Automorphism and cohomology I: Fano varieties of lines and cubics

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ABSTRACT

In this paper, we show that the automorphism groups of smooth cubic threefolds or fourfolds act on their cohomology groups faithfully. We also prove that the same result holds for their Fano varieties of lines and Lefschetz cubic threefolds and fourfolds. As an application, we classify the automorphism groups of smooth cubic fourfolds with small middle Picard numbers over the complex numbers.

1. Introduction

An important question in algebraic geometry is whether the Torelli theorem holds for varieties, namely, whether some algebraic structures (such as Hodge structures, F-isocrystals) on the cohomology groups of varieties can recover the geometry of varieties. Burns, Rapoport, Michael, Shafarevich, and others proved that the Torelli theorem holds for K3 surfaces over the complex numbers $\mathbb{C}$, and Ogus proved that the Torelli theorem holds for supersingular K3 surfaces. The Torelli theorem says that

an isomorphism $\varphi: H^*(X) \simeq H^*(X')$ between the cohomology groups which preserves the algebraic structures is induced by an isomorphism $\psi: X \simeq X'$ between the varieties.

It is natural to ask the following question.

Question 1.1. Is the map ψ which satisfies $\psi^* = \varphi$ unique (up to a sign)?

The question is often reduced to whether the automorphism group $\text{Aut}(X)$ acts faithfully on the cohomology group $H^*(X)$:

$$\text{The natural map } \text{Aut}(X) \rightarrow \text{Aut}(H^*(X)) \text{ is injective.} \quad (1.1)$$

For example, algebraic curves of genus at least 2 and algebraic K3 surfaces over an algebraically closed field have this property (1.1); see [DM69, Ogu79] and [Huy16, Chapter 15]. On the other hand, Mukai, Namikawa and Dolgachev proved that some Enriques surfaces do not have property (1.1); see [Dol13, MN84].

In this paper, we first prove the following faithfulness theorem for cubic threefolds and their Fano varieties of lines, in Section 2.

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THEOREM 1.2. *Let X_3 be a smooth projective cubic threefold over an algebraically closed field k , and let $F(X_3)$ be the Fano variety of lines of X_3 . If the characteristic $\text{char}(k)$ of the field k is not 2, then the natural maps*

$$\text{Aut}(X_3) \rightarrow \text{Aut}(H_{\text{ét}}^3(X_3, \mathbb{Q}_l)) \quad \text{and} \quad \text{Aut}(F(X_3)) \rightarrow \text{Aut}(H_{\text{ét}}^1(F(X_3), \mathbb{Q}_l))$$

are injective.

The proof of Theorem 1.2 is based on the Lefschetz fixed-point theorem and the theory of the Albanese variety of $F(X_3)$.

For smooth projective cubic fourfolds and their Fano varieties of lines, we prove a similar result in Section 4.

THEOREM 1.3. *Let X_3 be a smooth projective cubic fourfold over an algebraically closed field k , and let $F(X_3)$ be the Fano variety of lines of X_3 . The natural maps*

$$\text{Aut}(X_3) \rightarrow \text{Aut}(H_{\text{ét}}^4(X_3, \mathbb{Q}_l)) \quad \text{and} \quad \text{Aut}(F(X_3)) \rightarrow \text{Aut}(H_{\text{ét}}^2(F(X_3), \mathbb{Q}_l))$$

are injective.

To show Theorem 1.3, we use deformation techniques. In characteristic zero, we deform automorphisms of X_3 by the holomorphic symplectic structure on the Fano variety of lines $F(X_3)$ for a smooth cubic fourfold X_3 and techniques of projective geometry. In positive characteristic, we use a criterion to lift automorphisms of smooth projective varieties from positive characteristic to characteristic zero and reduce the theorem to the case of characteristic zero.

In Section 5, we show the following theorem by reducing to the faithfulness theorem for automorphisms of algebraic curves and $K3$ surfaces.

THEOREM 1.4. *Let X_3 be a projective cubic threefold or fourfold with one ordinary double point over an algebraically closed field k . If the characteristic $\text{char}(k)$ of the field k is not 2 or 3, then the natural map*

$$\text{Aut}(X_3) \rightarrow \text{Aut}(H_{\text{ét}}^3(X_3, \mathbb{Q}_l)) \quad (\text{respectively, } \text{Aut}(X_3) \rightarrow \text{Aut}(H_{\text{ét}}^4(X_3, \mathbb{Q}_l)))$$

is injective.

In Section 6, we use the theory of Milnor fibers and vanishing cycles to show that there is a smooth cubic fourfold of middle Picard number 1 with an automorphism of order 3. In Section 7, we use lattice theory to classify the automorphism group of a complex cubic fourfold whose middle Picard number is small. Combining the results proved in these two sections, we can show the following theorem.

THEOREM 1.5. *Let X_3 be a smooth projective cubic fourfold over $\mathbb{C}$. The automorphism group $\text{Aut}(X_3)$ is*

- trivial or $\mathbb{Z}/3\mathbb{Z}$ if the middle Picard number is 1,
- trivial if the middle Picard number is 2.

2. Automorphisms of Fano surfaces

In this section, we start with some general results on automorphisms and the cohomology of hypersurfaces. We will use these results in the following sections.

LEMMA 2.1 ([Poo05, Theorems 1.1 and 1.3]). Suppose that X_d is a smooth hypersurface of degree d in $\mathbb{P}^{n+1}$ over an algebraically closed field k . If $n \geq 1$, $d \geq 3$, and (n, d) is neither $(1, 3)$ nor $(2, 4)$, then $\text{Aut}(X_d) = \text{Aut}_L(X_d)$, where $\text{Aut}_L(X_d)$ is the linear automorphism group. Moreover, $\text{Aut}(X_d)$ is a finite group.

LEMMA 2.2. Let X_d be a smooth hypersurface of degree d in $\mathbb{P}^{n+1}$ over an algebraically closed field k of positive characteristic and f be a linear automorphism of X_d . If $n \geq 1$, $d \geq 3$, and (n, d) is neither $(1, 3)$ nor $(2, 4)$, then the following are equivalent:

- (i) We have $f^* = \text{Id}$ on $H_{\text{ét}}^i(X_d, \mathbb{Q}_l)$,
- (ii) We have $f^* = \text{Id}$ on $H_{\text{cris}}^i(X_d/W)_K$, where K is the fraction field of the Witt ring $W = W(k)$.

Proof. By Lemma 2.1, we know that the order of f is finite. Therefore, the induced map f^* on the cohomology group $H_{\text{ét}}^i(X_d, \overline{\mathbb{Q}_l})$ (respectively, $H_{\text{cris}}^i(X_d/W)_{\overline{K}}$) is diagonalizable. In particular, the map f^* is the identity on the cohomology group if and only if all the eigenvalues of f^* are 1. On the other hand, we have

$$\det(\text{Id} - f^*t, H_{\text{cris}}^i(X_d/W)_K) = \det(\text{Id} - f^*t, H_{\text{ét}}^i(X_d, \mathbb{Q}_l));$$

see [KM74, Theorem 2] and [Ill75, (3.7.3)]. We have proved the lemma. $\square$

The following lemma is well known.

LEMMA 2.3. Let X_d be a smooth hypersurface in $\mathbb{P}^{n+1}$ over $\mathbb{C}$. Then, the singular cohomology $H_{\text{sing}}^i(X_d, \mathbb{Z})$ is torsion free.

Proof. It is clear that we have $H_i(X_d, \mathbb{Z})_{\text{tor}} = H_{\text{sing}}^{2n-i}(X_d, \mathbb{Z})_{\text{tor}}$ by Poincaré duality. Using the Lefschetz hyperplane theorem for homology and cohomology, we conclude that the cohomology group $H_{\text{sing}}^i(X_d, \mathbb{Z})$ is torsion free. $\square$

LEMMA 2.4. Let X_d be a smooth hypersurface in $\mathbb{P}^{n+1}$ over an algebraically closed field k of positive characteristic. Then, the crystalline cohomology $H_{\text{cris}}^i(X_d/W)$ is torsion free.

Proof. We lift X_d to a smooth hypersurface $\overline{X_d}$ over the Witt ring $W(k) = W$ of k . By [DK73, Exposé IX, Theorem 1.5], the Hodge–de Rham spectral sequence degenerates and the Hodge numbers are invariant; that is, $h^{p,q}(X_d) = h^{p,q}(\overline{X_d}/K)$. We conclude that

$$\dim_k H_{\text{DR}}^*(X_d/k) = \dim_K H_{\text{DR}}^*(\overline{X_d}/K).$$

According to the universal coefficient theorem of crystalline cohomology [Ber74], we have

$$0 \rightarrow H_{\text{cris}}^i(X_d/W) \otimes_W k \rightarrow H_{\text{DR}}^i(X_d/k) \rightarrow \text{Tor}_1^W(H_{\text{cris}}^{i+1}(X_d/W), k) \rightarrow 0. \quad (2.1)$$

Since $\dim_k H_{\text{cris}}^i(X_d/W) \otimes_W k \geq \dim_K H_{\text{cris}}^i(X_d/W)_K$, we have

$$\dim_k H_{\text{DR}}^i(X_d/k) \geq \dim_k H_{\text{cris}}^i(X_d/W) \otimes k \geq \dim_K H_{\text{cris}}^i(X_d/W)_K.$$

Moreover, it follows from the comparison theorem that $H_{\text{DR}}^*(\overline{X_d}/K) = H_{\text{cris}}^*(X_d/W)_K$. Therefore, the inequalities above are equalities. In particular, we have $\text{Tor}_1^W(H_{\text{cris}}^{i+1}(X_d/W), k) = 0$ for every i . In other words, the cohomology group $H_{\text{cris}}^{i+1}(X_d/W)$ is torsion free. $\square$

Now, we start some preparation for proving Theorem 1.2. Let us denote by S the Fano surface $F(X_3)$ of lines of the smooth cubic threefold X_3 .

LEMMA 2.5. With the notation as above, we have $H_{\text{ét}}^2(S, \mathbb{Q}_l) = \wedge^2 H_{\text{ét}}^1(S, \mathbb{Q}_l)$.

Proof. Over the complex numbers, it is well known that

- (1) $\dim(H_{\text{ét}}^1(S, \mathbb{Q}_l)) = 10$ and
- (2) $\chi(S, \mathbb{Q}_l) = 1 - 10 + 45 - 10 + 1 = 27$; see [AK77] and [CG72, Corollary 9.5 and (9.14)].

Assume that $\text{char}(k)$ is positive. We lift X_3 to a smooth cubic X over the Witt ring $W(k)$ of k . We have the relative Fano scheme $f: F \rightarrow \mathbf{Spec}(W(k))$ of lines of X over $W(k)$. It is clear that $F|_{\mathbf{Spec}(k)} = S$. It follows from the base change theorems that the same results (1) and (2) above hold for S .

