \mathrm{Soc} \, \omega_S \simeq k.$$

Since k is a simple S -module, this implies that $I_q \omega_S = \mathrm{Soc} \, \omega_S$, which proves (3.4.1). $\square$

4. Families over Artinian local rings

We will frequently use the following notation.

Notation 4.1. Let A be a noetherian ring, $(R, \mathfrak{m})$ a noetherian local A -algebra, $I \subset R$ a nilpotent ideal and $(T, \mathfrak{n}) := (R/I, \mathfrak{m}/I)$ with natural morphism $\alpha: R \twoheadrightarrow T$.

DEFINITION 4.2. Recall from Definition 1.1 that we say that $(T, \mathfrak{n})$ has *liftable local cohomology over A* if for any $(R, \mathfrak{m})$ as in Notation 4.1, the induced homomorphism on local cohomology $H_{\mathfrak{m}}^i(R) \twoheadrightarrow H_{\mathfrak{n}}^i(T)$ is surjective for all i .

We extend this definition to schemes: Let (X, x) be a local scheme over a noetherian ring A . Then we say that (X, x) has *liftable local cohomology over A* if $\mathcal{O}_{X,x}$ has liftable local cohomology over A . If $f: X \rightarrow Z$ is a morphism of schemes, then we say that X has *liftable local cohomology over Z* if (X, x) has liftable local cohomology over A for each $x \in X$ and for each open affine neighborhood $\mathrm{Spec} A \subseteq Z$ of $f(x) \in Z$.

LEMMA 4.3. Let $X \rightarrow Y \rightarrow Z$ be morphisms schemes. If X has liftable local cohomology over Z , then X has liftable local cohomology over Y as well.

In particular, if X has liftable local cohomology over a field k , then it has liftable local cohomology over any other k -scheme to which it admits a morphism. In addition, if $\mathrm{char} k = 0$, then X has liftable local cohomology.

Proof. This follows from the definitions and Remark 1.2. $\square$

Next we need a simple lemma regarding liftable local cohomology.

LEMMA 4.4. Using Notation 4.1, let M be an R -module such that there exists a surjective R -module homomorphism $\phi: M \twoheadrightarrow T$. Assume that the induced natural homomorphism $H_{\mathfrak{m}}^i(R) \twoheadrightarrow H_{\mathfrak{n}}^i(T)$ is surjective for some $i \in \mathbb{N}$. Then the induced homomorphism on local cohomology

$$H_{\mathfrak{m}}^i(M) \twoheadrightarrow H_{\mathfrak{m}}^i(T) \simeq H_{\mathfrak{n}}^i(T) \quad (4.4.1)$$

is surjective for the same i . In particular, if $(T, \mathfrak{n})$ has liftable local cohomology over A , then the homomorphism in (4.4.1) is surjective for every $i \in \mathbb{N}$.

Proof. Let $t \in M$ be such that $\phi(t) = 1 \in T$, and let $\beta: R \rightarrow M$ be defined by $1 \mapsto t$. Then $\phi \circ \beta = \alpha: R \rightarrow T$ is the natural quotient morphism; hence, the surjective morphism $H_{\mathfrak{m}}^i(R) \rightarrow H_{\mathfrak{m}}^i(T)$ factors through $H_{\mathfrak{m}}^i(M)$, which proves the statement. $\square$

PROPOSITION 4.5. *Let $(S, \mathfrak{m}, k)$ be an Artinian local ring and $f: (X, x) \rightarrow (\operatorname{Spec} S, \mathfrak{m})$ a flat local morphism. Let N be a finite S -module with a filtration as in (3.1), and assume that $(X_{\mathfrak{m}}, x)$, where $X_{\mathfrak{m}}$ is the fiber of f over the closed point of $\operatorname{Spec} S$, has liftable local cohomology over S . Then for all i, j , the natural sequence of morphisms induced by the embeddings $N_{j+1} \hookrightarrow N_j$ forms a short exact sequence*

$$0 \longrightarrow H_x^i(f^*N_{j+1}) \longrightarrow H_x^i(f^*N_j) \longrightarrow H_x^i\left(f^*\left(N_j/N_{j+1}\right)\right) \simeq H_x^i(\mathcal{O}_{X_{\mathfrak{m}}}) \longrightarrow 0.$$

Proof. Since $\operatorname{ann}(N_j/N_{j+1}) = \mathfrak{m}$, there is a natural surjective morphism

$$f^*N_j \otimes \mathcal{O}_{X_{\mathfrak{m}}} \rightarrow f^*\left(N_j/N_{j+1}\right).$$

By Lemma 4.4 and (3.1.1), the natural homomorphism

$$H_x^i(f^*N_j) \twoheadrightarrow H_x^i\left(f^*\left(N_j/N_{j+1}\right)\right) \simeq H_x^i(\mathcal{O}_{X_{\mathfrak{m}}}) \quad (4.5.1)$$

is surjective for all i . Since f is flat, we have a short exact sequence for every $j > 0$:

$$0 \longrightarrow f^*N_{j+1} \longrightarrow f^*N_j \longrightarrow f^*\left(N_j/N_{j+1}\right) \longrightarrow 0,$$

and hence the statement follows from (4.5.1). $\square$

4.6. THE EXCEPTIONAL INVERSE IMAGE OF THE STRUCTURE SHEAVES. Let $(S, \mathfrak{m}, k)$ be an Artinian local ring with a filtration by ideals as in Section 3.2. Further, let $f: X \rightarrow \operatorname{Spec} S$ be a flat morphism which is essentially of finite type and $f_j = f|_{X_j}: X_j := X \times_{\operatorname{Spec} S} \operatorname{Spec} S_j \rightarrow \operatorname{Spec} S_j$, where $S_j = S/I_j$ as defined in Section 3.2, for example, $X_{q+1} = X$ and $X_1 = X_{\mathfrak{m}}$, the fiber of f over the closed point of S . By a slight abuse of notation, we will denote $\omega_{\operatorname{Spec} S}$ by ω_S as well, but it will be clear from the context which one is meant at any given time.

Using the description of the exceptional inverse image functor via the residual/dualizing complexes [Con00, (3.3.6)] (cf. [Har66, Corollary VII.3.4(a)], [Sta20, Tag 0E9L])

$$f^! = \mathcal{R}\mathcal{H}om_X(\mathcal{L}f^*\mathcal{R}\mathcal{H}om_S(_, \omega_S^\bullet), \omega_X^\bullet) \quad (4.6.1)$$

and the facts that S is Artinian and f is flat, we have that

$$\omega_{X_j/S_j}^\bullet \simeq f_j^!\mathcal{O}_{\operatorname{Spec} S_j} \simeq \mathcal{R}\mathcal{H}om_{X_j}(f_j^*\omega_{S_j}, \omega_{X_j}^\bullet).$$

By Grothendieck duality,

$$\mathcal{R}\mathcal{H}om_{X_j}(f_j^*\omega_{S_j}, \omega_{X_j}^\bullet) \simeq \mathcal{R}\mathcal{H}om_X(f_j^*\omega_{S_j}, \omega_X^\bullet),$$

and as $f_j^*\omega_{S_j} = f^*\omega_{S_j}$ and $\omega_{S_j} \simeq \operatorname{Hom}_S(S_j, \omega_S) \simeq \mathcal{R}\mathcal{H}om_S(\mathcal{O}_{\operatorname{Spec} S_j}, \omega_S^\bullet)$, we obtain that

$$\omega_{X_j/S_j}^\bullet \simeq \mathcal{R}\mathcal{H}om_X(f^*\omega_{S_j}, \omega_X^\bullet) \simeq f^!\mathcal{O}_{\operatorname{Spec} S_j}, \quad (4.6.2)$$

and, in particular, that

$$\omega_{X_{\mathfrak{m}}}^\bullet \simeq f^!k \simeq \mathcal{R}\mathcal{H}om_X(f^*\operatorname{Hom}_S(k, \omega_S), \omega_X^\bullet) \simeq \mathcal{R}\mathcal{H}om_X(\mathcal{O}_{X_{\mathfrak{m}}}, \omega_X^\bullet). \quad (4.6.3)$$

The following statement is standard. We include it for ease of reference.

