All-genus open-closed mirror symmetry for affine toric Calabi–Yau 3-orbifolds

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ABSTRACT

The remodeling conjecture proposed by Bouchard–Klemm–Mariño–Pasquetti relates all-genus open and closed Gromov–Witten invariants of a semi-projective toric Calabi–Yau 3-manifold/3-orbifold $\mathcal{X}$ to the Eynard–Orantin invariants of the mirror curve of $\mathcal{X}$. In this paper, we present a proof of the remodeling conjecture for open-closed orbifold Gromov–Witten invariants of an arbitrary affine toric Calabi–Yau 3-orbifold relative to a framed Aganagic–Vafa Lagrangian brane. This can be viewed as an all-genus open-closed mirror symmetry for affine toric Calabi–Yau 3-orbifolds.

1. Introduction

1.1 Background and motivation

Mirror symmetry relates the A-model topological string theory on a Calabi–Yau 3-fold $\mathcal{X}$ to the B-model topological string theory on another Calabi–Yau 3-fold $\check{\mathcal{X}}$, the mirror of $\mathcal{X}$. The genus g free energy of the topological A-model on $\mathcal{X}$ is mathematically defined as a generating function $F_g^{\mathcal{X}}$ of genus g Gromov–Witten invariants of $\mathcal{X}$ which is a function on a (formal) neighborhood of the large radius limit in the (complexified) Kähler moduli of $\mathcal{X}$. The genus g free energy of the topological B-model on $\check{\mathcal{X}}$ is a section of $\mathbb{L}^{2-2g}$, where $\mathbb{L}$ is a line bundle over the complex moduli of $\check{\mathcal{X}}$, so locally it is a function $\check{F}_g^{\check{\mathcal{X}}}$ on the complex moduli of $\check{\mathcal{X}}$. A mathematical consequence of mirror symmetry is $\check{F}_g^{\check{\mathcal{X}}} = F_g^{\mathcal{X}} + \delta_{g,0}a_0 + \delta_{g,1}a_1$ under the mirror map, where a_0 (respectively, a_1) is a cubic (respectively, linear) function in Kähler parameters. The mirror map and $\check{F}_0^{\check{\mathcal{X}}}$ are determined by period integrals of a holomorphic 3-form on $\check{\mathcal{X}}$. Period integrals of Calabi–Yau complete intersections in toric manifolds can be expressed in terms of explicit hypergeometric functions.

Let Q be the quintic 3-fold, which is a Calabi–Yau hypersurface in $\mathbb{P}^4$, and let $\check{Q}$ be the mirror of Q . Candelas, de la Ossa, Green, and Parkes computed $\check{F}_0^{\check{Q}}$ and the mirror map explicitly and obtained a conjectural formula for the number of rational curves of arbitrary degree in Q ; this formula was first proved independently by Givental [Giv96] and Lian–Liu–Yau [LLY97], who later extended their results to Calabi–Yau complete intersections in projective toric manifolds [Giv98, LLY99a, LLY99b]. Their proofs rely on a good understanding of genus zero Gromov–Witten theory of Q . The mirror formula for F_1^Q was conjectured by Bershadsky–Cecotti–Ooguri–

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Vafa (BCOV for short) [BCOV93] and first proved by Zinger [Zin09]. Combining the techniques of BCOV, results of Yamaguchi–Yau [YY04], and boundary conditions, Huang–Klemm–Quacken [HKQ09] proposed a mirror conjecture on F_g^Q up to $g = 51$. Maulik–Pandharipande provided a mathematical determination of Gromov–Witten invariants of Q in all genera and degrees [MP06]. In general, it is very difficult to evaluate higher-genus Gromov–Witten invariants of a compact Calabi–Yau 3-fold.

In contrast, higher-genus Gromov–Witten invariants of *toric* Calabi–Yau 3-folds (which must be non-compact) are much better understood. In general, Gromov–Witten invariants are defined for projective manifolds (or, more generally, compact almost-Kähler manifolds), but in the toric case one may use localization to define Gromov–Witten invariants of certain non-compact toric manifolds. By virtual localization [GP99], all-genus Gromov–Witten invariants of toric manifolds can be reduced to Hodge integrals, which can be evaluated by effective algorithms. When the toric manifold $\mathcal{X}$ is a Calabi–Yau 3-fold, the topological vertex [AKMV05, LLLZ09, MOOP11] provides a much more efficient algorithm for computing Gromov–Witten invariants of $\mathcal{X}$, as well as open Gromov–Witten invariants of $\mathcal{X}$ relative to an Aganagic–Vafa Lagrangian brane $\mathcal{L}$ (defined in [KL01, DF05, Liu02, LLLZ09] in several ways), in all genera and degrees. The topological B-model on the mirror $\check{\mathcal{X}}$ of a smooth semi-projective toric Calabi–Yau 3-fold $\mathcal{X}$ can be reduced to a theory on the mirror curve of $\mathcal{X}$. Under mirror symmetry, $F_0^{\mathcal{X}}$ corresponds to integrals of 1-forms on the mirror curve along loops, whereas the generating function $F_{0,1}^{\mathcal{X},\mathcal{L}}$ of genus zero open Gromov–Witten invariants (which count holomorphic disks in $\mathcal{X}$ bounded by $\mathcal{L}$) corresponds to integrals of 1-forms on the mirror curve along paths [AV00, AKV02]. On the basis of the work of Eynard–Orantin [EO07] and Mariño [Mar08], Bouchard–Klemm–Mariño–Pasquetti [BKMP09] proposed a new formalism of the topological B-model on $\mathcal{X}$ in terms of the Eynard–Orantin invariants $\omega_{g,n}$ of the mirror curve and conjectured a precise correspondence, known as the remodeling conjecture, between $\omega_{g,n}$ (where $n > 0$) and the generating function $F_{g,n}^{\mathcal{X},\mathcal{L}}$ of open Gromov–Witten invariants counting holomorphic maps from bordered Riemann surfaces with g handles and n holes to $\mathcal{X}$ with boundaries in $\mathcal{L}$. This can be viewed as a version of all-genus open mirror symmetry; the closed sector of the remodeling conjecture relates $\omega_{g,0}$ to $F_g^{\mathcal{X}}$. The open string part of the remodeling conjecture for $\mathbb{C}^3$ was proved independently by Chen [Che18] and Zhou [Zho09a, Zho09b]. The free energy part of the remodeling conjecture for $\mathbb{C}^3$ [BS12] was proved independently by Bouchard–Catuneanu–Marchal–Sulkowski [BCMS13] and Zhu [Zhu15]. Eynard and Orantin provided a proof of the remodeling conjecture for general smooth semi-projective toric Calabi–Yau 3-folds in [EO15].

Bouchard–Klemm–Mariño–Pasquetti have extended the remodeling conjecture to toric Calabi–Yau 3-orbifolds [BKMP10]. Topological string theory on orbifolds was constructed decades ago by physicists [DHVW85, DHVW86], and many works followed in both mathematics and physics (for example, [HV87, DFMS87, BR87, Kni87, Roa90, CV92]). Zaslow discussed orbifold quantum cohomology [Zas93] along with many examples, in both abelian and non-abelian quotients. Later, the mathematical definition of orbifold Gromov–Witten theory and quantum cohomology was laid by Chen–Ruan [CR02] in the symplectic setting and Abramovich–Graber–Vistoli [AGV02, AGV08] in the algebraic setting. Orbifold Gromov–Witten invariants of a toric Calabi–Yau 3-orbifold $\mathcal{X}$ and open orbifold Gromov–Witten invariants of $\mathcal{X}$ relative to an Aganagic–Vafa Lagrangian brane in $\mathcal{X}$ can be expressed in terms of the orbifold Gromov–Witten vertex, which is a generating function of abelian Hurwitz–Hodge integrals [Ros14]. The algorithm for the topological vertex is equivalent to the Gromov–Witten (GW)/Donaldson–Thomas (DT) correspondence

for smooth toric Calabi–Yau 3-folds [MNOP06]. The GW/DT correspondence has been conjectured for Calabi–Yau 3-orbifolds satisfying the hard Lefschetz condition [BCY12] and proved for toric Calabi–Yau 3-orbifolds with transverse A_n -singularities [Zon15, RZ13, RZ15, Ros15]. This provides an efficient algorithm for computing closed and open Gromov–Witten invariants of toric Calabi–Yau 3-orbifolds with transverse A_n -singularities, in all genera and degrees. The precise statement of the GW/DT correspondence for toric Calabi–Yau 3-orbifolds which do not satisfy the hard Lefschetz condition (for example, $[\mathbb{C}^3/\mu_3]$, where μ_3 acts diagonally) is not known even conjecturally. The remodeling conjecture provides a recursive algorithm to compute closed and open Gromov–Witten invariants of all semi-projective toric Calabi–Yau 3-orbifolds in all genera. Explicit predictions of open orbifold Gromov–Witten invariants of $[\mathbb{C}^3/\mu_3]$ are given in [BKMP10].

1.2 Statement of the main result and outline of the proof

In this paper, we study the remodeling conjecture for all affine toric Calabi–Yau 3-orbifolds $[\mathbb{C}^3/G]$, where G can be any finite subgroup of the maximal torus of $\mathrm{SL}(3, \mathbb{C})$. We consider a general framed Aganagic–Vafa brane $(\mathcal{L}, f)$, where $\mathcal{L} \cong [(S^1 \times \mathbb{C})/G]$ and $f \in \mathbb{Z}$. We define open-closed orbifold Gromov–Witten invariants of $\mathcal{X} = [\mathbb{C}^3/G]$ relative to $(\mathcal{L}, f)$ as certain equivariant relative orbifold invariants of a toric Calabi–Yau 3-orbifold $\mathcal{Y}_f$ relative to a divisor $\mathcal{D}_f \cong [\mathbb{C}^2/\mu_m]$, where $\mu_m \cong \mathbb{Z}_m$ is the stabilizer of the G -action on $S^1 \times \{0\} \subset S^1 \times \mathbb{C}$, and the coarse moduli space $\mathbb{C}^2/\mu_m$ of $\mathcal{D}_f$ is the A_{m-1} surface singularity. We define generating functions of open-closed Gromov–Witten invariants

$$F_{g,n}^{\mathcal{X},(\mathcal{L},f)}(\tau; X_1, \dots, X_n),$$

which are $H_{\mathrm{CR}}^*(\mathcal{B}\mu_m; \mathbb{C})^{\otimes n}$ -valued formal power series in A-model closed string coordinates $\tau = (\tau_1, \dots, \tau_p)$ and A-model open string coordinates $X_1, \dots, X_n$; here $H_{\mathrm{CR}}^*(\mathcal{B}\mu_m; \mathbb{C}) \cong \mathbb{C}^m$ is the Chen–Ruan orbifold cohomology of the classifying space $\mathcal{B}\mu_m$ of μ_m . We use the Eynard–Orantin invariants $\omega_{g,n}$ of the framed mirror curve to define B-model potentials

$$\tilde{F}_{g,n}(\tau; X_1, \dots, X_n),$$

which are $H_{\mathrm{CR}}^*(\mathcal{B}\mu_m; \mathbb{C})^{\otimes n}$ -valued functions in B-model closed string flat coordinates $\tau = (\tau_1, \dots, \tau_p)$ and B-model open string coordinates $X_1, \dots, X_n$, analytic in an open neighborhood of the origin in $\mathbb{C}^p \times \mathbb{C}^n$. The mirror map relates the B-model flat coordinates $(\tau_1, \dots, \tau_p)$ to the complex parameters $(q_1, \dots, q_p)$ of the framed mirror curve. Our main result is the following.

THEOREM 7.19. *For any $g \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{> 0}$,*

$$\tilde{F}_{g,n}(\tau; X_1, \dots, X_n) = (-1)^{g-1+n} F_{g,n}^{\mathcal{X},(\mathcal{L},f)}(\tau; X_1, \dots, X_n). \quad (1.1)$$

This is indeed more general than the original conjecture in [BKMP10], which covers the $m = 1$ case, that is, the case when $\mathcal{L}$ is on an effective leg.

We now give an outline of our proof of Theorem 7.19. For simplicity, we consider the stable case $2g - 2 + n > 0$ in this outline. (The unstable cases $(g, n) = (0, 1), (0, 2)$ will be treated separately.) The proof consists of three steps:

1. (*A-model graph sum*) In [FLZ16], we used Tseng’s orbifold quantum Riemann–Roch theorem [Tse10] to write down a graph sum formula for the total descendant equivariant Gromov–Witten potential of $\mathcal{X}$. We use this formula and localization to derive a graph

sum formula for the A-model potential $F_{g,n}^{\mathcal{X},(\mathcal{L},f)}$:

$$F_{g,n}^{\mathcal{X},(\mathcal{L},f)} = \sum_{\vec{\Gamma} \in \Gamma_{g,n}(\mathcal{X})} \frac{w_A(\vec{\Gamma})}{|\text{Aut}(\vec{\Gamma})|},$$

where $\Gamma_{g,n}(\mathcal{X})$ is a certain set of decorated graphs, $\text{Aut}(\vec{\Gamma})$ is the automorphism group of the decorated graph $\vec{\Gamma}$, and $w_A(\vec{\Gamma})$ is the A-model weight of the decorated graph $\vec{\Gamma}$ defined by (5.9).

2. (*B-model graph sum*) The Eynard–Orantin invariants $\omega_{g,n}$ can be expressed as sums over labeled graphs [KO10, Eyn11, Eyn14, DOSS14]. We use special geometry to obtain the Taylor series expansion of the graph sum formula in [DOSS14, Theorem 3.7] in B-model closed string flat coordinates $\tau = (\tau_1, \dots, \tau_p)$ at $\tau = 0$ and derive a graph sum formula for the B-model potential $\check{F}_{g,n}$:

$$\check{F}_{g,n} = \sum_{\vec{\Gamma} \in \Gamma_{g,n}(\mathcal{X})} \frac{w_B(\vec{\Gamma})}{|\text{Aut}(\vec{\Gamma})|},$$

where $w_B(\vec{\Gamma})$ is the B-model weight of the decorated graph $\vec{\Gamma}$ defined by (7.14).

3. (*Comparison of weights*) For each decorated graph $\vec{\Gamma} \in \Gamma_{g,n}(\mathcal{X})$, we prove the following identity relating A-model and B-model weights:

$$w_B(\vec{\Gamma}) = (-1)^{g-1+n} w_A(\vec{\Gamma}).$$

1.3 Remarks on the BKMP remodeling conjecture in the general case

In [FLZ19], the authors provide a proof of the BKMP remodeling conjecture for all semi-projective toric Calabi–Yau 3-orbifolds. The proof in [FLZ19] relies on (i) the quantization formula for the total descendant potential of equivariant GW theory of GKM orbifolds [Zon16], (ii) the B-model graph sum formula in [DOSS14, Theorem 3.7], (iii) the genus zero mirror theorem for toric DM stacks [CCIT15, CCK15], and (iv) the genus zero open mirror theorem for semi-projective toric Calabi–Yau 3-orbifolds [FLT12]. Item (i) is used to derive the A-model graph sum formula, whereas items (iii) and (iv) are used to match the A-model and B-model graph sums. In the affine case, the proof in this paper is more direct than the specialization of the proof in [FLZ19] to the affine case: the proof in this paper relies on Tseng’s orbifold quantum Riemann–Roch theorem [Tse10] and item (ii), but not on items (i), (iii), (iv). The proof in [FLZ19] also relies on the computation of oscillating integrals on a mirror curve in this paper; we obtain the desired result by integrating in the Landau–Ginzburg model and dimensional reduction (Theorem 7.8).

1.4 Overview of the paper

In Section 2, we describe affine toric Calabi–Yau 3-orbifolds and their mirror curves. In Section 3, for each affine toric Calabi–Yau 3-orbifold $\mathcal{X}$ and a framed Aganagic–Vafa A-brane $(\mathcal{L}, f)$, we construct a relative Calabi–Yau 3-orbifold $(\mathcal{Y}_f, \mathcal{D}_f)$, where $\mathcal{X} = \mathcal{Y}_f \setminus \mathcal{D}_f$. In Section 4, we describe the Chen–Ruan orbifold cohomology of the classifying space $\mathcal{B}G$ of the finite abelian group G and the $\tilde{T}$ -equivariant Chen–Ruan orbifold cohomology of $\mathcal{X}$ and $\mathcal{Y}_f$. In Section 5, we give the precise definition of the A-model partition functions $F_{g,n}^{\mathcal{X},(\mathcal{L},f)}$ as generating functions of equivariant relative Gromov–Witten invariants of $(\mathcal{Y}_f, \mathcal{D}_f)$ and derive the A-model graph sum formula (Theorem 5.5). In Section 6, we study the geometry and topology of the mirror curve.

In particular, we clarify the choice of A-cycles and B-cycles and the definition of the B-model flat coordinates. In Section 7, we give the precise definition of the B-model partition functions $\check{F}_{g,n}$, derive the B-model graph sum formula (Theorem 7.15), and complete the proof of the main result (Theorem 7.19).

2. Affine toric Calabi–Yau 3-orbifolds and their mirrors

2.1 The A-model geometry

Let $\mathbb{T} = (\mathbb{C}^*)^3$ and $N = \text{Hom}(\mathbb{C}^*, \mathbb{T}) \cong \mathbb{Z}^3$, and let $M = \text{Hom}(\mathbb{T}, \mathbb{C}^*) = \text{Hom}(N, \mathbb{Z})$. Let $\sigma \subset N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^3$ be a simplicial cone spanned by $b_1, b_2, b_3 \in N$ such that the simplicial affine toric variety $X_{\sigma} := \text{Spec } \mathbb{C}[\sigma^{\vee} \cap M]$ has trivial canonical divisor. Then there exists a $u \in M$ such that $\langle u, b_1 \rangle = \langle u, b_2 \rangle = \langle u, b_3 \rangle = 1$. We may choose a $\mathbb{Z}$ -basis u_1, u_2, u_3 of M such that $u_3 = u$. Let e_1, e_2, e_3 be the dual $\mathbb{Z}$ -basis of N . Then

$$b_1 = re_1 - se_2 + e_3, \quad b_2 = me_2 + e_3, \quad b_3 = e_3,$$

where r and m are positive integers and $s \in \{0, 1, \dots, r-1\}$.

We have a short exact sequence of abelian groups

$$1 \rightarrow G \rightarrow \tilde{T} = (\mathbb{C}^*)^3 \xrightarrow{\phi} \mathbb{T} = (\mathbb{C}^*)^3 \rightarrow 1,$$

where $\phi(\tilde{t}_1, \tilde{t}_2, \tilde{t}_3) = (\tilde{t}_1^r, \tilde{t}_1^{-s} \tilde{t}_2^m, \tilde{t}_1 \tilde{t}_2 \tilde{t}_3)$.

For $i = 1, 2, 3$, let $\chi_i: G \rightarrow \mathbb{C}^*$ be the projection to the i th factor. Then $\chi_1, \chi_2, \chi_3 \in G^* = \text{Hom}(G, \mathbb{C}^*)$, and $\chi_1 \chi_2 \chi_3 = 1$. Given a positive integer r , let $\mu_r := \{z \in \mathbb{C}^*: z^r = 1\} \cong \mathbb{Z}_r$ be the group of r th roots of unity. The image of χ_1 is μ_r , and the kernel of χ_1 is isomorphic to μ_m . We have a short exact sequence of finite abelian groups

$$1 \rightarrow \mu_m \rightarrow G \xrightarrow{\chi_1} \mu_r \rightarrow 1.$$

Then $X_{\sigma} = \mathbb{C}^3/G$, which is the coarse moduli space of $\mathcal{X} := [\mathbb{C}^3/G]$. Let $N_{\sigma} = \mathbb{Z}b_1 \oplus \mathbb{Z}b_2 \oplus \mathbb{Z}b_3$. Define

$$\text{Box}(\sigma) = \{v \in N: v = c_1 b_1 + c_2 b_2 + c_3 b_3, 0 \leq c_i < 1\}.$$

There is a bijection $\text{Box}(\sigma) \rightarrow N/N_{\sigma}$ given by $v \mapsto v + N_{\sigma}$; there is a bijection $\text{Box}(\sigma) \rightarrow G$ given by

$$c_1 b_1 + c_2 b_2 + c_3 b_3 \mapsto (e^{2\pi\sqrt{-1}c_1}, e^{2\pi\sqrt{-1}c_2}, e^{2\pi\sqrt{-1}c_3}). \quad (2.1)$$

For example, for $j \in \{1, \dots, m-1\}$, we have

$$(0, j, 1) = \frac{j}{m} b_2 + \left(1 - \frac{j}{m}\right) b_3 \in \text{Box}(\sigma), \quad (1, e^{2\pi\sqrt{-1}j/m}, e^{-2\pi\sqrt{-1}j/m}) \in G.$$

Define $\text{age}: \text{Box}(\sigma) \rightarrow \{0, 1, 2\}$ by

$$c_1 b_1 + c_2 b_2 + c_3 b_3 \mapsto c_1 + c_2 + c_3.$$

Then we have a disjoint union

$$\text{Box}(\sigma) = \{0\} \cup \{v \in \text{Box}(\sigma): \text{age}(v) = 1\} \cup \{v \in \text{Box}(\sigma): \text{age}(v) = 2\}.$$

Given $h \in G$ and $i \in \{1, 2, 3\}$, define $c_i(h) \in [0, 1)$ and $\text{age}(h) \in \{0, 1, 2\}$ by

$$e^{2\pi\sqrt{-1}c_i(h)} = \chi_i(h), \quad (2.2)$$

$$\text{age}(h) = c_1(h) + c_2(h) + c_3(h). \quad (2.3)$$

The inverse map $G \rightarrow \text{Box}(\sigma)$ of (2.1) is given by $h \mapsto c_1(h)b_1 + c_2(h)b_2 + c_3(h)b_3$.

2.2 The B-model geometry

There exist $(m_1, n_1), \dots, (m_p, n_p) \in \mathbb{Z}^2$ such that

$$\{v \in \text{Box}(\sigma) : \text{age}(v) = 1\} = \{(m_a, n_a, 1) : a = 1, \dots, p\}.$$

We define $b_{3+a} = (m_a, n_a, 1)$ for $a = 1, \dots, p$ and define

$$H_f(X, Y, q) = X^r Y^{-s-rf} + Y^m + 1 + \sum_{a=1}^p q_a X^{m_a} Y^{n_a-m_a f}.$$

The mirror of $\mathcal{X}$ is a hypersurface in $\mathbb{C}^2 \times (\mathbb{C}^*)^2$:

$$\check{\mathcal{X}}_q = \{(u, v, X, Y) \in \mathbb{C}^2 \times (\mathbb{C}^*)^2 : H_f(X, Y, q) - uv = 0\}.$$

This is a family of non-compact Calabi–Yau 3-folds parametrized by $q = (q_1, \dots, q_p)$. Different framings $f \in \mathbb{Z}$ give isomorphic $\check{\mathcal{X}}_q$ but different superpotentials $X : \check{\mathcal{X}}_q \rightarrow \mathbb{C}^*$. The framed mirror curve is

$$\Sigma_q = \{(X, Y) \in (\mathbb{C}^*)^2 : H_f(X, Y, q) = 0\}.$$

Then

- (i) the hypersurface $\check{\mathcal{X}}_q$ is smooth if and only if Σ_q is smooth;
- (ii) all the branch points of $X : \Sigma_q \rightarrow \mathbb{C}^*$ are simple if and only if the critical points of $X : \check{\mathcal{X}}_q \rightarrow \mathbb{C}^*$ are isolated and non-degenerate.

Note that $\Sigma_0 = \{(X, Y) \in (\mathbb{C}^*)^2 : X^r Y^{-s-rf} + Y^m + 1 = 0\}$ is smooth and all the branch points of $X : \Sigma_0 \rightarrow \mathbb{C}^*$ are simple. So properties (i) and (ii) hold for small enough q .

Remark 2.1. When $m = 1$, $G = \mu_r$, and $q_a = 0$, the framed mirror curve is given by $X^r Y^{-s-rf} + Y + 1 = 0$ or, equivalently,

$$Y^{s+rf}(1+Y) + X^r = 0. \quad (2.4)$$

Up to the sign, which is a matter of convention, (2.4) agrees with the following framed mirror curve given by equation (3.28) of [BHL14]:

$$Y^{s+rf}(1-Y) - X^r = 0.$$

In this paper, we assume that the framing f is a positive integer.

We define

$$w_1 = \frac{1}{r}, \quad w_2 = \frac{s+rf}{rm}, \quad w_3 = -w_1 - w_2 = \frac{-s-rf-m}{rm}. \quad (2.5)$$

For later convenience, we fix two explicit bijections (which are not necessarily group homomorphisms) $\iota : \mathbb{Z}_r \times \mathbb{Z}_m \rightarrow G$ and $\iota^* : G^* \rightarrow \mathbb{Z}_r \times \mathbb{Z}_m$:

- $\iota : \mathbb{Z}_r \times \mathbb{Z}_m \rightarrow G$, $(j, \ell) \mapsto \eta_1^j \eta_2^\ell$, where

$$\eta_1 = (e^{2\pi\sqrt{-1}w_1}, e^{2\pi\sqrt{-1}w_2}, e^{2\pi\sqrt{-1}w_3}), \quad \eta_2 = (1, e^{2\pi\sqrt{-1}/m}, e^{-2\pi\sqrt{-1}/m}). \quad (2.6)$$

- $\iota^* : \mathbb{Z}_r \times \mathbb{Z}_m \rightarrow G^*$, $(j, \ell) \mapsto \chi_1^j \chi_2^\ell$, where $\chi_1, \chi_2 \in G^*$ are defined as in Section 2.1.

3. The 1-leg framed orbifold topological vertex

Let $L = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 - |z_2|^2 = c, |z_1|^2 - |z_3|^2 = c, \Im(z_1 z_2 z_3) = 0\}$, where $c > 0$ and $\Im(z)$ means the imaginary part of z . Then L is a special Lagrangian of $\mathbb{C}^3$ (Harvey–Lawson [HL82]), and $\mathcal{L} := [L/G]$ is a special Lagrangian sub-orbifold of the affine toric Calabi–Yau 3-orbifold $\mathcal{X} = [\mathbb{C}^3/G]$. A framed Aganagic–Vafa A-brane of $\mathcal{X}$ is a pair $(\mathcal{L}, f)$, where $\mathcal{L} \subset \mathcal{X}$ is as above and $f \in \mathbb{Z}$. The 1-leg framed orbifold topological vertex is a relative toric Calabi–Yau 3-orbifold $(\mathcal{Y}_f, \mathcal{D}_f)$ associated with $(\mathcal{X}, \mathcal{L}, f)$.