Therefore, we have $\dim_{\mathbb{Q}_l}(\wedge^2 H_{\text{ét}}^1(S)) = \dim_{\mathbb{Q}_l}(H_{\text{ét}}^2(S))$. To prove the lemma, it suffices to show that the cup product

$$\bigcup: \bigwedge^2 H_{\text{ét}}^1(S, \mathbb{Q}_l) \rightarrow H_{\text{ét}}^2(S, \mathbb{Q}_l)$$

is injective. It follows from [CG72, Lemma 9.13] that the cup product is injective when k is of characteristic zero. Assume that $\text{char}(k)$ is positive. From the construction of the cup product for étale cohomology [Mil80, Section VI.8], we have the relative cup product

$$R^1 f_*(\mathbb{Q}_l) \times R^1 f_*(\mathbb{Q}_l) \rightarrow R^2 f_*(\mathbb{Q}_l).$$

Since f is smooth and proper, the higher direct image $R^i f_*(\mathbb{Q}_l)$ is a lisse ℓ -adic sheaf for every i ; see [DA73, Chapter XVI] and [Del77, Arcata V2,3]. Since $W(k)$ is strictly henselian, the sheaves $R^i f_*(\mathbb{Q}_l)$ are constant. By the proper base change theorem, to prove that the cup product $\bigcup$ is injective over k , it suffices to show that it is injective over K , where K is the fraction field of $W(k)$. The injectivity of the cup product $\bigcup$ over K follows from [CG72, Lemma 9.13]. $\square$

Let $q: L \rightarrow X_3$ be the universal lines over the Fano surface S . Let $p: L \rightarrow S$ be the structure morphism. Note that X_3 can be lifted to a smooth cubic threefold over characteristic zero if the characteristic of the field k is positive. One can show that the Abel–Jacobi map

$$p_* q^*: H_{\text{ét}}^3(X_3, \mathbb{Q}_l) \rightarrow H_{\text{ét}}^1(S, \mathbb{Q}_l) \quad (2.2)$$

is an isomorphism when k is of positive characteristic by reducing to the case of characteristic zero [CG72, Theorem 11.19].

LEMMA 2.6. *With the assumptions of Theorem 1.2, we have a natural diagram*

$$\begin{array}{ccc} \text{Aut}(X_3) & \xrightarrow{j} & \text{Aut}(S) \\ \downarrow & \square & \downarrow \\ \text{Aut}(H_{\text{ét}}^3(X_3, \mathbb{Q}_l)) & = & \text{Aut}(H_{\text{ét}}^1(S, \mathbb{Q}_l)), \end{array} \quad (2.3)$$

where the identification at the bottom of the diagram is induced by the Abel–Jacobi map (2.2). Moreover, the morphism j is injective.

Proof. Let g be an automorphism of X_3 . By Lemma 2.1, the automorphism g of X_3 is linear. Therefore, the automorphism g maps lines to lines. It gives rise to a natural morphism $j: \text{Aut}(X_3) \rightarrow \text{Aut}(S)$.

The proof of the commutativity of the diagram is similar to the proof of Lemma 4.3, so we postpone the proof to Section 5. On the other hand, using an argument similar to that for [CS09, Lemma 2.1], we can show that for a point s of X_3 , the space of lines contained in X_3 and passing through s can be viewed as the intersection of a quadratic and cubic form in the projectivized

tangent space of X_3 at s . Therefore, we conclude that for any point $s \in X_3$, there are at least two lines l_1 and l_2 passing through s (that is, $s = l_1 \cap l_2$). If $j(g)$ is the identity, then we have

$$g(s) = j(g)(l_1) \cap j(g)(l_2) = l_1 \cap l_2 = s.$$

It follows that the map g is the identity. Therefore, the map j is injective. $\square$

Proof of Theorem 1.2. Recall that we denote by S the Fano surface $F(X_3)$ of lines of the smooth cubic threefold X_3 . First, we claim that an automorphism g of S has a fixed point if it acts trivially on the first ℓ -adic cohomology group. From Lemma 2.6 and Poincaré duality, it follows that g acts trivially on $H_{\text{ét}}^i(S, \mathbb{Q}_\ell)$ for all i . Suppose that g has no fixed points. By the Lefschetz fixed-point theorem, we have a contradiction as follows:

$$0 = \#(\text{Fix}(g)) = \sum_{i=0}^4 (-1)^i \text{Tr}(g^*, H_{\text{ét}}^i(S, \mathbb{Q}_\ell)) = 1 - 10 + 45 - 10 + 1 = 27.$$

Suppose that $s_0 \in S$ is a fixed point of $g: S \rightarrow S$. Since $\text{char}(k) \neq 2$, the natural morphism $\lambda: S \rightarrow \text{Alb}(S)$ is an immersion, where $\text{Alb}(S)$ is the Albanese variety of S ; see [Bea82]. Moreover, we have $H_{\text{ét}}^1(S, \mathbb{Q}_\ell) = H_{\text{ét}}^1(\text{Alb}(S), \mathbb{Q}_\ell)$ via the pullback λ^* . Assume $\lambda(s_0) = 0$. By the universal property of $\text{Alb}(S)$, we have a morphism $h: \text{Alb}(S) \rightarrow \text{Alb}(S)$ such that $\lambda \circ g = h \circ \lambda$ and $h(\lambda(s_0)) = \lambda(s_0)$. Therefore, the morphism h is an automorphism of the group scheme $\text{Alb}(S)$ [Mum08, Chapter II]. Obviously, we have $h^* = \text{Id}$ on $H_{\text{ét}}^1(\text{Alb}(S), \mathbb{Q}_\ell)$. By a theorem of Tate [Mum08, Chapter IV], we know that $h = \text{Id}_{\text{Alb}(S)}$. This implies that g is the identity Id_S . Therefore, we have proved the faithfulness theorem for the Fano variety of lines. The faithfulness theorem for X_3 also follows by Lemma 2.6. $\square$

3. The first de Rham cohomology groups

Our main result of this section is Proposition 3.3. The proof is tedious. We suggest the reader to skip it during the first reading.

Suppose that X_3 is a smooth cubic fourfold in $\mathbb{P}^5$ over an algebraically closed field k . Denote by $F(X_3)$ the Fano variety of lines of X_3 . We use the notation of [AK77]. Let V be a vector space of dimension 6 over k . Denote the projective space $\mathbb{P}(V) = \mathbb{P}^5$ by P . Let G be the Grassmannian $\text{Grass}_2(V)$. The fundamental sequence on G is

$$0 \rightarrow M \rightarrow V_G \rightarrow Q \rightarrow 0, \quad (3.1)$$

where M is the universal subbundle and Q is the quotient bundle. Denote by Γ the incidence correspondence subscheme of $P \times G$; see [AK77, Setup 5.2, p. 36]. In particular, we have $\Gamma = \{(x, l) \mid x \in l\}$ and two maps p and q

$$G \xleftarrow{q} \mathbb{P}(Q) = \Gamma = \mathbb{P}(K) \xrightarrow{p} P,$$

where $K = \Omega_P^1(1)$. Note that there is a short exact sequence

$$0 \rightarrow K \rightarrow V_P \rightarrow \mathcal{O}_P(1) \rightarrow 0,$$

where $V_P = \mathcal{O}_P^{\oplus 6}$. Let R (respectively, N) be the fundamental sheaf $\mathcal{O}_{\mathbb{P}(Q)}(1) = p^* \mathcal{O}_P(1)$ (respectively, $\mathcal{O}_{\mathbb{P}(K)}(1)$). We denote $q^* Q$ by Q_Γ . We summarize some results from [AK77] in the following proposition.

PROPOSITION 3.1 ([AK77, Sections 5.2.2–5.4.2]). We have a short exact sequence $0 \rightarrow N \rightarrow Q_\Gamma \rightarrow R \rightarrow 0$. Therefore, we have $\wedge^2 Q_\Gamma = N \otimes R$. Moreover, the dualizing sheaf $\omega_{\Gamma/k}$ is $N^{-5} \otimes R^{-7}$. We have

$$\wedge^j E(n) = \wedge^{(4-j)}(\text{Sym}_3(Q)) \otimes (\wedge^2 Q)^{\otimes(n-6)},$$

where $E = \text{Sym}_3(Q)^\vee$.

LEMMA 3.2. Some cohomology groups are zero, as follows:

- (i) $H^i(\Gamma, N^s \otimes R^t) = 0$ if $-1 \geq s \geq -4$, or if $s = 0$ and $i \notin \{0, 5\}$, or if $s = 0, i = 0$ and $t \leq -1$, or if $s = 1$ and $i \neq 1$, or if $s = 1, i = 1$ and $t \neq -1$, or if $s = -5$ and $i \notin \{4, 9\}$, or if $s = -6$ and $i \notin \{4, 8, 9\}$;
- (ii) $H^i(\Gamma, Q_\Gamma \otimes N^s \otimes R^t) = 0$ if $-2 \geq s \geq -4$, or if $s = -5$ and $i \notin \{4, 9\}$;
- (iii) $H^i(\Gamma, \text{Sym}_2(Q_\Gamma) \otimes N^s \otimes R^t) = 0$ if $s = -3$ or $s = -4$, or if $s = -5, i \notin \{9, 4\}$, or if $s = -5, i = 4$ and $t \leq -4$, or if $s = -5, i = 9$ and $t \geq -8$, or if $s = -6$ and $i \notin \{8, 9, 4\}$;
- (iv) $H^i(\Gamma, \text{Sym}_3(Q_\Gamma) \otimes N^s \otimes R^t) = 0$ if $s = -5$ and $i \notin \{4, 9\}$, or if $s = -5, i = 4$ and $t \leq -5$, or if $s \in \{-6, -7\}$ and $i \in \{2, 3\}$;
- (v) $H^i(\Gamma, \wedge^2 \text{Sym}_2(Q_\Gamma) \otimes N^s \otimes R^t) = 0$ if $s \in \{-4, -5\}$, or if $s = -3$ and $i \notin \{0, 5\}$, or if $s = -3, i = -5$ and $t \geq -6$;
- (vi) $H^i(\Gamma, \wedge^2 \text{Sym}_3(Q_\Gamma) \otimes N^s \otimes R^t) = 0$ if $s = -5$ and $i \notin \{0, 5\}$, or if $s = -5, i = 5$ and $t \geq -6$, or if $s = -6, i \notin \{4, 9\}$, or if $s = -6, i = 4$ and $t \leq -5$, or if $s = -6, i = 9$ and $t \geq -9$ or if $s = -7$ and $i \in \{1, 2\}$.

Proof. The lemma follows from Proposition 3.1, the formulas [AK77, (5.7.1) and (5.7.2)], Serre duality and the Bott vanishing theorem. The proof is straightforward but tedious.

(i) By [AK77, (5.7.1)], the cohomology groups vanish if $-1 \geq s \geq -4$. If $s = 0$ and $i \notin \{0, 5\}$, or if $s = i = 0$ and $t \leq -1$, then, by [AK77, (5.7.1)], we have $H^i(\Gamma, N^s \otimes R^t) = H^i(P, \mathcal{O}_P(t)) = 0$. If $s = 1$ then, by [AK77, (5.7.1) and (5.7.2)], we have $H^i(\Gamma, N^s \otimes R^t) = H^i(P, \Omega_P^1(t+1)) = 0$ if $i \neq 1$, or if $i = 1$ and $t \neq -1$, where we use the Bott vanishing theorem (see [Hua01, Bott's formula]).