PROPOSITION 4.7. *Let $f: X \rightarrow B$ be a flat morphism of schemes that admits a relative dualizing complex, and let $Z \rightarrow B$ be a morphism. Then for each $i \in \mathbb{Z}$, there exists a natural base change morphism*

$$\varrho_Z^{-i}: h^{-i}(\omega_{X/B}^\bullet) \otimes_B \mathcal{O}_Z \longrightarrow h^{-i}(\omega_{X_Z/Z}^\bullet).$$

Proof. For any complex $A^\bullet$, tensoring with an object induces a natural morphism $h^i(A^\bullet) \otimes M \rightarrow h^i(A^\bullet \stackrel{L}{\otimes} M)$. Applying this to the dualizing complex gives a natural map

$$\varrho_Z^{-i}: h^{-i}(\omega_{X/B}^\bullet) \otimes_B \mathcal{O}_Z \longrightarrow h^{-i}\left(\omega_{X/B}^\bullet \stackrel{L}{\otimes}_B \mathcal{O}_Z\right).$$

But $\omega_{X/B}^\bullet \stackrel{L}{\otimes}_B \mathcal{O}_Z \simeq \omega_{X_Z/Z}^\bullet$ by the base change property of dualizing complexes [Sta20, Tag 0E2Y], so the statement follows. $\square$

4.8. NATURAL MORPHISMS OF DUALIZING COMPLEXES. We will continue using the notation from Subsection 4.6. Applying $f^!$ to the natural surjective morphism $S_{j+1} \twoheadrightarrow S_j$ gives a natural morphism

$$\varrho_j: \omega_{X_{j+1}/S_{j+1}}^\bullet \longrightarrow \omega_{X_j/S_j}^\bullet. \quad (4.8.1)$$

Notice that ϱ_j is Grothendieck dual to $f^* \varsigma_j$, defined in Section 3.2. Indeed, ς_j is obtained by applying $\mathrm{Hom}_S(_, \omega_S)$ to the morphism $S_{j+1} \twoheadrightarrow S_j$, and then ϱ_j is obtained by applying f^* and then $\mathcal{R}\mathrm{Hom}_X(_, \omega_X^\bullet)$. Notice further that $h^{-i}(\varrho_j)$ factors through the natural base change morphism of Proposition 4.7 for each $i \in \mathbb{Z}$.

The composition of the surjective morphisms $S_{j+1} \twoheadrightarrow S_j$ for all j is the natural surjective morphism $S \twoheadrightarrow S/\mathfrak{m} \simeq k$, and hence the composition of the ϱ_j gives the natural morphism

$$\varrho := \varrho_1 \circ \cdots \circ \varrho_g: \omega_{X/S}^\bullet \longrightarrow \omega_{X_1/S_1}^\bullet = \omega_{X_m}^\bullet, \quad (4.8.2)$$

which is then Grothendieck dual to $f^* \varsigma := f^*(\varsigma_g \circ \cdots \circ \varsigma_1)$; moreover, $h^{-i}(\varrho_j)$ factors through the natural base change morphism of Proposition 4.7 for each $i \in \mathbb{Z}$.

In the rest of this section, we will use the following notation and assumptions.

ASSUMPTIONS 4.9. Let $(S, \mathfrak{m}, k)$ be an Artinian local ring and $f: (X, x) \rightarrow (\mathrm{Spec} S, \mathfrak{m})$ a flat local morphism which is essentially of finite type. Assume that (X_m, x) , where X_m is the fiber of f over the closed point of $\mathrm{Spec} S$, has liftable local cohomology over S . Note that, by definition, $x \in X_m$; we will keep using the notation introduced in (4.8.1) and (4.8.2).

THEOREM 4.10. *For each $i, j \in \mathbb{N}$,*

- (i) *the natural morphism $h^{-i}(\varrho_j): h^{-i}(\omega_{X_{j+1}/S_{j+1}}^\bullet) \twoheadrightarrow h^{-i}(\omega_{X_j/S_j}^\bullet)$ is surjective,*
- (ii) *the natural morphism $h^{-i}(\varrho): h^{-i}(\omega_{X/S}^\bullet) \twoheadrightarrow h^{-i}(\omega_{X_m}^\bullet)$ is surjective,*
- (iii) *the natural morphisms induced by ϱ_j form a short exact sequence*

$$0 \longrightarrow h^{-i}(\omega_{X_m}^\bullet) \longrightarrow h^{-i}(\omega_{X_{j+1}/S_{j+1}}^\bullet) \xrightarrow{h^{-i}(\varrho_j)} h^{-i}(\omega_{X_j/S_j}^\bullet) \longrightarrow 0,$$

- (iv) $\ker h^{-i}(\varrho_j) = I_j h^{-i}(\omega_{X_{j+1}/S_{j+1}}^\bullet) \simeq I_j h^{-i}(\omega_{X/S}^\bullet) / I_{j+1} h^{-i}(\omega_{X/S}^\bullet),$
- (v) $h^{-i}(\omega_{X_j/S_j}^\bullet) \simeq h^{-i}(\omega_{X/S}^\bullet) / I_j h^{-i}(\omega_{X/S}^\bullet) \simeq h^{-i}(\omega_{X/S}^\bullet) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_j},$ and
- (vi) $\ker h^{-i}(\varrho) = \mathfrak{m} h^{-i}(\omega_{X/S}^\bullet).$

Proof. Let $N = \omega_S$, and consider the filtration on N given by $\omega_{S_j} = N_{q+1-j}$; cf. Section 3.3 and (3.3.1). Further, let $(\)^\wedge$ denote the completion at x (the closed point of X). Then by Proposition 4.5, for all $i, j \in \mathbb{N}$, there exists a short exact sequence

$$0 \longrightarrow H_x^i(f^*\omega_{S_j}) \longrightarrow H_x^i(f^*\omega_{S_{j+1}}) \longrightarrow H_x^i\left(f^*\left(\omega_{S_{j+1}}/\omega_{S_j}\right)\right) \longrightarrow 0. \quad (4.10.1)$$

Applying local duality [Har66, Corollary V.6.5] to (4.10.1) gives the short exact sequence

$$0 \longrightarrow \mathcal{E}\mathcal{H}_X^{-i}\left(f^*\left(\omega_{S_{j+1}}/\omega_{S_j}\right), \omega_X^\bullet\right)^\wedge \longrightarrow \mathcal{E}\mathcal{H}_X^{-i}(f^*\omega_{S_{j+1}}, \omega_X^\bullet)^\wedge \longrightarrow \mathcal{E}\mathcal{H}_X^{-i}(f^*\omega_{S_j}, \omega_X^\bullet)^\wedge \longrightarrow 0$$

for all $i, j \in \mathbb{N}$. Since completion is faithfully flat [Sta20, Tag 00MC], this implies that there are short exact sequences

$$\begin{aligned} 0 \longrightarrow \mathcal{E}\mathcal{H}_X^{-i}\left(f^*\left(\omega_{S_{j+1}}/\omega_{S_j}\right), \omega_X^\bullet\right) &\longrightarrow \\ &\longrightarrow \mathcal{E}\mathcal{H}_X^{-i}(f^*\omega_{S_{j+1}}, \omega_X^\bullet) \longrightarrow \mathcal{E}\mathcal{H}_X^{-i}(f^*\omega_{S_j}, \omega_X^\bullet) \longrightarrow 0. \end{aligned} \quad (4.10.2)$$

Recall that $\mathcal{E}\mathcal{H}_X^{-i}(f^*\omega_{S_j}, \omega_X^\bullet) \simeq \mathfrak{h}^{-i}(\omega_{X_j/S_j}^\bullet)$ for each i, j , by (4.6.2). Further observe that the surjective morphism in (4.10.2) is the $-i$ th cohomology sheaf of the Grothendieck dual of $f^*\varsigma_j$, and hence via the above isomorphisms, it corresponds to $\mathfrak{h}^{-i}(\varrho_j)$. Therefore, (4.10.2) implies assertion (i). By (3.3.1), we have $f^*(\omega_{S_{j+1}}/\omega_{S_j}) \simeq \mathcal{O}_{X_m}$, and hence $\mathcal{E}\mathcal{H}_X^{-i}(f^*(\omega_{S_{j+1}}/\omega_{S_j}), \omega_X^\bullet) \simeq \mathfrak{h}^{-i}(\omega_{X_m}^\bullet)$, so (4.10.2) also implies assertion (iii). Composing the surjective morphisms in (4.10.2) for all j implies that the natural morphism

$$\mathfrak{h}^{-i}(\omega_{X/S}^\bullet) \simeq \mathcal{E}\mathcal{H}_X^{-i}(f^*\omega_S, \omega_X^\bullet) \xrightarrow{\mathfrak{h}^{-i}(\varrho)} \mathcal{E}\mathcal{H}_X^{-i}(f^*\omega_{S_q}, \omega_X^\bullet) \simeq \mathfrak{h}^{-i}(\omega_{X_m}^\bullet)$$

is surjective, and hence assertion (ii) follows as well.