3.1 The toric graph

The lattice $N_\sigma = \mathbb{Z}b_1 \oplus \mathbb{Z}b_2 \oplus \mathbb{Z}b_3$ is a sublattice of $N = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3$ of index rm , and $\{b_1, b_2, b_3\}$ is a $\mathbb{Q}$ -basis of $N_\mathbb{Q} := N \otimes_\mathbb{Z} \mathbb{Q}$. Let $\{w_1, w_2, w_3\}$ be the dual $\mathbb{Q}$ -basis of $M_\mathbb{Q} := M \otimes_\mathbb{Z} \mathbb{Q}$. Then

$$w_1 = \frac{1}{r}u_1, \quad w_2 = \frac{s}{rm}u_1 + \frac{1}{m}u_2, \quad w_3 = -\frac{s+m}{rm}u_1 - \frac{1}{m}u_2 + u_3.$$

Note that

$$w_1 + w_2 + w_3 = u_3$$

is the weight of the $\mathbb{T}$ -action on $T_{\mathbf{p}}\mathcal{X}$, where $\mathbf{p} = \mathcal{B}G$ is the unique $\mathbb{T}$ -fixed point in the toric 3-orbifold $\mathcal{X}$. Let $\mathbb{T}'$ be the kernel of $u_3 \in M = \text{Hom}(\mathbb{T}, \mathbb{C}^*)$.

The relative formal toric Calabi–Yau (FTCY) graph of the 1-leg framed orbifold topological vertex is shown in Figure 1. This generalizes the 1-leg framed topological vertex [LLLZ09].

FIGURE 1. The toric graph of the 1-leg framed orbifold topological vertex

3.2 The fan

The toric graph in Figure 1 defines a relative toric Calabi–Yau 3-orbifold $\mathcal{Y}_f$ which is a partial compactification of $\mathcal{X}$. We now describe the fan Σ_f defining the coarse moduli space Y_f of $\mathcal{Y}_f$. Let

$$v_1 = b_1 = re_1 - se_2 + e_3, \quad v_2 = b_2 = me_2 + e_3, \quad v_3 = b_3 = e_3, \quad v_4 = -e_1 - fe_2.$$

Let σ be the 3-cone spanned by v_1, v_2, v_3 , as before; let σ' be the 3-cone spanned by v_2, v_3, v_4 . Let τ be the 2-cone $\sigma \cap \sigma'$, and let ρ_i be the 1-cone spanned by v_i . Then

$$\Sigma_f(1) = \{\rho_1, \rho_2, \rho_3, \rho_4\}, \quad \Sigma_f(3) = \{\sigma, \sigma'\}.$$

The coarse moduli space D_f is the $\mathbb{T}$ -invariant divisor in Y_f associated with ρ_4 . The pair $(\mathcal{Y}_f, \mathcal{D}_f)$ is a relative toric Calabi–Yau 3-orbifold, where the relative Calabi–Yau condition is $K_{\mathcal{Y}_f} + \mathcal{D}_f = 0$.

Let $\mathcal{D}_i$ be the $\mathbb{T}$ -invariant divisor associated with ρ_i . Then

$$K_{\mathcal{Y}_f} = -\mathcal{D}_1 - \mathcal{D}_2 - \mathcal{D}_3 - \mathcal{D}_4, \quad K_{\mathcal{Y}_f} + \mathcal{D}_f = -\mathcal{D}_1 - \mathcal{D}_2 - \mathcal{D}_3.$$

Note that $K_{\mathcal{X}_f} + \mathcal{D}_f$ is the principal $\mathbb{T}$ -divisor associated with $\chi^{u_3} \in \mathbb{C}[M]$.

3.3 The root construction

In this subsection, we give another description of the relative toric 3-orbifold $(\mathcal{Y}_f, \mathcal{D}_f)$. Let $\mathfrak{p}_0 = [0, 1]$ and $\mathfrak{p}_\infty = [1, 0]$ be the two torus-fixed points in $\mathbb{P}(1, r)$, where $\mathbb{P}(1, r)$ is the weighted projective line with an orbifold point of order r . Then $\mathfrak{p}_0$ is the unique stacky point in $\mathbb{P}(1, r)$, and any $\mathbb{C}^*$ -equivariant line bundle on $\mathbb{P}(1, r)$ is of the form

$$\mathcal{O}_{\mathbb{P}(1, r)}(a\mathfrak{p}_0 + b\mathfrak{p}_\infty),$$

where $a, b \in \mathbb{Z}$. We have

$$\deg \mathcal{O}_{\mathbb{P}(1, r)}(a\mathfrak{p}_0 + b\mathfrak{p}_\infty) = \frac{a}{r} + b.$$

Let $p: \mathfrak{l} \rightarrow \mathbb{P}(1, r)$ be the $\mathbb{Z}_m$ -gerbe obtained by applying the m th root construction to the equivariant line bundle $\mathcal{O}_{\mathbb{P}(1, r)}(s\mathfrak{p}_0 + f\mathfrak{p}_\infty)$. Let $\mathcal{L}_2$ be the tautological line bundle over $\mathfrak{l}$, so that

$$\mathcal{L}_2^{\otimes m} = p^* \mathcal{O}_{\mathbb{P}(1, r)}(s\mathfrak{p}_0 + f\mathfrak{p}_\infty).$$

Then $\deg \mathcal{L}_2 = s/rm + f/m = w_2$. Let

$$\mathcal{L}_3 := \mathcal{L}_2^{-1} \otimes p^* \mathcal{O}_{\mathbb{P}(1, r)}(-\mathfrak{p}_0).$$

Then $\deg \mathcal{L}_3 = -s/rm - f/m - 1/r = w_3$. The toric 3-orbifold $\mathcal{Y}_f$ is the total space of the rank 2 vector bundle

$$V_f := \mathcal{L}_2 \oplus \mathcal{L}_3 \rightarrow \mathfrak{l}.$$

The divisor $\mathcal{D}_f$ is the fiber of V_f over the stacky point $p^{-1}(\mathfrak{p}_\infty)$ in $\mathfrak{l}$.

When G is trivial, we have $r = m = 1$, $s = 0$, $\mathfrak{l} = \mathbb{P}(1, r) = \mathbb{P}^1$, and $V_f = \mathcal{O}_{\mathbb{P}^1}(f) \oplus \mathcal{O}_{\mathbb{P}^1}(-f-1) \rightarrow \mathbb{P}^1$.

4. Chen–Ruan orbifold cohomology

In this section, we describe the Chen–Ruan orbifold cohomology [CR04] of the classifying space $\mathcal{B}G$ of the finite abelian group G and the $\tilde{T}$ -equivariant Chen–Ruan orbifold cohomology of $\mathcal{X} = [\mathbb{C}^3/G]$ and $\mathcal{Y}_f$. The Chen–Ruan orbifold cohomology of the classifying space of any finite group is described in [JK02].

4.1 Chen–Ruan orbifold cohomology of $\mathcal{B}G$

The inertia stack of $\mathcal{B}G$ is

$$\mathcal{IB}G = \bigcup_{h \in G} (\mathcal{B}G)_h, \quad \text{where} \quad (\mathcal{B}G)_h = [\{h\}/G] \cong \mathcal{B}G.$$

As a graded vector space over $\mathbb{C}$,

$$H_{\text{CR}}^*(\mathcal{B}G; \mathbb{C}) = H^*(\mathcal{IB}G; \mathbb{C}) = \bigoplus_{h \in G} H^0((\mathcal{B}G)_h; \mathbb{C}),$$

where $H^0((\mathcal{B}G)_h; \mathbb{C}) = \mathbb{C}\mathbf{1}_h$. The orbifold Poincaré pairing of $H_{\text{CR}}^*(\mathcal{B}G; \mathbb{C})$ is given by

$$\langle \mathbf{1}_h, \mathbf{1}_{h'} \rangle = \frac{\delta_{h^{-1}, h'}}{|G|}.$$

The orbifold cup product of $H_{\text{CR}}^*(\mathcal{B}G; \mathbb{C})$ is given by $\mathbf{1}_h \star \mathbf{1}_{h'} = \mathbf{1}_{hh'}$.

We now define a canonical basis for the semi-simple algebra $H_{\text{CR}}^*(\mathcal{B}G; \mathbb{C})$. Given a character $\gamma \in G^* = \text{Hom}(G, \mathbb{C}^*)$, define

$$\phi_\gamma := \frac{1}{|G|} \sum_{h \in G} \chi_\gamma(h^{-1}) \mathbf{1}_h.$$

Then

$$H_{\text{CR}}^*(\mathcal{B}G; \mathbb{C}) = \bigoplus_{h \in G} \mathbb{C}\mathbf{1}_h = \bigoplus_{\gamma \in G^*} \mathbb{C}\phi_\gamma.$$

Recall that we have the orthogonality of characters:

- (1) For any $\gamma, \gamma' \in G^*$, we have $\frac{1}{|G|} \sum_{h \in G} \chi_\gamma(h^{-1}) \chi_{\gamma'}(h) = \delta_{\gamma, \gamma'}$,
- (2) For any $h, h' \in G$, we have $\frac{1}{|G|} \sum_{\gamma \in G^*} \chi_\gamma(h^{-1}) \chi_\gamma(h') = \delta_{h, h'}$.

Therefore, $\langle \phi_\gamma, \phi_{\gamma'} \rangle = \delta_{\gamma, \gamma'} / |G|^2$ and $\phi_\gamma \star \phi_{\gamma'} = \delta_{\gamma, \gamma'} \phi_\gamma$. Then $\{\phi_\gamma : \gamma \in G^*\}$ is a canonical basis of $H_{\text{CR}}^*(\mathcal{B}G; \mathbb{C})$.

4.2 Equivariant Chen–Ruan orbifold cohomology of $\mathcal{X} = [\mathbb{C}^3/G]$

Given any $h \in G$, define $c_i(h) \in [0, 1) \cap \mathbb{Q}$ and $\text{age}(h) \in \{0, 1, 2\}$ by equations (2.2) and (2.3) in Section 2.2, respectively. Let $(\mathbb{C}^3)^h$ denote the h -invariant subspace of $\mathbb{C}^3$. Then

$$\dim_{\mathbb{C}} (\mathbb{C}^3)^h = \sum_{i=1}^3 \delta_{c_i(h), 0}.$$

The inertial stack of $\mathcal{X}$ is

$$\mathcal{IX} = \bigcup_{h \in G} \mathcal{X}_h, \quad \text{where } \mathcal{X}_h = [(\mathbb{C}^3)^h / G].$$

In particular, $\mathcal{X}_1 = [\mathbb{C}^3 / G] = \mathcal{X}$. As a graded vector space over $\mathbb{C}$,

$$H_{\text{CR}}^*(\mathcal{X}; \mathbb{C}) = \bigoplus_{h \in G} H^*(\mathcal{X}_h; \mathbb{C})[2 \text{age}(h)] = \bigoplus_{h \in G} \mathbb{C}\mathbf{1}_h,$$

where $\deg(\mathbf{1}_h) = 2 \text{age}(h) \in \{0, 2, 4\}$. The orbifold Poincaré pairing of the (non-equivariant) Chen–Ruan orbifold cohomology $H_{\text{CR}}^*(\mathcal{X}; \mathbb{C})$ is given by

$$\langle \mathbf{1}_h, \mathbf{1}_{h'} \rangle_{\mathcal{X}} = \frac{1}{|G|} \delta_{h^{-1}, h'} \cdot \delta_{0, \dim_{\mathbb{C}} \mathcal{X}_h}.$$

Let $\mathcal{R} = H^*(\mathcal{BT}; \mathbb{C}) = \mathbb{C}[\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3]$, where $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are the first Chern classes of the universal line bundles over $\mathcal{BT}$. The $\tilde{T}$ -equivariant Chen–Ruan orbifold cohomology $H_{\text{CR}, \tilde{T}}^*([\mathbb{C}^3/G]; \mathbb{C})$ is an $\mathcal{R}$ -module. Given $h \in G$, define

$$\mathbf{e}_h := \prod_{i=1}^3 \mathbf{w}_i^{\delta_{c_i(h), 0}} \in \mathcal{R}.$$

In particular, $\mathbf{e}_1 = \mathbf{w}_1 \mathbf{w}_2 \mathbf{w}_3$. Then the $\tilde{T}$ -equivariant Euler class of $\mathbf{0}_h := [0/G]$ in $\mathcal{X}_h = [(\mathbb{C}^3)^h/G]$ is

$$e_{\tilde{T}}(T_{\mathbf{0}_h} \mathcal{X}_h) = \mathbf{e}_h \mathbf{1}_h \in H_{\tilde{T}}^*(\mathcal{X}_h; \mathbb{C}) = \mathcal{R} \mathbf{1}_h.$$

Let $\chi_1, \chi_2, \chi_3 \in G^*$ be defined as in Section 2.1. The image of $\chi_i: G \rightarrow \mathbb{C}^*$ is μ_{r_i} for some positive integer r_i . In particular, $r_1 = r$. Define

$$\mathcal{R}' = \mathbb{C}[\mathbf{w}_1^{1/r_1}, \mathbf{w}_2^{1/r_2}, \mathbf{w}_3^{1/r_3}],$$

which is a finite extension of $\mathcal{R}$. Let

$$\mathcal{Q} = \mathbb{C}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) \quad \text{and} \quad \mathcal{Q}' = \mathbb{C}(\mathbf{w}_1^{1/r_1}, \mathbf{w}_2^{1/r_2}, \mathbf{w}_3^{1/r_3})$$

be the fractional fields of $\mathcal{R}$ and $\mathcal{R}'$, respectively. The $\tilde{T}$ -equivariant Poincaré pairing of $H_{\tilde{T}, \text{CR}}^*(\mathcal{X}; \mathbb{C}) \otimes_{\mathcal{R}} \mathcal{Q}$ (which is isomorphic to $H_{\text{CR}}^*(\mathcal{X}; \mathcal{Q})$ as a vector space over $\mathcal{Q}$) is given by

$$\langle \mathbf{1}_h, \mathbf{1}_{h'} \rangle_{\mathcal{X}} = \frac{1}{|G|} \cdot \frac{\delta_{h^{-1}, h'}}{\mathbf{e}_h} \in \mathcal{Q}.$$

The $\tilde{T}$ -equivariant orbifold cup product of $H_{\tilde{T}, \text{CR}}^*(\mathcal{X}; \mathbb{C}) \otimes_{\mathcal{R}} \mathcal{Q}$ is given by

$$\mathbf{1}_h \star_{\mathcal{X}} \mathbf{1}_{h'} = \left(\prod_{i=1}^3 \mathbf{w}_i^{c_i(h) + c_i(h') - c_i(hh')} \right) \mathbf{1}_{hh'}.$$

Define

$$\bar{\mathbf{1}}_h := \frac{\mathbf{1}_h}{\prod_{i=1}^3 \mathbf{w}_i^{c_i(h)}} \in H_{\tilde{T}, \text{CR}}^*(\mathcal{X}; \mathbb{C}) \otimes_{\mathcal{R}} \mathcal{Q}'. \quad (4.1)$$

Then

$$\langle \bar{\mathbf{1}}_h, \bar{\mathbf{1}}_{h'} \rangle_{\mathcal{X}} = \frac{\delta_{h^{-1}, h'}}{|G| \mathbf{w}_1 \mathbf{w}_2 \mathbf{w}_3}, \quad \bar{\mathbf{1}}_h \star_{\mathcal{X}} \bar{\mathbf{1}}_{h'} = \bar{\mathbf{1}}_{hh'}.$$

We now define a canonical basis for the semi-simple algebra $H_{\tilde{T}, \text{CR}}^*(\mathcal{X}; \mathbb{C}) \otimes_{\mathcal{R}} \mathcal{Q}'$. Given $\gamma \in G^*$, define

$$\bar{\phi}_{\gamma} := \frac{1}{|G|} \sum_{h \in G} \chi_{\gamma}(h^{-1}) \bar{\mathbf{1}}_h. \quad (4.2)$$

Then

$$\langle \bar{\phi}_{\gamma}, \bar{\phi}_{\gamma'} \rangle_{\mathcal{X}} = \frac{\delta_{\gamma\gamma'}}{|G|^2 \mathbf{w}_1 \mathbf{w}_2 \mathbf{w}_3}, \quad \bar{\phi}_{\gamma} \star_{\mathcal{X}} \bar{\phi}_{\gamma'} = \delta_{\gamma\gamma'} \bar{\phi}_{\gamma}.$$

So $\{\bar{\phi}_{\gamma} : \gamma \in G^*\}$ is a canonical basis of $H_{\tilde{T}, \text{CR}}^*(\mathcal{X}; \mathbb{C}) \otimes_{\mathcal{R}} \mathcal{Q}'$.

4.3 Equivariant Chen–Ruan orbifold cohomology of $\mathcal{Y}_f$

We use the notation in Sections 3.2 and 3.3: the coarse moduli space Y_f of the toric 3-orbifold $\mathcal{Y}_f$ is defined by a simplicial fan Σ_f with two 3-dimensional cones σ and σ' . Let $\mathcal{X} = [\mathbb{C}^3/G]$ and $\mathcal{X}' = [\mathbb{C}^3/\mu_m]$ be the affine toric sub-orbifolds of $\mathcal{Y}_f$ associated with σ and σ' , respectively. As $\mathcal{Q}$ -vector spaces,

$$H_{\tilde{T}, \text{CR}}^*(\mathcal{Y}_f; \mathbb{C}) \otimes_{\mathcal{R}} \mathcal{Q} = H_{\text{CR}}^*(\mathcal{Y}_f; \mathcal{Q}) = H_{\text{CR}}^*(\mathcal{X}; \mathcal{Q}) \oplus H_{\text{CR}}^*(\mathcal{X}'; \mathcal{Q}),$$

where

$$H_{\text{CR}}^*(\mathcal{X}; \mathcal{Q}) = \bigoplus_{h \in G} \mathcal{Q} \mathbf{1}_h, \quad H_{\text{CR}}^*(\mathcal{X}'; \mathcal{Q}) = \mathcal{Q} \mathbf{1}' \oplus \bigoplus_{k=1}^{m-1} \mathcal{Q} \mathbf{1}'_{k/m}.$$

We have the following isomorphisms of $\mathcal{Q}$ -vector spaces (which do not preserve the grading and the orbifold cup product):

$$\begin{aligned} H_{\text{CR}}^*(\mathcal{Y}_f; \mathcal{Q}) &\cong H_{\text{CR}}^*(\mathfrak{l}; \mathcal{Q}), & H_{\text{CR}}^*(\mathcal{X}; \mathcal{Q}) &\cong H_{\text{CR}}^*(\mathcal{B}G; \mathcal{Q}), \\ H_{\text{CR}}^*(\mathcal{X}'; \mathcal{Q}) &\cong H_{\text{CR}}^*(\mathcal{D}_f; \mathcal{Q}) \cong H_{\text{CR}}^*(\mathfrak{p}_\infty; \mathcal{Q}) = H_{\text{CR}}^*(\mathcal{B}\mu_m; \mathcal{Q}). \end{aligned}$$

5. A-model topological string

5.1 Relative orbifold Gromov–Witten invariants of $(\mathcal{Y}_f, \mathcal{D}_f)$

In this section, we define open-closed orbifold Gromov–Witten invariants of $\mathcal{X}$ as certain equivariant relative orbifold Gromov–Witten invariants of $(\mathcal{Y}_f, \mathcal{D}_f)$. Recall from Section 3.3 that the orbifold $\mathcal{Y}_f$ can be viewed as the total space of the rank 2 vector bundle $V = \mathcal{L}_2 \oplus \mathcal{L}_3 \rightarrow \mathfrak{l}$, where $\mathfrak{l}$ is a μ_m -gerbe over $\mathbb{P}(1, r)$, so equivariant relative orbifold Gromov–Witten invariants of $(\mathcal{Y}_f, \mathcal{D}_f)$ are twisted equivariant relative orbifold Gromov–Witten invariants of $(\mathfrak{l}, \mathfrak{p}_\infty)$.

Relative orbifold Gromov–Witten theory has been constructed in algebraic geometry by Abramovich–Fantechi [AF16] and in symplectic geometry by Chen–Li–Sun–Zhao [CLSZ11]. We refer to [AF16, CLSZ11] for the precise definitions of moduli spaces of relative stable maps to smooth DM stacks.

We first introduce some notation. We fix non-negative integers g and ℓ and a vector $\vec{\mu} = ((\mu_1, k_1), \dots, (\mu_n, k_n))$, where $\mu_j \in \mathbb{Z}_{>0}$ and $k_j \in \{0, 1, \dots, m-1\}$.

(1) Let $\overline{\mathcal{M}}_{g,\ell}(\mathcal{Y}_f/\mathcal{D}_f, \vec{\mu})$ and $\overline{\mathcal{M}}_{g,\ell}(\mathfrak{l}/\mathfrak{p}_\infty, \vec{\mu})$ be the moduli spaces of genus g relative stable maps to $(\mathcal{Y}_f, \mathcal{D}_f)$ and $(\mathfrak{l}, \mathfrak{p}_\infty)$, respectively, with ℓ marked points and n relative points, where the relative points are ordered, and the ramification index of the j th relative point is μ_j and its monodromy is k_j .

(2) Let $\text{ev}_i: \overline{\mathcal{M}}_{g,\ell}(\mathcal{Y}_f/\mathcal{D}_f, \vec{\mu}) \rightarrow \mathcal{I}\mathcal{Y}_f$ and $\text{ev}_i: \overline{\mathcal{M}}_{g,\ell}(\mathfrak{l}/\mathfrak{p}_\infty, \vec{\mu}) \rightarrow \mathcal{I}\mathfrak{l}$ be the evaluation maps at the i th marked point, where $i = 1, \dots, \ell$.

(3) Let $\pi: \mathcal{U} \rightarrow \overline{\mathcal{M}}_{g,\ell}(\mathfrak{l}/\mathfrak{p}_\infty, \vec{\mu})$ be the universal curve, $\mathcal{T} \rightarrow \overline{\mathcal{M}}_{g,\ell}(\mathfrak{l}/\mathfrak{p}_\infty, \vec{\mu})$ be the universal target, and $F: \mathcal{U} \rightarrow \mathcal{T}$ be the universal map.

(4) There is a contraction map $\mathcal{T} \rightarrow \mathfrak{l}$. Let $\hat{F}: \mathcal{U} \rightarrow \mathfrak{l}$ be the composition of this contraction map and the universal map $F: \mathcal{U} \rightarrow \mathcal{T}$.

(5) Let $D = \bigcup_{\substack{1 \leq j \leq n \\ k_j = 0}} D_j \subset \mathcal{U}$ be the universal relative points corresponding to $\vec{\mu}$, and let $D_0 = \bigcup_{1 \leq j \leq n} D_j \subset D$.

(6) Let $\mathbb{T}_f = \{(t, t^f, 1): t \in \mathbb{C}^*\} \subset \mathbb{T} \cong (\mathbb{C}^*)^3$. This induces

$$\phi_f: H^*(B\mathbb{T}; \mathbb{Q}) = \mathbb{Q}[u_1, u_2, u_3] = \mathbb{Q}[w_1, w_2, w_3] \rightarrow H^*(B\mathbb{T}_f; \mathbb{Q}) = \mathbb{Q}[v],$$

$$u_1 \mapsto v, \quad u_2 \mapsto fv, \quad u_3 \mapsto 0$$

or, equivalently,

$$w_1 \mapsto \frac{1}{r}v = w_1v, \quad w_2 \mapsto \frac{s+rf}{rm}v = w_2v, \quad w_3 \mapsto \frac{-s-rf-m}{rm}v = w_3v.$$

We extend ϕ_f to $\phi_f: \mathcal{Q} = \mathbb{C}(u_1, u_2, u_3) = \mathbb{C}(w_1, w_2, w_3) \rightarrow \mathbb{C}(v)$.

For any

$$\gamma_1, \dots, \gamma_\ell \in H_{\text{CR}}^*(\mathcal{X}; \mathcal{Q}) = \bigoplus_{h \in G} \mathcal{Q}1_h \subset H_{\text{CR}}^*(\mathcal{Y}_f; \mathcal{Q}) \cong H_{\text{CR}}^*(\mathfrak{l}; \mathcal{Q}),$$

we define the $\tilde{T}$ -equivariant relative orbifold Gromov–Witten invariant

$$\begin{aligned} \langle \gamma_1, \dots, \gamma_\ell \rangle_{g, \vec{\mu}}^{\mathcal{X}, (\mathcal{L}, f)} &:= \frac{1}{|\mathrm{Aut}(\vec{\mu})|} \cdot \phi_f \left(\int_{[\overline{\mathcal{M}}_{g, \ell}(\mathcal{X}_f / \mathcal{D}_f, \vec{\mu})]^{\tilde{T}} \mathrm{vir}} \frac{(\mathbf{u}_2 - f\mathbf{u}_1)^{\sum_{j=1}^n \delta_{k_j, 0}}}{e_{\tilde{T}}(N^{\mathrm{vir}})} \prod_{i=1}^{\ell} \mathrm{ev}_i^* \gamma_i \right) \\ &= \frac{1}{|\mathrm{Aut}(\vec{\mu})|} \cdot \phi_f \left(\int_{[\overline{\mathcal{M}}_{g, \ell}(\mathbb{I}/\mathfrak{p}_\infty, \vec{\mu})]^{\tilde{T}} \mathrm{vir}} \frac{e_{\tilde{T}}(-R\pi_*(\hat{F}^* \mathcal{L}_2(-D_0) \oplus \hat{F}^* \mathcal{L}_3))}{e_{\tilde{T}}(N^{\mathrm{vir}})} \prod_{i=1}^{\ell} \mathrm{ev}_i^* \gamma_i \right) \\ &\in \mathbb{Q}(\mathbf{v}). \end{aligned}$$

Note that $\phi_f(\mathbf{u}_2 - f\mathbf{u}_1) = 0$. The factor $(\mathbf{u}_2 - f\mathbf{u}_1)^{\sum_{j=1}^n \delta_{k_j, 0}}$ is included to cancel such a factor in $e_{\tilde{T}}(N^{\mathrm{vir}})$.

If $\gamma_1, \dots, \gamma_\ell$ are homogeneous, then

$$\langle \gamma_1, \dots, \gamma_\ell \rangle_{g, \vec{\mu}}^{\mathcal{X}, (\mathcal{L}, f)} = \mathbf{v}^{\sum_{i=1}^{\ell} (\deg \gamma_i / 2 - 1)} \cdot \langle \gamma_1, \dots, \gamma_\ell \rangle_{g, \vec{\mu}}^{\mathcal{X}, (\mathcal{L}, f)} \Big|_{\mathbf{v}=1}. \quad (5.1)$$

5.2 Descendant orbifold Gromov–Witten invariants of $\mathcal{X}$

In this subsection, we use localization to compute $\langle \gamma_1, \dots, \gamma_\ell \rangle_{g, \vec{\mu}}^{\mathcal{X}, (\mathcal{L}, f)}$ and obtain an expression in terms of cubic abelian Hurwitz–Hodge integrals.