For $s = -5$ and $i \notin \{4, 9\}$, we conclude that $H^i(N^{-5} \otimes R^t) = 0$ by Serre duality and [AK77, (5.7.1)]. In fact, we have $H^i(N^{-5} \otimes R^t) = H^{9-i}(\Gamma, \omega_\Gamma \otimes N^5 R^{-t}) = H^{9-i}(P, \mathcal{O}_P(-t-7)) = 0$.

For $s = -6$ and $i \notin \{4, 8, 9\}$, we conclude that $H^i(N^s \otimes R^t) = 0$ by Serre duality and [AK77, (5.7.1)]. In fact, we have $H^i(N^{-6} \otimes R^t) = H^{9-i}(\Gamma, \omega_\Gamma \otimes N^6 R^{-t}) = H^{9-i}(P, \Omega_P^1(-t-6)) = 0$ by [AK77, (5.7.1)] and the Bott vanishing theorem.

In summary, we have proved part (i).

(ii) Applying the short exact sequence in Proposition 3.1 and [AK77, (5.7.1)], we have a long exact sequence

$$\cdots \rightarrow (0 =) H^i(\Gamma, N^{s+1} \otimes R^t) \rightarrow H^i(\Gamma, Q_\Gamma \otimes N^s \otimes R^t) \rightarrow H^i(\Gamma, N^s \otimes R^{t+1}) (= 0) \rightarrow \cdots$$

if $-2 \geq s \geq -4$.

For $s = -5$ and $i \notin \{9, 4\}$, it follows from part (i) that $H^i(\Gamma, N^{s+1} \otimes R^t) = 0$ and $H^i(\Gamma, N^s \otimes R^{t+1}) = 0$. In summary, we have proved part (ii).

(iii) By [AK77, (5.11.5)], we have a short exact sequence

$$0 \rightarrow Q_\Gamma \otimes N^{s+1} \otimes R^t \rightarrow \text{Sym}_2(Q_\Gamma) \otimes N^s \otimes R^t \rightarrow N^s \otimes R^{t+2} \rightarrow 0. \quad (3.2)$$

Using part (ii), we have $H^i(\Gamma, \text{Sym}_2(Q_\Gamma) \otimes N^s \otimes R^t) = 0$ if $s = -3$ or -4 .

For $s = -5$ and $i \notin \{4, 9\}$, we have the long exact sequence associated with (3.2)

$$\cdots \rightarrow H^i(Q_\Gamma \otimes N^{s+1}R^t) \rightarrow H^i(\text{Sym}_2(Q_\Gamma) \otimes N^sR^t) \rightarrow H^i(N^sR^{t+2}) \rightarrow \cdots,$$

where $H^i(N^{-5}R^{t+2}) = 0$ and $H^i(Q_\Gamma \otimes N^{-4}R^t) = 0$ by parts (i) and (ii).

For $s = -5$, $i = 4$ and $t \leq -4$, we have $H^i(Q_\Gamma \otimes N^{-4}R^t) = 0$ by part (ii) and $H^i(N^{-5}R^{t+2}) = H^{9-i}(R^{-t-9}) = H^5(P, \mathcal{O}_P(-t-9)) = 0$ by [AK77, (5.7.1)].

For $s = -5$, $i = 9$ and $t \geq -8$, we know that $H^i(Q_\Gamma \otimes N^{-4}R^t) = 0$ and $H^i(N^{-5}R^{t+2}) = H^{9-i}(R^{-t-9}) = H^0(P, \mathcal{O}_P(-t-9)) = 0$ by [AK77, (5.7.1)].

Similarly, for $s = -6$ and $i \notin \{4, 8, 9\}$, it follows from parts (i) and (ii) that $H^i(Q_\Gamma \otimes N^{-5}R^t) = 0$ and $H^i(\Gamma, N^{-6}R^{t+2}) = 0$. We have proved part (iii).

(iv) By [AK77, Proof of Proposition 5.11, p. 42], we have a short exact sequence

$$0 \rightarrow \text{Sym}_2(Q_\Gamma) \otimes N \rightarrow \text{Sym}_3(Q_\Gamma) \rightarrow R^3 \rightarrow 0. \quad (3.3)$$

It gives rise to a long exact sequence after tensoring by N^sR^t :

$$\cdots \rightarrow H^i(\text{Sym}_2(Q_\Gamma) \otimes N^{s+1}R^t) \rightarrow H^i(\text{Sym}_3(Q_\Gamma) \otimes N^sR^t) \rightarrow H^i(N^s \otimes R^{t+3}) \rightarrow \cdots.$$

If $s = -5$ and $i \notin \{4, 9\}$, then we have $H^i(N^s \otimes R^{t+3}) = 0$ and $H^i(\text{Sym}_2(Q_\Gamma) \otimes N^{s+1}R^t) = 0$ by parts (i) and (iii). If $s = -5$, $i = 4$ and $t \leq -6$, then we have $H^i(\text{Sym}_2(Q_\Gamma) \otimes N^{s+1}R^t) = 0$ by part (iii) and the fact that

$$H^i(N^s \otimes R^{t+3}) = H^{9-i}(\Gamma, R^{-t-10}) = H^5(P, \mathcal{O}_P(-t-10)) = H^0(P, \mathcal{O}_P(t+4)) = 0,$$

where we use Serre duality and [AK77, (5.7.1)].

For $s \in \{-6, -7\}$ and $i \in \{2, 3\}$, the exact sequence (3.3) gives rise to an exact sequence

$$H^i(\text{Sym}_2(Q_\Gamma) \otimes N^{s+1} \otimes R^t) \rightarrow H^i(\text{Sym}_3(Q_\Gamma) \otimes N^s \otimes R^t) \rightarrow H^i(N^s \otimes R^{t+3}),$$

where the first term is zero by part (iii). We claim that the last term is also zero. In fact, this follows from part (i) for $s = -6$. For $s = -7$, we have

$$H^i(\Gamma, N^s \otimes R^{t+3}) = H^{9-i}(\Gamma, N^{-s-5} \otimes R^{-t-10}) = H^{9-i}(\mathbb{P}^5, \text{Sym}_2(K)(-t-9)) = 0$$

by Serre duality and [AK77, (5.7.1)]. Therefore, we have proved part (iv).

(v) According to [AK77, Lemma 2.10 and (5.11.5)], we have a short exact sequence [AK77, display after (5.11.5)]

$$0 \rightarrow N^{s+3} \otimes R^{t+1} \rightarrow \wedge^2 \text{Sym}_2(Q_\Gamma) \otimes N^s \otimes R^t \rightarrow Q_\Gamma \otimes N^{s+1} \otimes R^{t+2} \rightarrow 0.$$

This gives rise to a long exact sequence

$$\cdots \rightarrow H^i(N^{s+3}R^{t+1}) \rightarrow H^i(\wedge^2 \text{Sym}_2(Q_\Gamma) \otimes N^sR^t) \rightarrow H^i(Q_\Gamma \otimes N^{s+1}R^{t+2}) \rightarrow \cdots.$$

We notice that if $s = -3$, $i = 5$ and $t \geq -6$, then $H^i(N^{s+3} \otimes R^{t+1}) = H^i(P, \mathcal{O}_P(t+1)) = 0$ by [AK77, (5.7.1)]. By parts (i) and (ii), we conclude that $H^i(\Gamma, N^{s+3}R^{t+1}) = H^i(\Gamma, Q_\Gamma \otimes N^{s+1}R^{t+2}) = 0$ under the hypotheses of part (v). Therefore, we have $H^i(\Gamma, \wedge^2 \text{Sym}_2(Q_\Gamma) \otimes N^sR^t) = 0$. We have proved part (v).

(vi) As in [AK77, Proof of Proposition 5.11, p. 42], we have the short exact sequence

$$0 \rightarrow \wedge^2 \text{Sym}_2(Q_\Gamma) \otimes N^{s+2} \otimes R^t \rightarrow \wedge^2 \text{Sym}_3(Q_\Gamma) \otimes N^s \otimes R^t \rightarrow \text{Sym}_2(Q_\Gamma) \otimes N^{s+1} \otimes R^{t+1} \rightarrow 0,$$

which gives rise to a long exact sequence

$$\cdots \rightarrow H^i(\wedge^2 \text{Sym}_2 \otimes N^{s+2}R^t) \rightarrow H^i(\wedge^2 \text{Sym}_3 \otimes N^sR^t) \rightarrow H^i(\text{Sym}_2 \otimes N^{s+1}R^{t+1}) \rightarrow \cdots.$$

By parts (iii) and (v), we conclude that

$$H^i(\Gamma, \wedge^2 \text{Sym}_2(Q_\Gamma) \otimes N^{s+2} \otimes R^t) \quad \text{and} \quad H^i(\Gamma, \text{Sym}_2(Q_\Gamma) \otimes N^{s+1} \otimes R^{t+1})$$

are zero under the hypotheses of part (vi). We have proved part (vi). $\square$

PROPOSITION 3.3. *With the notation as above, we have*

$$H^1(F(X_3), \mathcal{O}_{F(X_3)}) = H^0(F(X_3), \Omega_{F(X_3)}^1) = H^3(F(X_3), \mathcal{O}_{F(X_3)}) = 0.$$

Therefore, the Néron–Severi group $\text{NS}(F(X_3))$ of $F(X_3)$ is torsion free. Moreover, we have $H^2(F(X_3), \mathcal{O}_{F(X_3)}) = k$.

Proof. First of all, we show that $H^1(F(X_3), \mathcal{O}_{F(X_3)}) = H^3(F(X_3), \mathcal{O}_{F(X_3)}) = 0$. In fact, the Fano variety $F(X_3)$ of X_3 is defined by a regular section s of the vector bundle $\text{Sym}_3(Q) = E^\vee$, which is of rank 4; see [AK77, Theorem 1.3]. So we have the exact sequence

$$0 \longrightarrow \wedge^4 E \longrightarrow \wedge^3 E \longrightarrow \wedge^2 E \longrightarrow E \xrightarrow{s} \mathcal{O}_G \longrightarrow \mathcal{O}_F \longrightarrow 0. \quad (3.4)$$

By the syzygy argument, in order to prove that $H^1(F, \mathcal{O}_F) = 0$, it suffices to prove that

$$H^1(G, \mathcal{O}_G) = H^2(G, E) = H^3(G, \wedge^2 E) = H^4(G, \wedge^3 E) = H^5(G, \wedge^4 E) = 0.$$

This is verified by [AK77, Theorem 5.1]. Similarly, to prove that $H^3(F, \mathcal{O}_F) = 0$, it suffices to prove that

$$H^3(G, \mathcal{O}_G) = H^4(G, E) = H^5(G, \wedge^2 E) = H^6(G, \wedge^3 E) = H^7(G, \wedge^4 E) = 0,$$

which is also verified by [AK77, Theorem 5.1]. To prove that $H^2(F, \mathcal{O}_F) = k$, it suffices to prove that

$$\begin{aligned} H^2(\mathcal{O}_G) &= H^3(\mathcal{O}_G) = H^2(E) = H^3(E) = H^4(\wedge^3 E) = H^5(\wedge^3 E) \\ &= H^5(\wedge^4 E) = H^6(\wedge^4 E) = 0 \end{aligned}$$

and $H^4(G, \wedge^2 E) = k$, which are verified by [AK77, Theorem 5.1].