Similarly, composing the injective maps in (4.10.1) for all j shows that the embedding $\varsigma: \omega_{S_1} \hookrightarrow \omega_S$ induces an embedding on local cohomology:

$$H_x^i(f^*\omega_{S_1}) \subseteq H_x^i(f^*\omega_S). \quad (4.10.3)$$

Next, we prove assertion (iv) first for $j = q$. Since $\mathfrak{h}^{-i}(\omega_{X_q/S_q}^\bullet)$ is supported on X_q , it follows that

$$I_q \mathfrak{h}^{-i}(\omega_{X/S}^\bullet) \subseteq K := \ker \mathfrak{h}^{-i}(\varrho_q).$$

Recall from Section 3.2 that there exists a $t_q \in I_q$ such that $I_q = St_q \simeq S/\mathfrak{m}$ and from Lemma 3.4 that $I_q \omega_S = \text{Soc } \omega_S$. It follows that for $j = q$, the short exact sequence of (3.3.1) takes the form

$$0 \longrightarrow \omega_{S_q} \longrightarrow \omega_S \xrightarrow{\tau} \text{Soc } \omega_S \longrightarrow 0, \quad (4.10.4)$$

where $\tau: \omega_S \rightarrow \text{Soc } \omega_S \subset \omega_S$ may be identified with multiplication by t_q on ω_S . Applying f^* and taking local cohomology, we obtain the short exact sequence

$$0 \longrightarrow H_x^i(f^*\omega_{S_q}) \longrightarrow H_x^i(f^*\omega_S) \xrightarrow{H_x^i(\tau)} H_x^i(f^*\text{Soc } \omega_S) \longrightarrow 0, \quad (4.10.5)$$

which is of course just (4.10.1) for $j = q$. Clearly, the morphism $H_x^i(\tau)$ may also be identified with multiplication by t_q on $H_x^i(f^*\omega_S)$. By Lemma 3.4 and (4.10.3), the natural morphism $H_x^i(\varsigma): H_x^i(f^*\text{Soc } \omega_S) = H_x^i(I_q f^*\omega_S) = H_x^i(f^*\omega_{S_1}) \rightarrow H_x^i(f^*\omega_S)$ is injective. Since $H_x^i(\tau)$, that

is, multiplication by t_q on $H_x^i(f^*\omega_S)$, is surjective onto $H_x^i(f^*\text{Soc}\omega_S)$, it follows that

$$H_x^i(f^*\text{Soc}\omega_S) \xrightarrow[\widetilde{H_x^i(\zeta)}]{\simeq} \text{im } H_x^i(\zeta) = I_q H_x^i(f^*\omega_S) \hookrightarrow H_x^i(f^*\omega_S); \quad (4.10.6)$$

that is, $H_x^i(f^*\text{Soc}\omega_S)$ coincides with $I_q H_x^i(f^*\omega_S)$ as submodules of $H_x^i(f^*\omega_S)$. Next, let E be an injective hull of $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$, and consider a morphism $\phi: H_x^i(f^*\text{Soc}\omega_S) \rightarrow E$. As E is injective, ϕ extends to a morphism $\widetilde{\phi}: H_x^i(f^*\omega_S) \rightarrow E$. If $a \in H_x^i(f^*\omega_S)$, then $t_q a \in I_q H_x^i(f^*\omega_S) = H_x^i(f^*\text{Soc}\omega_S)$, so

$$t_q \widetilde{\phi}(a) = \widetilde{\phi}(t_q a) = \phi(t_q a) = (\phi \circ H_x^i(\tau))(a).$$

Therefore, $\phi \circ H_x^i(\tau) = t_q \widetilde{\phi}$. Similarly, if $\psi: H_x^i(f^*\omega_S) \rightarrow E$ is an arbitrary morphism, then setting $\phi = \psi|_{H_x^i(f^*\text{Soc}\omega_S)}: H_x^i(f^*\text{Soc}\omega_S) \rightarrow E$ and applying the same computation as above, with $\widetilde{\phi}$ replaced by ψ , shows that $\phi \circ H_x^i(\tau) = t_q \psi$. It follows that the embedding induced by $H_x^i(\tau)$,

$$\alpha: \text{Hom}_{\mathcal{O}_{X,x}}(H_x^i(f^*\text{Soc}\omega_S), E) \hookrightarrow \text{Hom}_{\mathcal{O}_{X,x}}(H_x^i(f^*\omega_S), E), \quad (4.10.7)$$

identifies $\text{Hom}_X(H_x^i(f^*\text{Soc}\omega_S), E)$ with $I_q \text{Hom}_X(H_x^i(f^*\omega_S), E)$. By local duality, this implies that

$$\left(\ker [\mathfrak{h}^{-i}(\varrho_q): \mathfrak{h}^{-i}(\omega_{\dot{X}/S}) \rightarrow \mathfrak{h}^{-i}(\omega_{\dot{X}_q/S_q})] / I_q \mathfrak{h}^{-i}(\omega_{\dot{X}/S}) \right) \otimes \widehat{\mathcal{O}}_{X,x} = 0,$$

and hence, since completion is faithfully flat, this implies assertion (iv) in the case $j = q$. Running through the same argument with S replaced by S_{j+1} gives the equality in assertion (iv) for all j . In addition, assertion (iv) for $j = q$ also implies assertion (v) for $j \geq q$. Assuming that assertion (v) holds for $j = r + 1$ implies the isomorphism in assertion (iv) for $j = r$. In turn, the entire assertion (iv) for $j = r$, combined with assertion (v) for $j = r + 1$, implies assertion (v) for $j = r$. Therefore, assertions (iv) and (v) follow by descending induction on j , and then assertion (vi) follows from assertion (iv) and the definition of ϱ . $\square$

Next, we need a simple lemma.

LEMMA 4.11. *Let R be a ring, M an R -module, $t \in R$ and $J = (t) \subseteq R$. Assume that $(0 : J)_M = (0 : J)_R \cdot M$. Then the natural morphism $J \otimes_R M \xrightarrow{\simeq} JM$ is an isomorphism.*

Proof. This natural morphism is always surjective. Suppose that $m \in M$ is such that $t \otimes m \mapsto 0$ via this morphism, in other words, such that $tm = 0$. This means, by definition, that $m \in (0 : J)_M$ and hence, by assumption, there exist $y \in (0 : J)_R \subseteq R$ and $m' \in M$ such that $m = ym'$. Then $t \otimes m = t \otimes ym' = yt \otimes m' = 0$ since $yt = 0$. This proves the claim. $\square$

PROPOSITION 4.12. *Using the same notation as above,*

- (i) $I_j \otimes \mathfrak{h}^{-i}(\omega_{\dot{X}/S}) \simeq I_j \mathfrak{h}^{-i}(\omega_{\dot{X}/S})$,
- (ii) for any $l \in \mathbb{N}$, we have that $I_j / I_{j+l} \otimes \mathfrak{h}^{-i}(\omega_{\dot{X}/S}) \simeq I_j \mathfrak{h}^{-i}(\omega_{\dot{X}/S}) / I_{j+l} \mathfrak{h}^{-i}(\omega_{\dot{X}/S})$, and
- (iii) for any $l \in \mathbb{N}$, we have that $\mathfrak{m}^l / \mathfrak{m}^{l+1} \otimes \mathfrak{h}^{-i}(\omega_{\dot{X}/S}) \simeq \mathfrak{m}^l \mathfrak{h}^{-i}(\omega_{\dot{X}/S}) / \mathfrak{m}^{l+1} \mathfrak{h}^{-i}(\omega_{\dot{X}/S})$.