We first give some definitions. For $i = 1, 2, 3$, let $\chi_i \in G^* = \mathrm{Hom}(G, \mathbb{C}^*)$ be the characters of G defined in Section 2.1, and let L_{χ_i} be the line bundle over $\mathcal{B}G$ associated with the irreducible character $\chi_i \in G^*$. Then $\mathcal{X} = [\mathbb{C}^3/G]$ is the total space of

$$\bigoplus_{i=1}^3 L_{\chi_i} \rightarrow \mathcal{B}G.$$

Let $\overline{\mathcal{M}}_{g, n}(\mathcal{B}G)$ denote the moduli space of genus g , n -pointed twisted stable maps to $\mathcal{B}G$. Let $\pi: \mathcal{C}_{g, n} \rightarrow \overline{\mathcal{M}}_{g, n}(\mathcal{B}G)$ be the universal curve, and let $F: \mathcal{C}_{g, n} \rightarrow \mathcal{B}G$ be the universal map. Define

$$\mathbf{e}_{g, n} := \phi_f \left(e_{\tilde{T}} \left(R\pi_* F^* \left(\bigoplus_{i=1}^3 L_{\chi_i} \right) \right) \right) = \prod_{i=1}^3 (w_i \mathbf{v})^{(\mathrm{rank} R\pi_* F^* L_{\chi_i})} c_{1/w_i \mathbf{v}}(R\pi_* F^* L_{\chi_i}), \quad (5.2)$$

where $c_t(E) = 1 + tc_1(E) + t^2 c_2(E) + \dots$ denotes the Chern polynomial of a complex vector bundle E (or, more generally, an element in the K -theory). The rank of $R\pi_* F^* L_{\chi_i}$ is constant on a connected component of $\overline{\mathcal{M}}_{g, n}(\mathcal{B}G)$ but might be different on different connected components of $\overline{\mathcal{M}}_{g, n}(\mathcal{B}G)$.

A moduli point in $\overline{\mathcal{M}}_{g, \ell}(\mathbb{I}/\mathfrak{p}_\infty, \vec{\mu})$ is represented by a morphism

$$\rho: (\mathcal{C}, x_1, \dots, x_\ell, y_1, \dots, y_n) \rightarrow \mathbb{I}[k] = \mathbb{I} \cup \Delta_1 \cup \dots \cup \Delta_k,$$

where each Δ_i is a trivial μ_m -gerbe over $\mathbb{P}^1$. Let

$$\overline{\mathcal{M}}_{g, \ell}^0(\mathbb{I}/\mathfrak{p}_\infty, \vec{\mu}) \subset \overline{\mathcal{M}}_{g, \ell}(\mathbb{I}/\mathfrak{p}_\infty, \vec{\mu})$$

be the substack where the target is $\mathbb{I}[0] = \mathbb{I}$. Computations in [RZ13] show that the contribution from a $\tilde{T}$ -fixed locus outside $\overline{\mathcal{M}}_{g, \ell}^0(\mathbb{I}/\mathfrak{p}_\infty, \vec{\mu})$ vanishes. Therefore,

$$\langle \gamma_1, \dots, \gamma_\ell \rangle_{g, \vec{\mu}}^{\mathcal{X}, (\mathcal{L}, f)} = \frac{1}{|\mathrm{Aut}(\vec{\mu})|} \cdot \phi_f \left(\int_{[\overline{\mathcal{M}}_{g, \ell}^0(\mathbb{I}/\mathfrak{p}_\infty, \vec{\mu})]^{\tilde{T}} \mathrm{vir}} \frac{e_{\tilde{T}}(-R\pi_*(\hat{F}^* \mathcal{L}_2(-D_0) \oplus \hat{F}^* \mathcal{L}_3))}{e_{\tilde{T}}(N^{\mathrm{vir}})} \prod_{i=1}^{\ell} \mathrm{ev}_i^* \gamma_i \right).$$

Suppose that $\rho: (\mathcal{C}, x_1, \dots, x_\ell, y_1, \dots, y_n) \rightarrow \mathfrak{l}$ represents a point in $\overline{\mathcal{M}}_{g,n+\ell}^0(\mathfrak{l}/\mathfrak{p}_\infty, \vec{\mu})^T$. Then

$$\mathcal{C} = \mathcal{C}_0 \cup \bigcup_{j=1}^n \mathcal{C}_j,$$

where $\mathcal{C}_0$ is an orbicurve of genus g , the components $\mathcal{C}_1, \dots, \mathcal{C}_n$ are 1-dimensional toric orbifolds, $x_1, \dots, x_\ell \in \mathcal{C}_0$ and $y_j \in \mathcal{C}_j$, and $\mathcal{C}_0$ and $\mathcal{C}_j$ intersect at a node z_j for $j = 1, \dots, n$. We view z_j as a point on $\mathcal{C}_j$ in order to determine its monodromy. Let $\rho_j := \rho|_{\mathcal{C}_j}: \mathcal{C}_j \rightarrow \mathfrak{l}$. Then ρ_0 is a constant map to $\mathfrak{p}_0 = [0, 1]$, and we have the following diagram for $1 \leq j \leq n$:

$$\begin{array}{ccc} \mathcal{C}_j & \xrightarrow{\rho_j} & \mathfrak{l} \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \xrightarrow{\bar{\rho}_j} & \mathbb{P}^1, \end{array} \quad (5.3)$$

where $\bar{\rho}_j([x, y]) = [x^{\mu_j}, y^{\mu_j}]$ is the map between coarse moduli spaces, the monodromy around y_j is $e^{2\pi\sqrt{-1}k_j/m} \in \boldsymbol{\mu}_m$, and the monodromy around z_j is

$$\left(e^{2\pi\sqrt{-1}\mu_j w_1}, e^{2\pi\sqrt{-1}(\mu_j w_2 - k_j/m)}, e^{2\pi\sqrt{-1}(\mu_j w_3 + k_j/m)} \right) = \eta_1^{\mu_j} \eta_2^{-k_j} \in G.$$

For $(d_0, k) \in \mathbb{Z} \times \mathbb{Z}_m$, define $h(d_0, k) := \eta_1^{d_0} \eta_2^{-k} \in G$, and define $D'(d_0, k)$ to be

$$-\frac{r}{\mathbf{v}}(-1)^{\lfloor d_0 w_3 + k/m \rfloor} \left(\frac{\mathbf{v}}{d_0} \right)^{\text{age}(h(d_0, k)) - 1} \cdot \frac{\Gamma(d_0(w_1 + w_2) + c_3(h(d_0, k)))}{\Gamma(d_0 w_1 - c_1(h(d_0, k)) + 1) \Gamma(d_0 w_2 - c_2(h(d_0, k)) + 1)}. \quad (5.4)$$

Then $D'(d_0, k) \in \mathbb{C} \mathbf{v}^{\text{age}(h(d_0, k)) - 2}$. Up to the sign, $D'(d_0, k)$ is essentially $|G|$ times the disk function in [Ros14, Section 3.3]. Note that

$$(-1)^{\lfloor d_0 w_3 + k/m \rfloor} = (-1)^{d_0 w_3 - c_3(h(d_0, k)) + k/m}.$$

By virtual localization, we have the following result.

PROPOSITION 5.1. *We have*

$$\begin{aligned} \langle \rangle_{0, (d_0, k)}^{\mathcal{X}, (\mathcal{L}, f)} &= \frac{\delta_{1, h(d_0, k)}}{|G|} D'(d_0, k) \cdot \left(\frac{\mathbf{v}}{d_0} \right)^2, \\ \langle \gamma \rangle_{0, (d_0, k)}^{\mathcal{X}, (\mathcal{L}, f)} &= \langle \gamma, \mathbf{e}_{h(d_0, k)} \mathbf{1}_{h(d_0, k)^{-1}} \rangle_{\mathcal{X}} \cdot D'(d_0, k) \cdot \frac{\mathbf{v}}{d_0}, \\ \langle \rangle_{0, (\mu_1, k_1), (\mu_2, k_2)}^{\mathcal{X}, (\mathcal{L}, f)} &= \langle \mathbf{e}_{h(\mu_1, k_1)} \mathbf{1}_{h(\mu_1, k_1)^{-1}}, \mathbf{e}_{h(\mu_2, k_2)} \mathbf{1}_{h(\mu_2, k_2)^{-1}} \rangle_{\mathcal{X}} \\ &\quad \cdot \frac{D'(\mu_1, k_1) D'(\mu_2, k_2)}{|\text{Aut}((\mu_1, k_1), (\mu_2, k_2))|} \frac{\mathbf{v}}{\mu_1 + \mu_2}. \end{aligned}$$

If $\vec{\mu} = ((\mu_1, k_1), \dots, (\mu_n, k_n))$ and $2g - 2 + \ell + n > 0$, then

$$\langle \gamma_1, \dots, \gamma_\ell \rangle_{g, \vec{\mu}}^{\mathcal{X}, (\mathcal{L}, f)} = \frac{1}{|\text{Aut}(\vec{\mu})|} \prod_{j=1}^n D'(\mu_j, k_j) \int_{\overline{\mathcal{M}}_{g, \ell+n}(BG)} \frac{\prod_{i=1}^\ell \text{ev}_i^* \gamma_i}{\mathbf{e}_{g, \ell+n}} \prod_{j=1}^n \frac{\text{ev}_{\ell+j}^* (\mathbf{e}_{h(\mu_j, k_j)} \mathbf{1}_{h(\mu_j, k_j)^{-1}})}{1 - (\mu_j / \mathbf{v}) \bar{\psi}_{\ell+j}}.$$

Introduce variables $X_1, \dots, X_n$, and let

$$\boldsymbol{\tau} = \sum_{a=1}^p \tau_a \mathbf{1}_{h_a},$$

where $h_a \in G$ corresponds to $(m_a, n_a, 1) \in \{v \in \text{Box}(\sigma) : \text{age}(v) = 1\}$. For $k \in \{0, 1, \dots, m-1\}$, let $\mathbf{1}'_{-k/m} := \mathbf{1}'_{1-\delta_{0,k}-k/m} \in H_{\text{CR}}^*(\mathcal{B}\mu_m; \mathbb{C}) = \bigoplus_{k=0}^{m-1} \mathbb{C}\mathbf{1}'_{k/m}$. Define

$$F_{g,n}^{\mathcal{X},(\mathcal{L},f)}(\tau; X_1, \dots, X_n) = \sum_{\mu_1, \dots, \mu_n > 0} \sum_{k_1, \dots, k_n=0}^{m-1} \sum_{\ell \geq 0} \frac{\langle \tau^\ell \rangle_{g,(\mu_1,k_1), \dots, (\mu_n,k_n)}^{\mathcal{X},(\mathcal{L},f)}}{\ell!} \cdot \prod_{j=1}^n (X_j)^{\mu_j} \cdot (-1)^{-k_1/m} \mathbf{1}'_{-k_1/m} \otimes \dots \otimes (-1)^{-k_n/m} \mathbf{1}'_{-k_n/m},$$

which is an $H_{\text{CR}}^*(\mathcal{B}\mu_m; \mathbb{C})^{\otimes n}$ -valued function, where $H_{\text{CR}}^*(\mathcal{B}\mu_m; \mathbb{C}) = \bigoplus_{k=0}^{m-1} \mathbb{C}\mathbf{1}'_{k/m}$.

Remark 5.2. For each $k \in \{0, 1, \dots, m-1\}$, the set $\{\mu_j : k_j = k\}$ determines a partition $\mu^k = (\mu_1^k \geq \dots \geq \mu_{\ell(\mu^k)}^k > 0)$. We have

$$\text{Aut}((\mu_1, k_1), \dots, (\mu_n, k_n)) = \prod_{k=0}^{m-1} \text{Aut}(\mu^k),$$

$$\ell(\mu^0) + \dots + \ell(\mu^{m-1}) = n.$$

The correlator $\langle \tau^\ell \rangle_{g,(\mu_1,k_1), \dots, (\mu_n,k_n)}^{\mathcal{X},(\mathcal{L},f)}$ is invariant under permutation of $(\mu_1, k_1), \dots, (\mu_n, k_n)$, so it depends only on the m -tuple of partitions $\mu^0, \dots, \mu^{m-1}$. So $F_{g,n}^{\mathcal{X},(\mathcal{L},f)}(\tau; X_1, \dots, X_n)$ is a symmetric function in $X_1, \dots, X_n$ and can be rewritten as

$$F_{g,n}^{\mathcal{X},(\mathcal{L},f)}(\tau; X_1, \dots, X_n) = \sum_{\substack{\mu^0, \dots, \mu^{m-1} \in \mathcal{P} \\ \ell(\mu^0) + \dots + \ell(\mu^{m-1}) = n}} \sum_{\ell \geq 0} \frac{\langle \tau^\ell \rangle_{g, \mu^0, \dots, \mu^{m-1}}^{\mathcal{X},(\mathcal{L},f)}}{\ell!} X_{\mu^0, \dots, \mu^{m-1}},$$

where $\mathcal{P}$ is the set of partitions and

$$X_{\mu^0, \dots, \mu^{m-1}} = \sum_{\sigma \in S_n} \prod_{k=0}^{m-1} \prod_{j=\ell(\mu^0)+\dots+\ell(\mu^{k-1})+1}^{\ell(\mu^0)+\dots+\ell(\mu^k)} X_{\sigma(j)}^{\mu_j^k} ((-1)^{-k/m} \mathbf{1}'_{-k/m})^{\otimes \ell(\mu^k)}.$$

We introduce some notation.

(1) Given $h \in G$, define

$$\begin{aligned} \Phi_0^h(X) &:= \frac{1}{|G|} \sum_{\substack{(d_0, k) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_m \\ h(d_0, k) = h}} D'(d_0, k) X^{d_0} (-1)^{-k/m} \mathbf{1}'_{-k/m} \\ &= - \sum_{\substack{(d_0, k) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_m \\ h(d_0, k) = h}} \frac{1}{m\nu} e^{\sqrt{-1}\pi(d_0 w_3 - c_3(h))} \left(\frac{\nu}{d_0} \right)^{\text{age}(h)-1} \\ &\quad \cdot \frac{\Gamma(d_0(w_1 + w_2) + c_3(h))}{\Gamma(d_0 w_1 - c_1(h) + 1) \Gamma(d_0 w_2 - c_2(h) + 1)} X^{d_0} \mathbf{1}'_{-k/m}. \end{aligned}$$

Then $\Phi_0^h(X)$ takes values in $\bigoplus_{k=0}^{m-1} \mathbb{C}\nu^{\text{age}(v)-2} \mathbf{1}'_{k/m}$. For $a \in \mathbb{Z}$ and $h \in G$, we define

$$\Phi_a^h(X) := \frac{1}{|G|} \sum_{\substack{d_0 > 0 \\ h(d_0, k) = h}} D'(d_0, k) \left(\frac{d_0}{\nu} \right)^a X^{d_0} (-1)^{-k/m} \mathbf{1}'_{-k/m}.$$

Then $\Phi_a^h(X)$ takes values in $\bigoplus_{k=0}^{m-1} \mathbb{C} \mathbf{v}^{\text{age}(v)-2-a} \mathbf{1}'_{k/m}$, and

$$\Phi_{a+1}^h(X) = \left(\frac{1}{\mathbf{v}} X \frac{d}{dX} \right) \Phi_a^h(X).$$

(2) For $a \in \mathbb{Z}$ and $\gamma \in G^*$, we define

$$\tilde{\xi}_a^\gamma(X) := |G| \sum_{h \in G} \chi_\gamma(h^{-1}) \left(\prod_{i=1}^3 (w_i \mathbf{v})^{1-c_i(h)} \right) \Phi_a^h(X).$$

Then $\tilde{\xi}_a^\gamma(X)$ takes values in $\bigoplus_{k=0}^{m-1} \mathbb{C} \mathbf{v}^{1-a} \mathbf{1}'_{k/m}$.

(3) Given $\gamma_1, \gamma_2 \in H_{\text{CR}}^*(\mathcal{X}; \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}(\mathbf{v})$ and $a_1, a_2 \in \mathbb{Z}_{\geq 0}$, define

$$\begin{aligned} \langle \tau_{a_1}(\gamma_1) \rangle_{0,1}^{\mathcal{X}, \mathbb{T}_f} &= \delta_{a_1,0} \langle 1, \gamma_1 \rangle_{\mathcal{X}} \cdot \mathbf{v}^2, \\ \langle \tau_{a_1}(\gamma_1) \tau_{a_2}(\gamma_2) \rangle_{0,2}^{\mathcal{X}, \mathbb{T}_f} &= \delta_{a_1,0} \delta_{a_2,0} \cdot \mathbf{v} \langle \gamma_1, \gamma_2 \rangle_{\mathcal{X}}. \end{aligned}$$

(4) Given $\gamma_1, \dots, \gamma_n \in H_{\text{CR}}^*(\mathcal{X}; \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}(\mathbf{v})$ and $a_1, \dots, a_n \in \mathbb{Z}_{\geq 0}$, define

$$\langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \rangle_{g,n}^{\mathcal{X}, \mathbb{T}_f} = \int_{\overline{\mathcal{M}}_{g,n}(\mathcal{B}G)} \frac{1}{\mathbf{e}_{g,n}} \prod_{i=1}^n (\text{ev}_i^*(\gamma_i) \bar{\psi}_i^{a_i}).$$

(5) Given $h \in G$, define $\mathbf{1}_h^* = |G| \phi_f(\mathbf{e}_h \mathbf{1}_{h^{-1}})$.

(6) In the rest of this paper, we consider $\mathbb{T}_f$ -equivariant cohomology and write ϕ_γ and $\bar{\phi}_\gamma$ instead of $\phi_f(\phi_\gamma)$ and $\phi_f(\bar{\phi}_\gamma)$, respectively. (Recall that $\phi_f(\mathbf{w}_i) = w_i \mathbf{v}$.)

PROPOSITION 5.3. *We have the following equations:*

(i) (Disk invariants)

$$\begin{aligned} F_{0,1}^{\mathcal{X}, (\mathcal{L}, f)}(\boldsymbol{\tau}; X) &= \Phi_{-2}^1(X) + \sum_{a=1}^p \tau_a \Phi_{-1}^{h_a}(X) + \sum_{a \in \mathbb{Z}_{\geq 0}} \sum_{h \in G} \sum_{\ell \geq 2} \frac{\langle \boldsymbol{\tau}^\ell, \tau_a(\mathbf{1}_h^*) \rangle_{0, \ell+1}^{\mathcal{X}, \mathbb{T}_f}}{\ell!} \Phi_a^h(X) \\ &= \frac{1}{|G|^2 w_1 w_2 w_3} \left(\sum_{\gamma \in G^*} \tilde{\xi}_{-2}^\gamma(X) + \sum_{a=1}^p \tau_a \prod_{i=1}^3 w_i^{c_i(h_a)} \sum_{\gamma \in G^*} \chi_\gamma(h_a) \tilde{\xi}_{-1}^\gamma(X) \right) \Big|_{\mathbf{v}=1} \\ &\quad + \sum_{a \in \mathbb{Z}_{\geq 0}} \sum_{\gamma \in G^*} \sum_{\ell \geq 2} \frac{\langle \boldsymbol{\tau}^\ell, \tau_a(\bar{\phi}_\gamma) \rangle_{0, \ell+1}^{\mathcal{X}, \mathbb{T}_f}}{\ell!} \tilde{\xi}_a^\gamma(X). \end{aligned}$$

(ii) (Annulus invariants)

$$\begin{aligned} F_{0,2}^{\mathcal{X}, (\mathcal{L}, f)}(\boldsymbol{\tau}; X_1, X_2) - F_{0,2}^{\mathcal{X}, (\mathcal{L}, f)}(0; X_1, X_2) \\ &= \sum_{a_1, a_2 \in \mathbb{Z}_{\geq 0}} \sum_{h_1, h_2 \in G} \sum_{\ell \geq 1} \frac{\langle \boldsymbol{\tau}^\ell, \tau_{a_1}(\mathbf{1}_{h_1}^*) \tau_{a_2}(\mathbf{1}_{h_2}^*) \rangle_{0, \ell+2}^{\mathcal{X}, \mathbb{T}_f}}{\ell!} \Phi_{a_1}^{h_1}(X_1) \Phi_{a_2}^{h_2}(X_2) \\ &= \sum_{a_1, a_2 \in \mathbb{Z}_{\geq 0}} \sum_{\gamma_1, \gamma_2 \in G^*} \sum_{\ell \geq 1} \frac{\langle \boldsymbol{\tau}^\ell, \tau_{a_1}(\bar{\phi}_{\gamma_1}) \tau_{a_2}(\bar{\phi}_{\gamma_2}) \rangle_{0, \ell+2}^{\mathcal{X}, \mathbb{T}_f}}{\ell!} \tilde{\xi}_{a_1}^{\gamma_1}(X_1) \tilde{\xi}_{a_2}^{\gamma_2}(X_2), \end{aligned}$$

where

$$\begin{aligned} \left(X_1 \frac{\partial}{\partial X_1} + X_2 \frac{\partial}{\partial X_2} \right) F_{0,2}^{\mathcal{X},(\mathcal{L},f)}(0; X_1, X_2) &= |G| \left(\sum_{h \in G} \mathbf{e}_h \Phi_0^h(X_1) \Phi_0^{h^{-1}}(X_2) \right) \Big|_{v=1} \\ &= \frac{1}{|G|^2 w_1 w_2 w_3} \left(\sum_{\gamma \in G^*} \tilde{\xi}_0^\gamma(X_1) \tilde{\xi}_0^\gamma(X_2) \right) \Big|_{v=1}. \end{aligned} \quad (5.5)$$

(iii) For $2g - 2 + n > 0$,

$$\begin{aligned} F_{g,n}^{\mathcal{X},(\mathcal{L},f)}(\boldsymbol{\tau}; X_1, \dots, X_n) &= \sum_{a_1, \dots, a_n \in \mathbb{Z}_{\geq 0}} \sum_{h_1, \dots, h_n \in G} \sum_{\ell \geq 0} \frac{\langle \boldsymbol{\tau}^\ell, \tau_{a_1}(\mathbf{1}_{h_1}^*) \cdots \tau_{a_n}(\mathbf{1}_{h_n}^*) \rangle_{g, \ell+n}^{\mathcal{X}, \mathbb{T}_f}}{\ell!} \prod_{j=1}^n \Phi_{a_j}^{h_j}(X_j) \\ &= \sum_{a_1, \dots, a_n \in \mathbb{Z}_{\geq 0}} \sum_{\gamma_1, \dots, \gamma_n \in G^*} \sum_{\ell \geq 0} \frac{\langle \boldsymbol{\tau}^\ell, \tau_{a_1}(\bar{\phi}_{\gamma_1}) \cdots \tau_{a_n}(\bar{\phi}_{\gamma_n}) \rangle_{g, \ell+n}^{\mathcal{X}, \mathbb{T}_f}}{\ell!} \prod_{j=1}^n \tilde{\xi}_{a_j}^{\gamma_j}(X_j). \end{aligned}$$

Remark 5.4. The function $F_{0,2}^{\mathcal{X},(\mathcal{L},f)}(0; X_1, X_2)$ is an $H^*(\mathcal{B}\boldsymbol{\mu}_m; \mathbb{C})^{\otimes 2}$ -valued power series in X_1 and X_2 which vanishes at $(X_1, X_2) = (0, 0)$, so it is determined by (5.5).

5.3 A-model graph sum

Introduce formal variables

$$\hat{\mathbf{u}} = \sum_{a \geq 0} \hat{u}_a z^a = \sum_{a \geq 0} \sum_{\beta \in G^*} u_a^\beta \bar{\phi}_\beta z^a = \sum_{\beta \in G^*} \mathbf{u}^\beta(z) \bar{\phi}_\beta,$$

where

$$u_a^\beta \in \mathcal{Q}, \quad \hat{u}_a = \sum_{\beta \in G^*} u_a^\beta \bar{\phi}_\beta \in H_{\text{CR}}^*(\mathcal{X}; \mathcal{Q}), \quad \mathbf{u}^\beta(z) = \sum_{a \geq 0} u_a^\beta z^a.$$

For $j = 1, \dots, n$, define $\hat{\mathbf{u}}_j$ and $\mathbf{u}_j^\beta(z)$ similarly. Define

$$\langle \hat{\mathbf{u}}^\ell, \hat{\mathbf{u}}_1, \dots, \hat{\mathbf{u}}_n \rangle_{g, \ell+n}^{\mathcal{X}, \mathbb{T}_f} = \sum_{a_1, \dots, a_\ell, b_1, \dots, b_n \in \mathbb{Z}_{\geq 0}} \langle \tau_{a_1}(\hat{u}_{a_1}) \cdots \tau_{a_\ell}(\hat{u}_{a_\ell}) \tau_{b_1}((\hat{u}_1)_{b_1}) \cdots \tau_{b_n}((\hat{u}_n)_{b_n}) \rangle_{g, \ell+n}^{\mathcal{X}, \mathbb{T}_f}.$$

By [FLZ16, Theorem 4.2], the term $(1/\ell!) \langle \hat{\mathbf{u}}^\ell, \hat{\mathbf{u}}_1, \dots, \hat{\mathbf{u}}_n \rangle_{g, \ell+n}^{\mathcal{X}, \mathbb{T}_f}$ can be written as a graph sum. By Proposition 5.3,

$$F_{g,n}^{\mathcal{X},(\mathcal{L},f)}(\boldsymbol{\tau}; X_1, \dots, X_n) = \sum_{\ell \geq 0} \frac{1}{\ell!} \langle \boldsymbol{\tau}^\ell, \tilde{\xi}(X_1), \dots, \tilde{\xi}(X_n) \rangle_{g, \ell+n}^{\mathcal{X}, \mathbb{T}_f}, \quad (5.6)$$

where $\tilde{\xi}(X) = \sum_{a \in \mathbb{Z}_{\geq 0}} \sum_{\beta \in G^*} \tilde{\xi}_a^\beta(X) \bar{\phi}_\beta z^a$. Therefore, a graph sum of $F_{g,n}^{\mathcal{X},(\mathcal{L},f)}(\boldsymbol{\tau}; X_1, \dots, X_n)$ can be obtained by the following specialization of [FLZ16, Theorem 2]:

$$\hat{\mathbf{u}} = \boldsymbol{\tau} = \sum_{a=1}^p \tau_a \mathbf{1}_{h_a}, \quad \hat{\mathbf{u}}_j = \tilde{\xi}(X_j), \quad w_i = w_i v.$$

By (5.1), we may let $v = 1$; that is, $w_i = w_i$. In the rest of this subsection, we give the precise statement of this graph sum.

Given a connected graph Γ , we introduce the following notation:

- (1) $V(\Gamma)$ is the set of vertices in Γ .

- (2) $E(\Gamma)$ is the set of edges in Γ .
- (3) $H(\Gamma)$ is the set of half-edges in Γ .
- (4) $L^O(\Gamma)$ is the set of open leaves in Γ . The open leaves are ordered: $L^O(\Gamma) = \{l_1, \dots, l_n\}$, where n is the number of open leaves. (Open leaves correspond to *ordered* ordinary leaves in [FLZ16].)
- (5) $L^o(\Gamma)$ is the set of primary leaves in Γ . The primary leaves are unordered. (Primary leaves correspond to *unordered* ordinary leaves in [FLZ16].)
- (6) $L^1(\Gamma)$ is the set of dilaton leaves in Γ . The dilaton leaves are not ordered.