Apply Lemmas 3.4 and 3.6 to the long exact sequence of cohomology groups associated with the short exact sequence

$$0 \rightarrow E|_{F(X_3)} \rightarrow \Omega_{G|F(X_3)}^1 \rightarrow \Omega_{F(X_3)}^1 \rightarrow 0.$$

Note that $E|_{F(X_3)}$ is the conormal bundle of $F(X_3)$ in G . We conclude that $H^0(F(X_3), \Omega_{F(X_3)}^1) = 0$.

In summary, we conclude that $H_{\text{DR}}^1(F(X_3)) = 0$ since the Hodge numbers $H^1(F(X_3), \mathcal{O}_{F(X_3)}) = H^0(F(X_3), \Omega_{F(X_3)}^1)$ are zero.

By the universal coefficient theorem of crystalline cohomology, we know that $H_{\text{cris}}^2(F(X_3)/W)$ is torsion free. By a theorem of Illusie and Deligne, see [Del81, Remark 3.5] and [Ill79], we have an injection

$$\text{NS}(F(X_3)) \otimes \mathbb{Z}_p \hookrightarrow H_{\text{cris}}^2(F(X_3)/W).$$

We conclude that $\text{NS}(F(X_3))$ is p -torsion free. On the other hand, we have an injection [Mil80]

$$\text{NS}(F(X_3)) \otimes \mathbb{Z}_l \hookrightarrow H_{\text{ét}}^2(F(X_3), \mathbb{Z}_l(1)).$$

We claim that $H_{\text{ét}}^2(F(X_3), \mathbb{Z}_l(1))$ is torsion free. Therefore, the group $\text{NS}(F(X_3))$ is torsion free.

In fact, it follows from [BD85, Propositions 1 and 2] that the variety $F(X_3)_{\mathbb{C}}$ is simply

connected. By the universal coefficient theorem, we have

$$\begin{aligned} H_{\text{sing}}^2(F(X_3)_{\mathbb{C}}, \mathbb{Z}_l) &= \text{Hom}(H_2(F(X_3)_{\mathbb{C}}, \mathbb{Z}), \mathbb{Z}_l) = \lim_{\leftarrow n} \text{Hom}(H_2(F(X_3)_{\mathbb{C}}, \mathbb{Z}), \mathbb{Z}/l^n \mathbb{Z}) \\ &= \lim_{\leftarrow n} H_{\text{ét}}^2(F(X_3)_{\mathbb{C}}, \mathbb{Z}/l^n \mathbb{Z}) = H_{\text{ét}}^2(F(X_3)_{\mathbb{C}}, \mathbb{Z}_l). \end{aligned}$$

Since $H_{\text{sing}}^2(F(X_3)_{\mathbb{C}}, \mathbb{Z})$ is torsion free, the group $H_{\text{sing}}^2(F(X_3)_{\mathbb{C}}, \mathbb{Z}_l)$ is torsion free. We show that the claim holds in characteristic zero. Assume $\text{char}(k) > 0$. Since $F(X_3)$ over k has a lifting from positive characteristic to characteristic zero, the claim holds for $F(X_3)$ over k . $\square$

LEMMA 3.4. *With the notation as above, we have $H^1(F, E|_F) = 0$.*

Proof. By the exact sequence (3.4) and the syzygy argument, it suffices to prove that

$$H^5(G, \wedge^4 E \otimes E) = H^4(\wedge^3 E \otimes E) = H^3(\wedge^2 E \otimes E) = H^2(E \otimes E) = H^1(G, E) = 0.$$

(1) We claim $H^5(G, \wedge^4 E \otimes E) = 0$. In fact, we have $\wedge^4 E = (\wedge^2 Q)^{-6}$ by Proposition 3.1. It follows from [AK77, Proof of Lemma 5.12, p. 43] that $E = R^1 q_*(R^{-5}) \otimes \wedge^2 Q$. By the equality $R^0 q_*(R^{-5}) = 0$ and Proposition 3.1, we conclude that $\wedge^4 E \otimes E$ is

$$R^1 q_*(R^{-5}) \otimes (\wedge^2 Q)^{-5} = R q_*(R^{-5} \otimes q^*(\wedge^2 Q)^{-5}) = R q_*(R^{-10} \otimes N^{-5}).$$

Therefore,

$$H^5(G, \wedge^4 E \otimes E) = H^5(\Gamma, R^{-10} \otimes N^{-5}) = H^4(\Gamma, R^3) = H^4(P, \mathcal{O}_P(3)) = 0.$$

(2) We claim that $H^4(\wedge^3 E \otimes E) = 0$. In fact, by Proposition 3.1 and [AK77, Proof of Lemma 5.12,], we conclude that $\wedge^3 E \otimes E$ is

$$\begin{aligned} \text{Sym}_3(Q) \otimes (\wedge^2 Q)^{-5} \otimes R^1 q_*(R^{-5}) &= \text{Sym}_3(Q) \otimes (\wedge^2 Q)^{-5} \otimes R q_*(R^{-5}) \\ &= R q_*(\text{Sym}_3(Q_{\Gamma}) \otimes N^{-5} R^{-10}). \end{aligned}$$

Therefore, by Lemma 3.2(iv), we conclude that

$$H^4(G, \wedge^3 E \otimes E) = H^4(\Gamma, \text{Sym}_3(Q_{\Gamma}) \otimes N^{-5} R^{-10}) = 0.$$

(3) We claim that $H^3(G, \wedge^2 E \otimes E) = 0$. In fact, as before,

$$\begin{aligned} \wedge^2 E \otimes E &= \wedge^2 \text{Sym}_3(Q) \otimes (\wedge^2 Q)^{-5} \otimes R q_*(R^{-5}) \\ &= R q_*(\wedge^2 \text{Sym}_3(Q_{\Gamma}) \otimes N^{-5} \otimes R^{-10}). \end{aligned}$$

It follows that $H^3(G, \wedge^2 E \otimes E) = H^3(\Gamma, \wedge^2 \text{Sym}_3(Q_{\Gamma}) \otimes N^{-5} \otimes R^{-10}) = 0$. The last equality $H^3(\Gamma, \wedge^2 \text{Sym}_3(Q_{\Gamma}) \otimes N^{-5} \otimes R^{-10}) = 0$ follows from Lemma 3.2(vi).

(4) We claim that $H^2(G, E \otimes E) = 0$. In fact, as before, we have that

$$\begin{aligned} E \otimes E &= R q_*(R^{-5} \otimes \wedge^2 Q_{\Gamma} \otimes q^* E) \\ &= R q_*(R^{-4} \otimes N \otimes q^* E) = R q_*(R^{-4} \otimes N \otimes \text{Sym}_3(Q_{\Gamma})^{\vee}), \end{aligned}$$

where the last equality follows from $q^* E = \text{Sym}_3(Q_{\Gamma})^{\vee}$. Therefore, we have $H^2(G, E \otimes E) = H^2(\Gamma, \text{Sym}_3(Q_{\Gamma})^{\vee} \otimes N \otimes R^{-4})$. The short exact sequence (3.3) gives rise to a short exact sequence

$$0 \rightarrow N \otimes R^{-7} \rightarrow \text{Sym}_3(Q_{\Gamma})^{\vee} \otimes N \otimes R^{-4} \rightarrow \text{Sym}_2(Q_{\Gamma})^{\vee} \otimes R^{-4} \rightarrow 0.$$

By Lemma 3.2(i), we have $H^2(\Gamma, N \otimes R^{-7}) = 0$. To prove the claim, it suffices to prove that $H^2(\Gamma, \text{Sym}_2(Q_{\Gamma})^{\vee} \otimes R^{-4}) = 0$. In fact, by the exact sequence [AK77, (5.11.5)], we have an exact sequence

$$0 \rightarrow R^{-6} \rightarrow \text{Sym}_2(Q_{\Gamma})^{\vee} \otimes R^{-4} \rightarrow Q_{\Gamma}^{\vee} \otimes N^{-1} \otimes R^{-4} \rightarrow 0.$$

By Lemma 3.2(i), we have $H^2(\Gamma, R^{-6}) = 0$.

In the following, we show that $H^2(\Gamma, Q_\Gamma^\vee \otimes N^{-1} \otimes R^{-4}) = 0$. In fact, by Proposition 3.1, we have a short exact sequence

$$0 \rightarrow N^{-1} \otimes R^{-5} \rightarrow Q_\Gamma \otimes N^{-1} \otimes R^{-4} \rightarrow N^{-2} \otimes R^{-4} \rightarrow 0.$$

We conclude that $H^2(\Gamma, Q_\Gamma^\vee \otimes N^{-1} \otimes R^{-4}) = 0$ since $H^2(N^{-1} \otimes R^{-5}) = H^2(N^{-2} \otimes R^{-4}) = 0$ by Lemma 3.2(i). Therefore, we have shown that $H^2(\Gamma, \text{Sym}_2(Q_\Gamma)^\vee \otimes R^{-4}) = 0$.

(5) The equality $H^1(G, E) = 0$ follows from [AK77, Theorem 5.1]. $\square$

LEMMA 3.5. *For any vector bundle V on G , we have $H^i(\Gamma, q^*V) = H^i(G, V)$.*

Proof. The lemma follows from the projection formula and the equality $R^i q_* \mathcal{O}_\Gamma = 0$ for $i > 0$. In fact, we have

$$H^i(G, V) = H^i(G, V \otimes q_* \mathcal{O}_G) = H^i(G, q_* q^* V) = H^i(G, Rq_* q^* V) = H^i(\Gamma, q^* V),$$

where the second equality follows from $q_* \mathcal{O}_\Gamma = \mathcal{O}_G$. $\square$

LEMMA 3.6. *With the notation as above, we have $H^0(F, \Omega_G^1|_F) = 0$.*

Proof. By the exact sequence (3.4), we have an exact sequence

$$\wedge^4 E \otimes \Omega_G \hookrightarrow \wedge^3 E \otimes \Omega_G \rightarrow \wedge^2 E \otimes \Omega_G \rightarrow E \otimes \Omega_G \rightarrow \Omega_G \twoheadrightarrow \Omega_G|_F.$$

By the syzygy argument, to prove the lemma, it suffices to show that

$$H^4(\wedge^4 E \otimes \Omega_G) = H^3(\wedge^3 E \otimes \Omega_G) = H^2(\wedge^2 E \otimes \Omega_G) = H^1(E \otimes \Omega_G) = H^0(G, \Omega_G) = 0.$$

We first notice that $\Omega_G^1 = Q^\vee \otimes M$.

(1) By the exact sequence (3.1), we have the exact sequence

$$0 \rightarrow Q^\vee \otimes M \rightarrow Q^\vee \rightarrow Q^\vee \otimes Q \rightarrow 0.$$

We claim that $H^0(G, Q^\vee) = 0$. Therefore, we have $H^0(G, \Omega_G^1) = H^0(G, Q^\vee \otimes M) = 0$. In fact, by the exact sequence in Proposition 3.1, we have a short exact sequence $0 \rightarrow N^{-1} \rightarrow Q_\Gamma^\vee \rightarrow R^{-1} \rightarrow 0$. It follows from Lemma 3.2(i) that $H^0(\Gamma, R^{-1}) = H^0(\Gamma, N^{-1}) = 0$. We have proved the claim.