Proof. Notice that since $H_x^i(f^*\text{Soc}\omega_S)$ is both a quotient and a submodule of $H_x^i(f^*\omega_S)$, there are two natural maps between $\text{Hom}_{\mathcal{O}_{X,x}}(H_x^i(f^*\text{Soc}\omega_S), E)$ and $\text{Hom}_{\mathcal{O}_{X,x}}(H_x^i(f^*\omega_S), E)$. Regarding $H_x^i(f^*\text{Soc}\omega_S)$ as a quotient module via $H_x^i(\tau)$, we get the embedding $\alpha = (_) \circ H_x^i(\tau)$ in (4.10.7),

and considering it a submodule, we get the restriction map

$$\begin{aligned} \beta: \operatorname{Hom}_{\mathcal{O}_{X,x}}(H_x^i(f^*\omega_S), E) &\longrightarrow \operatorname{Hom}_{\mathcal{O}_{X,x}}(H_x^i(f^*\operatorname{Soc}\omega_S), E). \\ \phi &\longmapsto \phi|_{H_x^i(f^*\operatorname{Soc}\omega_S)} \end{aligned}$$

These maps are of course not inverses to each other. In fact, we have already established (cf. (4.10.7)) that $\phi|_{H_x^i(f^*\operatorname{Soc}\omega_S)} \circ H_x^i(\tau) = t_q\phi$, and hence the composition $\alpha \circ \beta$ is just multiplication by t_q :

$$\begin{array}{ccc} \phi \in \operatorname{Hom}_{\mathcal{O}_{X,x}}(H_x^i(f^*\omega_S), E) & \xrightarrow{\beta} & \operatorname{Hom}_{\mathcal{O}_{X,x}}(H_x^i(f^*\operatorname{Soc}\omega_S), E) \\ & \searrow \alpha \circ \beta & \downarrow \alpha \\ & & t_q\phi \in I_q \operatorname{Hom}_{\mathcal{O}_{X,x}}(H_x^i(f^*\omega_S), E). \end{array} \quad (4.12.1)$$

This implies (cf. (4.10.3) and (4.10.6)) that $h^{-i}(\varrho)$ may be identified with multiplication by t_q on $h^{-i}(\omega_{X/S}^\bullet)$. Together with Theorem 4.10(vi), this implies that

$$(0 : I_q)_{h^{-i}(\omega_{X/S}^\bullet)} = \ker h^{-i}(\varrho) = \mathfrak{m}h^{-i}(\omega_{X/S}^\bullet) = (0 : I_q)_S \cdot h^{-i}(\omega_{X/S}^\bullet),$$

and hence the natural morphism

$$I_q \otimes_S h^{-i}(\omega_{X/S}^\bullet) \xrightarrow{\simeq} I_q h^{-i}(\omega_{X/S}^\bullet) \quad (4.12.2)$$

is an isomorphism by Lemma 4.11. Now assume, by induction, that assertion (i) holds for S_q in place of S . In particular, keeping in mind that $S_q = S/I_q$, the natural map

$$I_j/I_q \otimes_{S_q} h^{-i}(\omega_{X_q/S_q}^\bullet) \xrightarrow{\simeq} (I_j/I_q) h^{-i}(\omega_{X_q/S_q}^\bullet) \quad (4.12.3)$$

is an isomorphism for all j . Consider the short exact sequence (cf. Theorem 4.10(v))

$$0 \longrightarrow I_q h^{-i}(\omega_{X/S}^\bullet) \longrightarrow h^{-i}(\omega_{X/S}^\bullet) \longrightarrow h^{-i}(\omega_{X_q/S_q}^\bullet) \longrightarrow 0$$

and apply $I_j/I_q \otimes_S (_)$. The image of $I_j/I_q \otimes_S I_q h^{-i}(\omega_{X/S}^\bullet)$ in $I_j/I_q \otimes_S h^{-i}(\omega_{X/S}^\bullet)$ is 0, and hence by (4.12.3), the natural map

$$\begin{aligned} \boxed{I_j/I_q \otimes_S h^{-i}(\omega_{X/S}^\bullet)} &\simeq I_j/I_q \otimes_{S_q} h^{-i}(\omega_{X_q/S_q}^\bullet) \xrightarrow{\simeq} (I_j/I_q) h^{-i}(\omega_{X_q/S_q}^\bullet) \simeq \\ &\simeq (I_j/I_q) h^{-i}(\omega_{X/S}^\bullet) / I_q h^{-i}(\omega_{X/S}^\bullet) \simeq \boxed{I_j h^{-i}(\omega_{X/S}^\bullet) / I_q h^{-i}(\omega_{X/S}^\bullet)} \end{aligned}$$

is an isomorphism. This, combined with (4.12.2) and the 5-lemma, implies assertion (i). Then assertion (ii) is a direct consequence of assertion (i) and the fact that tensor product is right exact.

Finally, recall that the choice of filtration in Section 3.2 was fairly unrestricted. In particular, we may assume that the filtration $I_\bullet$ of S is chosen such that for all $l \in \mathbb{N}$, there exists a $j(l)$ such that $I_{j(l)} = \mathfrak{m}^l$. Applying assertion (ii) for this filtration implies assertion (iii). $\square$

The following theorem is an easy combination of the results of this section.

THEOREM 4.13. *Let $(S, \mathfrak{m}, k)$ be an Artinian local ring and $f: (X, x) \rightarrow \operatorname{Spec} S$ a flat local morphism which is essentially of finite type. If $(X_{\mathfrak{m}}, x)$ has liftable local cohomology over S , then $\mathbf{h}^{-i}(\omega_{X/S}^\bullet)$ is flat over $\operatorname{Spec} S$ for each i .*

Proof. This follows directly from Proposition 4.12(iii) and [Sta20, Tag 0AS8]. $\square$

5. Flatness and base change

In this section, we prove a rather general flatness and base change theorem for the cohomology sheaves of the relative dualizing complex. The main essential assumption is that the relative dualizing complex exists.

DEFINITIONS AND NOTATION 5.1. For morphisms $f: X \rightarrow B$ and $\vartheta: Z \rightarrow B$, the symbol X_Z will denote $X \times_B Z$ and $f_Z: X_Z \rightarrow Z$ the induced morphism. In particular, for $b \in B$, we write $X_b = f^{-1}(b)$.

Let $f: X \rightarrow B$ be a morphism of locally noetherian schemes. Then f is *embeddable into a smooth morphism of dimension N* if there exist a smooth morphism $\pi: P \rightarrow B$ of pure relative dimension N over B and a closed embedding $j: X \hookrightarrow P$ such that $f = \pi \circ j$. Furthermore, f is *locally embeddable into a smooth morphism* if each $x \in X$ has a neighborhood $x \in U_x \subseteq X$ such that $f|_{U_x}$ is embeddable into a smooth morphism of dimension N for some $N \in \mathbb{N}$.

Note that if $f: X \rightarrow B$ is a flat morphism which is essentially of finite type, then it is locally embeddable into a smooth morphism and that if f is flat and locally embeddable into a smooth morphism, then it admits a relative dualizing complex by [Sta20, Tag 0E2X].

LEMMA 5.2. *Let (B, b) be a local scheme and $f: X \rightarrow B$ a flat morphism embeddable into a smooth morphism $P \rightarrow B$ of relative dimension N . Then*

$$\mathbf{h}^{-i}(\omega_{X/B}^\bullet) \simeq \mathcal{E}xt_P^{N-i}(\mathcal{O}_X, \omega_{P/B}) \quad \text{and} \quad \mathbf{h}^{-i}(\omega_{X_b}^\bullet) \simeq \mathcal{E}xt_{P_b}^{N-i}(\mathcal{O}_{X_b}, \omega_{P_b}).$$

Proof. Since P/B is an N -dimensional smooth morphism, $\omega_{P/B}^\bullet = \omega_{P/B}[N]$ is a relative dualizing complex. By Grothendieck duality [Har66, Corollary VII.3.4],

$$\mathbf{h}^{-i}(\omega_{X/B}^\bullet) \simeq \mathbf{h}^{-i}(\mathcal{R}Hom_P(\mathcal{O}_X, \omega_{P/B}[N])) \simeq \mathcal{E}xt_P^{N-i}(\mathcal{O}_X, \omega_{P/B}).$$

The same argument implies the equivalent statement for $\mathbf{h}^{-i}(\omega_{X_b}^\bullet)$. $\square$

TERMINOLOGY 5.3. Let $f: X \rightarrow B$ be a flat morphism of schemes and $\vartheta: Z \rightarrow B$ a morphism. Then for an $i \in \mathbb{Z}$, we will say that $\mathbf{h}^{-i}(\omega_{X/B}^\bullet)$ *commutes with base change to Z* if the natural base change morphism ϱ_Z^{-i} of Proposition 4.7 is an isomorphism.