With the above notation, we introduce the following labels:

- (1) (genus) $g: V(\Gamma) \rightarrow \mathbb{Z}_{\geq 0}$,
- (2) (marking) $\alpha: V(\Gamma) \rightarrow G^*$ (This induces

$$\alpha: L(\Gamma) = L^O(\Gamma) \cup L^o(\Gamma) \cup L^1(\Gamma) \rightarrow G^*$$

as follows: if $l \in L(\Gamma)$ is a leaf attached to a vertex $v \in V(\Gamma)$, define $\alpha(l) = \alpha(v)$.),

- (3) (height) $k: H(\Gamma) \rightarrow \mathbb{Z}_{\geq 0}$.

Given a vertex $v \in V(\Gamma)$, let $H(v)$ denote the set of half-edges emanating from v . The valency $\text{val}(v)$ of the vertex v is equal to the size $|H(v)|$ of the set $H(v)$. A labeled graph $\vec{\Gamma} = (\Gamma, g, \alpha, k)$ is *stable* if $2g(v) - 2 + \text{val}(v) > 0$ for all $v \in V(\Gamma)$.

Let $\mathbf{\Gamma}(\mathcal{BG})$ denote the set of all stable labeled graphs $\vec{\Gamma} = (\Gamma, g, \alpha, k)$. The genus of a stable labeled graph $\vec{\Gamma}$ is defined to be

$$g(\vec{\Gamma}) := \sum_{v \in V(\Gamma)} g(v) + |E(\Gamma)| - |V(\Gamma)| + 1 = \sum_{v \in V(\Gamma)} (g(v) - 1) + \sum_{e \in E(\Gamma)} 1 + 1.$$

Define

$$\mathbf{\Gamma}_{g, \ell, n}(\mathcal{BG}) = \{ \vec{\Gamma} = (\Gamma, g, \alpha, k) \in \mathbf{\Gamma}(\mathcal{BG}) : g(\vec{\Gamma}) = g, |L^o(\Gamma)| = \ell, |L^O(\Gamma)| = n \}.$$

Let

$$R(z) = \prod_{i=1}^3 \exp \left(\sum_{m \geq 1} \frac{(-1)^m}{m(m+1)} A_{m+1}^i \left(\frac{z}{w_i} \right)^m \right),$$

where A_m^i is the operator defined by

$$A_m^i : H^*(\mathcal{IBG}; \mathbb{C}) \rightarrow H^*(\mathcal{IBG}; \mathbb{C}), \quad \mathbf{1}_h \mapsto B_m(c_i(h)) \mathbf{1}_h$$

and $B_m(x)$ is the Bernoulli polynomial. Then, relative to the basis $\{\phi_\gamma : \gamma \in G^*\}$,

$$R(z)_\beta^\alpha = \frac{1}{|G|} \sum_{h \in G} \chi_\alpha(h) \chi_\beta(h^{-1}) \prod_{i=1}^3 \exp \left(\sum_{m=1}^{\infty} \frac{(-1)^m}{m(m+1)} B_{m+1}(c_i(h)) \left(\frac{z}{w_i} \right)^m \right). \quad (5.7)$$

Given $\beta \in G^*$, define

$$\tilde{\xi}^\beta(z, X) = \sum_{a \geq 0} z^a \tilde{\xi}_a^\beta(X) \Big|_{v=1}.$$

We assign weights to leaves, edges, and vertices of a labeled graph $\vec{\Gamma} \in \mathbf{\Gamma}(\mathcal{Y})$ as follows:

(1) (*Open leaves*) To each open leaf $l_j \in L^O(\Gamma)$ with $\alpha(l_j) = \alpha \in G^*$ and $k(l_j) = k \in \mathbb{Z}_{\geq 0}$, we assign

$$(\mathcal{L}^{\tilde{\xi}})_k^\alpha(l_j) = [z^k] \left(\frac{1}{|G|\sqrt{w_1 w_2 w_3}} \sum_{\beta \in G^*} R(-z)_\alpha^{\tilde{\xi}^\beta}(z, X_j) \right).$$

(2) (*Primary leaves*) To each primary leaf $l \in L^O(\Gamma)$ with $\alpha(l) = \alpha \in G^*$ and $k(l) = k \in \mathbb{Z}_{\geq 0}$, we assign

$$(\mathcal{L}^\tau)_k^\alpha(l) = [z^k] \left(\frac{1}{|G|\sqrt{w_1 w_2 w_3}} \sum_{\beta \in G^*} \sum_{a=1}^p \prod_{i=1}^3 w_i^{c_i(h_a)} R(-z)_\alpha^\beta \chi_\beta(h_a) \tau_a \right). \quad (5.8)$$

By the orthogonality of the characters, we have

$$\begin{aligned} & \sum_{\beta \in G^*} \sum_{a=1}^p \prod_{i=1}^3 w_i^{c_i(h_a)} R(-z)_\alpha^\beta \chi_\beta(h_a) \tau_a \\ &= \sum_{a=1}^p \prod_{i=1}^3 w_i^{c_i(h_a)} \chi_\alpha(h_a) \tau_a \exp \left(\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m(m+1)} B_{m+1}(c_i(h_a)) \left(\frac{z}{w_i} \right)^m \right). \end{aligned}$$

Here we used the facts that $B_m(1-x) = (-1)^m B_m(x)$, that $B_m = 0$ when m is odd, and that $c_i(h) + c_i(h^{-1}) = 1 - \delta_{c_i(h), 0}$.

(3) (*Dilaton leaves*) To each dilaton leaf $l \in L^1(\Gamma)$ with $\alpha(l) = \alpha \in G^*$ and $2 \leq k(l) = k \in \mathbb{Z}_{\geq 0}$, we assign

$$(\mathcal{L}^1)_k^\alpha(l) = [z^{k-1}] \left(- \frac{1}{|G|\sqrt{w_1 w_2 w_3}} \sum_{\beta \in G^*} R(-z)_\alpha^\beta \right).$$

(4) (*Edges*) To an edge connecting a vertex marked by $\alpha \in G^*$ and a vertex marked by $\beta \in G^*$, and with heights k and l at the corresponding half-edges, we assign

$$\mathcal{E}_{k,l}^{\alpha,\beta}(e) = [z^k w^l] \left(\frac{1}{z+w} \left(\delta_{\alpha,\beta} - \sum_{\gamma \in G^*} R(-z)_\alpha^\gamma R(-w)_\beta^\gamma \right) \right).$$

(5) (*Vertices*) To a vertex v with genus $g(v) = g \in \mathbb{Z}_{\geq 0}$ and with marking $\alpha(v) = \gamma \in G^*$, with n primary or open leaves and half-edges attached to it with heights $k_1, \dots, k_n \in \mathbb{Z}_{\geq 0}$ and m dilaton leaves with heights $k_{n+1}, \dots, k_{n+m} \in \mathbb{Z}_{\geq 0}$, we assign

$$(|G|\sqrt{w_1 w_2 w_3})^{2g-2+\text{val}(v)} \int_{\mathcal{M}_{g,n+m}} \psi_1^{k_1} \dots \psi_{n+m}^{k_{n+m}}.$$

Given a labeled graph $\vec{\Gamma} \in \mathbf{\Gamma}_{g,\ell,n}(\mathcal{BG})$ with $L^O(\Gamma) = \{l_1, \dots, l_n\}$, we define its A-model weight to be

$$\begin{aligned} w_A(\vec{\Gamma}) &= \prod_{v \in V(\Gamma)} (|G|\sqrt{w_1 w_2 w_3})^{2g-2+\text{val}(v)} \left\langle \prod_{h \in H(v)} \tau_{k(h)} \right\rangle_{g(v)} \prod_{e \in E(\Gamma)} \mathcal{E}_{k(h_1(e)), k(h_2(e))}^{\alpha(v_1(e)), \alpha(v_2(e))}(e) \\ &\cdot \prod_{l \in L^O(\Gamma)} (\mathcal{L}^\tau)_{k(l)}^{\alpha(l)}(l) \prod_{j=1}^n (\mathcal{L}^{\tilde{\xi}})_{k(l_j)}^{\alpha(l_j)}(l_j) \prod_{l \in L^1(\Gamma)} (\mathcal{L}^1)_{k(l)}^{\alpha(l)}(l). \end{aligned} \quad (5.9)$$

Setting $\hat{\mathbf{u}} = \boldsymbol{\tau}$, $\hat{\mathbf{u}}_j = \tilde{\xi}(X_j)$, and $w_i = w_i$ in [FLZ16, Theorem 4.2], we obtain

$$\frac{1}{\ell!} \langle \boldsymbol{\tau}^\ell, \tilde{\xi}(X_1), \dots, \tilde{\xi}(X_n) \rangle_{g, \ell+n}^{\mathcal{X}} = \sum_{\vec{\Gamma} \in \Gamma_{g, \ell, n}(\mathcal{B}G)} \frac{w_A(\vec{\Gamma})}{|\text{Aut}(\vec{\Gamma})|}. \quad (5.10)$$

Let $\Gamma_{g, n}(\mathcal{X}) := \bigcup_{\ell \geq 0} \Gamma_{g, \ell, n}(\mathcal{B}G)$. Equations (5.6) and (5.10) and Proposition 5.3 imply the following A-model graph sum formulae.

THEOREM 5.5 (A-model graph sum). *We have the following equations:*

(i) (Disk invariants)

$$F_{0,1}^{\mathcal{X}, (\mathcal{L}, f)}(\boldsymbol{\tau}; X) = \Phi_{-2}^1(X) + \sum_{a=1}^p \tau_a \Phi_{-1}^{h_a}(X) + \sum_{\vec{\Gamma} \in \Gamma_{0,1}(\mathcal{X})} \frac{w_A(\vec{\Gamma})}{|\text{Aut}(\vec{\Gamma})|},$$

where

$$\begin{aligned} & \Phi_{-2}^1(X) + \sum_{a=1}^p \tau_a \Phi_{-1}^{h_a}(X) \\ &= \frac{1}{|G|^2 w_1 w_2 w_3} \left(\sum_{\gamma \in G^*} \tilde{\xi}_{-2}^\gamma(X) + \sum_{a=1}^p \tau_a \prod_{i=1}^3 w_i^{c_i(h_a)} \sum_{\gamma \in G^*} \chi_\gamma(h_a) \tilde{\xi}_{-1}^\gamma(X) \right) \Big|_{v=1}. \end{aligned} \quad (5.11)$$

(ii) (Annulus invariants)

$$F_{0,2}^{\mathcal{X}, (\mathcal{L}, f)}(\boldsymbol{\tau}; X_1, X_2) = F_{0,2}^{\mathcal{X}, (\mathcal{L}, f)}(0; X_1, X_2) + \sum_{\vec{\Gamma} \in \Gamma_{0,2}(\mathcal{X})} \frac{w_A(\vec{\Gamma})}{|\text{Aut}(\vec{\Gamma})|},$$

where $F_{0,2}^{\mathcal{X}, (\mathcal{L}, f)}(0; X_1, X_2)$ is determined by equation (5.5).

(iii) For $2g - 2 + n > 0$,

$$F_{g,n}^{\mathcal{X}, (\mathcal{L}, f)}(\boldsymbol{\tau}; X_1, \dots, X_n) = \sum_{\vec{\Gamma} \in \Gamma_{g,n}(\mathcal{X})} \frac{w_A(\vec{\Gamma})}{|\text{Aut}(\vec{\Gamma})|}.$$

6. The mirror curve and its compactification

The equation of the framed mirror curve is given by

$$X^r Y^{-s-rf} + Y^m + 1 + \sum_{a=1}^p q_a X^{m_a} Y^{n_a - m_a f} = 0,$$

where

$$m_a = r c_1(h_a), \quad n_a = -s c_1(h_a) + m c_2(h_a).$$

Let $\triangle$ be the triangle on $\mathbb{R}^2$ with vertices $(r, -s)$, $(0, m)$, and $(0, 0)$. The compactified mirror curve $\bar{\Sigma}_q$ is embedded in the toric surface $S_\triangle$ defined by the polytope $\triangle$. In this section, we assume that $q_1, \dots, q_p \in \mathbb{C}$ are sufficiently small so that the compactified mirror curve $\bar{\Sigma}_q$ is a smooth compact Riemann surface. Let g be the genus of $\bar{\Sigma}_q$, and let n be the number of points in $\bar{\Sigma}_q \setminus \Sigma_q$.

6.1 The Liouville 1-form

Let $\tilde{\Sigma}_q := \pi^{-1}(\Sigma_q) \subset \mathbb{C}^2$ be the preimage of $\Sigma_q \subset (\mathbb{C}^*)^2$ under the (universal) covering map $\pi: \mathbb{C}^2 \rightarrow (\mathbb{C}^*)^2$ defined by $(x, y) \mapsto (X = e^{-x}, Y = e^{-y})$. Then $\pi_q := \pi|_{\tilde{\Sigma}_q}: \tilde{\Sigma}_q \rightarrow \Sigma_q$ is a regular $\mathbb{Z}^2$ -cover. The 1-form $\log Y dX/X$ is multi-valued on $(\mathbb{C}^*)^2$, but $\pi^*(\log Y dX/X) = ydx$ is a well-defined holomorphic 1-form on $\mathbb{C}^2$; indeed, it is the Liouville 1-form of the cotangent bundle $T^*\mathbb{C} = (\mathbb{C})^2$:

$$d(ydx) = dy \wedge dx.$$

We define

$$\Phi_q := \left(\log Y \frac{dX}{X} \right) \Big|_{\Sigma_q},$$

which is a multi-valued 1-form on Σ_q . Then $\pi_q^* \Phi_q = ydx|_{\tilde{\Sigma}_q}$ is a well-defined holomorphic 1-form on $\tilde{\Sigma}_q$.

Given a $q \in \mathbb{C}^p$ such that Σ_q is smooth, we choose a base point $p_q \in \Sigma_q$ and a point $\tilde{p}_q \in \pi_q^{-1}(p_q)$. We have a commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(\tilde{\Sigma}_q, \tilde{p}_q) & \xrightarrow{\pi_{q*}} & \pi_1(\Sigma_q, p_q) & \xrightarrow{I_{q*}} & \pi_1((\mathbb{C}^*)^2, p_q) \longrightarrow 1 \\ & & \tilde{\alpha} \downarrow & & \alpha \downarrow & & \bar{\alpha} \downarrow \\ & & H_1(\tilde{\Sigma}_q; \mathbb{Z}) & \xrightarrow{\pi_{q*}} & H_1(\Sigma_q; \mathbb{Z}) & \xrightarrow{I_{q*}} & H_1((\mathbb{C}^*)^2; \mathbb{Z}) \longrightarrow 0. \end{array} \quad (6.1)$$

In the above diagram, the following hold:

- The map I_{q*} is induced by the inclusion map $I_q: \Sigma_q \hookrightarrow (\mathbb{C}^*)^2$.
- The first row is a short exact sequence of multiplicative groups.
- The second row is exact at $H_1(\Sigma_q; \mathbb{Z}) = \mathbb{Z}^{2g+n-1}$ and $H_1((\mathbb{C}^*)^2; \mathbb{Z}) = \mathbb{Z}^2$.
- The maps $\tilde{\alpha}$ and α are surjective group homomorphisms given by abelianization, and $\bar{\alpha}$ is a group isomorphism.
- The group $\pi_1(\Sigma_q, p_q)$ is a free group generated by $2g + n - 1$ elements.

We define

$$K_1(\Sigma_q; \mathbb{Z}) := \text{Ker}(I_{q*}: H_1(\Sigma_q; \mathbb{Z}) \rightarrow H_1((\mathbb{C}^*)^2; \mathbb{Z})) = \text{Im}(\pi_{q*}: H_1(\tilde{\Sigma}_q; \mathbb{Z}) \rightarrow H_1(\Sigma_q; \mathbb{Z})).$$

Then $K_1(\Sigma_q; \mathbb{Z}) \cong \mathbb{Z}^{2g+n-3}$. There is a well-defined map

$$\pi_1(\tilde{\Sigma}_q, \tilde{p}_q) \rightarrow \mathbb{C}, \quad \tilde{A} \mapsto \int_{\tilde{A}} \pi_q^* \Phi_q.$$

Given any $A \in K_1(\Sigma_q; \mathbb{Z})$, there exists an $\tilde{A} \in \pi_1(\tilde{\Sigma}_q, \tilde{p}_q)$ such that $A = \pi_{q*} \circ \tilde{\alpha}(\tilde{A}) = \alpha \circ \pi_{q*}(\tilde{A})$. If $\tilde{A}_1, \tilde{A}_2 \in \pi_1(\tilde{\Sigma}_q, \tilde{p}_q)$ and $\alpha \circ \pi_{q*}(\tilde{A}_1) = \alpha \circ \pi_{q*}(\tilde{A}_2) = A$, then $\pi_{q*}(\tilde{A}_2) = \pi_{q*}(\tilde{A}_1) B_1 B_2 B_1^{-1} B_2^{-1}$ for some $B_1, B_2 \in \pi_1(\Sigma_q, p_q)$. If $I_*(B_1) = (m_1, n_1)$ and $I_*(B_2) = (m_2, n_2)$, where $m_1, n_1, m_2, n_2 \in \mathbb{Z}$, then

$$\int_{\tilde{A}_2} \pi_q^* \Phi = \int_{\tilde{A}_1} \pi_q^* \Phi + (2\pi\sqrt{-1})^2 (m_1 n_2 - m_2 n_1).$$

So we have a map

$$K_1(\Sigma_q; \mathbb{Z}) \rightarrow \mathbb{C}/(2\pi\sqrt{-1})^2 \mathbb{Z}, \quad A \mapsto \int_{\tilde{A}} \pi_q^* \Phi_q + (2\pi\sqrt{-1})^2 \mathbb{Z}, \quad (6.2)$$

where $\tilde{A}$ is any element in $(\pi_{q*} \circ \tilde{\alpha})^{-1}(A) \subset \pi_1(\tilde{\Sigma}_q, \tilde{p}_q)$. Let $\int_A \Phi$ denote the image of $A \in K_1(\Sigma_q; \mathbb{Z})$ under the above map. Then $\int_A \Phi_q \in \mathbb{C}/(2\pi\sqrt{-1})^2\mathbb{Z}$ is independent of choice of p_q and $\tilde{p}_q$.

Let B_ϵ denote the open ball in $\mathbb{C}^p$ with radius $\epsilon > 0$ and center 0. There exists an $\epsilon > 0$ such that Σ_q is smooth for all $q \in B_\epsilon$. For $q \in B_\epsilon$, there are canonical isomorphisms

$$H_1(\Sigma_q; \mathbb{Z}) \cong H_1(\Sigma_0; \mathbb{Z}), \quad K_1(\Sigma_q; \mathbb{Z}) \cong K_1(\Sigma_0; \mathbb{Z}).$$

Given an $A \in K_1(\Sigma_0; \mathbb{Z}) \cong K_1(\Sigma_q; \mathbb{Z})$, the integral $\int_A \Phi_q$ can be viewed as an element in $\Gamma(B_\epsilon, \mathbb{C})/(2\pi\sqrt{-1})^2\mathbb{Z}$, where $\Gamma(B_\epsilon, \mathbb{C})$ is the space of holomorphic functions on B_ϵ and $(2\pi\sqrt{-1})^2\mathbb{Z}$ is identified with the set of constant functions from B_ϵ to $(2\pi\sqrt{-1})^2\mathbb{Z}$.

The inclusion $J_q: \Sigma_q \hookrightarrow \bar{\Sigma}_q$ induces a surjective map

$$J_{q*}: H_1(\Sigma_q; \mathbb{C}) \cong \mathbb{C}^{2\mathfrak{g}+\mathfrak{n}-1} \rightarrow H_1(\bar{\Sigma}_q; \mathbb{C}) \cong \mathbb{C}^{2\mathfrak{g}},$$

which restricts to a surjective $\mathbb{C}$ -linear map

$$K_1(\Sigma_q; \mathbb{C}) := K_1(\Sigma_q; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}^{2\mathfrak{g}+\mathfrak{n}-3} \rightarrow H_1(\bar{\Sigma}_q; \mathbb{C}) \cong \mathbb{C}^{2\mathfrak{g}}. \quad (6.3)$$

The complex vector spaces $K_1(\Sigma_q; \mathbb{C})$, $H_1(\Sigma_q; \mathbb{C})$, and $H_1(\bar{\Sigma}_q; \mathbb{C})$ form flat complex vector bundles $\mathcal{K}$, $\mathcal{H}$, and $\bar{\mathcal{H}}$ over B_ϵ of rank $2\mathfrak{g}+\mathfrak{n}-3$, $2\mathfrak{g}+\mathfrak{n}-1$, and $2\mathfrak{g}$, respectively. Given $A \in K_1(\Sigma_q; \mathbb{C})$ which is a flat section of $\mathcal{K}$ with respect to the Gauss–Manin connection, $\int_A \Phi_q$ can be viewed as an element in $\Gamma(B_\epsilon, \mathbb{C})/\mathbb{C}$, where $\mathbb{C}$ is identified with the space of constant functions from B_ϵ to $\mathbb{C}$.

6.2 Holomorphic 1-forms on Σ_q

For any integers m, n ,

$$\varpi_{m,n} := \operatorname{Res}_{H_f=0} \frac{X^m Y^{n-fm}}{H_f} \cdot \frac{dX}{X} \wedge \frac{dY}{Y} = \frac{-X^m Y^{n-fm}}{Y \partial H_f(X, Y, q) / \partial Y} \frac{dX}{X}$$

is a holomorphic 1-form on Σ_q . We view X as a flat coordinate independent of q and use implicit differentiation to obtain

$$\frac{\partial Y}{\partial q_a} = \frac{-X^{m_a} Y^{n_a - f m_a}}{\partial H / \partial Y}.$$

We define

$$\nabla_{\partial/\partial q_a} \Phi := \left. \frac{\partial \Phi}{\partial q_a} \right|_{X=\text{constant}} = \frac{\partial Y}{\partial q_a} \frac{dX}{YX}.$$

Then $\nabla_{\partial/\partial q_a} \Phi = \varpi_{m_a, n_a}$ for $a=1, \dots, p$.

6.3 Differentials of the first kind on $\bar{\Sigma}_q$

Recall that a differential of the first kind is a holomorphic 1-form. By results in [BC94], the holomorphic 1-form $\varpi_{m,n}$ on Σ_q extends to a holomorphic 1-form on $\bar{\Sigma}_q$ if and only if (m, n) is in the interior of the triangle Δ with vertices $(r, -s)$, $(0, m)$, $(0, 0)$. Without loss of generality, we may assume that (m_a, n_a) is in the interior of Δ for $1 \leq a \leq \mathfrak{g}$ and is on the boundary of Δ for $\mathfrak{g}+1 \leq a \leq p$; note that $\mathfrak{g}$ can be zero. Let $\mathfrak{n} = p - \mathfrak{g} + 3$ be the number of lattice points on the boundary of Δ . Then Σ_q is a Riemann surface of genus $\mathfrak{g}$ with $\mathfrak{n}$ punctures, and $\bar{\Sigma}_q$ is a compact Riemann surface of genus $\mathfrak{g}$. We have

$$p = \dim_{\mathbb{C}} H_{\text{CR}}^2(\mathcal{X}; \mathbb{C}), \quad \mathfrak{g} = \dim_{\mathbb{C}} H_{\text{CR}}^4(\mathcal{X}; \mathbb{C}),$$

and

$$-\chi(\Sigma_q) = 2\mathfrak{g} - 2 + \mathfrak{n} = 1 + p + \mathfrak{g} = |G| = \dim_{\mathbb{C}} H_{\text{CR}}^*(\mathcal{X}; \mathbb{C}).$$

A basis of $H^0(\bar{\Sigma}_q, \omega_{\bar{\Sigma}_q}) = H^{1,0}(\bar{\Sigma}_q, \mathbb{C})$ is given by $\{\nabla_{\partial/\partial q_a} \Phi = \varpi_{m_a, n_a} : a = 1, \dots, \mathfrak{g}\}$.

Let $B_\epsilon = \{(q_1, \dots, q_p) \in \mathbb{C}^p : |q| < \epsilon\}$ be an open ball in $\mathbb{C}^p$ with radius $\epsilon > 0$ centered at the origin. Let $\Sigma_\epsilon = \{(X, Y, q) \in \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^p : H_f(X, Y, q) = 0\}$ be the family of mirror curves over B_ϵ , and let $\pi_\epsilon : \Sigma_\epsilon \rightarrow B_\epsilon$ be given by $(X, Y, q) \mapsto q$, so that $\pi_\epsilon^{-1}(q) = \Sigma_q$. Then G^* acts on the total space Σ_ϵ by

$$\alpha \cdot (X, Y, q_a) = (\chi_\alpha(\eta_1)X, \chi_\alpha(\eta_2)Y, \chi_\alpha(h_a^{-1})q_a), \quad (6.4)$$

where $\alpha \in G^*$, $\chi_\alpha : G \rightarrow \mathbb{C}^*$ is the associated character, $\eta_1, \eta_2 \in G$ are defined as in equation (2.6), and $h_a \in G$ corresponds to $(m_a, n_a, 1) \in \text{Box}(\sigma)$ for $a = 1, \dots, p$. The G^* -action on Σ_ϵ extends to the family of compactified mirror curve $\bar{\pi}_\epsilon : \bar{\Sigma}_\epsilon \rightarrow B_\epsilon$, where $\bar{\pi}_\epsilon^{-1}(q) = \bar{\Sigma}_q$. In particular, G^* acts on $\bar{\Sigma}_0$, and the induced G^* -action on $H_1(\bar{\Sigma}_0; \mathbb{C})$ preserves the intersection pairing $\cdot$ on $H_1(\bar{\Sigma}_0; \mathbb{C})$. We choose a symplectic basis $\bar{A}_1, \bar{B}_1, \dots, \bar{A}_\mathfrak{g}, \bar{B}_\mathfrak{g}$ of $H_1(\bar{\Sigma}_0; \mathbb{C})$ such that

$$\bar{A}_a \cdot \bar{A}_b = \bar{B}_a \cdot \bar{B}_b = 0, \quad \bar{A}_a \cdot \bar{B}_b = \delta_{ab}, \quad a, b = 1, \dots, \mathfrak{g}$$

and

$$\frac{1}{2\pi\sqrt{-1}} \int_{\bar{A}_a} (\nabla_{\partial/\partial q_b} \Phi)|_{q=0} = \delta_{ab}, \quad a, b = 1, \dots, \mathfrak{g}.$$

The G^* -action on $\bar{A}_a$ is given by $\alpha \cdot \bar{A}_a = \chi_\alpha(h_a) \bar{A}_a$, where $\alpha \in G^*$ and $1 \leq a \leq \mathfrak{g}$. We may choose $\bar{B}_a$ such that $\alpha \cdot \bar{B}_a = \chi_\alpha(h_a^{-1}) \bar{B}_a$ for $\alpha \in G^*$ and $1 \leq a \leq \mathfrak{g}$.

