Corrigendum

Integral cohomology of the generalized Kummer fourfold

(Algebraic Geometry 5, no. 5 (2018), 523–567)

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ABSTRACT

B. Totaro pointed out that the reference used for Theorem 5.2 in [*Integral cohomology of the generalized Kummer fourfold*, Algebr. Geom. **5** (2018), no. 5, 523–567] is inappropriate. Here we show that the theorem holds at least in dimension 4; that is, the integral cohomology of the generalized Kummer fourfold is torsion free. All other results of the article in question remain unaffected.

We use notation from [KM18] and put $H^*(X) := H^*(X, \mathbb{Z})$. Let A be a 2-dimensional complex torus and $K_2(A)$ the generalized Kummer fourfold. The embedding $j: A \times A \hookrightarrow A \times A \times A$ defined by $(x, y) \mapsto (x, y, -x - y)$ gives rise to an $\mathfrak{S}_3$ -action on $A \times A$; the fourfold $K_2(A)$ is a resolution of $(A \times A)/\mathfrak{S}_3$.

THEOREM 1. *The cohomology $H^*(K_2(A))$ is torsion free.*

By Poincaré duality and the universal coefficient theorem, it suffices to show that we have $\text{tors } H^3(K_2(A)) = \text{tors } H^5(K_2(A)) = 0$. Let $W_\tau \subset K_2(A)$ be the locus of subschemes supported at $\tau \in A[3]$. By [HT13, Section 4], we have $W_\tau \simeq \mathbb{P}(1, 1, 3)$. Let $\text{pt}_\tau \in W_\tau$ be the singular point and $W_\tau^* := W_\tau \setminus \text{pt}_\tau$. Put

$$U := K_2(A) \setminus \bigcup_{\tau \in A[3]} W_\tau, \quad U' := K_2(A) \setminus \bigcup_{\tau \in A[3]} \text{pt}_\tau.$$

LEMMA 1. *We have $\text{tors } H^3(U) = \text{tors } H^3(K_2(A))$ and an injection*

$$\text{tors } H^5(K_2(A)) \hookrightarrow \text{tors } H^5(U).$$

Proof. The fourfold $K_2(A)$ is smooth in pt_τ . Applying Thom’s isomorphism to the long exact sequence of the relative cohomology of $(K_2(A), U')$, we obtain

$$H^3(K_2(A)) = H^3(U') \quad \text{and} \quad H^5(K_2(A)) = H^5(U'). \quad (1)$$

Received 28 May 2019, accepted in final form 26 June 2019.

2010 Mathematics Subject Classification 14J17, 14J35, 14F43.

Keywords: integral cohomology, generalized Kummer fourfolds, torsion in cohomology, cohomology of quotients. This journal is © [Foundation Compositio Mathematica](#) 2020. This article is distributed with Open Access under the terms of the [Creative Commons Attribution Non-Commercial License](#), which permits non-commercial reuse, distribution, and reproduction in any medium, provided that the original work is properly cited. For commercial re-use, please contact the [Foundation Compositio Mathematica](#).

This work was supported by the ERC-ALKAGE, grant No. 670846.

Moreover, we have the exact sequence

$$\begin{aligned} H^3(U', U) \rightarrow H^3(U') \longrightarrow H^3(U) \longrightarrow H^4(U', U) \rightarrow H^4(U') \rightarrow \\ \longrightarrow H^4(U) \rightarrow H^5(U', U) \longrightarrow H^5(U') \longrightarrow H^5(U). \end{aligned} \quad (2)$$

By Thom's isomorphism, $H^3(U', U) = 0$, $H^4(U', U) = \bigoplus_{\tau \in A[3]} H^0(W_\tau^*)$, and $H^5(U', U) = \bigoplus_{\tau \in A[3]} H^1(W_\tau^*) = 0$ since W_τ^* is the quotient of $\mathbb{P}^2 \setminus \{\text{pt}\}$ by an automorphism of order 3. It follows from (2) that

$$\text{tors } H^3(U') = \text{tors } H^3(U) \quad \text{and} \quad \text{tors } H^5(U') \hookrightarrow \text{tors } H^5(U). \quad \square$$

Consider $V := (A \times A) \setminus A[3]$, with $A[3]$ embedded diagonally. Let $\tilde{V} \rightarrow V$ be the blow-up in $\Delta := \{(x, x) \in V \mid x \in A\}$, $S_1 := \{(x, -2x) \in V \mid x \in A\}$ and $S_2 := \{(-2x, x) \in V \mid x \in A\}$. We have $U \simeq \tilde{V}/\mathfrak{S}_3$.