Remark 5.4. Since the base change morphism is defined naturally, it can be checked locally whether it is an isomorphism. In other words, if $\mathbf{h}^{-i}(\omega_{X/B}^\bullet)$ commutes with base change to Z locally on X , then it commutes with base change to Z .

Remark 5.5. A simple case when the condition in (5.3) holds is if $f: X \rightarrow B$ has Cohen–Macaulay fibers. In that case, the only non-zero cohomology sheaf of the relative dualizing complex is $\mathbf{h}^{-m}(\omega_{X/B}^\bullet) \simeq \omega_{X/B}$, where $m = \dim X - \dim B$ by [Con00, Theorem 3.5.1], and it commutes with base change by [Con00, Theorem 3.6.1]. In moduli theory, one typically has to deal with non-Cohen–Macaulay fibers. The next example shows that for these, not even $\omega_{X/B}$ commutes with base change. However, we see in Lemma 5.8 that the $\mathbf{h}^{-i}(\omega_{X/B}^\bullet)$ commute with inverse limits.

Example 5.6. Let Y be a normal quasi-projective threefold with isolated singularities and a trivial canonical divisor. Assume that Y is S_2 but not S_3 . For instance, a non-Cohen–Macaulay normal threefold such as a cone over an abelian surface in characteristic zero satisfies these conditions; cf. Example 2.3. Consider a general projection of Y to a line and resolve the indeterminacies of the projection map. Let X denote the blow-up of Y on which this rational map becomes a morphism, and let $\pi: X \rightarrow \mathbb{A}^1$ denote the resulting morphism. Note that since the projection was general, we may assume that the birational morphism $X \rightarrow Y$ is locally isomorphic near their singular points. In particular, we may assume that X is a normal affine threefold with isolated singularities and a trivial canonical divisor, which is S_2 but not S_3 . Observe that then $\mathbf{h}^{-2}(\omega_{X/\mathbb{A}^1}^\bullet) \simeq \omega_{X/\mathbb{A}^1} \simeq \mathcal{O}_X$ by construction. Next, let $z \in \mathbb{A}^1$ be the image of a non- S_3 point of X . Then X_z and hence $\mathcal{O}_{X_z}$ is not S_2 since otherwise X would be S_3 along X_z . At the same time, $\mathbf{h}^{-2}(\omega_{X_z/\{z\}}^\bullet) \simeq \omega_{X_z}$ is an S_2 sheaf (cf. [KM98, Corollary 5.69], [Kov17, Lemma 3.22]) and hence cannot be isomorphic to $\mathcal{O}_{X_z}$. This implies that $\mathbf{h}^{-2}(\omega_{X/\mathbb{A}^1}^\bullet)$ does not commute with base change for the morphism $\pi: X \rightarrow \mathbb{A}^1$.

Notation 5.7. Let $f: (X, x) \rightarrow (B, b) = (\mathrm{Spec} S, \mathfrak{m})$ be a local morphism. Let $q \in \mathbb{N}$, $S_q := S/\mathfrak{m}^q$, $\mathfrak{m}_q = \mathfrak{m}/\mathfrak{m}^q$ its (unique) maximal ideal, $B_q = \mathrm{Spec} S_q$, $X_q := X \times_B B_q$ and $f_q: (X_q, x) \rightarrow (B_q, b)$ the induced local morphism. Further, let $\widehat{B} := \mathrm{Spec}(\varprojlim S_q)$, the completion of B at b , and $\widehat{X} := X \times_B \widehat{B}$.

LEMMA 5.8. *Let $f: (X, x) \rightarrow (B, b)$ be a flat local morphism that admits a relative dualizing complex. Fix an $i \in \mathbb{Z}$, and assume that the inverse system $(\mathbf{h}^{-i-1}(\omega_{X_q/B_q}^\bullet))$ satisfies the Mittag-Leffler condition [Sta20, Tag 0595]. Then the natural base change morphism (cf. Proposition 4.7) induces an isomorphism*

$$\varprojlim (\mathbf{h}^{-i}(\omega_{X/B}^\bullet) \otimes_X \mathcal{O}_{X_q}) \xrightarrow{\simeq} \varprojlim \mathbf{h}^{-i}(\omega_{X_q/B_q}^\bullet).$$

Remark 5.9. If the local scheme (X_b, x) has liftable local cohomology over B , then the inverse system $(\mathbf{h}^{-i-1}(\omega_{X_q/B_q}^\bullet))$ satisfies the Mittag-Leffler condition by Theorem 4.10(i).

Proof of Lemma 5.8. By the base change property of dualizing complexes [Sta20, Tag 0E2Y], there exist natural restriction morphisms

$$\omega_{X_{q+1}/B_{q+1}}^\bullet \longrightarrow \omega_{X_{q+1}/B_{q+1}}^\bullet \overset{L}{\otimes}_{X_{q+1}} \mathcal{O}_{X_q} \simeq \omega_{X_q/B_q}^\bullet,$$

so $(\omega_{X_q/B_q}^\bullet)$ forms an inverse system in $D^b(X)$, and hence $\mathcal{R}\lim \omega_{X_q/B_q}^\bullet$, the derived limit of the inverse system $(\omega_{X_q/B_q}^\bullet)$, exists [Sta20, Tag 0CQD]. Since the inverse system $(\mathbf{h}^{-i-1}(\omega_{X_q/B_q}^\bullet))$ satisfies the Mittag-Leffler condition, $\mathcal{R}^1 \lim \mathbf{h}^{-i-1}(\omega_{X_q/B_q}^\bullet) = 0$ by [Sta20, Tag 091D(3)]. Combined with [Sta20, Tag 0CQE], this implies that the natural base change morphism of Proposition 4.7 induces an isomorphism

$$\mathbf{h}^{-i}(\mathcal{R}\lim \omega_{X_q/B_q}^\bullet) \xrightarrow{\simeq} \varprojlim \mathbf{h}^{-i}(\omega_{X_q/B_q}^\bullet).$$

The base change property of dualizing complexes also applies to $\omega_{X/B}^\bullet$, and hence the natural restricting morphisms induce isomorphisms

$$\omega_{X/B}^\bullet \longrightarrow \omega_{X/B}^\bullet \overset{L}{\otimes}_X \mathcal{O}_{X_q} \simeq \omega_{X_q/B_q}^\bullet.$$

Then the derived completion of $\omega_{X/B}^\bullet$ with respect to the ideal

$$\mathcal{J} := f^* \mathfrak{m}_{B,b} = \mathcal{J}_{X_b \subseteq X} \subseteq \mathcal{O}_{X,x}$$

[Sta20, Tag 0BKH] is isomorphic to $\mathcal{R}\lim \omega_{X_q/B_q}^\bullet$ constructed above. The statement then follows by [Sta20, Tag 0A06]. $\square$

Remark 5.10. In the proof above, it is important to consider the derived limit $\mathcal{R}\lim \omega_{X_q/B_q}^\bullet$ as a derived completion over X and not over B , because for the cited results, the $\mathrm{h}^{-i}(\omega_{X/B}^\bullet)$ need to be finite modules. They are finite over $\mathcal{O}_{X,x}$ but not necessarily over $\mathcal{O}_{B,b}$.

Next, we prove our main flatness and base change statement.