The inclusion $I_0 : \Sigma_0 \rightarrow (\mathbb{C}^*)^2$ is G^* -equivariant, where $\alpha \in G^*$ acts on $(\mathbb{C}^*)^2$ by $\alpha \cdot (X, Y) = (\chi_\alpha(\eta_1)X, \chi_\alpha(\eta_2)Y)$. So the G^* -actions on $H_1((\mathbb{C}^*)^2; \mathbb{Z})$ and $H_1((\mathbb{C}^*)^2; \mathbb{C})$ are trivial, and the G^* -action on $H_1(\Sigma_0; \mathbb{C})$ preserves $K_1(\Sigma_0; \mathbb{C}) = \text{Ker}(I_* : H_1(\Sigma_0; \mathbb{C}) \rightarrow H_1((\mathbb{C}^*)^2; \mathbb{Z}))$.

We extend $\bar{A}_1, \bar{B}_1, \dots, \bar{A}_\mathfrak{g}, \bar{B}_\mathfrak{g}$ to a symplectic basis of $H_1(\bar{\Sigma}_q; \mathbb{C})$ such that $\bar{A}_i$ and $\bar{B}_i$ are flat sections of $\bar{\mathcal{H}}$ with respect to the Gauss–Manin connection. Note that the composition

$$K_1(\Sigma_q; \mathbb{C}) \rightarrow H_1(\Sigma_q; \mathbb{C}) \xrightarrow{J_*} H_1(\bar{\Sigma}_q; \mathbb{C})$$

is surjective. For $a = 1, \dots, \mathfrak{g}$, we lift $\bar{A}_a, \bar{B}_a \in H_1(\bar{\Sigma}_q; \mathbb{C})$ to $A_a, B_a \in K_1(\Sigma_q; \mathbb{C})$. We choose $A_a, B_a \in K_1(\Sigma_q; \mathbb{C})$ such that they are flat sections of $\mathcal{K}$ with respect to the Gauss–Manin connection and are eigenvectors of the G^* -action on $K_1(\Sigma_0; \mathbb{C})$ at $q = 0$.

6.4 Differentials of the third kind on $\bar{\Sigma}_q$

Recall that a differential of the third kind is a meromorphic 1-form with only simple poles.

Let E_1, E_2, E_3 be the edges of the triangle Δ opposite to the vertices $(r, -s), (0, m), (0, 0)$, respectively. Let $\mathfrak{n}_i + 1$ be the number of lattice points on E_i (including the end points). Then $G_i := \{h \in G : \chi_i(h) = 1\} \cong \mu_{\mathfrak{n}_i}$; in particular, $\mathfrak{n}_1 = m$. For $i = 1, 2, 3$, we have short exact sequences of abelian groups

$$1 \rightarrow G_i = \mu_{\mathfrak{n}_i} \rightarrow G \xrightarrow{\chi_i} \mu_{r_i} \rightarrow 1, \quad 1 \rightarrow \mu_{r_i}^* \rightarrow G^* \rightarrow G_i^* \rightarrow 1.$$

The G^* -action on $\bar{\Sigma}_0$ preserves the finite set $\bar{\Sigma}_0 \cap D_i$, where $D_i \subset S_\Delta$ is the torus-invariant divisor associated with the edge E_i . The G^* -action on $\bar{\Sigma}_0 \cap D_i$ induces a free and transitive G_i^* -action on $\bar{\Sigma}_0 \cap D_i$. So we may label the point in $\bar{\Sigma}_0 \cap D_i$ (non-canonically) by elements in $G_i^* = \mu_{\mathfrak{n}_i}^*$; that is, $\bar{\Sigma}_0 \cap D_i = \{\bar{p}_\ell^i : \ell \in \mu_{\mathfrak{n}_i}^*\}$.

If $\mathfrak{g} + 1 \leq a \leq p$, then $h_a \in G_i \setminus \{1\} \cong \mu_{\mathfrak{n}_i} \setminus \{1\}$ for some $i \in \{1, 2, 3\}$. The meromorphic 1-form $\nabla_{\partial/\partial q_a} \Phi$ has a simple pole at each point in $\{\bar{p}_\ell^i : \ell \in G_i^*\}$ and is holomorphic elsewhere

on the compactified mirror curve $\bar{\Sigma}_q$, so it is a differential of the third kind on $\bar{\Sigma}_q$. We have

$$\operatorname{Res}_{p \rightarrow \bar{p}_\ell^i} (\nabla_{\partial/\partial q_a} \Phi)|_{q=0} = \frac{e^{\pi\sqrt{-1}k/\mathbf{n}_i}}{\mathbf{n}_i} \chi_\ell(k),$$

where $h_a = e^{2\pi\sqrt{-1}k/\mathbf{n}_i} \in \mu_{\mathbf{n}_i}$ for $1 \leq k \leq \mathbf{n}_i - 1$ and $\chi_\ell: G_i \rightarrow \mathbb{C}^*$ is the character associated with $\ell \in G_i^*$. Let $C_\ell^i \in H_1(\Sigma_0; \mathbb{Z})$ be the class of a small circle around $\bar{p}_\ell^i$. Then there exists an $A_a \in K_1(\Sigma_0; \mathbb{C})$ ($a = \mathfrak{g} + 1, \dots, p$) such that

- A_a is an eigenvector of the G^* -action on $K_1(\Sigma_0; \mathbb{C})$;
- A_a is a $\mathbb{C}$ -linear combination of C_ℓ^i , with $\ell \in G_i^*$;
- for $a, b = 1, \dots, p$,

$$\frac{1}{2\pi\sqrt{-1}} \int_{A_a} (\nabla_{\partial/\partial q_b} \Phi)|_{q=0} = \delta_{ab}.$$

We extend $A_{\mathfrak{g}+1}, \dots, A_p$ to flat sections of $K_1(\Sigma_q; \mathbb{C})$. Then for $a, b \in \{1, \dots, p\}$,

$$\frac{1}{2\pi\sqrt{-1}} \int_{A_a} \nabla_{\partial/\partial q_b} \Phi = \delta_{ab} + O(|q|).$$

DEFINITION 6.1 (B-model closed string flat coordinates). For $a = 1, \dots, p$, let $\tau_a(q)$ be the unique function in $\Gamma(B_\epsilon, \mathbb{C})$ such that $\tau_a(0) = 0$ and

$$\frac{1}{2\pi\sqrt{-1}} \int_{A_a} \Phi = \tau_a(q) + \mathbb{C} \in \Gamma(B_\epsilon, \mathbb{C})/\mathbb{C}.$$

Then $\tau_a(q) = q_a + O(|q|^2)$ and

$$\frac{1}{2\pi\sqrt{-1}} \int_{A_a} \nabla_{\partial/\partial \tau_b} \Phi = \delta_{ab}, \quad a, b = 1, \dots, p.$$

The B-model closed string flat coordinates are related to q_a for $a = 1, \dots, p$ via the following hypergeometric formulae:

$$\tau_a(q) = q_a \left(\sum_{\substack{d_b \in \mathbb{Z}_{\geq 0} \\ \{\sum_{b=1}^p d_b c_i(h_b)\} = 0, i=1,2,3}} \prod_{i=1}^3 \frac{\Gamma(-\{c_i(h_a)\} + 1)}{\Gamma(-c_i(h_a) - \sum_{b=1}^p d_b c_i(h_b) + 1)} \cdot \frac{1}{\prod_{b=1}^p d_b!} \prod_{b=1}^p q_b^{d_b} \right). \quad (6.5)$$

Here $\{x\}$ denotes the fractional part of x . This mirror map is obtained by solving the Gelfand–Kapranov–Zelevinsky-type (GKZ-type for short) Picard–Fuchs equations. Iritani explains these GKZ-operators and explicitly writes down the mirror map for general toric orbifolds [Iri09]. The integrals over A-cycles on mirror curves as flat coordinates and their hypergeometric expressions for toric Calabi–Yau 3-folds are discussed in [CKYZ99] together with genus zero closed Gromov–Witten mirror symmetry.

Let $D_q^\infty = \bar{\Sigma}_q \setminus \Sigma_q = \bigcup_{i=1}^3 \{\bar{p}_1^i, \dots, \bar{p}_{\mathbf{n}_i}^i\}$. By Lefschetz duality, there is a perfect pairing

$$H_1(\Sigma_q; \mathbb{C}) \times H_1(\bar{\Sigma}_q, D_q^\infty; \mathbb{C}) \rightarrow \mathbb{C}. \quad (6.6)$$

The inclusion $\Sigma_q \subset \bar{\Sigma}_q$ induces a surjective $\mathbb{C}$ -linear map

$$H_1(\Sigma_q; \mathbb{C}) \cong \mathbb{C}^{2\mathfrak{g}+\mathbf{n}-1} \rightarrow H_1(\bar{\Sigma}_q; \mathbb{Z}) \cong \mathbb{C}^{2g}.$$

Our choice of $A_1, B_1, \dots, A_{\mathfrak{g}}, B_{\mathfrak{g}}$ gives a splitting

$$j_1: H_1(\bar{\Sigma}_q; \mathbb{C}) \rightarrow H_1(\Sigma_q; \mathbb{C}). \quad (6.7)$$

The long exact sequence of relative homology groups of the pair $(\bar{\Sigma}_q, D_q^\infty)$ gives an injective $\mathbb{C}$ -linear map

$$j_2: H_1(\bar{\Sigma}_q; \mathbb{C}) \rightarrow H_1(\bar{\Sigma}_q, D_q^\infty; \mathbb{C}). \quad (6.8)$$

Under the inclusion maps j_1 and j_2 , the perfect pairing (6.6) restricts to the intersection pairing

$$H_1(\bar{\Sigma}_q; \mathbb{C}) \times H_1(\bar{\Sigma}_q; \mathbb{C}) \rightarrow \mathbb{C},$$

which is perfect by Poincaré duality. For $\mathfrak{g} + 1 \leq a \leq p$, choose $B_a \in H_1(\bar{\Sigma}_q, D_q^\infty; \mathbb{C})$ such that

- $(A_1, \dots, A_p, B_1, \dots, B_{\mathfrak{g}})$ and $(B_1, \dots, B_p, -A_1, \dots, -A_{\mathfrak{g}})$ are dual under the pairing (6.6);
- $\alpha \cdot B_a = \chi_\alpha(h_a^{-1})B_a$ for $\alpha \in G^*$, and $\mathfrak{g} + 1 \leq a \leq p$.

Then for $a = 1, \dots, p$,

$$(\nabla_{\partial/\partial\tau_a}\Phi)(z) = \int_{z' \in B_a} B(z', z),$$

where $B(z', z)$ is the fundamental normalized differential of the second kind on $\bar{\Sigma}_q$ (see Section 6.6).

6.5 Critical points and Lefschetz thimbles

We choose a framing f such that $X: \Sigma_0 \rightarrow \mathbb{C}^*$ has only simple branch points. Then $X: \Sigma_q \rightarrow \mathbb{C}^*$ has only simple branch points for q sufficiently small.

The critical points of $X: \Sigma_0 = \{(X, Y) \in (\mathbb{C}^*)^2: X^r Y^{-s-rf} + Y^m + 1 = 0\} \rightarrow \mathbb{C}^*$ are

$$\{(X_{j\ell}, Y_{j\ell}): j \in \{0, 1, \dots, r-1\}, \ell \in \{0, 1, \dots, m-1\}\},$$

where

$$\begin{aligned} X_{j\ell} &= \exp\left(2\pi\sqrt{-1}\left(\frac{j}{r} + \ell\frac{s+rf}{rm}\right)\right) m^{1/r}(s+rf)^{(s+rf)/rm}(-m-s-rf)^{(-m-s-rf)/rm} \\ &= (\chi_1^j \chi_2^\ell)(\eta_1) \prod_{i=1}^3 (|G|w_i)^{w_i}, \\ Y_{j\ell} &= \exp\left(2\pi\sqrt{-1}\frac{\ell}{m}\right) \left(\frac{s+rf}{-m-s-rf}\right)^{1/m} = (\chi_1^j \chi_2^\ell)(\eta_2) \left(\frac{w_2}{w_3}\right)^{1/m}. \end{aligned}$$

If $\alpha = \chi_1^j \chi_2^\ell \in G^*$, then we define $(X_\alpha, Y_\alpha) = (X_{j\ell}, Y_{j\ell})$. The critical points of $X: \Sigma_0 \rightarrow \mathbb{C}^*$ are $\{(X_\alpha, Y_\alpha): \alpha \in G^*\}$. We define

$$a_\alpha := -\log(X_\alpha) = -\sum_{i=1}^3 w_i \log w_i - \log(\chi_\alpha(\eta_1)), \quad (6.9)$$

$$b_\alpha := -\log(Y_\alpha) = \frac{1}{m} \log(w_3/w_2) - \log(\chi_\alpha(\eta_2)). \quad (6.10)$$

In equation (6.9), we use the identity $\sum_{i=1}^3 w_i \log(|G|w_i) = \sum_{i=1}^3 w_i \log w_i$, which holds since $\sum_{i=1}^3 w_i = 0$. Note that

$$\log(\chi_\alpha(\eta_1)) = \sqrt{-1}\vartheta_\alpha, \quad \log(\chi_\alpha(\eta_2)) = \sqrt{-1}\varphi_\alpha$$

for some $\vartheta_\alpha, \varphi_\alpha \in \mathbb{R}$. We may assume $\vartheta_\alpha, \varphi_\alpha \in [0, 2\pi)$.

Let $X = e^{-x}$ and $Y = e^{-y}$. Around each critical point p_α , we set up the following local

coordinates

$$x = a_\alpha(q) + \zeta_\alpha(q)^2, \quad y = b_\alpha(q) + \sum_{d=1}^{\infty} h_d^\alpha(q) \zeta_\alpha(q)^d,$$

where

$$h_1^\alpha(q) = \sqrt{\frac{2}{(d^2 x/dy^2)(b_\alpha(q))}}.$$

Let γ_α be the Lefschetz thimble of the superpotential $x = -\log X: \Sigma_q \rightarrow \mathbb{C}$, so that $x(\gamma_\alpha) = a_\alpha + \mathbb{R}^+$. Then $\{\gamma_\alpha: \alpha \in G^*\}$ is a basis of the relative homology group

$$H_1(\Sigma_q, \{(X, Y) \in \Sigma_q: \log X \ll 0\}; \mathbb{Z}) \cong \mathbb{Z}^{|G|}.$$

6.6 Differentials of the second kind on $\bar{\Sigma}_q$

Recall that a differential of the second kind is a meromorphic 1-form with no residues.

We choose a symplectic basis $\{A_1, B_1, \dots, A_{\mathfrak{g}}, B_{\mathfrak{g}}\}$ of $H_1(\bar{\Sigma}_q, \mathbb{C})$ as in Section 6.3. Let $B(p_1, p_2)$ be the fundamental normalized differential of the second kind on $\bar{\Sigma}_q$ (see, for example, [Fay73]), which is characterized by the following properties:

- (1) The differential $B(p_1, p_2)$ is bilinear symmetric meromorphic on $\bar{\Sigma}_q \times \bar{\Sigma}_q$.
- (2) The differential $B(p_1, p_2)$ is holomorphic everywhere except for a double pole along the diagonal $p_1 = p_2$. If z_1, z_2 are local coordinates on $\bar{\Sigma}_q \times \bar{\Sigma}_q$, then

$$B(z_1, z_2) = \left(\frac{1}{(z_1 - z_2)^2} + f(z_1, z_2) \right) dz_1 dz_2,$$

where $f(z_1, z_2)$ is holomorphic.

- (3) $\int_{p_1 \in A_i} B(p_1, p_2) = 0$ for $i = 1, \dots, \mathfrak{g}$.

Let $p_\alpha(q) = (X_\alpha(q), Y_\alpha(q))$ be the branch point of $X: \bar{\Sigma}_q \rightarrow \mathbb{C}^*$ such that

$$\lim_{q \rightarrow 0} (X_\alpha(q), Y_\alpha(q)) = (X_\alpha, Y_\alpha).$$

Following [Eyn14, EO15], given $\alpha \in G^*$ and $d \in \mathbb{Z}_{\geq 0}$, define

$$\theta_d^\alpha(p) := -(2d-1)!! 2^{-d} \operatorname{Res}_{p' \rightarrow p_\alpha} B(p, p') \zeta_\alpha^{-2d-1}. \quad (6.11)$$

(In this paper, we use the symbol θ_d^α instead of the symbol $d\xi_{\alpha,d}$ in [Eyn11, EO15] because the 1-form defined by the right-hand side of (6.11) is not necessarily exact.) Then θ_d^α satisfies the following properties:

- (1) The form θ_d^α is a meromorphic 1-form on $\bar{\Sigma}_q$ with a single pole of order $2d+2$ at p_α .
- (2) In local coordinate ζ_α near p_α ,

$$\theta_d^\alpha = \left(-\frac{-(2d+1)!!}{2^d \zeta_\alpha^{2d+2}} + f(\zeta_\alpha) \right) d\zeta_\alpha,$$

where $f(\zeta_\alpha)$ is analytic around p_α . The residue of θ_d^α at p_α is zero, so θ_d^α is a differential of the second kind.

- (3) $\int_{A_i} \theta_d^\alpha = 0$ for $i = 1, \dots, \mathfrak{g}$.

The meromorphic 1-form θ_d^α is characterized by the above properties; θ_d^α can be viewed as a section in $H^0(\bar{\Sigma}_q, \omega_{\bar{\Sigma}_q}((2d+2)p_\alpha))$.

LEMMA 6.2. Suppose that f is a meromorphic function on $\bar{\Sigma}_q$ with simple poles at the ramification points $\{p_\beta: \beta \in G^*\}$ and is holomorphic on $\bar{\Sigma}_q \setminus \{p_\beta: \beta \in G^*\}$. Then

$$df = \sum_{\beta \in G^*} c_\beta \theta_0^\beta, \quad \text{where} \quad c_\beta = \lim_{p \rightarrow p_\beta} f \zeta_\beta.$$

Proof. Let $\Delta\omega = df - \sum_{\beta \in G^*} c_\beta \theta_0^\beta$. Then $\Delta\omega$ is a holomorphic 1-form on $\bar{\Sigma}_q$ and $\int_{A_i} \Delta\omega = 0$ for $i = 1, \dots, g$. So $\Delta\omega = 0$. $\square$

Remark 6.3. In Lemma 6.2, different choice of A -cycles $A_1, \dots, A_g$ will give different meromorphic 1-forms θ_0^β , but the equality

$$df = \sum_{\beta \in G^*} c_\beta \theta_0^\beta$$

still holds, where the coefficients $c_\beta = \lim_{p \rightarrow p_\beta} f \zeta_\beta$ do not depend on the choice of A -cycles.

We make the following observations:

- (1) The form $dx = -dX/X$ is a meromorphic 1-form on $\bar{\Sigma}_q$ which is holomorphic on Σ_q . It has a simple zero at each of the $|G|$ ramification points $\{p_\alpha: \alpha \in G^*\}$ and a simple pole at each of the n punctures $\bar{p}_1, \dots, \bar{p}_n$.
- (2) The form $dy = -dY/Y$ is a meromorphic 1-form on $\bar{\Sigma}_q$ and a holomorphic 1-form on Σ_q . It is non-zero at each of the $|G|$ ramification points and has at most a simple pole at each of the n punctures.
- (3) For $a = 1, \dots, g$, the form $\nabla_{\partial/\partial q_a} \Phi$ is a holomorphic 1-form on $\bar{\Sigma}_q$ which is non-zero at each of the $|G|$ ramification point.
- (4) For $a = g+1, \dots, p$, the form $\nabla_{\partial/\partial q_a} \Phi$ is a meromorphic 1-form on $\bar{\Sigma}_q$ which is holomorphic on Σ_q . It is non-zero at each of the $|G|$ ramification points and has at most a simple pole at each of the n punctures.

On the basis of the above observations, the following are meromorphic function on $\bar{\Sigma}_q$ satisfying the assumption of Lemma 6.2:

$$\frac{dy}{dx}, \quad \frac{\nabla_{\partial/\partial q_a} \Phi}{dx} = \frac{\partial y}{\partial q_a},$$

where $a = 1, \dots, p$ and $\alpha \in G^*$.

PROPOSITION 6.4. We have

$$d\left(\frac{dy}{dx}\right) = \frac{1}{2} \sum_{\beta \in G^*} h_1^\beta(q) \theta_0^\beta. \quad (6.12)$$

For $a = 1, \dots, p$,

$$d\left(\frac{\nabla_{\partial/\partial q_a} \Phi}{dx}\right) = \frac{1}{2} \sum_{\beta \in G^*} \frac{X_\beta(q)^{m_a} Y_\beta(q)^{n_a - f m_a}}{(\partial H / \partial x)(X_\beta(q), Y_\beta(q), q)} h_1^\beta(q) \theta_0^\beta. \quad (6.13)$$

Equation (6.12) was proved in [EO15, Appendix D]; we include it for completeness.

Proof of Proposition 6.4. Near p_β ,

$$\begin{aligned} \frac{dy}{dx} \zeta_\beta &= \frac{\sum_{k=1}^{\infty} h_k^\beta(q) k (\zeta_\beta)^k d\zeta_\beta}{2\zeta_\beta d\zeta_\beta} = \frac{1}{2} \sum_{k=1}^{\infty} h_k^\beta(q) k (\zeta_\beta)^{k-1}, \\ \frac{\nabla_{\partial/\partial q_a} \Phi}{dx} \zeta_\beta &= \frac{-X^{m_a} Y^{n_a - f m_a}}{(\partial H / \partial y)(X, Y, q)} \zeta_\beta = \frac{X^{m_a} Y^{n_a - f m_a}}{(\partial H / \partial x)(X, Y, q)} \cdot \frac{dy}{dx} \zeta_\beta. \end{aligned}$$

So

$$\lim_{p \rightarrow p_\beta} \frac{dy}{dx} \zeta_\beta = \frac{h_1^\beta(q)}{2}, \quad \lim_{p \rightarrow p_\beta} \frac{\nabla_{\partial/\partial q_a} \Phi}{dx} \zeta_\beta = \frac{X_\beta(q)^{m_a} Y_\beta(q)^{n_a - f m_a}}{(\partial H / \partial x)(X_\beta(q), Y_\beta(q), q)} \frac{h_1^\beta(q)}{2}.$$

The proposition follows from Lemma 6.2. $\square$

PROPOSITION 6.5. We have

$$d \left(\frac{\nabla_{\partial/\partial \tau_a} \Phi}{dx} \right) = \frac{1}{2} \sum_{\beta \in G^*} \left(\frac{\partial H}{\partial \tau_a} / \frac{\partial H}{\partial x} \right) \Big|_{X=X_\beta(q), Y=Y_\beta(q)} \cdot h_1^\beta(q) \theta_0^\beta, \quad (6.14)$$

$$\nabla_{\partial/\partial \tau_a} \nabla_{\partial/\partial \tau_b} \Phi = \frac{1}{2} \sum_{\beta \in G^*} \left(\frac{\partial H}{\partial \tau_a} / \frac{\partial H}{\partial x} \right) \cdot \left(\frac{\partial H}{\partial \tau_b} / \frac{\partial H}{\partial x} \right) \Big|_{X=X_\beta(q), Y=Y_\beta(q)} \cdot h_1^\beta(q) \theta_0^\beta. \quad (6.15)$$

Proof. Equation (6.14) follows from equation (6.13). It remains to prove (6.15).

By a special geometry property of the topological recursion ([EO15, Theorem 4.4], proved in [EO07]),

$$(\nabla_{\partial/\partial \tau_a} \nabla_{\partial/\partial \tau_b} \Phi)(p) = \int_{p_1 \in B_a} \int_{p_2 \in B_b} \omega_{0,3}(p_1, p_2, p),$$

where (see Example 7.13)

$$\omega_{0,3}(p_1, p_2, p) = \sum_{\beta \in G^*} \frac{-1}{2h_1^\beta} \theta_0^\beta(p_1) \theta_0^\beta(p_2) \theta_0^\beta(p).$$

So $\nabla_{\partial/\partial \tau_a} \nabla_{\partial/\partial \tau_b} \Phi$ is a linear combination of θ_0^β :

$$\nabla_{\partial/\partial \tau_a} \nabla_{\partial/\partial \tau_b} \Phi = \sum_{\beta \in G^*} c_\beta \theta_0^\beta,$$

where the coefficient c_β is given by $c_\beta = \text{Res}_{\zeta_\beta \rightarrow 0} (\zeta_\beta \cdot \nabla_{\partial/\partial \tau_a} \nabla_{\partial/\partial \tau_b} \Phi)$. We have

$$\nabla_{\partial/\partial \tau_a} \nabla_{\partial/\partial \tau_b} \Phi = \sum_{c,d} \frac{\partial q_c}{\partial \tau_a} \frac{\partial q_d}{\partial \tau_b} \nabla_{\partial/\partial q_c} \nabla_{\partial/\partial q_d} \Phi + \sum_c \frac{\partial^2 q_c}{\partial \tau_a \partial \tau_b} \nabla_{\partial/\partial q_c} \Phi,$$

where $\nabla_{\partial/\partial q_c} \Phi$ is holomorphic on Σ_q , and

$$\nabla_{\partial/\partial q_c} \nabla_{\partial/\partial q_d} \Phi = \left(-\frac{\partial H}{\partial q_a} \frac{\partial H}{\partial q_b} \frac{\partial^2 H}{\partial y^2} / \left(\frac{\partial H}{\partial y} \right)^3 + \left(\frac{\partial^2 H}{\partial q_b \partial y} \frac{\partial H}{\partial q_a} + \frac{\partial^2 H}{\partial q_a \partial y} \frac{\partial H}{\partial q_b} \right) / \left(\frac{\partial H}{\partial y} \right)^2 \right) dx.$$

So

$$\begin{aligned} c_\beta &= \frac{\partial H}{\partial \tau_a} \frac{\partial H}{\partial \tau_b} \Big|_{X=X_\beta(q), Y=Y_\beta(q)} \cdot \text{Res}_{\zeta_\beta \rightarrow 0} \left(\zeta_\beta \left(-\frac{\partial^2 H}{\partial y^2} \right) / \left(\frac{\partial H}{\partial y} \right)^3 dx \right) \\ &= \frac{1}{2} \left(\frac{\partial H}{\partial \tau_a} / \frac{\partial H}{\partial x} \right) \cdot \left(\frac{\partial H}{\partial \tau_b} / \frac{\partial H}{\partial x} \right) \Big|_{X=X_\beta(q), Y=Y_\beta(q)} \cdot h_1^\beta(q). \end{aligned} \quad \square$$

Given $\alpha, \beta \in G^*$ and $k, l \in \mathbb{Z}_{\geq 0}$, define $B_{k,l}^{\alpha,\beta}$ to be the coefficients of the expansion of $B(p_1, p_2)$ near $(p_\alpha, p_\beta) \in \bar{\Sigma}_q \times \bar{\Sigma}_q$ in coordinates $\zeta_1 = \zeta_\alpha(p_1)$ and $\zeta_2 = \zeta_\beta(p_2)$. Then

$$B(p_1, p_2) = \left(\frac{\delta_{\alpha,\beta}}{(\zeta_1 - \zeta_2)^2} + \sum_{k,l \in \mathbb{Z}_{\geq 0}} B_{k,l}^{\alpha,\beta} \zeta_1^k \zeta_2^l \right) d\zeta_1 d\zeta_2.$$

We define

$$\check{B}_{k,l}^{\alpha,\beta} = \frac{(2k-1)!!(2l-1)!!}{2^{k+l+1}} B_{2k,2l}^{\alpha,\beta}. \quad (6.16)$$

The following lemma is the differential of [Eyn11, Equation (D.4)] and holds globally on the compactified mirror curve.