LEMMA 2. *The groups $H^3(U)$ and $H^5(U)$ can only have 3-torsion.*

Proof. Let $S' := \{\xi \in A^{[2]} \mid \text{Supp } \xi = \{x, -2x\}, x \in A\}$. This is isomorphic to the blow-up $\tilde{A} \rightarrow A$ in $A[3]$. Let $r: \tilde{A}^{[2]} \rightarrow A^{[2]}$ be the blow-up in S' . For $\tau \in A[3]$, we have Hirzebruch surfaces

$$\Sigma_\tau := r^{-1}(\{\xi \in A^{[2]} \mid \text{Supp } \xi = \{\tau\}\}).$$

We also consider

$$\tilde{\mathcal{W}} := \tilde{A}^{[2]} \setminus \bigcup_{\tau \in A[3]} \Sigma_\tau \simeq \tilde{V}/\mathfrak{S}_2,$$

which provides a triple cover: $\pi: \tilde{\mathcal{W}} \rightarrow U$. By [AP06, Theorem 5.4], we need $\text{tors } H^3(\tilde{\mathcal{W}}) = \text{tors } H^5(\tilde{\mathcal{W}}) = 0$. By [Tot16, Theorem 2.2], we have $\text{tors } H^*(A^{[2]}) = 0$; by [DL09, Theorem 4.1], we have $\text{tors } H^*(\tilde{A}^{[2]}) = 0$. Consider the exact sequence

$$\begin{aligned} H^3(\tilde{A}^{[2]}, \tilde{\mathcal{W}}) \rightarrow H^3(\tilde{A}^{[2]}) \longrightarrow H^3(\tilde{\mathcal{W}}) \longrightarrow H^4(\tilde{A}^{[2]}, \tilde{\mathcal{W}}) \rightarrow H^4(\tilde{A}^{[2]}) \longrightarrow H^4(\tilde{\mathcal{W}}) \longrightarrow \\ \longrightarrow H^5(\tilde{A}^{[2]}, \tilde{\mathcal{W}}) \longrightarrow H^5(\tilde{A}^{[2]}) \longrightarrow H^5(\tilde{\mathcal{W}}) \rightarrow H^6(\tilde{A}^{[2]}, \tilde{\mathcal{W}}). \end{aligned}$$

By Thom's isomorphism,

$$H^k(\tilde{A}^{[2]}, \tilde{\mathcal{W}}) \simeq \bigoplus_{\tau \in A[3]} H^{k-4}(\Sigma_\tau).$$

Since Σ_τ is an Hirzebruch surface, $H^3(\tilde{A}^{[2]}, \tilde{\mathcal{W}}) = H^5(\tilde{A}^{[2]}, \tilde{\mathcal{W}}) = 0$ and $\text{tors } H^4(\tilde{A}^{[2]}, \tilde{\mathcal{W}}) = \text{tors } H^6(\tilde{A}^{[2]}, \tilde{\mathcal{W}}) = 0$. It follows that $\text{tors } H^3(\tilde{\mathcal{W}}) = \text{tors } H^5(\tilde{\mathcal{W}}) = 0$. $\square$

We also consider the double cover $\tilde{V}/\mathfrak{A}_3 \rightarrow U$. Applying [AP06, Theorem 5.4], it suffices to prove the following.

LEMMA 3. *We have $\text{tors } H^3(\tilde{V}/\mathfrak{A}_3) = \text{tors } H^5(\tilde{V}/\mathfrak{A}_3) = 0$.*

First, we show that $\text{tors } H^3(V/\mathfrak{A}_3) = \text{tors } H^5(V/\mathfrak{A}_3) = 0$. Since the $\mathfrak{A}_3$ -action on V is free, we compute the equivariant cohomology as in [Men18, Section 4], using the Boissière–Sarti–Nieper-Wißkirchen invariants; see [BNS13, Section 2].

LEMMA 4. *The Boissière–Sarti–Nieper-Wißkirchen invariants for the $\mathfrak{A}_3$ -action on $A \times A$ are*

- (i) $\ell_1^1(A \times A) = \ell_3^1(A \times A) = 0$ and $\ell_2^1(A \times A) = 4$;
- (ii) $\ell_1^2(A \times A) = 10$, $\ell_2^2(A \times A) = 0$, and $\ell_3^2(A \times A) = 6$;
- (iii) $\ell_1^3(A \times A) = 0$, $\ell_2^3(A \times A) = 16$, and $\ell_3^3(A \times A) = 8$;
- (iv) $\ell_1^4(A \times A) = 19$, $\ell_2^4(A \times A) = 0$, and $\ell_3^4(A \times A) = 17$.