(2) We claim that $H^1(G, E \otimes \Omega_G^1) = 0$. As in [AK77, Proof of Lemma 5.12], we have $E = R^1 q_*(R^{-5}) \otimes \wedge^2 Q$. Denote by M_Γ the pullback of M via q . We conclude that

$$H^1(E \otimes \Omega_G) = H^1(G, Rq_*(R^{-5} \otimes \wedge^2 Q_\Gamma \otimes Q_\Gamma^\vee \otimes M_\Gamma)) = H^1(\Gamma, R^{-5} \otimes Q_\Gamma \otimes M_\Gamma),$$

where the last equality follows from $Q_\Gamma = \wedge^2 Q_\Gamma \otimes Q_\Gamma^\vee$. By the short exact sequence

$$0 \rightarrow M_\Gamma \otimes Q_\Gamma \otimes R^{-5} \rightarrow Q_\Gamma^{\oplus 6} \otimes R^{-5} \rightarrow Q_\Gamma \otimes Q_\Gamma \otimes R^{-5} \rightarrow 0,$$

to prove the claim, it suffices to prove that $H^0(\Gamma, Q_\Gamma \otimes Q_\Gamma \otimes R^{-5}) = H^1(\Gamma, Q_\Gamma \otimes R^{-5}) = 0$. The latter follows from Lemma 3.7(i) below.

(3) We claim that $H^2(G, \wedge^2 E \otimes \Omega_G) = 0$. By [AK77, (5.4.2)] and Lemma 3.5, we have $H^2(G, \wedge^2 E \otimes \Omega_G) = H^2(\Gamma, \wedge^2 \text{Sym}_3(Q_\Gamma) \otimes (\wedge^2 Q_\Gamma)^{-6} \otimes Q_\Gamma^\vee \otimes M_\Gamma)$. From the short exact sequence $0 \rightarrow M_\Gamma \rightarrow V_\Gamma \rightarrow Q_\Gamma \rightarrow 0$, we have the exact cohomology sequence

$$\begin{aligned} H^1(\wedge^2 \text{Sym}_3(Q_\Gamma) \otimes (\wedge^2 Q_\Gamma)^{-6} \otimes Q_\Gamma^\vee \otimes Q_\Gamma) &\rightarrow H^2(\wedge^2 \text{Sym}_3(Q_\Gamma) \otimes (\wedge^2 Q_\Gamma)^{-6} \otimes Q_\Gamma^\vee \otimes M_\Gamma) \\ &\rightarrow H^2(\wedge^2 \text{Sym}_3(Q_\Gamma) \otimes (\wedge^2 Q_\Gamma)^{-6} \otimes Q_\Gamma^\vee)^{\oplus 6}. \end{aligned}$$

By Lemma 3.7(ii),(iii) below, we know that the first and last terms of this cohomology sequence are zero. The claim follows.

(4) We claim that $H^3(\wedge^3 E \otimes \Omega_G) = 0$. In fact, by [AK77, (5.4.2)], Proposition 3.1 and Lemma 3.5, we have $H^3(G, \wedge^3 E \otimes \Omega_G) = H^3(\Gamma, \text{Sym}_3(Q_\Gamma) \otimes N^{-6} \otimes R^{-6} \otimes Q_\Gamma^\vee \otimes M_\Gamma)$. From the short exact sequence (3.1), we have an exact cohomology sequence

$$\begin{aligned} H^2(\text{Sym}_3(Q_\Gamma) \otimes N^{-6} \otimes R^{-6} \otimes Q_\Gamma^\vee \otimes Q_\Gamma) &\rightarrow H^3(\Gamma, \text{Sym}_3(Q_\Gamma) \otimes N^{-6} \otimes R^{-6} \otimes Q_\Gamma^\vee \otimes M_\Gamma) \\ &\rightarrow H^3(\text{Sym}_3(Q_\Gamma) \otimes N^{-6} \otimes R^{-6} \otimes Q_\Gamma^\vee)^{\oplus 6}. \end{aligned}$$

The claim follows from the fact that first term and last term of this sequence are zero by Lemma 3.7(iv),(v).

(5) We claim that $H^4(\wedge^4 E \otimes \Omega_G) = 0$. By [AK77, (5.4.2)] and Lemma 3.5, we have $H^4(\wedge^4 E \otimes \Omega_G) = H^4(\Gamma, N^{-6} \otimes R^{-6} \otimes Q_\Gamma^\vee \otimes M_\Gamma)$. The exact sequence (3.1) gives rise to an exact cohomology sequence

$$H^3(N^{-6} \otimes R^{-6} \otimes Q_\Gamma \otimes Q_\Gamma^\vee) \rightarrow H^4(N^{-6} \otimes R^{-6} \otimes M_\Gamma \otimes Q_\Gamma^\vee) \rightarrow H^4(Q_\Gamma \otimes N^{-6} \otimes R^{-6})^{\oplus 6},$$

where the first and last terms are zero by Lemma 3.7(vii)–(viii) below. The claim follows. $\square$

LEMMA 3.7. *With the notation as before, we have*

- (i) $H^0(\Gamma, Q_\Gamma \otimes Q_\Gamma \otimes R^{-5}) = H^1(\Gamma, Q_\Gamma \otimes R^{-5}) = 0$,
- (ii) $H^2(\Gamma, \wedge^2 \text{Sym}_3(Q_\Gamma) \otimes (\wedge^2 Q_\Gamma)^{-6} \otimes Q_\Gamma^\vee) = 0$,
- (iii) $H^1(\Gamma, \wedge^2 \text{Sym}_3(Q_\Gamma) \otimes (\wedge^2 Q_\Gamma)^{-6} \otimes Q_\Gamma^\vee \otimes Q_\Gamma) = 0$,
- (iv) $H^3(\Gamma, \text{Sym}_3(Q_\Gamma) \otimes N^{-6} \otimes R^{-6} \otimes Q_\Gamma^\vee) = 0$,
- (v) $H^2(\Gamma, \text{Sym}_3(Q_\Gamma) \otimes N^{-6} \otimes R^{-6} \otimes Q_\Gamma^\vee \otimes Q_\Gamma) = 0$,
- (vi) $H^5(\mathbb{P}^5, \text{Sym}_2(K)(-1)) = 0$,
- (vii) $H^4(\Gamma, Q_\Gamma \otimes N^{-6} \otimes R^{-6}) = 0$,
- (viii) $H^3(\Gamma, N^{-6} \otimes R^{-6} \otimes Q_\Gamma \otimes Q_\Gamma^\vee) = 0$.

Proof. For part (i), by Proposition 3.1, we have a short exact sequence

$$0 \rightarrow N \otimes Q_\Gamma \otimes R^{-5} \rightarrow Q_\Gamma \otimes Q_\Gamma \otimes R^{-5} \rightarrow R^{-4} \otimes Q_\Gamma \rightarrow 0.$$

We claim that $H^0(R^{-4} \otimes Q_\Gamma) = 0$. This follows from the exact sequence

$$(0 =) H^0(\mathbb{P}^5, \Omega_{\mathbb{P}^5}(-3)) = H^0(N \otimes R^{-4}) \rightarrow H^0(R^{-4} \otimes Q_\Gamma) \rightarrow H^0(R^{-3})(=0),$$

where the first term is equal to zero by the Bott vanishing theorem and [AK77, (5.7.1)] and the last term is equal to zero by Lemma 3.2(i). Therefore, we conclude that $H^0(Q_\Gamma \otimes Q_\Gamma \otimes R^{-5}) = 0$ since $H^0(R^{-4} \otimes Q_\Gamma) = 0$.

The equality $H^0(Q_\Gamma \otimes N \otimes R^{-5}) = 0$ follows from the exact sequence

$$H^0(N^2 \otimes R^{-5}) \rightarrow H^0(Q_\Gamma \otimes N \otimes R^{-5}) \rightarrow H^0(N \otimes R^{-4}),$$

where the last term is zero by Lemma 3.2(i) and the first term is zero by the equality $H^0(N^2 \otimes R^{-5}) = H^0(\mathbb{P}^5, \text{Sym}_2(K)(-5)) = 0$. In the following, we show that $H^0(\mathbb{P}^5, \text{Sym}_2(K)(-5)) = 0$.

The fundamental sequence [AK77, Setup 5.2] gives rise to an exact sequence

$$0 \rightarrow H^0(K \otimes K(-5)) \rightarrow H^0(K(-5)^{\oplus 6}) (= H^0(\Omega^1(-4))^{\oplus 6} = 0).$$

It follows that $H^0(K \otimes K(-5)) = 0$. Moreover, we have a short exact sequence

$$0 \rightarrow \wedge^2 K(-5) \rightarrow K \otimes K(-5) \rightarrow \text{Sym}_2(K)(-5) \rightarrow 0, \quad (3.5)$$

which gives rise to an exact sequence

$$(0 =)H^0(K \otimes K(-5)) \rightarrow H^0(\mathrm{Sym}_2(K)(-5)) \rightarrow H^1(\wedge^2 K(-5)) (= H^1(\Omega_{\mathbb{P}^5}^2(-3)) = 0),$$

where the last equality $H^1(\Omega_{\mathbb{P}^5}^2(-3)) = 0$ follows from the Bott vanishing theorem. Therefore, we have proved that $H^0(\mathbb{P}^5, \mathrm{Sym}_2(K)(-5)) = 0$.

For part (ii), by Proposition 3.1, we have $\wedge^2 Q_\Gamma = N \otimes R$ and a short exact sequence

$$0 \rightarrow R^{-1} \rightarrow Q_\Gamma^\vee \rightarrow N^{-1} \rightarrow 0; \quad (3.6)$$

the latter gives rise to an exact sequence

$$\begin{aligned} (0 =)H^2(\wedge^2 \mathrm{Sym}_3 \otimes N^{-6} \otimes R^{-7}) &\rightarrow H^2(\wedge^2 \mathrm{Sym}_3(Q_\Gamma) \otimes (\wedge^2 Q_\Gamma)^{-6} \otimes Q_\Gamma^\vee) \\ &\rightarrow H^2(\wedge^2 \mathrm{Sym}_3 \otimes N^{-7} \otimes R^{-6})(= 0), \end{aligned}$$

where the last term is equal to zero by Lemma 3.2(vi). We have proved part (ii).