THEOREM 5.11. *Let $X \rightarrow B$ be a flat morphism locally embeddable into a smooth morphism. Fix an $i \in \mathbb{Z}$ and assume that for any Artinian scheme Z and morphism $Z \rightarrow B$, the sheaf $\mathrm{h}^{-i}(\omega_{X_Z/Z}^\bullet)$ is flat over Z and commutes with any base change to a closed subscheme of Z . Then $\mathrm{h}^{-i}(\omega_{X/B}^\bullet)$ is flat over B and commutes with arbitrary base change.*

Proof. Since the base change morphism ϱ_Z^{-i} is natural, the statement is local on B , so we may replace B with a local scheme (B, b) . Furthermore, since both flatness and whether or not ϱ_Z^{-i} is an isomorphism can be tested locally on X , we may also replace X with a local scheme (X, x) , use the notation established in Notation 5.7, assume that $f: (X, x) \rightarrow (B, b)$ is embeddable into a smooth morphism and apply Lemma 5.2.

Let $M_q := \mathrm{h}^{-i}(\omega_{X_q/B_q}^\bullet)$. Then by assumption, M_q is flat over B_q for every $q \in \mathbb{N}$, and the natural base change morphism is an isomorphism:

$$M_{q+1} \otimes_{B_{q+1}} \mathcal{O}_{B_q} \xrightarrow{\simeq} M_q.$$

In particular, the induced natural morphism $M_{q+1} \rightarrow M_q$ is surjective, and hence (M_q) satisfies the Mittag-Leffler condition, and $\varprojlim M_q$ is flat over B by the first statement of [Sta20, Tag 0912]. Furthermore, let $Q = \mathcal{O}_{X_j}$ for a fixed $j \in \mathbb{N}$. Then $M_q \otimes_X Q \simeq M_j$ for any $q \geq j$ by assumption, and hence $\varprojlim (M_q \otimes_X Q) \simeq M_j$. Then by the second statement of [Sta20, Tag 0912],

$$\left(\varprojlim M_q \right) \otimes_X \mathcal{O}_{X_j} = \left(\varprojlim M_q \right) \otimes_X Q \simeq \varprojlim (M_q \otimes_X Q) \simeq M_j. \quad (5.11.1)$$

On the other hand, $\varprojlim M_q \simeq \varprojlim (\mathrm{h}^{-i}(\omega_{X/B}^\bullet) \otimes_X \mathcal{O}_{X_q})$ by Lemma 5.8, and so by [Sta20, Tag 031C],

$$\left(\varprojlim M_q \right) \otimes_X \mathcal{O}_{X_j} \simeq \mathrm{h}^{-i}(\omega_{X/B}^\bullet) \otimes_X \mathcal{O}_{X_j}. \quad (5.11.2)$$

Comparing (5.11.1) and (5.11.2) shows that $\mathrm{h}^{-i}(\omega_{X/B}^\bullet)$ commutes with base change to B_q for every $q \in \mathbb{N}$, and then $\mathrm{h}^{-i}(\omega_{X/B}^\bullet)$ commutes with arbitrary base change by Lemma 5.2 and [AK80, Theorem 1.9]. Using that $M_q = \mathrm{h}^{-i}(\omega_{X_q/B_q}^\bullet)$ is flat over B_q for every $q \in \mathbb{N}$, it follows that $\mathrm{h}^{-i}(\omega_{X/B}^\bullet)$ is flat over B by [Sta20, Tag 0523]. $\square$

COROLLARY 5.12. *Let $f: (X, x) \rightarrow (B, b)$ be a flat local morphism which is essentially of finite type. If (X_b, x) has liftable local cohomology over B , then $\mathrm{h}^{-i}(\omega_{X/B}^\bullet)$ is flat over B and commutes with arbitrary base change for each $i \in \mathbb{Z}$.*

Proof. By Theorems 4.13, 4.10(i) and 4.10(v), the morphism f satisfies the assumptions of Theorem 5.11 and hence the statement follows from Theorem 5.11. $\square$

Now we are ready to prove Theorem 1.4.

THEOREM 5.13 (Theorem 1.4). *Let $f: X \rightarrow B$ be a flat morphism of schemes which is essentially of finite type, and let $b \in B$ be such that X_b has liftable local cohomology over B . Then there exists an open neighborhood $X_b \subset U \subset X$ such that $\mathbf{h}^{-i}(\omega_{U/B}^\bullet)$ is flat over B and commutes with base change for each $i \in \mathbb{Z}$.*

Proof. Let $x \in X_b$, and temporarily replace $f: X \rightarrow B$ with the induced local morphism $(X, x) \rightarrow (B, b)$. Then $\mathbf{h}^{-i}(\omega_{X/B}^\bullet)$ is flat over B and commutes with arbitrary base change by Corollary 5.12.

Since localization is an exact functor, we obtain that for the original $f: X \rightarrow B$ and any $x \in X_b$, the localized cohomology sheaves $\mathbf{h}^{-i}(\omega_{X/B}^\bullet)_x$ are flat over B and commute with base change for each $i \in \mathbb{Z}$. Both of these properties are open on X , and hence there is an open neighborhood of x where they hold. The union of these neighborhoods for all $x \in X_b$ provides an open neighborhood of X_b where these properties hold. $\square$

Now Theorem 5.13 and Proposition 6.1 imply Corollary 1.5(i), and Theorem 5.13 and Proposition 7.2 imply Corollary 1.5(ii).

6. Du Bois singularities

In characteristic zero, the optimal setting for deformation invariance of cohomology seems to be the class of Du Bois singularities, introduced by Steenbrink [Ste83]. For a proper complex variety with Du Bois singularities, the natural morphism

$$H^i(X, \mathbb{C}_X) \longrightarrow H^i(X, \mathcal{O}_X) \quad (\star)$$

is surjective, and one should think of Du Bois singularities as the largest class for which this holds; cf. [Kov12]. This surjectivity enables one to use topological arguments to control the sheaf cohomology groups $H^i(X, \mathcal{O}_X)$ in flat families as in [DJ74].

The proof of Corollary 1.6 for projective morphisms in [KK10] very much relied on global duality, hence properness. Our first hope was that one can localize the proofs by replacing $(\star)$ with the analogous map between local cohomology groups

$$H_x^i(X, \mathbb{C}_X) \longrightarrow H_x^i(X, \mathcal{O}_X).$$

However, this turned out to be too simplistic; one needs to consider instead the map

$$H_x^i(X, \mathbb{C}_X) \longrightarrow \mathbb{H}_x^i(X, \underline{\Omega}_X^0),$$

where $\underline{\Omega}_X^0$ denotes the 0th associated graded Du Bois complex of X . For the construction of the Du Bois complex, see [DuB81, GNPP88], and for its relevance to higher-dimensional geometry, see [Kol13, § 6]. The surjectivity in $(\star)$ seems simple, but it is a key element of Kodaira-type vanishing theorems [Kol87], [Kol95, § 12], [Kov00], [KSS10] and leads to various results on deformations of Du Bois schemes [DJ74, KK10, KS16b]. Eventually, we understood that for our purposes, the key property is liftable local cohomology, cf. Theorem 5.13, and that Du Bois singularities have liftable local cohomology.

For the definition of Du Bois singularities, the reader is referred to [Kol13, § 6]. A scheme defined over a field of characteristic zero is said to have *Du Bois singularities* if its base extension to $\mathbb{C}$ does. In addition to the properties mentioned above, recall that Du Bois singularities are invariant under small deformation by [KS16a, Theorem 4.1].

PROPOSITION 6.1 ([MSS17, Lemma 3.3]). *Let X be a scheme, essentially of finite type over a field of characteristic zero. Assume that X is Du Bois. Then X has liftable local cohomology.*

Remark 6.2. Our formulation is slightly different since we are stating that X has liftable local cohomology (over $\mathbb{Z}$) while [MSS17, Lemma 3.3] states this over $\mathbb{C}$. This is a minor difference in view of Remark 1.2. In any case, Proposition 6.1 is a direct consequence of the following.

THEOREM 6.3 ([KS16a, KS16b, MSS17]). *Let X be a scheme, essentially of finite type over a field of characteristic zero. Then the natural morphism*

$$h^i(\underline{\omega}_X^\bullet) \hookrightarrow h^i(\omega_X^\bullet)$$

is injective for every $i \in \mathbb{Z}$.

Remark 6.4. Theorem 6.3 has its roots in [Kov99, Lemma 2.2]. It was first proved essentially in this form in [KS16a, Theorem 3.3]. A version for pairs, with almost the same proof, was given in [KS16b, Theorem B]. Both of these were stated for reduced schemes even though the proof does not need that assumption. This was noticed and carefully confirmed in [MSS17, Lemma 3.2], where the proof is carried out in detail for the not-necessarily-reduced case.