LEMMA 6.6. *We have*

$$\theta_{k+1}^\alpha = -d \left(\frac{\theta_k^\alpha}{dx} \right) - \sum_{\beta \in G^*} \check{B}_{k,0}^{\alpha,\beta} \theta_0^\beta.$$

Proof. We have the following Laurent series expansion of the meromorphic function θ_k^α/dx near p_β in the local coordinate ζ_β :

$$\frac{\theta_k^\alpha}{dx} = \delta_{\alpha,\beta} \frac{-(2d+1)!!}{2^{d+1} \zeta_\beta^{2d+3}} - \frac{\check{B}_{k,0}^{\alpha,\beta}}{\zeta_\beta} + h(\zeta_\beta),$$

where $h(\zeta_\beta)$ is a power series in ζ_β . Set $\Delta\omega = \theta_{k+1}^\alpha + d(\theta_k^\alpha/dx) + \sum_{\beta \in G^*} \check{B}_{k,0}^{\alpha,\beta} \theta_0^\beta$. Then $\Delta\omega$ is a holomorphic 1-form on $\bar{\Sigma}_q$, and $\int_{A_i} \Delta\omega = 0$ for $i = 1, \dots, g$. So $\Delta\omega = 0$. $\square$

Given $\alpha \in G^*$ and $k \in \mathbb{Z}_{>0}$, we define

$$\hat{\xi}_{\alpha,k} := (-1)^k \left(\frac{d}{dx} \right)^{k-1} \left(\frac{\theta_0^\alpha}{dx} \right) = \left(X \frac{d}{dX} \right)^{k-1} \left(X \frac{\theta_0^\alpha}{dX} \right), \quad (6.17)$$

which is a meromorphic function on $\bar{\Sigma}_q$. We define $d\hat{\xi}_{\alpha,0} := \theta_0^\alpha$.

LEMMA 6.7. *We have*

$$\theta_k^\alpha = d\hat{\xi}_{\alpha,k} - \sum_{i=0}^{k-1} \sum_{\beta \in G^*} \check{B}_{k-1-i,0}^{\alpha,\beta} d\hat{\xi}_{\beta,i}.$$

Proof. We prove the result by induction on k . When $k = 0$, we have $\theta_0^\alpha = d\hat{\xi}_{\alpha,0}$ by definition. Suppose that the lemma holds for $k = d$. By Lemma 6.6,

$$\begin{aligned} \theta_{d+1}^\alpha &= -d \left(\frac{\theta_d^\alpha}{dx} \right) - \sum_{\beta \in G^*} \check{B}_{d,0}^{\alpha,\beta} \theta_0^\beta \\ &= -d \left(\frac{d\hat{\xi}_{\alpha,d}}{dx} - \sum_{i=0}^{d-1} \sum_{\beta \in G^*} \check{B}_{d-1-i,0}^{\alpha,\beta} \frac{d\hat{\xi}_{\beta,i}}{dx} \right) - \sum_{\beta \in G^*} \check{B}_{d,0}^{\alpha,\beta} d\hat{\xi}_{\beta,0} \\ &= d\hat{\xi}_{\alpha,d+1} - \sum_{i=0}^{d-1} \sum_{\beta \in G^*} \check{B}_{d-1-i,0}^{\alpha,\beta} d\hat{\xi}_{\beta,i+1} - \sum_{\beta \in G^*} \check{B}_{d,0}^{\alpha,\beta} d\hat{\xi}_{\beta,0} \\ &= d\hat{\xi}_{\alpha,d+1} - \sum_{i=0}^d \sum_{\beta \in G^*} \check{B}_{d-i,0}^{\alpha,\beta} d\hat{\xi}_{\beta,i}. \end{aligned}$$

The second equality follows from the induction hypothesis. So the lemma holds for $k = d+1$. $\square$

7. B-model topological string

7.1 Laplace transform

The Laplace transform of a meromorphic 1-form λ along a Lefschetz thimble γ_α is given by $\int_{z \in \gamma_\alpha} e^{-ux} \lambda$.

DEFINITION 7.1. We have $f_\beta^\alpha(u, q) := \frac{e^{ua_\alpha}}{2\sqrt{\pi}u} \int_{z \in \gamma_\alpha} e^{-ux} \theta_0^\beta$.

It is straightforward to check that $f_\beta^\alpha(u, q) = \delta_{\alpha\beta} + O(u^{-1})$.

The Eynard–Orantin invariants $\{\omega_{g,n}\}$ of the mirror curve can be expressed as a graph sum involving $f_\beta^\alpha(u, q)$, θ_d^β , and descendant integrals over moduli spaces of stable curves [KO10, Eyn11, Eyn14, DOSS14]. The goal of this subsection and the next two subsections is to relate $f_\beta^\alpha(u, q)$ and θ_d^β to terms in the A-model graph sum. The following is our strategy:

1. In this section (Section 7.1), we relate $f_\beta^\alpha(u, q)$ to the following Laplace transforms:

$$\int_{\gamma_\alpha} e^{-ux} \Phi, \quad \int_{\gamma_\alpha} e^{-ux} \nabla_{\partial/\partial\tau_a} \Phi, \quad \int_{\gamma_\alpha} e^{-ux} \nabla_{\partial/\partial\tau_a} \nabla_{\partial/\partial\tau_b} \Phi. \quad (7.1)$$

2. In Section 7.2, we evaluate the oscillatory integrals in the Landau–Ginzburg mirror of $\mathcal{X} = [\mathbb{C}^3/G]$ for any small q . These oscillatory integrals can be identified with the Laplace transforms in (7.1) by dimensional reduction.
3. In Section 7.3, we expand an antiderivative of θ_0^β near $X = 0$ and relate it to $\tilde{\xi}_0^\beta$ from Section 5.2.

By equation (6.12),

$$\begin{aligned} \int_{z \in \gamma_\alpha} e^{-ux} \Phi &= u^{-2} \int_{z \in \gamma_\alpha} e^{-ux} d\left(\frac{dy}{dx}\right) = \frac{1}{2} u^{-2} \sum_{\beta \in G^*} h_1^\beta \int_{z \in \gamma_\alpha} e^{-ux} d\xi_{\beta,0} \\ &= \sqrt{\pi} u^{-3/2} e^{-ua_\alpha} \sum_{\beta \in G^*} h_1^\beta f_\beta^\alpha(u, q). \end{aligned}$$

By equation (6.14),

$$\begin{aligned} \int_{z \in \gamma_\alpha} e^{-ux} \nabla_{\partial/\partial\tau_a} \Phi &= -u^{-1} \int_{z \in \gamma_\alpha} e^{-ux} d\left(\frac{\nabla_{\partial/\partial\tau_a} \Phi}{dx}\right) \\ &= -\frac{1}{2} u^{-1} \sum_{\beta \in G^*} \left(\frac{\partial H}{\partial\tau_a} / \frac{\partial H}{\partial x}\right) \Big|_{X=X_\beta, Y=Y_\beta} \cdot h_1^\beta \int_{z \in \gamma_\alpha} e^{-ux} \theta_0^\beta \\ &= -\sqrt{\pi} u^{-1/2} e^{-ua_\alpha} \sum_{\beta \in G^*} \left(\frac{\partial H}{\partial\tau_a} / \frac{\partial H}{\partial x}\right) \Big|_{X=X_\beta, Y=Y_\beta} \cdot h_1^\beta(q) f_\beta^\alpha(u, q). \end{aligned}$$

By equation (6.15),

$$\begin{aligned} \int_{z \in \gamma_\alpha} e^{-ux} \nabla_{\partial/\partial\tau_b} \nabla_{\partial/\partial\tau_a} \Phi &= \frac{1}{2} \sum_{\beta \in G^*} \left(\frac{\partial H}{\partial\tau_a} / \frac{\partial H}{\partial x}\right) \cdot \left(\frac{\partial H}{\partial\tau_b} / \frac{\partial H}{\partial x}\right) \Big|_{X=X_\beta, Y=Y_\beta} \cdot h_1^\beta \int_{z \in \gamma_\alpha} e^{-ux} \theta_0^\beta \\ &= \sqrt{\pi} u^{1/2} e^{-ua_\alpha} \sum_{\beta \in G^*} \left(\frac{\partial H}{\partial\tau_a} / \frac{\partial H}{\partial x}\right) \cdot \left(\frac{\partial H}{\partial\tau_b} / \frac{\partial H}{\partial x}\right) \Big|_{X=X_\beta, Y=Y_\beta} \cdot h_1^\beta(q) f_\beta^\alpha(u, q). \end{aligned}$$

In the remainder of this subsection, we consider the limit $q \rightarrow 0$. We have

$$\lim_{q \rightarrow 0} h_1^\beta(q) = \frac{1}{|G|} \sqrt{\frac{-2}{w_1 w_2 w_3}}, \quad (7.2)$$

$$\lim_{q \rightarrow 0} \left(\frac{\partial H}{\partial \tau_a} / \frac{\partial H}{\partial x} \right) \Big|_{X=X_\beta, Y=Y_\beta} = \chi_\beta(h_a) \prod_{i=1}^3 w_i^{c_i(h_a)}, \quad (7.3)$$

where $h_a \in G$ corresponds to $(m_a, n_a, 1) \in \text{Box}(\sigma)$, so that $\text{age}(h_a) = 1$.

We introduce some notation. For $h \in G$, let

$$\nabla_h \Phi = \begin{cases} \Phi, & h = 1, \\ \nabla_{\partial/\partial \tau_a} \Phi, & h = h_a, \\ \nabla_{\partial/\partial \tau_b} \nabla_{\partial/\partial \tau_a} \Phi, & h = h_a h_b \text{ and } \text{age}(h) = 2. \end{cases}$$

Then

$$\begin{aligned} \lim_{q \rightarrow 0} \int_{z \in \gamma_\alpha} e^{-ux} \nabla_h \Phi &= \frac{\sqrt{-2\pi}(-1)^{\text{age}(h)}}{|G|} u^{\text{age}(h)-3/2} \left(\prod_{i=1}^3 w_i^{c_i(h)-1/2} \right) \\ &\cdot e^{u(\sqrt{-1}\vartheta_\alpha + \sum_{i=1}^3 w_i \log w_i)} \sum_{\beta \in G^*} f_\beta^\alpha(u, 0) \chi_\beta(h). \end{aligned} \quad (7.4)$$

We conclude the following.

PROPOSITION 7.2.

$$\begin{aligned} f_\beta^\alpha(u, 0) &= \frac{1}{\sqrt{-2\pi}} \exp \left(- \left(\sqrt{-1}\vartheta_\alpha + \sum_{i=1}^3 w_i \log w_i \right) u \right) \\ &\cdot \sum_{h \in G} \chi_\beta(h^{-1}) \cdot (-1)^{\text{age}(h)} u^{3/2-\text{age}(h)} \prod_{i=1}^3 w_i^{-c_i(h)+1/2} \cdot \lim_{q \rightarrow 0} \int_{\gamma_\alpha} e^{-ux} \nabla_h \Phi. \end{aligned}$$

We will evaluate $\lim_{q \rightarrow 0} \int_{\gamma_\alpha} e^{-ux} \nabla_h \Phi$ in the next subsection.

7.2 Oscillatory integrals

The equivariant Landau–Ginzburg mirror of a general toric orbifolds has been studied by Iritani [Iri09].

The mirror B-model to $\mathcal{X} = [\mathbb{C}^3/G]$ is a Landau–Ginzburg model $W_q: (\mathbb{C}^*)^3 \rightarrow \mathbb{C}$, where

$$W_q = X_1^r X_2^{-s-rf} X_3 + X_2^m X_3 + X_3 + \sum_{a=1}^p q_a X_1^{m_a} X_2^{n_a-m_a f} X_3.$$

Define $H = W/X_3$. Following Iritani [Iri09], the equivariantly perturbed B-model superpotential $\widetilde{W}_q$ is

$$\widetilde{W}_q = W_q - u \log X_1.$$

We assume $w_1, w_2, u > 0$ and $w_3 < 0$ as usual. Define

$$t_1 = X_1^r X_2^{-s-rf} X_3, \quad t_2 = X_2^m X_3, \quad \hat{t}_1 = X_1^r X_2^{-s-rf}, \quad \hat{t}_2 = X_2^m.$$

We use the notation in Section 6.5. For each critical point $p_\alpha(q) = (a_\alpha(q), b_\alpha(q))$ on the mirror curve, we have

$$\Im(a_\alpha(0)) = -\pi w_3 - \vartheta_\alpha, \quad \Im(b_\alpha(0)) = \frac{\pi}{m} - \varphi_\alpha, \quad e^{\sqrt{-1}\vartheta_\alpha} = \chi_\alpha(\eta_1), \quad e^{\sqrt{-1}\varphi_\alpha} = \chi_\alpha(\eta_2),$$

where $\Im(z)$ is the imaginary part of z . Notice that we have a preferred choice of the labeling of branch points by the elements in G^* with $\vartheta_1 = \varphi_1 = 0$. Let $C = -1 + \sqrt{-1}\mathbb{R} \subset \mathbb{C}^*$, and define (Lagrangian) cycles $\Gamma_{\alpha,q}^{\text{red}} \subset (\mathbb{C}^*)^2$ and $\Gamma_{\alpha,q} \subset (\mathbb{C}^*)^3$ by

$$\Gamma_{\alpha,q}^{\text{red}} := (\mathbb{R}^+ e^{-a_\alpha(q)}) \times (\mathbb{R}^+ e^{-b_\alpha(q)}), \quad \Gamma_{\alpha,q} := \Gamma_{\alpha,q}^{\text{red}} \times C \subset (\mathbb{C}^*)^3.$$

In the perturbed superpotential $\widetilde{W}_q$, the logarithm is defined in the following way: when $q = 0$, we have $X_3 < 0$, $\Im(\log X_1) = \pi w_3 + \vartheta_\alpha$, $\Im(\log X_2) = -\pi/m + \varphi_\alpha$, $\Im(\log X_3) = \pi$. Since the cycle $\Gamma_{\alpha,q}$ is contractible and deforms continuously with respect to q , the choice is fixed by these conditions on $\Gamma_{\alpha,q}$.

Define the oscillatory integral of $\widetilde{W}_q$ to be

$$\tilde{I}_\alpha^{\mathcal{X}}(u) = \int_{\Gamma_{\alpha,q}} e^{-\widetilde{W}_q} \frac{dX_1}{X_1} \frac{dX_2}{X_2} \frac{dX_3}{X_3}.$$

Define $L_\alpha := \{(q_1, \dots, q_p) \in \mathbb{C}^p : q_a X_\alpha^{m_a} Y_\alpha^{n_a - m_a f} \in \mathbb{R}\}$, which is a totally real linear subspace of $\mathbb{C}^p$. We will view $\tilde{I}_\alpha^{\mathcal{X}}(u)$ as a function of $q \in L_\alpha \cong \mathbb{R}^p$ and consider the power series expansion at $q = 0$.

LEMMA 7.3. *If $q \in L_\alpha$ is sufficiently small, then $\Gamma_{\alpha,q}^{\text{red}} = \Gamma_{\alpha,0}^{\text{red}}$.*

Proof. Consider the change of variables $X = X_\alpha A$, $Y = Y_\alpha B$, and $q_a = X_\alpha^{-m_a} Y_\alpha^{-n_a + m_a f} \epsilon_a$ for $a = 1, \dots, p$. Then $q = (q_1, \dots, q_p) \in L_\alpha$ if and only if $\epsilon = (\epsilon_1, \dots, \epsilon_p) \in \mathbb{R}^p$. We have

$$\hat{t}_1 = \frac{w_1}{w_3} A^r B^{-s-rf}, \quad \hat{t}_2 = \frac{w_2}{w_3} B^m, \quad H(X, Y, q) = F(A, B, \epsilon), \quad Y \frac{\partial H}{\partial Y} = G(A, B, \epsilon),$$

where

$$F(A, B, \epsilon) = \frac{w_1}{w_3} A^r B^{-s-rf} + \frac{w_2}{w_3} B^m + 1 + \sum_{a=1}^p \epsilon_a A^{m_a} B^{n_a - f m_a},$$

$$G(A, B, \epsilon) = \frac{|G| w_1 w_2}{w_3} (-A^r B^{-s-rf} + B^m) + \sum_{a=1}^p (n_a - f m_a) \epsilon_a A^{m_a} B^{n_a - f m_a}$$

are $\mathbb{R}$ -valued real analytic functions on $\mathbb{R}^+ \times \mathbb{R}^p$. We have $F(1, 1, 0) = G(1, 1, 0) = 0$, and the Jacobian $|\partial(F, G)/\partial(A, B)|(1, 1, 0)| = -|G|^2 w_1 w_2$ is non-zero. By the implicit function theorem, there exist real analytic functions $c, d: U \rightarrow \mathbb{R}^+$, where U is an open neighborhood of 0 in $\mathbb{R}^p$, such that $c(0) = d(0) = 1$ and $F(c(\epsilon), d(\epsilon), \epsilon) = G(c(\epsilon), d(\epsilon), \epsilon) = 0$. If $\epsilon \in U$, then $e^{-a_\alpha(q)} = e^{-a_\alpha(0)} c(\epsilon)$ and $e^{-b_\alpha(q)} = e^{-b_\alpha(0)} d(\epsilon)$, so $\Gamma_{\alpha,q}^{\text{red}} = \Gamma_{\alpha,0}^{\text{red}}$. $\square$

By Lemma 7.3 and its proof, if $q \in L_\alpha$ and $(X_1, X_2) \in \Gamma_{\alpha,q}^{\text{red}}$ then $\hat{t}_1, \hat{t}_2 \in \mathbb{R}^-$ and $H \in \mathbb{R}$. (Recall that $w_1, w_2 > 0$ and $w_3 = -w_1 - w_2 < 0$.) We first integrate out $\hat{t}_1$ and $\hat{t}_2$, and then integrate out $X_3 \in C$. Recall that $h_1, \dots, h_p$ are age 1 elements in G . Given $\vec{h} = (r_1, \dots, r_p) \in \mathbb{Z}^{\oplus p}$, where $r_a \geq 0$, define $c_i(\vec{h}) = \sum_{a=1}^p r_a c_i(h_a) \in \mathbb{Q}$, and define $\chi_\alpha(\vec{h}) = \chi_\alpha\left(\prod_{a=1}^p h_a^{r_a}\right) \in U(1)$.

Then $\tilde{I}_\alpha^{\mathcal{X}}(u)$ is equal to

$$\begin{aligned}
 & \int_{\Gamma_{\alpha,q}} \left(\exp \left(- \sum_{a=1}^p q_a X_1^{m_a} X_2^{n_a - m_a f} X_3 - t_1 - t_2 - X_3 \right) (-\hat{t}_1)^{uw_1} (-\hat{t}_2)^{uw_2} \right. \\
 & \quad \cdot e^{u(w_1+w_2+w_3)(\log X_3 - \sqrt{-1}\pi)} e^{\sqrt{-1}(\vartheta_\alpha + \pi w_3)u} \left. \right) \frac{d(-\hat{t}_1)d(-\hat{t}_2)dX_3}{|G|(-\hat{t}_1)(-\hat{t}_2)X_3} \\
 &= \frac{-1}{|G|} e^{\sqrt{-1}(\vartheta_\alpha + \pi w_3)u} \sum_{\substack{\vec{h}=(r_1,\dots,r_p) \\ r_a \in \mathbb{Z}_{\geq 0}}} e^{\sqrt{-1}\pi c_3(\vec{h})} \chi_\alpha(\vec{h}) \prod_{a=1}^p \frac{(-q_a)^{r_a}}{r_a!} \\
 & \quad \cdot \int_{\Gamma_{\alpha,q}} \left((-\hat{t}_1)^{c_1(\vec{h})+uw_1-1} (-\hat{t}_2)^{c_2(\vec{h})+uw_2-1} \right. \\
 & \quad \cdot e^{(\log X_3 - \sqrt{-1}\pi)(c_1(\vec{h})+c_2(\vec{h})+c_3(\vec{h})+u(w_1+w_2+w_3)-1)} e^{-\hat{t}_1 X_3 - \hat{t}_2 X_3 - X_3} \left. \right) d(-\hat{t}_1)d(-\hat{t}_2)dX_3 \\
 &= \frac{-1}{|G|} e^{\sqrt{-1}(\vartheta_\alpha + \pi w_3)u} \sum_{\substack{\vec{h}=(r_1,\dots,r_p) \\ r_a \in \mathbb{Z}_{\geq 0}}} e^{\sqrt{-1}\pi c_3(\vec{h})} \chi_\alpha(\vec{h}) \prod_{a=1}^p \frac{(-q_a)^{r_a}}{r_a!} \\
 & \quad \cdot \int_{X_3 \in C} \left(\int_{-\hat{t}_1 > 0} e^{-X_3 \hat{t}_1} (-\hat{t}_1)^{c_1(\vec{h})+uw_1-1} d(-\hat{t}_1) \right) \left(\int_{-\hat{t}_2 > 0} e^{-\hat{t}_2} (-\hat{t}_2)^{c_2(\vec{h})+uw_2-1} d(-\hat{t}_2) \right) \\
 & \quad \cdot e^{-X_3} e^{(\log X_3 - \sqrt{-1}\pi)(c_1(\vec{h})+c_2(\vec{h})+c_3(\vec{h})+u(w_1+w_2+w_3)-1)} dX_3 \\
 &= \frac{-1}{|G|} e^{\sqrt{-1}(\vartheta_\alpha + \pi w_3)u} \sum_{\substack{\vec{h}=(r_1,\dots,r_p) \\ r_a \in \mathbb{Z}_{\geq 0}}} e^{\sqrt{-1}\pi c_3(\vec{h})} \chi_\alpha(\vec{h}) \prod_{a=1}^p \frac{(-q_a)^{r_a}}{r_a!} \Gamma(uw_1 + c_1(\vec{h})) \Gamma(uw_2 + c_2(\vec{h})) \\
 & \quad \cdot \int_{X_3 \in C} e^{-X_3} e^{(\log X_3 - \sqrt{-1}\pi)(c_3(\vec{h})+uw_3-1)} dX_3 \\
 &= \frac{2\pi\sqrt{-1}}{|G|} e^{\sqrt{-1}(\vartheta_\alpha + \pi w_3)u} \sum_{\substack{\vec{h}=(r_1,\dots,r_p) \\ r_a \in \mathbb{Z}_{\geq 0}}} e^{\pi\sqrt{-1}c_3(\vec{h})} \chi_\alpha(\vec{h}) \prod_{a=1}^p \frac{(-q_a)^{r_a}}{r_a!} \cdot \frac{\Gamma(uw_1 + c_1(\vec{h}))\Gamma(uw_2 + c_2(\vec{h}))}{\Gamma(-uw_3 - c_3(\vec{h}) + 1)}.
 \end{aligned}$$

Here we use the following identity.

LEMMA 7.4. *If $\Re z > 0$, then*

$$\frac{\sqrt{-1}}{2\pi} \left(\int_C e^{-z(\log(s) - \sqrt{-1}\pi)} e^{-s} ds \right) = \frac{1}{\Gamma(z)}.$$

Proof. Hankel's representation of the reciprocal Gamma function says

$$\frac{\sqrt{-1}}{2\pi} \left(\int_{C_\delta} e^{-z(\log(t) - \sqrt{-1}\pi)} e^{-t} dt \right) = \frac{1}{\Gamma(z)},$$

where C_δ is a Hankel contour (see Figure 2). The integrand is holomorphic on $\mathbb{C} \setminus [0, \infty)$, and

$$|e^{-z(\log(t) - \sqrt{-1}\pi)} e^{-t}| = e^{-\Re(z) \log |s| + \Im(z)(\arg(t) - \pi) - \Re t} \leq e^{\pi \Im z} |t|^{-\Re z} e^{-(\Re z)t}, \quad (7.5)$$

so we may deform C_δ to the contour $C_{a,b}$ (see Figure 2) without changing the value of the contour

integral. Therefore,

$$\frac{\sqrt{-1}}{2\pi} \left(\int_{C_{a,b}} e^{-z(\log(t) - \sqrt{-1}\pi)} e^{-t} dt \right) = \frac{1}{\Gamma(z)} \quad (7.6)$$

for any $a, b \in (0, \infty)$. The estimate (7.5) implies that contributions to the above contour integral from the two horizontal rays in $C_{a,b}$ tend to zero as $a, b \rightarrow +\infty$. The lemma follows from taking the limit of (7.6) as $a, b \rightarrow +\infty$. $\square$

FIGURE 2. A Hankel contour C_δ and the contour $C_{a,b}$ ($a, b > 0$)

Remark 7.5. Let $f(t)$ be the inverse Laplace transform of $F(s) = \Gamma(z)/s^z$, where $z \in \mathbb{C}$ is a constant with $\Re z > 0$. Lemma 7.4 implies that $f(t) = t^{z-1}$ for $t > 0$.

By Hori–Iqbal–Vafa [HIV00], this oscillatory integral could be reduced to a Laplace transform on the mirror curve. The Landau–Ginzburg model on $(\mathbb{C}^*)^3$ is equivalent to a 5-dimensional Landau–Ginzburg model, and the 5D model is again reduced to a Calabi–Yau threefold without potential. Further dimensional reduction reduces it to the mirror curve.