For part (iii), as in the proof of part (ii), the short exact sequence (3.6) gives rise to an exact cohomology sequence

$$\begin{aligned} (0 =)H^1(\wedge^2 \mathrm{Sym}_3(Q_\Gamma) \otimes N^{-6} \otimes R^{-7} \otimes Q_\Gamma) &\rightarrow H^1(\wedge^2 \mathrm{Sym}_3(Q_\Gamma) \otimes (\wedge^2 Q_\Gamma)^{-6} \otimes Q_\Gamma^\vee \otimes Q_\Gamma) \\ &\rightarrow H^1(\wedge^2 \mathrm{Sym}_3(Q_\Gamma) \otimes N^{-7} \otimes R^{-6} \otimes Q_\Gamma)(= 0), \end{aligned}$$

where the first term is zero by the following exact sequence associated with the short exact sequence in Proposition 3.1, see Lemma 3.2(vi):

$$\begin{aligned} (0 =)H^1(\wedge^2 \mathrm{Sym}_3(Q_\Gamma) \otimes N^{-5} \otimes R^{-7}) &\rightarrow H^1(\wedge^2 \mathrm{Sym}_3(Q_\Gamma) \otimes N^{-6} \otimes R^{-7} \otimes Q_\Gamma) \\ &\rightarrow H^1(\wedge^2 \mathrm{Sym}_3(Q_\Gamma) \otimes N^{-6} \otimes R^{-6})(= 0). \end{aligned}$$

Similarly, that the last term $H^1(\wedge^2 \mathrm{Sym}_3(Q_\Gamma) \otimes N^{-7} \otimes R^{-6} \otimes Q_\Gamma)$ is zero follows from the exact sequence

$$\begin{aligned} (0 =)H^1(\wedge^2 \mathrm{Sym}_3(Q_\Gamma) \otimes N^{-6} \otimes R^{-6}) &\rightarrow H^1(\wedge^2 \mathrm{Sym}_3(Q_\Gamma) \otimes N^{-7} \otimes R^{-6} \otimes Q_\Gamma) \\ &\rightarrow H^1(\wedge^2 \mathrm{Sym}_3(Q_\Gamma) \otimes N^{-7} \otimes R^{-5})(= 0), \end{aligned}$$

where the first and the last terms are zero by Lemma 3.2(vi). We have proved part (iii).

For part (iv), the exact sequence in Proposition 3.1 gives rise to an exact sequence

$$\begin{aligned} H^3(\Gamma, \mathrm{Sym}_3(Q_\Gamma) \otimes N^{-6} \otimes R^{-7}) &\rightarrow H^3(\Gamma, \mathrm{Sym}_3(Q_\Gamma) \otimes N^{-6} \otimes R^{-6} \otimes Q_\Gamma^\vee) \\ &\rightarrow H^3(\Gamma, \mathrm{Sym}_3(Q_\Gamma) \otimes N^{-7} \otimes R^{-6}), \end{aligned}$$

where the first and last terms are zero by Lemma 3.2(iv). Therefore, we have proved part (iv).

For part (v), Lemma 3.5 and the exact sequence in Proposition 3.1 give rise to an exact cohomology sequence (denote $\mathrm{Sym}_3(Q_\Gamma)$ by Sym_3)

$$\begin{aligned} H^2(\mathrm{Sym}_3 \otimes N^{-6} \otimes R^{-7} \otimes Q_\Gamma) &\rightarrow H^2(\mathrm{Sym}_3 \otimes N^{-6} \otimes R^{-6} \otimes Q_\Gamma \otimes Q_\Gamma^\vee) \\ &\rightarrow H^2(\mathrm{Sym}_3 \otimes N^{-7} \otimes R^{-6} \otimes Q_\Gamma) \rightarrow \cdots \end{aligned} \quad (3.7)$$

The term $H^2(\mathrm{Sym}_3 \otimes N^{-6} \otimes R^{-7} \otimes Q_\Gamma)$ is zero by the exact sequence

$$\begin{aligned} (0 =)H^3(\mathrm{Sym}_3 \otimes N^{-5} \otimes R^{-7}) &\rightarrow H^2(\mathrm{Sym}_3 \otimes N^{-6} \otimes R^{-7} \otimes Q_\Gamma) \\ &\rightarrow H^2(\mathrm{Sym}_3 \otimes N^{-6} \otimes R^{-6})(= 0), \end{aligned}$$

which follows from Lemma 3.2(iv), Lemma 3.5 and the short exact sequence in Proposition 3.1.

Similarly, the term $H^2(\text{Sym}_3 \otimes N^{-7} \otimes R^{-6} \otimes Q_\Gamma)$ in the sequence (3.7) is also zero by the exact cohomology sequence

$$H^2(\text{Sym}_3 \otimes N^{-6} \otimes R^{-6}) \rightarrow H^2(\text{Sym}_3 \otimes N^{-7} \otimes R^{-6} \otimes Q_\Gamma) \rightarrow H^2(\text{Sym}_3 \otimes N^{-7} \otimes R^{-5}).$$

It follows from Lemma 3.2(iv) that the first and the last terms of this cohomology sequence are zero.

For part (vi), an exact sequence similar to (3.5) gives rise to an exact sequence of cohomology groups

$$H^5(\mathbb{P}^5, (\wedge^2 K)(-1)) \rightarrow H^5(\mathbb{P}^5, (K \otimes K)(-1)) \rightarrow H^5(\mathbb{P}^5, \text{Sym}_2(K)(-1)) \rightarrow 0.$$

The first term $H^5(\mathbb{P}^5, (\wedge^2 K)(-1))$ is zero by the Bott vanishing theorem and the exact sequence

$$(0) = H^4(K) \rightarrow H^5(K \otimes K(-1)) \rightarrow H^5(K(-1))^{\oplus 6} (= 0),$$

which is associated with the short exact sequence $K \otimes K(-1) \rightarrow (K(-1))^6 \rightarrow K \rightarrow 0$. The latter follows from the Euler sequence $0 \rightarrow \Omega_{\mathbb{P}^5}^1 \rightarrow \mathcal{O}_{\mathbb{P}^5}(-1)^{\oplus 6} \rightarrow \mathcal{O}_{\mathbb{P}^5} \rightarrow 0$ and the fact that $K = \Omega_{\mathbb{P}^5}^1(1)$.

For part (vii), the exact sequence in Proposition 3.1 gives rise to the exact sequence of cohomology groups

$$H^4(\Gamma, N^{-6} \otimes R^{-7}) \rightarrow H^4(\Gamma, Q_\Gamma \otimes N^{-6} \otimes R^{-6}) \rightarrow H^4(\Gamma, N^{-7} \otimes R^{-6}),$$

where the first term is equal to $H^5(\Gamma, N)$ by Serre duality. It follows from [AK77, (5.7.1)] and the Bott vanishing theorem that $H^5(\Gamma, N) = H^5(\mathbb{P}^5, \Omega_{\mathbb{P}^5}^1(1)) = 0$. The last term $H^4(\Gamma, N^{-7} \otimes R^{-6})$ is $H^5(\mathbb{P}^5, \text{Sym}_2(K)(-1))$ by Serre duality and [AK77, (5.7.1)]. By part (vi), we know that $H^5(\mathbb{P}^5, \text{Sym}_2(K)(-1))$ is zero; therefore, we have proved part (vii).

For part (viii), the exact sequence in Proposition 3.1 gives rise to an exact sequence of cohomology groups

$$H^3(Q_\Gamma \otimes N^{-6} \otimes R^{-7}) \rightarrow H^3(N^{-6} \otimes R^{-6} \otimes Q_\Gamma \otimes Q_\Gamma^\vee) \rightarrow H^3(Q_\Gamma \otimes N^{-7} \otimes R^{-6}).$$

We claim that the first and last terms are zero. In fact, the exact sequence in Proposition 3.1 gives rise to the exact sequences of cohomology groups

$$\begin{aligned} H^3(N^{-5} \otimes R^{-7}) &\rightarrow H^3(Q_\Gamma \otimes N^{-6} \otimes R^{-7}) \rightarrow H^3(N^{-6} \otimes R^{-6}), \\ H^3(N^{-6} \otimes R^{-6}) &\rightarrow H^3(Q_\Gamma \otimes N^{-7} \otimes R^{-6}) \rightarrow H^3(N^{-7} \otimes R^{-5}). \end{aligned}$$

The first and the last terms of these two sequences are of the form $H^6(\Gamma, N^{\geq 0} \otimes R^*)$, and thus equal to $H^6(\mathbb{P}^5, -)$ by [AK77, (5.7.1)]. Therefore, they are zero. We have proved the claim and part (viii). $\square$

4. Deformations of automorphisms

Let X be a smooth projective scheme over the Witt ring $W(k) = W$ with relative dimension 4.

ASSUMPTION 4.1. We assume that the Hodge–de Rham spectral sequences of X/W degenerates at E_1 and the terms are locally free, so that the Hodge and de Rham cohomology sheaves commute with base change; see [Kat70, Theorem 8.0].

Under this assumption, we have the following theorem.

THEOREM 4.2 ([Pan16, Theorem 5.5]). *Let X be a smooth projective scheme over the Witt ring $W(k) = W$ with relative dimension 4. Let X_0 be the special fiber over k and f_0 be an automorphism of X_0 . Moreover, we assume that the map*

$$H_{\text{cris}}^4(f_0): H_{\text{cris}}^4(X_0/W) \rightarrow H_{\text{cris}}^4(X_0/W)$$

preserves the Hodge filtrations under the natural identification $H_{\text{cris}}^4(X_0/W) \cong H_{\text{DR}}^4(X/W)$. If Assumption 4.1 holds for X/W and the infinitesimal Torelli theorem holds for X_0 , that is, the cup product

$$H^1(X_0, T_{X_0}) \rightarrow \text{Hom}(H^q(X_0, \Omega_{X_0}^p), H^{q+1}(X_0, \Omega_{X_0}^{p-1}))$$

is injective for some p and q with $p + q = 4$, then one can lift the automorphism f_0 to an automorphism $f: X \rightarrow X$ over W .

Let X_3 be a smooth cubic fourfold. In the following lemma, we prove that one can deform the automorphisms of X_3 which are trivial on the cohomology.

LEMMA 4.3. *Over the complex numbers $\mathbb{C}$, the kernel $\text{Ker}(s)$ of the natural map*

$$s: \text{Aut}(X_3) \rightarrow \text{Aut}(H_{\text{sing}}^4(X_3, \mathbb{Q}))$$

is invariant for any smooth deformation of X_3 .

Proof. Denote by t the map $\text{Aut}(F(X_3)) \rightarrow \text{Aut}(H_{\text{sing}}^2(F(X_3), \mathbb{Q}))$. We claim that $\text{Ker}(s) \simeq \text{Ker}(t)$.

In fact, we have a canonical map

$$j: \text{Aut}(X_3) = \text{Aut}_L(X_3) \rightarrow \text{Aut}(F(X_3));$$

see Lemma 2.6. It is clear that we have a commutative diagram (cf. (2.3))

$$\begin{array}{ccccc} F(X_3) & \xleftarrow{q} & L & \xrightarrow{p} & X_3 \\ j(g) \downarrow & & \downarrow h & & \downarrow g \\ F(X_3) & \xleftarrow{q} & L & \xrightarrow{p} & X_3, \end{array}$$

where $g \in \text{Aut}(X_3) = \text{Aut}_L(X_3)$ and $q: L \rightarrow F(X_3)$ is the universal bundle of lines. We have a morphism $h: L \rightarrow L$ associating with a line l in X_3 the line $g(l)$ in X_3 . Note that the inverse map of $j(g)^*$ is $j(g)_*$. Therefore, we have $j(g)^* \circ (q_* p^*) = q_* \circ h^* \circ p^* = (q_* p^*) \circ g^*$ on $H_{\text{sing}}^4(X_3, \mathbb{Q})$. Since the Abel–Jacobi map $q_* p^*: H_{\text{sing}}^4(X_3, \mathbb{Q}) \rightarrow H_{\text{sing}}^2(F(X_3), \mathbb{Q})$ is an isomorphism [BD85], the following are equivalent:

- $j(g)^* = \text{Id}$ on $H_{\text{sing}}^2(F(X_3), \mathbb{Q})$,
- $g^* = \text{Id}$ on $H_{\text{sing}}^4(X_3, \mathbb{Q})$.