Applying local duality to Theorem 6.3 leads to the following useful consequence.

COROLLARY 6.5. *Let X be a scheme, essentially of finite type over a field of characteristic zero, and $x \in X$ a closed point. Then the natural morphism*

$$H_x^i(\mathcal{O}_X) \twoheadrightarrow \mathbb{H}_x^i(\underline{\Omega}_X^0)$$

is surjective for every $i \in \mathbb{Z}$.

Remark 6.6. An important aspect of Corollary 6.5 is that the local cohomology of $\mathcal{O}_X$ depends on the non-reduced structure, while that of $\underline{\Omega}_X^0$ does not. The left-hand side reflects the algebraic structure, while the right-hand side behaves as if it depends only on the topology (this is not entirely true!).

This behavior allows one to prove Proposition 6.1. The proof is based on the interplay between the non-reduced and reduced data. Essentially the same idea have been used in the proofs of [DJ74, Lemme 1], [KS16b, Theorem 5.1] and [MSS17, Lemma 3.3].

7. F -pure singularities

There is an intriguing correspondence between singularities of the minimal model program in characteristic zero and singularities defined by the action of the Frobenius morphism in positive characteristic. For more on this correspondence, the reader may consult [ST12, Appendix C] or [Koll13, §8.4]. Our goal here is to show that F -pure, or more generally F -anti-nilpotent, singularities have liftable local cohomology over their ground field.

DEFINITION 7.1. Let $(R, \mathfrak{m})$ be a noetherian local ring of characteristic $p > 0$ with the Frobenius endomorphism $F: R \rightarrow R; x \mapsto x^p$.

Recall that a homomorphism of R -modules $M \rightarrow M'$ is called *pure* if for every R -module N , the induced homomorphism $M \otimes_R N \rightarrow M' \otimes_R N$ is injective. The ring R is called *F -pure* if the Frobenius endomorphism is pure. The ring R is called *F -finite* if R is a finitely generated R -module via the Frobenius endomorphism F . For instance, if R is essentially of finite type over a perfect field, then it is F -finite. Further, note that if R is F -finite or complete, then it is F -pure if and only if the Frobenius endomorphism $F: R \rightarrow R$ has a left inverse [HR76, Corollary 5.3].

The ring R is called F -injective if the induced Frobenius action on $H_{\mathfrak{m}}^i(R)$ is injective for all $i \in \mathbb{N}$. This holds, for example, if R is F -pure [HR76, Lemma 2.2], and if R is Gorenstein, then it is F -pure if and only if it is F -injective [Fed83, Lemma 3.3].

A strengthening of the notion of F -injective was recently introduced in [EH08]: Consider the induced Frobenius action $F: H_{\mathfrak{m}}^i(R) \rightarrow H_{\mathfrak{m}}^i(R)$. A submodule $M \subseteq H_{\mathfrak{m}}^i(R)$ is called F -stable if $F(M) \subseteq M$, and R is called F -anti-nilpotent if for any F -stable submodule $M \subseteq H_{\mathfrak{m}}^i(R)$, the induced Frobenius action on the quotient $H_{\mathfrak{m}}^i(R)/M$ is injective. If R is F -anti-nilpotent, then it is F -injective since $\{0\} \subseteq H_{\mathfrak{m}}^i(R)$ is an F -stable submodule. Furthermore, if R is F -pure, then it is F -anti-nilpotent [Ma14, Theorem 3.8]. So we have the following implications:

$$F\text{-pure} \implies F\text{-anti-nilpotent} \implies F\text{-injective}. \quad (7.1.1)$$

Let (X, x) be a local scheme. Then we say that X has F -pure, F -anti-nilpotent or F -injective singularities if the local ring $\mathcal{O}_{X,x}$ has the corresponding property. An arbitrary scheme X of equicharacteristic $p > 0$ has F -pure, F -anti-nilpotent, or F -injective singularities if the local scheme (X, x) has the corresponding property for each $x \in X$.

These singularities are related to the singularities of the minimal model program. Normal $\mathbb{Q}$ -Gorenstein F -pure singularities are log canonical by [HW02], and it is conjectured that in some form, the converse also holds. Similarly, F -injective singularities correspond to Du Bois singularities: If X is essentially of finite type over a field of characteristic zero and its reduction mod p is F -injective for infinitely many primes p , then X has Du Bois singularities [Sch09b], and the converse of this is also conjectured to hold. So the (outside) implication in (7.1.1) is analogous to that log canonical singularities are Du Bois [KK10].

Curiously, we have this additional notion, F -anti-nilpotent, in between the more familiar F -pure and F -injective notions. It turns out that F -anti-nilpotent, and hence F -pure, rings have liftable local cohomology but F -injective rings in general do not. This suggests that, possibly, F -anti-nilpotent is a better analog of Du Bois singularities in positive characteristic than F -injective. Of course, this is far from conclusive evidence, and this issue will not be settled here.

These singularities, defined by the action of Frobenius, have been studied extensively through their local cohomology. So it is no surprise that the fact that F -anti-nilpotent singularities have liftable local cohomology is a relatively simple consequence of known results. The following statement is essentially proved in [MSS17, Remark 3.4], although their statement is slightly different, so we include a proof for completeness.

PROPOSITION 7.2 (Ma–Schwede–Shimomoto). *F -anti-nilpotent singularities have liftable local cohomology over their ground field.*

Proof. (Following the argument in [MSS17, Remark 3.4].) Using Notation 4.1, assume that $A = k$ is a field of characteristic $p > 0$ and that $(R, \mathfrak{m})$ has F -anti-nilpotent singularities. We need to prove that the induced morphism on local cohomology $H_{\mathfrak{m}}^i(R) \rightarrow H_{\mathfrak{m}}^i(T)$ is surjective for each i (cf. Definition 1.1).

Since the statement is about local cohomology, we may assume that R is complete and hence $R \simeq R'/J$, where R' is a complete regular local ring and $J \subseteq R'$ is an ideal. Note that, denoting the pre-image of $I \subseteq R$ in R' by I' , we have that $T \simeq R'/I'$ is also a quotient of R' .

Let $M := \text{im}[H_{\mathfrak{m}}^i(R) \rightarrow H_{\mathfrak{m}}^i(T)]$, which is an F -stable submodule of $H_{\mathfrak{m}}^i(T)$. Then M contains $F^e(H_{\mathfrak{m}}^i(T))$ for some $e > 0$ by [Lyu06, Lemma 2.2], and hence the Frobenius action on $H_{\mathfrak{m}}^i(T)/M$ is nilpotent. In particular, it is injective only if this quotient is 0. Therefore, if R is F -anti-nilpotent, then $H_{\mathfrak{m}}^i(R) \rightarrow H_{\mathfrak{m}}^i(T)$ is surjective, as desired. $\square$

COROLLARY 7.3 (Ma). *F-pure singularities have liftable local cohomology over their ground field.*

Proof. *F-pure singularities are F-anti-nilpotent by [Ma14, Theorem 3.8], so this is a direct consequence of Proposition 7.2. $\square$*

8. Degenerations of Cohen–Macaulay singularities with liftable local cohomology

PROPOSITION 8.1. *Let Z be a scheme that admits a dualizing complex $\omega_Z^\bullet$, let $z \in Z$ be a (not-necessarily-closed) point, and let $n \in \mathbb{N}$. Then Z is S_n at z if and only if for all $i \in \mathbb{Z}$,*

$$h^{-i}(\omega_Z^\bullet)_z = 0 \quad \text{for } i < \min(n, \dim_z Z) + \dim z. \quad (8.1.1)$$

In particular, if Z is an equidimensional local scheme, then Z is S_n if and only if for all $i \in \mathbb{Z}$ with $i < \dim Z$,

$$\dim \operatorname{supp} h^{-i}(\omega_Z^\bullet) \leq i - n. \quad (8.1.2)$$

(In this statement, we take $\dim \emptyset = -\infty$.)

Proof. Since $h^{-i}(\omega_Z^\bullet) = \mathcal{E}xt_Z^{-i}(\mathcal{O}_Z, \omega_Z^\bullet)$, equation (8.1.1) follows directly from [Kov11, Proposition 3.2].