Introduce two variables $v^+, v^- \in \mathbb{C}$ and the cycles

$$\tilde{\Gamma}_{\alpha,q} = \Gamma_{\alpha,q} \times \{v^+ = \overline{v^-}\}, \quad \tilde{\Gamma}_{\alpha,q}^{\text{red}} = \Gamma_{\alpha,q}^{\text{red}} \times \{v^+ = \overline{v^-}\}.$$

The equation $H(X_1, X_2, q_1, \dots, q_p) = 0$ prescribes the mirror curve. Define the holomorphic volume form

$$\Omega = \frac{dX_1}{X_1} \frac{dX_2}{X_2} \frac{dv^-}{v^-} = dx dy \frac{dv^-}{v^-}.$$

We reduce the oscillatory integral to the mirror curve as follows:

$$\begin{aligned} \tilde{I}_\alpha^{\mathcal{X}}(u) &= \frac{-1}{2\sqrt{-1}\pi} \int_{\tilde{\Gamma}_{\alpha,q}} e^{-X_3(H-v^+v^-)} e^{-ux} \frac{dX_1}{X_1} \frac{dX_2}{X_2} dX_3 dv^+ dv^- \\ &= \int_{\tilde{\Gamma}_{\alpha,q}^{\text{red}}} \delta(H - v^+v^-) e^{-ux} \frac{dX_1}{X_1} \frac{dX_2}{X_2} dv^+ dv^- \\ &= - \int_{\tilde{\Gamma}_{\alpha,q}^{\text{red}} \cap \{H-v^+v^-=0\}} e^{-ux} \frac{dX_1}{X_1} \frac{dX_2}{X_2} \frac{dv^-}{v^-}. \end{aligned}$$

The first identity comes from integrating out v^+ and v^- (notice that on C , we have $\text{Re}(X_3) < 0$), using

$$\int_{v^+=\overline{v^-}} e^{X_3 v^+ v^-} dv^+ dv^- = -\frac{2\pi\sqrt{-1}}{X_3}.$$

The second identity is the Fourier transform

$$\int_C e^{-X_3(H-v^+v^-)} dX_3 = \int_{-\infty}^{+\infty} e^{(1+s\sqrt{-1})(H-v^+v^-)} (-\sqrt{-1}) ds = -2\pi\sqrt{-1}\delta(H-v^+v^-).$$

For the third identity, fixing X_1 and X_2 such that $H \geq 0$, and letting $v^\pm = re^{\pm\sqrt{-1}\theta}$, we have

$$\int_{v^+=v^-} \delta(H-v^+v^-) dv^+ dv^- = 2\pi\sqrt{-1} \int_0^\infty \delta(H-r^2) 2r dr = 2\pi\sqrt{-1} = - \int_{|v^-|=H} \frac{dv^-}{v^-}.$$

This integration is further reduced to the mirror curve $H(e^{-x}, e^{-y}) = 0$ as follows:

$$\tilde{I}_\alpha^\mathcal{X}(u) = - \int_{\tilde{\Gamma}_{\alpha,q}^{\text{red}} \cap \{H-v^+v^-=0\}} e^{-ux} dx dy \frac{dv^-}{v^-} = 2\sqrt{-1}\pi \int_{\gamma_\alpha = \tilde{\Gamma}_{\alpha,q}^{\text{red}} \cap \{H=v^-=0\}} e^{-ux} y dx.$$

Notice that we use the fact that $d(e^{-ux} y dx (dv^-/v^-)) = -e^{-ux} \Omega$ near $\tilde{\Gamma}_{\alpha,q}^{\text{red}} \cap \{H-v^+v^-=0\}$. Therefore, we obtain the following formula.

THEOREM 7.6. *The integral $\int_{\gamma_\alpha} e^{-ux} \Phi$ is equal to*

$$\frac{1}{|G|} e^{\sqrt{-1}(\vartheta_\alpha + \pi w_3)u} \sum_{\substack{\vec{h}=(r_1, \dots, r_p) \\ r_a \in \mathbb{Z}_{\geq 0}}} e^{\pi\sqrt{-1}c_3(\vec{h})} \chi_\alpha(\vec{h}) \prod_{a=1}^p \frac{(-q_a)^{r_a}}{r_a!} \cdot \frac{\Gamma(uw_1 + c_1(\vec{h}))\Gamma(uw_2 + c_2(\vec{h}))}{\Gamma(-uw_3 - c_3(\vec{h}) + 1)}.$$

COROLLARY 7.7. *We have*

$$\lim_{q \rightarrow 0} \int_{\gamma_\alpha} e^{-ux} \nabla_h \Phi = \frac{e^{\sqrt{-1}(\vartheta_\alpha + \pi w_3)u}}{|G|} (-1)^{\text{age}(h)} e^{\sqrt{-1}\pi c_3(h)} \chi_\alpha(h) \frac{\Gamma(uw_1 + c_1(h))\Gamma(uw_2 + c_2(h))}{\Gamma(-uw_3 - c_3(h) + 1)}.$$

THEOREM 7.8. *We have*

$$f_\beta^\alpha(u, 0) = \sum_{h \in G} \frac{\chi_\alpha(h) \chi_\beta(h^{-1})}{|G|} \exp \left(\sum_{m \geq 1} \frac{(-1)^{m+1}}{m(m+1)} \sum_{i=1}^3 B_{m+1}(c_i(h)) (w_i u)^{-m} \right).$$

Proof. By Proposition 7.2 and Corollary 7.7,

$$f_\beta^\alpha(u, 0) = \sum_{h \in G} \frac{\chi_\alpha(h) \chi_\beta(h^{-1})}{|G|} \cdot C_h \cdot \frac{\Gamma(uw_1 + c_1(h))\Gamma(uw_2 + c_2(h))}{\Gamma(-uw_3 - c_3(h) + 1)},$$

where

$$C_h = \frac{1}{\sqrt{-2\pi}} \exp \left(-u \sum_{i=1}^3 w_i \log w_i \right) \cdot u^{3/2 - \text{age}(h)} \prod_{i=1}^3 w_i^{-c_i(h)+1/2} e^{\sqrt{-1}\pi(w_3 u + c_3(h))}.$$

By the Stirling formula [KP11],

$$\begin{aligned}
 & \log \frac{\Gamma(w_1 u + c_1(h)) \Gamma(w_2 u + c_2(h))}{\Gamma(-w_3 u + 1 - c_3(h))} \\
 &= \left(w_1 u + c_1(h) - \frac{1}{2} \right) \log(w_1 u) + \left(w_2 u + c_2(h) - \frac{1}{2} \right) \log(w_2 u) \\
 & \quad - \left(-w_3 u - c_3(h) + \frac{1}{2} \right) \log(-w_3 u) + \frac{1}{2} \log(2\pi) \\
 & \quad + \sum_{m \geq 1} \frac{(-1)^{m+1}}{m(m+1)} \sum_{i=1}^3 B_{m+1}(c_i(h)) (w_i u)^{-m} \\
 &= \log \sqrt{-2\pi} - \sqrt{-1} \pi (w_3 u + c_3(h)) + \left(\text{age}(h) - \frac{3}{2} \right) \log u + \sum_{i=1}^3 \left(c_i(h) - \frac{1}{2} \right) \log w_i \\
 & \quad + u \sum_{i=1}^3 w_i \log w_i + \sum_{m \geq 1} \frac{(-1)^{m+1}}{m(m+1)} \sum_{i=1}^3 B_{m+1}(c_i(h)) (w_i u)^{-m}, \\
 & \frac{\Gamma(w_1 u + c_1(h)) \Gamma(w_2 u + c_2(h))}{\Gamma(-w_3 u + 1 - c_3(h))} \\
 &= \sqrt{-2\pi} e^{-\sqrt{-1} \pi (w_3 u + c_3(h))} u^{\text{age}(h) - 3/2} \prod_{i=1}^3 w_i^{c_i(h) - 1/2} \exp \left(u \sum_{i=1}^3 w_i \log w_i \right) \\
 & \quad \cdot \exp \left(\sum_{m \geq 1} \frac{(-1)^{m+1}}{m(m+1)} \sum_{i=1}^3 B_{m+1}(c_i(h)) (w_i u)^{-m} \right) \\
 &= C_h^{-1} \cdot \exp \left(\sum_{m \geq 1} \frac{(-1)^{m+1}}{m(m+1)} \sum_{i=1}^3 B_{m+1}(c_i(h)) (w_i u)^{-m} \right).
 \end{aligned}$$

Therefore,

$$f_\beta^\alpha(u, 0) = \sum_{h \in G} \frac{\chi_\alpha(h) \chi_\beta(h^{-1})}{|G|} \exp \left(\sum_{m \geq 1} \frac{(-1)^{m+1}}{m(m+1)} \sum_{i=1}^3 B_{m+1}(c_i(h)) (w_i u)^{-m} \right). \quad \square$$

Define

$$\check{h}^\alpha(u, q) := \frac{u^{3/2}}{\sqrt{\pi}} e^{u a_\alpha} \int_{z \in \gamma_\alpha} e^{-u x} \Phi = \sum_{\beta \in G^*} f_\beta^\alpha(u, q) h_1^\beta. \quad (7.7)$$

COROLLARY 7.9.

$$\check{h}^\alpha(u, 0) = \frac{1}{|G|} \sqrt{\frac{-2}{w_1 w_2 w_3}} \exp \left(\sum_{m \geq 1} \frac{(-1)^{m+1}}{m(m+1)} \sum_{i=1}^3 B_{m+1}(w_i u)^{-m} \right).$$

Proof. Recall that

$$\lim_{q \rightarrow 0} h_1^\beta = \frac{1}{|G|} \sqrt{\frac{-2}{w_1 w_2 w_3}}$$

for any $\beta \in G^*$, so

$$\check{h}^\alpha(u, 0) = \frac{1}{|G|} \sqrt{\frac{-2}{w_1 w_2 w_3}} \sum_{\beta \in G^*} f_\beta^\alpha(u, 0).$$

By Theorem 7.8,

$$\sum_{\beta \in G^*} f_{\beta}^{\alpha}(u, 0) = \sum_{h \in G} \frac{\chi_{\alpha}(h)}{|G|} \left(\sum_{\beta \in G^*} \chi_{\beta}(h^{-1}) \right) \exp \left(\sum_{m \geq 1} \frac{(-1)^{m+1}}{m(m+1)} \sum_{i=1}^3 B_{m+1}(c_i(h)) (w_i u)^{-m} \right),$$

where

$$\sum_{\beta \in G^*} \chi_{\beta}(h^{-1}) = \sum_{\beta \in G^*} \chi_{\beta}(h^{-1}) \chi_{\beta}(1) = |G| \delta_{h^{-1}, 1}.$$

So

$$\begin{aligned} \check{h}^{\alpha}(u, 0) &= \frac{1}{|G|} \sqrt{\frac{-2}{w_1 w_2 w_3}} \chi_{\alpha}(1) \exp \left(\sum_{m \geq 1} \frac{(-1)^{m+1}}{m(m+1)} \sum_{i=1}^3 B_{m+1}(c_i(1)) (w_i u)^{-m} \right) \\ &= \frac{1}{|G|} \sqrt{\frac{-2}{w_1 w_2 w_3}} \exp \left(\sum_{m \geq 1} \frac{(-1)^{m+1}}{m(m+1)} \sum_{i=1}^3 B_{m+1}(w_i u)^{-m} \right). \end{aligned} \quad \square$$

7.3 Inverse Laplace transform and expansion at $X = 0$

Recall that on the Lefschetz thimble γ_{α} , the local coordinate is $\zeta_{\alpha}(q)$, and $x = a_{\alpha}(q) + \zeta_{\alpha}(q)^2$. We choose $\zeta_{\alpha}(q)$ such that when $\zeta_{\alpha}(0) \rightarrow -\infty$, the corresponding point on the mirror curve Σ_0 approaches a point with $X = 0$ and $Y^m = -1$ (this is consistent with our choice $h_1^{\alpha}(0) > 0$). Furthermore, we require that for any element $\alpha' \in G^*$, its action on the compactified mirror curve $\bar{\Sigma}_0$ moves the end point $\zeta_{\alpha} = -\infty$ on γ_{α} to the end point $\zeta_{\alpha'\alpha} = -\infty$ on $\gamma_{\alpha'\alpha}$. (Here $\alpha'\alpha \in G^*$ is the product of $\alpha, \alpha' \in G^*$; that is, $\chi_{\alpha'\alpha}(h) = \chi_{\alpha'}(h) \chi_{\alpha}(h)$ for all $h \in G$.)

We label the m points with $X = 0$ and $Y^m = -1$ on $\bar{\Sigma}_0$ by elements in $\mu_m^* \cong \mathbb{Z}_m$ as follows: $\bar{p}_{\ell} := (0, e^{\pi\sqrt{-1}(2\ell+1)/m})$ for $\ell \in \mathbb{Z}_m$. With our convention, if $\chi_{\alpha} = \chi_1^j \chi_2^{\ell}$, then the end point $\zeta_{\alpha} = -\infty$ is $\bar{p}_{\ell-1}$.

Every 1-form on γ_{α} is exact, so there exists a function $\xi_{\beta,0}^{\alpha}$ on γ_{α} such that $d\xi_{\beta,0}^{\alpha} = \theta_0^{\beta}|_{\gamma_{\alpha}}$. The Laplace transform of θ_0^{β} along γ_{α} is

$$\begin{aligned} f_{\beta}^{\alpha}(u) &= \frac{e^{u a_{\alpha}}}{2\sqrt{\pi}u} \int_{z \in \gamma_{\alpha}} e^{-ux(z)} \theta_0^{\beta} = \frac{\sqrt{u}}{2\sqrt{\pi}} \int_{z \in \gamma_{\alpha}} e^{-u(x(z)-a_{\alpha})} \xi_{\beta,0}^{\alpha} d(x(z)-a_{\alpha}) \\ &= \frac{\sqrt{u}}{2\sqrt{\pi}} \int_{x-a_{\alpha} \in \mathbb{R}^+} e^{-u(x-a_{\alpha})} (\xi_{\beta,0}^{\alpha+}(x) - \xi_{\beta,0}^{\alpha-}(x)) d(x-a_{\alpha}), \end{aligned}$$

where $\xi_{\beta,0}^{\alpha\pm}(x) = \xi_{\beta,0}^{\alpha}(x, y(\pm\sqrt{x-a_{\alpha}})) \in L^2(0, \infty)$. From the proof of Theorem 7.8,

$$\begin{aligned} f_{\beta}^{\alpha}(u, 0) &= \frac{e^{-u \sum_{i=1}^3 w_i \log w_i}}{\sqrt{-2\pi}} \sum_{h \in G} \left(e^{\sqrt{-1}\pi(w_3 u + c_3(h))} \frac{\chi_{\alpha}(h) \chi_{\beta}(h^{-1})}{|G|} \right. \\ &\quad \cdot u^{3/2 - \text{age}(h)} \cdot \prod_{i=1}^3 w_i^{1/2 - c_i(h)} \frac{\Gamma(w_1 u + c_1(h)) \Gamma(w_2 u + c_2(h))}{\Gamma(-w_3 u + 1 - c_3(h))} \Big). \end{aligned}$$

So the “classical Laplace transform” is

$$\begin{aligned} & \int_{x-a_\alpha \in \mathbb{R}^+} e^{-u(x-a_\alpha)} (\xi_{\beta,0}^\alpha|_{q=0}) d(x-a_\alpha) \\ &= -\sqrt{-2} e^{-u \sum_{i=1}^3 w_i \log w_i} \sum_{h \in G} \left(e^{\sqrt{-1}\pi(w_3 u + c_3(h))} \frac{\chi_\alpha(h) \chi_\beta(h^{-1})}{|G|} \right. \\ & \quad \cdot u^{1-\text{age}(h)} \prod_{i=1}^3 w_i^{1/2-c_i(h)} \frac{\Gamma(w_1 u + c_1(h)) \Gamma(w_2 u + c_2(h))}{\Gamma(-w_3 u + 1 - c_3(h))} \Big). \end{aligned}$$

By the inverse Laplace transform formula,

$$\begin{aligned} -\lim_{q \rightarrow 0} \xi_{\beta,0}^{\alpha-} &= -\sqrt{-2} \sum_{\substack{(d_0, h) \in \mathbb{Z} \times G \\ d_0/r - c_1(h) \in \mathbb{Z}_{\geq 0}}} \text{Res}_{u=-d_0} (\Gamma(w_1 u + c_1(h))) \left(e^{u(x-a_\alpha - \sum_{i=1}^3 w_i \log w_i)} e^{\sqrt{-1}\pi(w_3 u + c_3(h))} \right. \\ & \quad \cdot \frac{\chi_\alpha(h) \chi_\beta(h^{-1})}{|G|} u^{1-\text{age}(h)} \prod_{i=1}^3 w_i^{1/2-c_i(h)} \frac{\Gamma(w_2 u + c_2(h))}{\Gamma(-w_3 u + 1 - c_3(h))} \Big) \Big|_{u=-d_0}, \\ \lim_{q \rightarrow 0} \xi_{\beta,0}^{\alpha+} &= -\sqrt{-2} \sum_{\substack{(d_0, h) \in \mathbb{Z} \times G \\ w_2 d_0 - c_2(h) \in \mathbb{Z}_{\geq 0}}} \text{Res}_{u=-d_0} (\Gamma(w_2 u + c_2(h))) \left(e^{u(x-a_\alpha - \sum_{i=1}^3 w_i \log w_i)} e^{\sqrt{-1}\pi(w_3 u + c_3(h))} \right. \\ & \quad \cdot \frac{\chi_\alpha(h) \chi_\beta(h^{-1})}{|G|} u^{1-\text{age}(h)} \prod_{i=1}^3 w_i^{\frac{1}{2}-c_i(h)} \frac{\Gamma(w_1 u + c_1(h))}{\Gamma(-w_3 u + 1 - c_3(h))} \Big) \Big|_{u=-d_0}. \end{aligned}$$

Remark 7.10. We obtain $\xi_{\beta,0}^{\alpha+}$ and $\xi_{\beta,0}^{\alpha-}$ by taking residues around the poles of $\Gamma(w_1 u + c_1(h))$ and $\Gamma(w_2 u + c_2(h))$. When both $w_1 u + c_1(h)$ and $w_2 u + c_2(h)$ are non-positive integers, the sum of the respective terms in $\xi_{\beta,0}^{\alpha+} - \xi_{\beta,0}^{\alpha-}$ is the residue.

We are interested in the expansion of $\xi_{\beta,0}^\alpha$ at $\zeta = -\infty$, that is, the expansion of $\xi_{\beta,0}^{\alpha-}$ at $X = 0$,

$$\begin{aligned} \lim_{q \rightarrow 0} \xi_{\beta,0}^{\alpha-} &= \sqrt{-2} \sum_{\substack{(d_0, k) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_m \\ h=h(d_0, k)}} e^{-d_0 x} \chi_\alpha(\eta_1^{-d_0}) e^{-\sqrt{-1}\pi(d_0 w_3 - c_3(h))} \\ & \quad \cdot \frac{\chi_\alpha(h) \chi_\beta(h^{-1})}{|G|} (-d_0)^{1-\text{age}(h)} \prod_{i=1}^3 w_i^{1/2-c_i(h)} \frac{(-1)^{\lfloor d_0/r \rfloor} \Gamma(-d_0 w_2 + c_2(h))}{\Gamma(d_0 w_1 - c_1(h) + 1) \Gamma(d_0 w_3 + 1 - c_3(h))} \\ &= \sqrt{-2} \sum_{\substack{(d_0, k) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_m \\ h=h(d_0, k)}} X^{d_0} e^{-\sqrt{-1}\pi(d_0 w_3 - c_3(h))} \frac{\chi_\beta(h^{-1})}{m} \left(\frac{1}{d_0} \right)^{\text{age}(h)-1} \\ & \quad \cdot \prod_{i=1}^3 w_i^{1/2-c_i(h)} \frac{\Gamma(d_0(w_1 + w_2) + c_3(h))}{\Gamma(d_0 w_1 - c_1(h) + 1) \Gamma(d_0 w_2 - c_2(h) + 1)} \chi_\alpha(\eta_2^{-k}). \end{aligned}$$

This is the expansion of $\lim_{q \rightarrow 0} \xi_{\beta,0}^\alpha$ at $X = 0$ and $Y = e^{-\sqrt{-1}\pi/m} \chi_\alpha(\eta_2)$. If $\alpha = \chi_1^j \chi_2^\ell$, then

$$\chi_\alpha(\eta_2^{-k}) = e^{-2\sqrt{-1}\pi k \ell / m}$$

depends only on $\ell \in \{0, 1, \dots, m-1\}$. Let $\xi_{\beta,0}^{\ell-}$ be the expansion of $\xi_{\beta,0}^\alpha$ at $\bar{p}_\ell = (0, e^{\pi\sqrt{-1}(2\ell+1)/m})$

for $\alpha = \chi_1^j \chi_2^{\ell+1}$. Define

$$\psi_\ell := \frac{1}{m} \sum_{k=0}^{m-1} \omega_m^{-k\ell} \mathbf{1}'_{k/m}, \quad \ell = 0, 1, \dots, m-1,$$

where $\omega_m = e^{2\pi\sqrt{-1}/m}$. Then $\{\psi_0, \dots, \psi_{m-1}\}$ is a canonical basis of $H_{\text{CR}}^*(\mathcal{B}\mu_m; \mathbb{C})$. Define

$$\xi_0^\beta := \sum_{\ell=0}^{m-1} \xi_{\beta,0}^{\ell-} \psi_\ell, \quad (7.8)$$

which takes values in $H_{\text{CR}}^*(\mathcal{B}\mu_m; \mathbb{C})$. Compare with the definition of $\tilde{\xi}_0^\beta$ in Section 5.2; we obtain the following identity.

PROPOSITION 7.11. *We have*

$$\lim_{q \rightarrow 0} \xi_0^\beta = -\frac{1}{|G|} \sqrt{\frac{-2}{w_1 w_2 w_3}} \tilde{\xi}_0^\beta|_{v=1}.$$

7.4 Eynard–Orantin topological recursion

Let $\omega_{g,n}$ be defined recursively by the Eynard–Orantin topological recursion [EO07]:

$$\omega_{0,1} = 0, \quad \omega_{0,2} = B(z_1, z_2).$$

When $2g - 2 + n > 0$,

$$\begin{aligned} \omega_{g,n}(p_1, \dots, p_n) = & \sum_{\alpha \in G^*} \text{Res}_{p \rightarrow p_\alpha} \frac{\int_{\xi=p}^{\bar{p}} B(p_n, \xi)}{2(\Phi(p) - \Phi(\bar{p}))} \left(\omega_{g-1,n+1}(p, \bar{p}, p_1, \dots, p_{n-1}) \right. \\ & \left. + \sum_{g_1+g_2=g} \sum_{\substack{I \cup J = \{1, \dots, n-1\} \\ I \cap J = \emptyset}} \omega_{g_1, |I|+1}(p, p_I) \omega_{g_2, |J|+1}(\bar{p}, p_J) \right), \end{aligned}$$

where $p, \bar{p} \rightarrow p_\alpha$, $X(p) = X(\bar{p})$, and $p \neq \bar{p}$.

The B-model invariants $\omega_{g,n}$ can be expressed as sums over labeled graphs involving intersection numbers on moduli spaces of stable curves [KO10, Eyn11, Eyn14, DOSS14]. We will use the formula stated in [DOSS14, Theorem 3.7], which is equivalent to the formula in [Eyn11, Theorem 5.1]. To state this graph sum, we introduce some definitions. Following Eynard [Eyn11], define the Laplace transform of the Bergman kernel $B(z_1, z_2)$

$$\check{B}^{\alpha,\beta}(u, v, q) := \frac{uv}{u+v} \delta_{\alpha,\beta} + \frac{\sqrt{uv}}{2\pi} e^{ua_\alpha + va_\beta} \int_{z_1 \in \gamma_\alpha} \int_{z_2 \in \gamma_\beta} B(z_1, z_2) e^{-ux(z_1) - vx(z_2)}, \quad (7.9)$$

where $\alpha, \beta \in G^*$. By [Eyn11, equation (B.9)],

$$\check{B}^{\alpha,\beta}(u, v, q) = \frac{uv}{u+v} \left(\delta_{\alpha,\beta} - \sum_{\gamma \in G^*} f_\gamma^\alpha(u, q) f_\gamma^\beta(v, q) \right). \quad (7.10)$$

Let $\check{B}_{k,l}^{\alpha,\beta}(q)$ be defined by (6.16). Then $\check{B}^{\alpha,\beta}(u, v, q) = \sum_{k,l} \check{B}_{k,l}^{\alpha,\beta}(q) u^{-k} v^{-l}$. Let $\check{h}_k^\alpha(q)$ be defined by

$$\check{h}^\alpha(u, q) = \sum_k \check{h}_k^\alpha(q) u^{-k},$$

where $\check{h}^\alpha$ is defined by (7.7).

Given a labeled graph $\vec{\Gamma} = (\Gamma, g, \alpha, k) \in \mathbf{\Gamma}_{g,0,n}(\mathcal{BG})$ with $L^O(\Gamma) = \{l_1, \dots, l_n\}$, we define its weight to be

$$w(\vec{\Gamma}) = (-1)^{g(\vec{\Gamma})-1} \prod_{v \in V(\Gamma)} \left(\frac{h_1^\alpha}{\sqrt{-2}} \right)^{2-2g-\text{val}(v)} \left\langle \prod_{h \in H(v)} \tau_{k(h)} \right\rangle \prod_{g(v)} \prod_{e \in E(\Gamma)} \check{B}_{k(e), l(e)}^{\alpha(v_1(e)), \alpha(v_2(e))}(e) \\ \cdot \prod_{j=1}^n \frac{1}{\sqrt{-2}} \theta_{k(l_j)}^{\alpha(l_j)}(z_j) \prod_{l \in \mathcal{L}^1(\Gamma)} \left(-\frac{1}{\sqrt{-2}} \right) \check{h}_{k(l)}^{\alpha(l)}(l).$$

In our notation, [DOSS14, Theorem 3.7] is equivalent to the following.

THEOREM 7.12. *For $2g - 2 + n > 0$, we have*

$$\omega_{g,n} = \sum_{\Gamma \in \mathbf{\Gamma}_{g,0,n}(\mathcal{BG})} \frac{w(\vec{\Gamma})}{|\text{Aut}(\vec{\Gamma})|}.$$

Example 7.13 (Pair of pants).

$$\omega_{0,3}(p_1, p_2, p_3) = \sum_{\alpha \in G^*} \frac{1}{2h_1^\alpha} \theta_0^\alpha(p_1) \theta_0^\alpha(p_2) \theta_0^\alpha(p_3).$$

We now consider the unstable case $(g, n) = (0, 2)$. Recall that $dx = -dX/X$ is a meromorphic 1-form on $\bar{\Sigma}_q$, and $d/dx = -Xd/dX$ is a meromorphic vector field on $\bar{\Sigma}_q$. Define

$$C(z_1, z_2) := \left(-\frac{\partial}{\partial x}(z_1) - \frac{\partial}{\partial x}(z_2) \right) \left(\frac{\omega_{0,2}}{dx_1 dx_2} \right) (z_1, z_2) d(x(z_1))(dx(z_2)). \quad (7.11)$$

Then $C(z_1, z_2)$ is meromorphic on $(\bar{\Sigma}_q)^2$ and holomorphic on $(\bar{\Sigma}_q \setminus \{p_\alpha : \alpha \in G^*\})^2$. It only has double poles on $\{p_\alpha : \alpha \in G^*\} \times \bar{\Sigma}_q$ and $\bar{\Sigma}_q \times \{p_\beta : \beta \in G^*\}$.