In particular, the map j sends $\text{Ker}(s)$ into $\text{Ker}(t)$. We claim that this map is an isomorphism. The injectivity is obvious by the fact that for each point $x \in X_3$, there are at least two lines in X_3 passing through x .

We show that j is surjective as follows. Since $F(X_3)$ is simply connected, an automorphism $j(g)$ such that $j(g)^* = \text{Id}$ on $H_{\text{sing}}^2(F(X_3))$ must preserve the natural ample bundle $\mathcal{O}_{F(X_3)}(1)$ of $F(X_3)$ which is induced by the Plücker embedding.

We claim that an automorphism f of $F(X_3)$ preserving the natural polarization $\mathcal{O}_{F(X_3)}(1)$ comes from a morphism $g: X_3 \rightarrow X_3$; in other words, we have $j(g) = f$. We consider the natural

inclusions $F(X_3) \subseteq \mathbb{G}(1, 5) \subseteq \mathbb{P}^N$, where the last inclusion is the Plücker embedding. By [AK77, Theorem 1.16(iii)], the Grassmannian $\mathbb{G}(1, 5)$ is the intersection of the quadric hypersurfaces in $\mathbb{P}^N$ containing $F(X_3)$. So there is an automorphism $g': \mathbb{G}(1, 5) \rightarrow \mathbb{G}(1, 5)$ sending $F(X_3)$ to $F(X_3)$ and such that $g'|_{F(X_3)} = f$. By a theorem of Chow [Cho49], we have

$$\mathrm{Aut}(\mathbb{G}(1, 5)) = \mathrm{Aut}(\mathbb{P}^5).$$

So there is a linear automorphism g'' of $\mathbb{P}^5$ sending X_3 to X_3 . The automorphism $g''|_{X_3}$ induces the automorphism $f: F(X_3) \rightarrow F(X_3)$. It follows that $j(g''|_{X_3}) = f$. We have shown that $\mathrm{Ker}(s) \simeq \mathrm{Ker}(t)$.

Since $F(X_3)$ is a holomorphic symplectic manifold [BD85], the lemma follows from [HT13, Theorem 2.1]. $\square$

PROPOSITION 4.4. *Over the complex numbers $\mathbb{C}$, the map $\mathrm{Aut}(X_3) \rightarrow \mathrm{Aut}(\mathrm{H}_{\mathrm{sing}}^4(X_3, \mathbb{Q}))$ is injective.*

Proof. The proposition follows from Lemma 4.3 and the fact that the automorphism group of a general cubic fourfold is trivial; see [MM63]. For a different proof of the proposition, we refer to [PCZ15, Proposition 6.1]. $\square$

Remark 4.5. One can also use [Pan16, Theorem 3.2 and Corollary 3.3] to give a simple proof.

Proof of Theorem 1.3. By Proposition 4.4 and the proof of Lemma 4.3, the theorem holds in characteristic zero. Assume that the characteristic of k is positive. Let $f_0 \in \mathrm{Aut}(X_3)$ be an automorphism of X_3 such that $f_0^* = \mathrm{Id}$ on $\mathrm{H}_{\mathrm{et}}^4(X_3, \mathbb{Q}_l)$. It follows from Lemma 2.2 that $f_0^* = \mathrm{Id}$ on $\mathrm{H}_{\mathrm{cris}}^4(X_3/W)_K$, where K is the fraction field of the Witt ring $W = W(k)$. Therefore, by Lemma 2.4, we have $f_0^* = \mathrm{Id}$ on $\mathrm{H}_{\mathrm{cris}}^4(X_3/W)$. Let $\widehat{X}_3$ be a lifting of X_3 over W . By the comparison theorem of crystalline cohomology and de Rham cohomology, we have

$$f_0^* = \mathrm{Id}: \mathrm{H}_{\mathrm{DR}}^4(\widehat{X}_3/W) \rightarrow \mathrm{H}_{\mathrm{DR}}^4(\widehat{X}_3/W).$$

In particular, the action f_0^* preserves the Hodge filtrations of $\mathrm{H}_{\mathrm{DR}}^4(\widehat{X}_3/W) = \mathrm{H}_{\mathrm{cris}}^4(X_3/W)$. Note that Assumption 4.1 holds for $\widehat{X}_3/W$ [DK73, Exposé IX, Theorem 1.5]. By Theorem 4.2 and the infinitesimal Torelli theorem [Lev01, Theorem 3.4] for cubic fourfolds, one can lift the map f_0 to a map $\widehat{f}$ on $\widehat{X}_3/W$. By Proposition 4.4, we conclude that $(\widehat{f})_K: (\widehat{X}_3)_K \rightarrow (\widehat{X}_3)_K$ is the identity map, where K is the fraction field of $W(k)$. Therefore, its specialization f_0 is the identity map. We have proved the first assertion that the map $\mathrm{Aut}(X_3) \rightarrow \mathrm{Aut}(\mathrm{H}_{\mathrm{et}}^4(X_3, \mathbb{Q}_l))$ is injective.

For the second statement, by Proposition 3.3, we know that the Picard group $\mathrm{Pic}(F(X_3)) = \mathrm{NS}(F(X_3))$ of $F(X_3)$ is torsion free; hence, it is a subgroup of $\mathrm{H}_{\mathrm{et}}^2(F(X_3), \mathbb{Q}_l)$, where we omit the Tate twist. Hence, for an automorphism f of $F(X_3)$ such that f^* is the identity Id on $\mathrm{H}_{\mathrm{et}}^2(F(X_3), \mathbb{Q}_l)$, we have $f^* = \mathrm{Id}$ on the Picard group $\mathrm{Pic}(F(X_3))$. Let L be the very ample line bundle of $F(X_3)$ induced by the Plücker embedding. We have $f^*(L) = L$. By the proof of Lemma 4.3, we know that f is induced by an automorphism g of the cubic fourfold X_3 . To prove that f is the identity map, it suffices to show that g is the identity map. In fact, it follows from $\mathrm{H}_{\mathrm{et}}^2(F(X_3), \mathbb{Q}_l) \simeq \mathrm{H}_{\mathrm{et}}^4(X_3, \mathbb{Q}_l)$ [BD85] that $g^* = \mathrm{Id}$ on $\mathrm{H}_{\mathrm{et}}^4(X_3, \mathbb{Q}_l)$. Hence, we have $g = \mathrm{Id}$ by the first statement. $\square$

5. Lefschetz cubics

Let k be an algebraically closed field. We assume $\mathrm{char}(k) \neq 2, 3$. A cubic hypersurface is called a *Lefschetz cubic* if it has exactly one ordinary double point.

LEMMA 5.1. *Let X_d be a hypersurface in $\mathbb{P}^n$ over k . If the dimension of X_d is at least 3, then we have $\text{Aut}_L(X_d) = \text{Aut}(X_d)$.*

Proof. By [Har70, Chapter IV, Corollary 3.2], we know that $\text{Pic}(X_d) = \mathbb{Z}[\mathcal{O}_{X_d}(1)]$. It is clear that $H^0(X_d, \mathcal{O}_{X_d}(1)) = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. Therefore, we conclude that $\text{Aut}_L(X_d) = \text{Aut}(X_d)$. $\square$

Proof of Theorem 1.4. We give a proof for Lefschetz cubic fourfolds and sketch a proof for Lefschetz cubic threefolds. First of all, we blow up the node p of the cubic fourfold X_3 . Denote the blow-up by $\text{Bl}_p: \widetilde{X}_3 \rightarrow X_3$. Suppose $g \in \text{Aut}(X_3)$. It is clear that $g(p) = p$. Therefore, g induces a map g' from $\widetilde{X}_3$ to $\widetilde{X}_3$. We have the diagram

$$\begin{array}{ccccc} Q & \xrightarrow{i} & \widetilde{X}_3 & \xrightarrow{\text{Bl}_p} & X_3 \\ \downarrow g'|_Q & & \downarrow g' & & \downarrow g \\ Q & \xrightarrow{i} & \widetilde{X}_3 & \xrightarrow{\text{Bl}_p} & X_3, \end{array} \quad (5.1)$$

where Q is the exceptional divisor of the blow-up; it is a smooth quadric threefold [DK73, Exposé XII]. We apply the long exact sequence of ℓ -adic cohomology with compact support to the pair $(X_3, X_3 - p)$; see [Fu11, Theorem 7.4.4(iii)]. This gives rise to an exact sequence

$$(0 =) H_{\text{ét},c}^3(p) \longrightarrow H_{\text{ét},c}^4(U) \longrightarrow H_{\text{ét}}^4(X_3) \longrightarrow H_{\text{ét},c}^4(p) (= 0),$$

where U is $X_3 - p$. We conclude that $H_{\text{ét},c}^4(U) \simeq H_{\text{ét}}^4(X_3)$. On the other hand, we have an exact sequence

$$(0 =) H_{\text{ét}}^3(Q) \rightarrow H_{\text{ét}}^4(X_3) \rightarrow H_{\text{ét}}^4(\widetilde{X}_3) \oplus H_{\text{ét}}^4(p) \rightarrow H_{\text{ét}}^4(Q) \rightarrow H_{\text{ét}}^5(X_3)$$

by [Mil80, Chapter VI, Exercise 3.4]. It gives rise to an exact sequence

$$0 \rightarrow H_{\text{ét}}^4(X_3) \rightarrow H_{\text{ét}}^4(\widetilde{X}_3) \rightarrow H_{\text{ét}}^4(Q) (= \mathbb{Q}_l); \quad (5.2)$$

see [DK73, Section XII.3]. Let $N_{Q/\widetilde{X}_3}$ be the normal bundle of Q in $\widetilde{X}_3$. By [GBH⁺77, Chapter VII, Theorem 4.1], we have maps i_* and i^* such that

$$i^* i_*(-) = (-) \cup c_1(N_{Q/\widetilde{X}_3})$$

and

$$\mathbb{Q}_l = H_{\text{ét}}^2(Q) \xrightarrow[\substack{i_* \\ -\cup c_1(N_{Q/\widetilde{X}_3})}]{i^*} H_{\text{ét}}^4(\widetilde{X}_3) \xrightarrow{i^*} H_{\text{ét}}^4(Q) = \mathbb{Q}_l, \quad (5.3)$$

where $\mathbb{Q}_l = H_{\text{ét}}^2(Q)$ and $H_{\text{ét}}^4(Q) = \mathbb{Q}_l$ follow from [DK73, Exposé XII]. Let $\widetilde{\mathbb{P}^5}$ be the blow-up of $\mathbb{P}^5$ at p . Since $\widetilde{X}_3$ is the strictly transformation of X_3 in the blow-up $\widetilde{\mathbb{P}^5}$, it is easy to see that $N_{Q/\widetilde{X}_3} = \mathcal{O}_Q(-1)$. Therefore, the map $i^* i_*$ is an isomorphism which gives a splitting by (5.2). In particular, we have

$$H_{\text{ét}}^2(Q) \oplus H_{\text{ét}}^4(X_3) \xrightarrow[\simeq]{(i_*, \text{Bl}_p^*)} H_{\text{ét}}^4(\widetilde{X}_3).$$

If $g^* = \text{Id}$ on $H_{\text{ét}}^4(X_3)$, then, by Lemma 5.2 below and the equality $H_{\text{ét}}^2(Q) = \mathbb{Q}_l \langle c_1(\mathcal{O}_Q(1)) \rangle$ (see [DK73, Exposé XII]), we conclude that the map g'^* is the identity on $H_{\text{ét}}^4(\widetilde{X}_3)$.