Proof. We start by calculating the $\ell_q^1(A \times A)$ for $q \in \{1, 2, 3\}$. Applying the same idea as in [BNS13, Lemma 6.14], we then deduce the other invariants $\ell_q^k(A \times A)$ using that $H^k(A \times A) \simeq \wedge^k H^1(A \times A)$.

(i) For $a \in H^1(A)$ and a generator A of $H^0(A)$, we have $\sigma_{1,2,3}^*(A \otimes a) = -a \otimes A$ and $\sigma_{1,2,3}^*(a \otimes A) = -A \otimes a + a \otimes A$.

(ii) $\wedge^2(N_2^4) = (\wedge^2 N_2)^4 \oplus (N_2 \otimes N_2)^6 = N_1^4 \oplus (N_3 \oplus N_1)^6$.

(iii) We have $\wedge^3(N_2^4) = \underbrace{(N_2 \otimes N_2 \otimes N_2)^4}_{(1)} \oplus \underbrace{(\wedge^2 N_2 \otimes N_2)^{12}}_{(2)}$. Moreover,

$$(1) = (N_3 \oplus N_1) \otimes N_2 = N_3^2 \oplus N_2,$$

$$(2) = N_1 \otimes N_2 = N_2.$$

(iv) We have $\wedge^4(N_2^4) = \underbrace{N_2 \otimes N_2 \otimes N_2 \otimes N_2}_{(1)} \oplus \underbrace{(\wedge^2 N_2 \otimes \wedge^2 N_2)^6}_{(2)} \oplus \underbrace{(\wedge^2 N_2 \otimes N_2 \otimes N_2)^{12}}_{(3)}$.

Moreover,

$$(1) = (N_3 \oplus N_1) \otimes (N_3 \oplus N_1) = \underbrace{N_3^3}_{N_3^3} \oplus N_3^2 \oplus N_1 = N_3^5 \oplus N_1,$$

$$(2) = N_1,$$

$$(3) = N_1 \otimes (N_1 \oplus N_3) = N_1 \oplus N_3. \quad \square$$

Proof of Lemma 3. Since $H^k(A \times A) \simeq H^k(V)$ for all $k \leq 6$, we have $\ell_q^k(V) = \ell_q^k(A \times A)$ for all $q \in \{1, 2, 3\}$ and $k \leq 6$. By [Men18, Corollary 4.2] and Lemma 4, we have $H^p(\mathfrak{A}_3, H^q(V)) = 0$ for all p, q with $p \neq 0$ and either $p + q = 3$ or $p + q = 5$. Since $\mathfrak{A}_3$ acts freely on V , the spectral sequence of equivariant cohomology implies that $\text{tors } H^3(V/\mathfrak{A}_3) = \text{tors } H^5(V/\mathfrak{A}_3) = 0$. Moreover, $\tilde{V}/\mathfrak{A}_3$ is the blow-up of $V/\mathfrak{A}_3$ in the image $\overline{\Delta}$ of Δ . Since $V/\mathfrak{A}_3$ and $\overline{\Delta}$ are smooth, by [DL09, Theorem 4.1], we have an isomorphism of graded $\mathbb{Z}$ -modules

$$H^*(\tilde{V}) \simeq H^*(V) \oplus t \cdot H^*(\overline{\Delta}),$$

where t is a class of degree 2. Hence, in degrees 3 and 5, we obtain

$$\text{tors } H^3(\tilde{V}/\mathfrak{A}_3) = \text{tors } H^3(V/\mathfrak{A}_3) \oplus \text{tors } H^1(\overline{\Delta}) = 0,$$

$$\text{tors } H^5(\tilde{V}/\mathfrak{A}_3) = \text{tors } H^5(V/\mathfrak{A}_3) \oplus \text{tors } H^3(\overline{\Delta}) = \text{tors } H^3(\overline{\Delta}).$$

It remains to show that $\text{tors } H^3(\overline{\Delta}) = 0$. Since $\overline{\Delta} \simeq A \setminus A[3]$, this follows from the exact sequence of relative cohomology of the pair $(A, A \setminus A[3])$. $\square$

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