Next, let $i \in \mathbb{Z}$ with $i < \dim Z$ be such that $h^{-i}(\omega_Z^\bullet) \neq 0$, and let $z \in \operatorname{supp} h^{-i}(\omega_Z^\bullet)$ be a general point such that $\dim z = \dim \operatorname{supp} h^{-i}(\omega_Z^\bullet)$. If Z is S_n , then $i \geq \min(n, \dim_z Z) + \dim z$ by (8.1.1), and hence, since $i < \dim Z$, we must have $\min(n, \dim_z Z) = n$ (this is where Z being equidimensional is used), so indeed $\dim \operatorname{supp} h^{-i}(\omega_Z^\bullet) \leq i - n$.

In order to prove the other implication, let $i \in \mathbb{Z}$ with $i < \dim Z$ again be such that $h^{-i}(\omega_Z^\bullet) \neq 0$, but now choose an arbitrary point $z \in \operatorname{supp} h^{-i}(\omega_Z^\bullet)$. In this case, we only have that $\dim z \leq \dim \operatorname{supp} h^{-i}(\omega_Z^\bullet)$, but this will be enough. If $\dim \operatorname{supp} h^{-i}(\omega_Z^\bullet) \leq i - n < \dim Z - n$, then $n < \dim Z - \dim \operatorname{supp} h^{-i}(\omega_Z^\bullet) \leq \dim Z - \dim z = \dim_z Z$, that is, $\min(n, \dim_z Z) = n$. It also follows that $i \geq n + \dim \operatorname{supp} h^{-i}(\omega_Z^\bullet) \geq \min(n, \dim_z Z) + \dim z$, and hence Z is S_n at z by (8.1.1). We obtain that for all $i < \dim Z$, the support $\operatorname{supp} h^{-i}(\omega_Z^\bullet)$ is contained in the S_n -locus of Z . However, Z is Cohen–Macaulay and hence S_n at every point in $Z \setminus \bigcup_{i < \dim Z} \operatorname{supp} h^{-i}(\omega_Z^\bullet)$, which proves (8.1.2). $\square$

COROLLARY 8.2. *Let Z be an equidimensional local scheme that admits a dualizing complex $\omega_Z^\bullet$. If Z is S_n for some $n \in \mathbb{N}$, then $h^{-i}(\omega_Z^\bullet) = 0$ for $i < n$.*

THEOREM 8.3. *Let $f: X \rightarrow B$ be a flat morphism which is essentially of finite type with equidimensional fibers and locally embeddable into a smooth morphism. Assume that there exists a $b_0 \in B$ such that X_{b_0} has liftable local cohomology over B . If X_{b_0} is not S_n , then there exists an open subset $b_0 \in V \subseteq B$ such that X_b is not S_n for each $b \in V$.*

Proof. By Theorem 5.13, there exists an open neighborhood $X_b \subset U \subset X$ such that $h^{-i}(\omega_{U/B}^\bullet)$ is flat over B and commutes with base change for each $i \in \mathbb{Z}$. Then $\dim \operatorname{supp} h^{-i}(\omega_{U_b}^\bullet)$ is a locally constant function on the set $\{b \in B \mid h^{-i}(\omega_{X_b}^\bullet) \neq 0\}$, so the claim follows from (8.1.2). $\square$

8.A Deformations of local schemes

DEFINITION 8.4. Let $(A, \mathfrak{m}_1, \dots, \mathfrak{m}_r)$ be a semi-local ring. Then $(X, x_1, \dots, x_r)$ is called a *semi-local scheme*, where $X = \operatorname{Spec} A$ and $x_1 = \mathfrak{m}_1, \dots, x_r = \mathfrak{m}_r \in X$. If $r = 1$ and A is a local ring, then (X, x_1) is a *local scheme*.

A *family of semi-local schemes* consists of a pair (X, χ) , where $\chi \subseteq X$ is a closed subscheme, and a flat morphism $f: X \rightarrow B$ that is essentially of finite type such that $f|_{\chi}: \chi \rightarrow B$ is a dominant finite morphism and $(X_b, \text{red}(\chi_b))$ is an equidimensional semi-local scheme for any $b \in B$. By a slight abuse of notation, this family of semi-local schemes will be denoted by $f: (X, \chi) \rightarrow B$.

Let P be a local property of a scheme such as being Du Bois, F -pure, F -anti-nilpotent, S_n or Cohen–Macaulay. We will say that a semi-local scheme $(X, x_1, \dots, x_r)$ has property P if X is P at $x_1, \dots, x_r$. In particular, we will say that $(X, x_1, \dots, x_r)$ is a Du Bois semi-local scheme, etc. We use the analogous terminology for “local scheme” in place of “semi-local scheme.”

THEOREM 8.5. *Let $f: (X, \chi) \rightarrow B$ be a family of semi-local schemes which is essentially of finite type. Assume that*

- (i) *B is irreducible,*
- (ii) *the fibers of f are equidimensional,*
- (iii) *there exists a $b_0 \in B$ such that X_{b_0} has liftable local cohomology over B , and*
- (iv) *there exist a $b_1 \in B$ and an $n \in \mathbb{N}$ such that X_{b_1} is S_n (respectively, Cohen–Macaulay).*

Then there exists an open set $b_0 \in U \subseteq B$ such that X_b is S_n (respectively, Cohen–Macaulay) for each $b \in U$. In particular, X_{b_0} is S_n (respectively, Cohen–Macaulay).

Proof. It is enough to prove the statement for the S_n property. Since $f|_{\chi}$ is proper, the set $U := \{b \in B \mid X_b \text{ is } S_n\}$ is open in B by [EGAIV₃, Théorème 12.1.6]. By assertion (iv), it is non-empty, and hence it is dense in B . Then it must contain b_0 by Theorem 8.3, which proves the statement. $\square$

Remark 8.6. If X_{b_0} has Du Bois singularities, then we may even choose U such that X_b is S_n and has Du Bois singularities for all $b \in U$ by [KS16a, Theorem 4.1]. As we mentioned earlier, it is not known whether small deformations of F -anti-nilpotent singularities remain F -anti-nilpotent. It is also an interesting question whether the condition of having liftable local cohomology is invariant under small deformations.

It follows that Theorem 8.5 applies to families of semi-local schemes with Du Bois or F -anti-nilpotent singularities by Propositions 6.1 and 7.2 (cf. Lemma 4.3).

As a simple consequence we obtain a generalization of Example 2.3.

COROLLARY 8.7. *Let Z be a normal projective variety over a field k such that K_Z is $\mathbb{Q}$ -Cartier and numerically equivalent to 0. Let $\mathcal{L}$ be an ample line bundle on Z and $X = C_a(Z, \mathcal{L})$ the affine cone over Z with conormal bundle $\mathcal{L}$. If X has liftable local cohomology over k and admits an S_n deformation for some $n \in \mathbb{N}$, then $H^i(Z, \mathcal{O}_Z) = 0$ for $0 < i < n - 1$. In particular, a cone over an abelian variety (ordinary, if $\text{char } k > 0$) of dimension at least 2 does not admit an S_3 deformation.*

Proof. If X admits an S_n deformation for some $n \in \mathbb{N}$, then X itself is S_n by Theorem 8.5 and the first statement follows from (2.4.3). Then the second statement follows from Example 2.3 and Proposition 6.1 in characteristic zero and from Example 2.7 and Corollary 7.3 in positive characteristic. $\square$

Remark 8.8. As noted in the introduction, it is proved in [Som79] that the projective cone over an abelian variety of dimension at least 2 over $\mathbb{C}$ is not smoothable inside the ambient projective space. This is a strict special case of Corollary 8.7. In general, it is possible that a non-smoothable

projective variety is locally smoothable. An example of that is given in [CS18, Theorem 1.5, § 2.3]. Furthermore, Corollary 8.7 is also valid in positive characteristic.

ACKNOWLEDGEMENTS

We would like to thank Johan de Jong for comments and discussions from which we have greatly benefited and for supplying several results we needed in [Sta20]. We would also like to thank Linquan Ma and Karl Schwede for pointing us to [MSS17, Remark 3.4] and for numerous useful discussions about F -singularities, and the referee for suggesting several improvements to the exposition.

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