LEMMA 7.14. *We have*

$$C(z_1, z_2) = \frac{1}{2} \sum_{\gamma \in G^*} \theta_0^\gamma(z_1) \theta_0^\gamma(z_2).$$

Proof. First, we show that the Laplace transforms of both sides are asymptotically equal. For any $\alpha, \beta \in G^*$,

$$\int_{z_1 \in \gamma_\alpha} \int_{z_2 \in \gamma_\beta} e^{-u(x(z_1)-a_\alpha)-v(x(z_2)-a_\beta)} C(z_1, z_2) \\ = (-u - v) \int_{z_1 \in \gamma_\alpha} \int_{z_2 \in \gamma_\beta} e^{-u(x(z_1)-a_\alpha)-v(x(z_2)-a_\beta)} \omega_{0,2} \\ \sim 2\pi\sqrt{uv} \sum_{\gamma \in G^*} f_\gamma^\alpha(u, q) f_\gamma^\beta(v, q) \\ \sim \frac{1}{2} \int_{z_1 \in \gamma_\alpha} \int_{z_2 \in \gamma_\beta} e^{-u(x(z_1)-a_\alpha)-v(x(z_2)-a_\beta)} \theta_0^\gamma(z_1) \theta_0^\gamma(z_2).$$

Define their difference

$$\omega = C(z_1, z_2) - \frac{1}{2} \sum_{\gamma \in G^*} \theta_0^\gamma(z_1) \theta_0^\gamma(z_2);$$

its Laplace transform at $\gamma_\alpha \times \gamma_\beta$ for any $\alpha, \beta \in G^*$ vanishes. For $i = 1, \dots, g$,

$$\int_{p_2 \in A_i} \omega_{0,2}(p_1, p_2) = 0, \quad \int_{A_i} \theta_0^\alpha = 0.$$

Then

$$\int_{p_2 \in A_i} \omega(p_1, p_2) = 0, \quad \int_{p_2 \in A_i} \operatorname{Res}_{p_1 \rightarrow p_\alpha} \zeta_{1,\alpha}(p_1) \omega(p_1, p_2) = 0$$

for all $i = 1, \dots, \mathfrak{g}$. Notice that the 1-form

$$\operatorname{Res}_{p_1 \rightarrow p_\alpha} \zeta_{1,\alpha}(p_1) \omega(p_1, p_2) = [u^0] \frac{-1}{2\sqrt{\pi u}} \int_{p_1 \in \gamma_\alpha} e^{-u(x(p_1) - a_\alpha)} \omega(p_1, p_2)$$

is a well-defined holomorphic form on $\bar{\Sigma}_q$. It has no poles; otherwise, any possible double pole at p_β implies a non-zero Laplace transform of ω at $\gamma_\alpha \times \gamma_\beta$. It follows from the vanishing A-cycles integrals that

$$\operatorname{Res}_{p_1 \rightarrow p_\alpha} \zeta_{1,\alpha}(p_1) \omega(p_1, p_2) = 0$$

for any $\alpha \in G^*$, and then ω does not have any poles since it is symmetric and could only have double poles at the ramification points. Therefore, by the vanishing A-periods of ω , we know that $\omega = 0$. $\square$

7.5 B-model potentials

Choose $\delta > 0$ and $\epsilon > 0$ sufficiently small that for $|q| < \epsilon$, the meromorphic function $X: \bar{\Sigma}_q \rightarrow \mathbb{C} \cup \{\infty\}$ restricts to an isomorphism

$$X_q^\ell: D_q^\ell \rightarrow D_\delta = \{X \in \mathbb{C} : |X| < \delta\},$$

where D_q^ℓ is an open neighborhood of $\bar{p}_\ell \in X^{-1}(0)$ and $\ell = 0, \dots, m-1$. Define

$$\rho_q^{\ell_1, \dots, \ell_n} := (X_q^{\ell_1})^{-1} \times \dots \times (X_q^{\ell_n})^{-1}: (D_\delta)^n \rightarrow D_q^{\ell_1} \times \dots \times D_q^{\ell_n} \subset (\bar{\Sigma}_q)^n.$$

- (1) (Disk invariants) At $q = 0$, we have $Y(\bar{p}_\ell)^m = -1$ for $\ell = 0, \dots, m-1$. When ϵ and δ are sufficiently small, $Y(\rho_q^\ell(X)) \in \mathbb{C} \setminus [0, \infty)$. Choose a branch of logarithm $\log: \mathbb{C} \setminus [0, \infty) \rightarrow (0, 2\pi)$, and define $y_q^\ell(X) = -\log Y(\rho_q^\ell(X))$. The function $y_q^\ell(X)$ depends on the choice of logarithm, but $y_q^\ell(X) - y_q^\ell(0)$ does not. The form $dx = -dX/X$ is a meromorphic 1-form on $\mathbb{C}$ with a simple pole at $X = 0$, and $(y_q^\ell(X) - y_q^\ell(0))dx$ is a holomorphic 1-form on D_δ . Define the *B-model disk potential* by

$$\check{F}_{0,1}(q; X) := \sum_{\ell \in \mathbb{Z}_m} \int_0^X (y_q^\ell(X') - y_q^\ell(0)) \left(-\frac{dX'}{X'} \right) \cdot \psi_\ell;$$

this takes values in $H^*(\mathcal{B}\mu_m; \mathbb{C})$.

- (2) (Annulus invariants) The form

$$(\rho_q^{\ell_1, \ell_2})^* \omega_{0,2} - \frac{dX_1 dX_2}{(X_1 - X_2)^2}$$

is holomorphic on $D_\delta \times D_\delta$. Define the *B-model annulus potential* by

$$\check{F}_{0,2}(q; X_1, X_2) := \sum_{\ell_1, \ell_2 \in \mathbb{Z}_m} \int_0^{X_1} \int_0^{X_2} \left((\rho_q^{\ell_1, \ell_2})^* \omega_{0,2} - \frac{dX'_1 dX'_2}{(X'_1 - X'_2)^2} \right) \cdot \psi_{\ell_1} \otimes \psi_{\ell_2};$$

this takes values in $H^*(\mathcal{B}\mu_m; \mathbb{C})^{\otimes 2}$.

- (3) For $2g - 2 + n > 0$, the form $(\rho_q^{\ell_1, \dots, \ell_n})^* \omega_{g,n}$ is holomorphic on $(D_\delta)^n$. Define

$$\check{F}_{g,n}(q; X_1, \dots, X_n) := \sum_{\ell_1, \dots, \ell_n \in \mathbb{Z}_m} \int_0^{X_1} \dots \int_0^{X_n} (\rho_q^{\ell_1, \dots, \ell_n})^* \omega_{g,n} \cdot \psi_{\ell_1} \otimes \dots \otimes \psi_{\ell_n};$$

this takes values in $H^*(\mathcal{B}\mu_m; \mathbb{C})^{\otimes n}$.

For $g \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{> 0}$, the function $\check{F}_{g,n}(q; X_1, \dots, X_n)$ is holomorphic on $B_\epsilon \times (D_\delta)^n$ when $\epsilon, \delta > 0$ are sufficiently small. By construction, the power series expansion of $\check{F}_{g,n}(q; X_1, \dots, X_n)$ involves only positive powers of X_i .

For $\beta \in G^*$, let $\xi_{\beta,0}^{\ell-}$ be defined as in Section 7.3. Then $\xi_{\beta,0}^{\ell-}(X) = \int_0^X (\rho_q^\ell)^* \theta_0^\beta$.

7.6 Special geometry and B-model graph sum

By the special geometry property of the topological recursion ([EO15, Theorem 4.4], proved in [EO07]),

$$\nabla_{\partial/\partial\tau_a} \omega_{g,n}(z_1, \dots, z_n) = \int_{z_{n+1} \in B_a} \omega_{g,n+1}(z_1, \dots, z_{n+1}),$$

where B_a is defined as in Section 6.4. At $q = 0$,

$$\begin{aligned} \int_{z_{n+1} \in B_a} \theta_k^\alpha(z_{n+1}) &= [u^{-k}] \int_{z_{n+1} \in B_a} \left(-\frac{\sqrt{u} e^{ua_\alpha}}{\sqrt{\pi}} \right) \int_{z' \in \gamma_\alpha} B(z_{n+1}, z') e^{-ux(z')} \\ &= [u^{-k}] \left(-\frac{\sqrt{u} e^{ua_\alpha}}{\sqrt{\pi}} \right) \int_{z' \in \gamma_\alpha} (\nabla_{\partial/\partial\tau_a} \Phi)(z') e^{-ux(z')} \\ &= [u^{-k}] \frac{1}{|G|} \sqrt{\frac{-2}{w_1 w_2 w_3}} \prod_{i=1}^3 w_i^{c_i(h_a)} \sum_{\beta \in G^*} f_\beta^\alpha(u) \chi_\beta(h_a). \end{aligned}$$

Define the B-model primary leaf to be

$$(\check{h}\tau)_k^\alpha = [u^{-k}] \left(\frac{1}{|G|} \sqrt{\frac{-2}{w_1 w_2 w_3}} \sum_{\beta \in G^*} \sum_{a=1}^p \prod_{i=1}^3 w_i^{c_i(h_a)} f_\beta^\alpha(u, 0) \chi_\beta(h_a) \tau_a \right). \quad (7.12)$$

For any $\alpha \in G^*$, $k \in \mathbb{Z}_{\geq 0}$, and $\ell \in \mathbb{Z}_m$, the form $(\rho_0^\ell)^* \theta_k^\alpha$ is a holomorphic 1-form on D_δ for small enough $\delta > 0$. To each open leaf l_j of a labeled graph $\vec{\Gamma} \in \Gamma_n(\mathcal{X})$, we assign

$$(\mathcal{L}^\xi)_k^\alpha(l_j) = \frac{1}{\sqrt{-2}} \sum_{\ell \in \mathbb{Z}_m} \left(\int_0^{X_j} (\rho_0^\ell)^* \theta_k^\alpha \right) \psi_\ell. \quad (7.13)$$

The Taylor series expansion of $(\rho_{\vec{\tau}}^{\ell_1, \dots, \ell_n})^* \omega_{g,n}$ at $\tau = 0$ is

$$(\rho_{\vec{\tau}}^{\ell_1, \dots, \ell_n})^* \omega_{g,n} = \sum_{l_1, \dots, l_p \in \mathbb{Z}_{\geq 0}} (\rho_0^{\ell_1, \dots, \ell_n})^* \left((\nabla_{\partial/\partial\tau_1})^{l_1} \dots (\nabla_{\partial/\partial\tau_p})^{l_p} \omega_{g,n} \Big|_{\tau=0} \right) \cdot \frac{\tau_1^{l_1}}{l_1!} \dots \frac{\tau_p^{l_p}}{l_p!}.$$

Recall that $\Gamma_{g,n}(\mathcal{X})$ is the set of stable genus g graphs with n open leaves and any number of primary leaves. Given $\vec{\Gamma} = (\Gamma, g, \alpha, k) \in \Gamma_{g,n}(\mathcal{X})$ with open leaves $l_1, \dots, l_n$, we define its B-model weight to be

$$\begin{aligned} w_B(\vec{\Gamma}) &= (-1)^{g(\vec{\Gamma})-1} \prod_{v \in V(\Gamma)} \left(\frac{h_1^\alpha|_{\tau=0}}{\sqrt{-2}} \right)^{2-2g-\text{val}(v)} \left\langle \prod_{h \in H(v)} \tau_{k(h)} \right\rangle \prod_{g(v) e \in E(\Gamma)} \check{B}_{k(e), l(e)}^{\alpha(v_1(e)), \alpha(v_2(e))}(e) \Big|_{\tau=0} \\ &\cdot \prod_{l \in L^o(\Gamma)} \frac{1}{\sqrt{-2}} (\check{h}\tau)_{k(l)}^{\alpha(l)} \cdot \prod_{j=1}^n (\mathcal{L}^\xi)_k^\alpha(l_j) \prod_{l \in L^1(\Gamma)} \left(-\frac{1}{\sqrt{-2}} \right) \check{h}_{k(l)}^{\alpha(l)}(l) \Big|_{\tau=0}. \end{aligned} \quad (7.14)$$

We have the following B-model graph sum formulae.

THEOREM 7.15 (B-model graph sum). (i) (Disk invariants)

$$\check{F}_{0,1}(\boldsymbol{\tau}; X) = \check{F}_{0,1}(0; X) + \sum_{a=1}^p \tau_a \frac{\partial \check{F}_{0,1}}{\partial \tau_a}(0; X) + \sum_{\vec{\Gamma} \in \Gamma_{0,1}(\mathcal{X})} \frac{w_B(\vec{\Gamma})}{|\text{Aut}(\vec{\Gamma})|}.$$

(ii) (Annulus invariants)

$$\check{F}_{0,2}(\boldsymbol{\tau}; X_1, X_2) = \check{F}_{0,2}(0; X_1, X_2) + \sum_{\vec{\Gamma} \in \Gamma_{0,2}(\mathcal{X})} \frac{w_B(\vec{\Gamma})}{|\text{Aut}(\vec{\Gamma})|}.$$

(iii) For $2g - 2 + n > 0$,

$$\check{F}_{g,n}(\boldsymbol{\tau}; X_1, \dots, X_n) = \sum_{\vec{\Gamma} \in \Gamma_{g,n}(\mathcal{X})} \frac{w_B(\vec{\Gamma})}{|\text{Aut}(\vec{\Gamma})|}.$$

7.7 All-genus open-closed mirror symmetry

PROPOSITION 7.16. We have

$$\check{F}_{0,1}(0; X) + \sum_{a=1}^p \tau_a \frac{\partial \check{F}_{0,1}}{\partial \tau_a}(0; X) = \Phi_{1,-2}(X) + \sum_{a=1}^p \tau_a \Phi_{h_a,-1}(X).$$

Proof. From the definition,

$$-X \frac{d}{dX} \check{F}_{0,1}(0; X) = \sum_{\ell \in \mathbb{Z}_m} (y_0^\ell(X) - y_0^\ell(0)) \psi_\ell.$$

By Proposition 6.4, equations (6.13) and (7.2), and Proposition 7.11,

$$\left(X \frac{d}{dX}\right)^2 \check{F}_{0,1}(0; X) = \sum_{\beta \in G^*} \frac{1}{|G| \sqrt{-2w_1 w_2 w_3}} \lim_{q \rightarrow 0} \xi_0^\beta(X) = \sum_{\beta \in G^*} \frac{1}{|G|^2 w_1 w_2 w_3} \tilde{\xi}_0^\beta(X)|_{v=1},$$

$$\begin{aligned} \left(X \frac{d}{dX}\right) \frac{\partial \check{F}_{0,1}}{\partial \tau_a}(0; X) &= \frac{1}{|G| \sqrt{-2w_1 w_2 w_3}} \sum_{\beta \in G^*} \chi_\beta(h_a) \prod_{i=1}^3 w_i^{c_i(h_a)} \lim_{q \rightarrow 0} \xi_0^\beta(X) \\ &= \frac{1}{|G|^2 w_1 w_2 w_3} \sum_{\beta \in G^*} \chi_\beta(h_a) \prod_{i=1}^3 w_i^{c_i(h_a)} \tilde{\xi}_0^\beta(X)|_{v=1}. \end{aligned}$$

The functions $\check{F}_{0,1}(0; X)$ and $(\partial \check{F}_{0,1} / \partial \tau_a)(0; X)$ are power series in X with no constant term, so

$$\check{F}_{0,1}(0; X) = \frac{1}{|G|^2 w_1 w_2 w_3} \tilde{\xi}_{-2}^\beta(X)|_{v=1}, \quad (7.15)$$

$$\frac{\partial \check{F}_{0,1}}{\partial \tau_a}(0; X) = \frac{1}{|G|^2 w_1 w_2 w_3} \sum_{\beta \in G^*} \chi_\beta(h_a) \prod_{i=1}^3 w_i^{c_i(h_a)} \tilde{\xi}_{-1}^\beta(X)|_{v=1}. \quad (7.16)$$

The lemma follows from (7.15), (7.16), and (5.11). $\square$

PROPOSITION 7.17. We have $\check{F}_{0,2}(0; X_1, X_2) = -F_{0,2}^{\mathcal{X},(\mathcal{L},f)}(0; X_1, X_2)$.

Proof. Let $C = C(z_1, z_2)$ be defined by (7.11). Then

$$\left(X_1 \frac{\partial}{\partial X_1} + X_2 \frac{\partial}{\partial X_2}\right) \check{F}_{0,2}(0; X_1, X_2) = \sum_{\ell_1, \ell_2 \in \mathbb{Z}_m} \int_0^{X_1} \int_0^{X_2} (\rho_0^{\ell_1, \ell_2})^* C \psi_{\ell_1} \otimes \psi_{\ell_2}$$

$$\begin{aligned}
 &= \frac{1}{2} \sum_{\ell_1, \ell_2 \in \mathbb{Z}_m} \sum_{\gamma \in G^*} \int_0^{X_1} (\rho_0^{\ell_1})^* \theta_0^\gamma \int_0^{X_2} (\rho_0^{\ell_2})^* \theta_0^\gamma \psi_{\ell_1} \otimes \psi_{\ell_2} \\
 &= \lim_{q \rightarrow 0} \frac{1}{2} \sum_{\gamma \in G^*} \xi_0^\gamma(X_1) \xi_0^\gamma(X_2) \\
 &= -\frac{1}{|G|^2 w_1 w_2 w_3} \left(\sum_{\gamma \in G^*} \tilde{\xi}_0^\gamma(X_1) \tilde{\xi}_0^\gamma(X_2) \right) \Big|_{v=1} \\
 &= -\left(X_1 \frac{\partial}{\partial X_1} + X_2 \frac{\partial}{\partial X_2} \right) F_{0,2}^{\mathcal{X},(\mathcal{L},f)}(0; X_1, X_2),
 \end{aligned}$$

where the second equality follows from Lemma 7.14, the fourth equality follows from Proposition 7.11, and the last equality follows from (5.5). Both $W_{0,2}(0; X_1, X_2)$ and $F_{0,2}^{\mathcal{X},(\mathcal{L},f)}(0; X_1, X_2)$ are $H_{\text{CR}}^*(\mathcal{B}\mu_m; \mathbb{C})^{\otimes 2}$ -valued power series in X_1 and X_2 which vanish at $(X_1, X_2) = (0, 0)$, so

$$\check{F}_{0,2}(0; X_1, X_2) = -F_{0,2}^{\mathcal{X},(\mathcal{L},f)}(0; X_1, X_2). \quad \square$$

PROPOSITION 7.18. *For any $\vec{\Gamma} \in \mathbf{\Gamma}_{g,n}(\mathcal{X})$, we have*

$$w_B(\vec{\Gamma}) = (-1)^{g(\vec{\Gamma})-1+n} w_A(\vec{\Gamma}),$$

where $w_A(\vec{\Gamma})$ is defined in Section 5.3 and $w_B(\vec{\Gamma})$ is defined in Section 7.6.

Proof. We fix $\vec{\Gamma} \in \mathbf{\Gamma}_{g,n}(\mathcal{X})$. From (5.7) and Theorem 7.8, for any $\alpha, \beta \in G^*$,

$$R(-z)_\alpha^\beta = f_\beta^\alpha \left(\frac{1}{z}, 0 \right). \quad (7.17)$$

Here we have used the fact that $B_m(1-x) = (-1)^m B_m(x)$, $B_m = 0$ when m is odd and $c_i(h) + c_i(h^{-1}) = 1 - \delta_{c_i(h), 0}$.

(1) (*Vertex*) By equation (7.2), we have

$$\frac{\lim_{\tau \rightarrow 0} h_1^\alpha}{\sqrt{-2}} = \frac{1}{|G| \sqrt{w_1 w_2 w_3}}. \quad (7.18)$$

(2) (*Edge*) By equation (7.10) (that is, [Eyn11, equation (B.9)]), we have

$$\begin{aligned}
 \check{B}_{k,l}^{\alpha,\beta} \Big|_{\tau=0} &= [u^{-k} v^{-l}] \left(\frac{uv}{u+v} (\delta_{\alpha,\beta} - \sum_{\gamma \in G^*} f_\gamma^\alpha(u, 0) f_\gamma^\beta(v, 0)) \right) \\
 &= [z^k w^l] \left(\frac{1}{z+w} \left(\delta_{\alpha,\beta} - \sum_{\gamma \in G^*} f_\gamma^\alpha \left(\frac{1}{z}, 0 \right) f_\gamma^\beta \left(\frac{1}{w}, 0 \right) \right) \right).
 \end{aligned}$$

By definition,

$$\mathcal{E}_{k,l}^{\alpha,\beta} = [z^k w^l] \left(\frac{1}{z+w} \left(\delta_{\alpha,\beta} - \sum_{\gamma \in G^*} R(-z)_\alpha^\gamma R(-w)_\beta^\gamma \right) \right).$$

It follows from (7.17) that

$$\check{B}_{k,l}^{\alpha,\beta} \Big|_{\tau=0} = \mathcal{E}_{k,l}^{\alpha,\beta}. \quad (7.19)$$

(3) (*Primary leaf*) By equation (7.12),

$$\frac{1}{\sqrt{-2}} (\check{h}^\tau)_k^\alpha = [u^{-k}] \left(\frac{1}{|G| \sqrt{w_1 w_2 w_3}} \sum_{\beta \in G^*} \sum_{a=1}^p \prod_{i=1}^3 w_i^{c_i(h_a)} f_\beta^\alpha(u) \chi_\beta(h_a) \tau_a \right).$$

By equation (5.8),

$$(\mathcal{L}^\tau)_k^\alpha = [z^k] \left(\frac{1}{|G|\sqrt{w_1 w_2 w_3}} \sum_{\beta \in G^*} \sum_{a=1}^p \prod_{i=1}^3 w_i^{c_i(h_a)} R(-z)_\alpha^\beta \chi_\beta(h_a) \tau_a \right).$$

It follows from (7.17) that

$$\frac{1}{\sqrt{-2}} (\check{h}^\tau)_k^\alpha = (\mathcal{L}^\tau)_k^\alpha. \quad (7.20)$$

(4) (*Open leaf*) Given $\beta \in G^*$ and $k \in \mathbb{Z}_{\geq 0}$, define

$$\hat{\xi}_k^\beta(X) := \sum_{\ell=0}^{m-1} \int_0^X (\rho_0^\ell)^* (d\hat{\xi}_{\beta,k}) \psi_\ell, \quad (7.21)$$

which is an $H_{\text{CR}}^*(\mathcal{B}\mu_m; \mathbb{C})$ -valued holomorphic function on D_δ . Then $\hat{\xi}_k^\beta(X) = (X d/dX)^k \hat{\xi}_0^\beta(X)$ and $\hat{\xi}_0^\beta(X) = \lim_{q \rightarrow 0} \xi_0^\beta$, where ξ_0^β is defined by equation (7.8). By Proposition 7.11 and the definitions of $\hat{\xi}_k^\beta(X)$ and $\tilde{\xi}_k^\beta(X)$,

$$\hat{\xi}_k^\beta(X) = -\frac{1}{|G|} \sqrt{\frac{-2}{w_1 w_2 w_3}} \tilde{\xi}_k^\beta(X). \quad (7.22)$$

By (7.13), (7.21), (7.22), and Lemma 6.7,

$$(\mathcal{L}^\xi)_k^\alpha(l_j) = \frac{-1}{|G|\sqrt{w_1 w_2 w_3}} \left(\tilde{\xi}_k^\alpha - \sum_{i=0}^{k-1} \sum_{\beta \in G^*} (\check{B}_{k-1-i,0}^{\alpha,\beta} |_{\tau=0}) \tilde{\xi}_i^\beta(X_j) \right).$$

By item (2) above, for $k \in \mathbb{Z}_{\geq 0}$,

$$\check{B}_{k,0}^{\alpha,\beta} |_{\tau=0} = [z^k w^0] \left(\frac{1}{z+w} \left(\delta_{\alpha,\beta} - \sum_{\gamma \in G^*} R(-z)_\alpha^\gamma R(-w)_\gamma^\beta \right) \right) = [z^{k+1}] (-R(-z)_\alpha^\beta).$$

We also have $[z^0] (R(-z)_\alpha^\beta) = \delta_{\alpha,\beta}$. Therefore,

$$(\mathcal{L}^\xi)_k^\alpha(l_j) = [z^k] \left(\frac{-1}{|G|\sqrt{w_1 w_2 w_3}} \sum_{\beta \in G^*} R(-z)_\alpha^\beta \tilde{\xi}^\beta(z, X_j) \right).$$

Comparing the above expression with the definition of the A-model open leaf $(\mathcal{L}^\xi)_k^\alpha(l_j)$, we conclude that

$$(\mathcal{L}^\xi)_k^\alpha(l_j) = -(\mathcal{L}^{\tilde{\xi}})_k^\alpha(l_j). \quad (7.23)$$

(5) (*Dilaton leaf*)

$$\left(-\frac{1}{\sqrt{-2}} \right) \check{h}_k^\alpha |_{\tau=0} = [u^{1-k}] \left(-\frac{1}{\sqrt{-2}} \right) \check{h}^\alpha(u) |_{\tau=0} = [u^{1-k}] \frac{-1}{|G|\sqrt{w_1 w_2 w_3}} \sum_{\beta \in G^*} f_\beta^\alpha(u, 0),$$

where the second identity follows from Corollary 7.9. By definition,

$$(\mathcal{L}^1)_k^\alpha = [z^{k-1}] \frac{-1}{|G|\sqrt{w_1 w_2 w_3}} \sum_{\beta \in G^*} R_\alpha^\beta(-z).$$

It follows from (7.17) that

$$\left(-\frac{1}{\sqrt{-2}} \right) \check{h}_k^\alpha |_{\tau=0} = (\mathcal{L}^1)_k^\alpha. \quad (7.24)$$

By (5.9), (7.14), (7.18), (7.19), (7.20), (7.23), and (7.24),

$$w_B(\vec{\Gamma}) = (-1)^{g(\vec{\Gamma})-1+n} w_A(\vec{\Gamma}). \quad \square$$

Combining Theorem 5.5 (A-model graph sum), Theorem 7.15 (B-model graph sum), and Propositions 7.16, 7.17, and 7.18, we obtain the final result.

THEOREM 7.19 (All-genus open-closed mirror symmetry). *We have*

$$\check{F}_{g,n}(\tau; X_1, \dots, X_n) = (-1)^{g-1+n} F_{g,n}^{\mathcal{X},(\mathcal{L},f)}(\tau; X_1, \dots, X_n).$$

As a consequence, the $H^*(\mathcal{B}\mu_m)^{\otimes n}$ -valued formal power series $F_{g,n}^{\mathcal{X},(\mathcal{L},f)}(\tau; X_1, \dots, X_n)$ converges in an open neighborhood of the origin in $\mathbb{C}^p \times \mathbb{C}^n$.

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