Denote by S the space of lines in X_3 passing through p . Let

$$L \rightarrow S$$

be the tautological bundle of lines. We hope that there is no confusion if we also denote by L the union of lines in X_3 passing through p . In an affine coordinate system with the origin p , we can write down the equation for X_3 as follows:

$$0 = f_2(x_1, \dots, x_5) + f_3(x_1, \dots, x_5),$$

where f_2 and f_3 are polynomials of degree 2 and 3, respectively. Therefore, the surface S is a complete intersection in $\mathbb{P}^4$ of type $(2, 3)$ defined by the equations f_2 and f_3 . Since X_3 has exactly one node, the surface S is a smooth $K3$ surface by local calculations (here we use the assumption $\text{char}(k) \neq 2$ or 3). We project X_3 from p to $\mathbb{P}^4$. This gives rise to the following commutative diagram (see Lemma 5.1):

$$\begin{array}{ccccccc} \widetilde{X}_3 & \xrightarrow{\text{Bl}_p} & X_3 & \longleftarrow & X_3 - L & \xrightarrow{g|_{X_3-L}} & X_3 - L \\ \downarrow \simeq & & \downarrow \text{proj} & & \downarrow \simeq & & \downarrow \simeq \\ \text{Bl}_S(\mathbb{P}^4) & \xrightarrow{\text{Bl}_S} & \mathbb{P}^4 & \longleftarrow & \mathbb{P}^4 - S & \xrightarrow{g_0} & \mathbb{P}^4 - S, \end{array}$$

where proj is the projection from p to a projective subspace $\mathbb{P}^4$ and g_0 is induced by the vertical identification between $X_3 - L$ and $\mathbb{P}^4 - S$. Since S is of codimension 2, the map g_0 extends to an automorphism g_1 of $\mathbb{P}^4$ sending S to S . It induces a map

$$g_2: \text{Bl}_S(\mathbb{P}^4) \rightarrow \text{Bl}_S(\mathbb{P}^4).$$

Under the identification $\text{Bl}_S(\mathbb{P}^4) = \widetilde{X}_3$, the map $g': \widetilde{X}_3 \rightarrow \widetilde{X}_3$ coincides with the map g_2 over the open subset $\text{Bl}_S(\mathbb{P}^4) - E$, where E is the exceptional divisor of the blow-up $\text{Bl}_S(\mathbb{P}^4)$ along S . It follows that $g_2 = g'$. So we have the following diagram:

$$\begin{array}{ccccccc} E & \xrightarrow{j} & \text{Bl}_S(\mathbb{P}^4) & \xrightarrow{g_2} & \text{Bl}_S(\mathbb{P}^4) & \xleftarrow{j} & E \\ \downarrow \pi & & \downarrow \text{Bl}_S & & \downarrow \text{Bl}_S & & \downarrow \pi \\ S & \xrightarrow{\text{inc}} & \mathbb{P}^4 & \xrightarrow{g_1} & \mathbb{P}^4 & \xleftarrow{\text{inc}} & S. \end{array}$$

Consider the map

$$\varphi: H_{\text{ét}}^4(\mathbb{P}^4) \oplus H_{\text{ét}}^2(S) \oplus H_{\text{ét}}^0(S) \rightarrow H_{\text{ét}}^4(\text{Bl}_S(\mathbb{P}^4)) \quad (5.4)$$

that sends (x, y, z) to $\text{Bl}_S^*(x) + j_*\pi^*(y) + j_*((\pi^*z) \cup c_1(\mathcal{O}_E(1)))$. It follows from [Voi02, Theorem 7.3.1] and [GBH⁺77, Chapter VII] that the map φ is an isomorphism.

By Lemma 5.3 below, we have

$$g_2^* \circ \varphi = \varphi \circ (g_1^*, g_1|_S^*, g_1|_S^*). \quad (5.5)$$

From the discussion above, we know that $g_2^* = \text{Id}$ on $H_{\text{ét}}^4(\text{Bl}_S(\mathbb{P}^4))$ if the map g^* is Id on $H_{\text{ét}}^4(X_3)$. Therefore, we have $(g_1|_S)^* = \text{Id}$ on $H_{\text{ét}}^2(S, \mathbb{Q}_l)$. By [Riz06, Proposition 3.4.2], we conclude that $g_1|_S$ is the identity since $H_{\text{ét}}^2(S, \mathbb{Z}_l)$ is torsion free. Since S is a smooth complete intersection surface of type $(2, 3)$ in $\mathbb{P}^4$, we have

$$H^0(S, \mathcal{O}_S(1)) = \frac{c_1(\mathcal{O}_S(1)) \cdot c_1(\mathcal{O}_S(1))}{2} + \chi(\mathcal{O}_S) = 3 + 2 = 5$$

by the Riemann–Roch theorem. Therefore, we know that the inclusion inc is given by the complete linear system $|\mathcal{O}_S(1)|$

$$\begin{array}{ccccc} & & g_1|_S & & \\ & \searrow & \xrightarrow{\quad} & \swarrow & \\ S & \xrightarrow{\text{inc}} & \mathbb{P}^4 & \xrightarrow{g_1} & \mathbb{P}^4 & \xleftarrow{\text{inc}} & S. \end{array}$$

Therefore, we conclude that $g_1 = \text{Id}$ and $g_2 = \text{Id}$. In summary, we have proved that the theorem holds for Lefschetz cubic fourfolds. For the cubic threefolds, the proof is similar. We sketch a proof here.

- (1) The exact sequence (5.2) is replaced by $H_{\text{ét}}^3(X_3) \rightarrow H_{\text{ét}}^3(\widetilde{X}_3) \rightarrow 0$. Then $g_2^* = \text{Id}$ on $H_{\text{ét}}^3(\widetilde{X}_3) \cong H_{\text{ét}}^3(\text{Bl}_S \mathbb{P}^3)$.
- (2) The space S of lines is a smooth complete intersection curve of type $(2, 3)$ in $\mathbb{P}^3$. Instead of using [Riz06, Proposition 3.4.2], we use [DM69, Theorem 1.13] to show that $g_1 = \text{Id}$ and $g_2 = \text{Id}$.

Steps (1) and (2) imply that the theorem holds for Lefschetz cubic threefolds. $\square$

LEMMA 5.2. *Under the hypothesis as above, we have the following commutative diagram:*

$$\begin{array}{ccc} H_{\text{ét}}^4(X_3) \oplus H_{\text{ét}}^2(Q) & \xrightarrow[\simeq]{(\text{Bl}_p^*, i_*)} & H_{\text{ét}}^4(\widetilde{X}_3) \\ (g^*, g'^*|_Q) \downarrow & & \downarrow g^* \\ H_{\text{ét}}^4(X_3) \oplus H_{\text{ét}}^2(Q) & \xrightarrow[\simeq]{(\text{Bl}_p^*, i_*)} & H_{\text{ét}}^4(\widetilde{X}_3). \end{array}$$

Proof. By the diagram (5.1), we have $g'^* \circ \text{Bl}_p^* = \text{Bl}_p^* \circ g^*$ and $g'^* \circ i_* = i_* \circ g'|_Q^*$ since g'^* is the inverse of g'_* . $\square$

LEMMA 5.3. *With the same notation and hypotheses as above, we have $g_2^* \circ \varphi = \varphi \circ (g_1^*, g_1|_S^*, g_1|_S^*)$; see (5.5) and (5.4).*

Proof. It suffices to show that

$$g_2^*(\text{Bl}_S^*(x) + j_*\pi^*(y) + j_*((\pi^*z) \cup c_1(\mathcal{O}_E(1))))$$

is equal to

$$\text{Bl}_S^*g_1^*(x) + j_*(\pi^*(g_1|_S)^*(y)) + j_*[(\pi^*(g_1|_S)^*(z)) \cup c_1(\mathcal{O}_E(1))].$$

We claim that

- $g_2^* \circ \text{Bl}_S^*(x) = \text{Bl}_S^* \circ g_1^*(x)$,
- $g_2^* \circ j_* \circ \pi^*(y) = j_* \circ \pi^* \circ (g_1|_S)^*(y)$,
- $(g_2^* \circ j_*)(\pi^*z \cup c_1(\mathcal{O}_E(1))) = j_*((\pi^* \circ (g_1|_S)^*(z)) \cup c_1(\mathcal{O}_E(1)))$.

The first equality is obvious. The second equality follows from the equalities $g_2^* \circ j_*(\pi^*y) = (j_* \circ (g_2|_E)^*)(\pi^*y)$ and $g_{2*} = (g_2^*)^{-1}$. To show the third equality, it suffices to show that

$$(g_2^* \circ j_*)(\pi^*z \cup c_1(\mathcal{O}_E(1))) = j_*([(g_2|_E)^*(\pi^*(z))] \cup c_1(\mathcal{O}_E(1))).$$

In fact, that $g_{2*}j_*((g_2|_E)^*(\pi^*(z)) \cup c_1(\mathcal{O}_E(1)))$ is equal to

$$j_*((g_2|_E)_*([(g_2|_E)^*(\pi^*(z))] \cup c_1(\mathcal{O}_E(1)))) = j_*(\pi^*(z) \cup [(g_2|_E)_*(c_1(\mathcal{O}_E(1)))])$$

follows from the projection formula and the fact that $g_{2*} = (g_2^*)^{-1}$. Therefore, it suffices to show that

$$j_*(\pi^*z \cup c_1(\mathcal{O}_E(1))) = j_*(\pi^*(z) \cup [(g_2|_E)_*(c_1(\mathcal{O}_E(1)))]).$$

We claim that $(g_2|_E)_*(c_1(\mathcal{O}_E(1))) = c_1(\mathcal{O}_E(1))$. The lemma follows from this claim. To prove the latter, we just need to show that $(g_2|_E)^*(\mathcal{O}_E(1)) = \mathcal{O}_E(1)$. In fact, we have $\mathcal{O}_E(-1) = N_{E/\text{Bl}_S(\mathbb{P}^4)}$, where $N_{E/\text{Bl}_S(\mathbb{P}^4)}$ is the normal bundle of E in $\text{Bl}_S(\mathbb{P}^4)$. It follows that $(g_2|_E)^*(N_{E/\text{Bl}_S(\mathbb{P}^4)}) = N_{E/\text{Bl}_S(\mathbb{P}^4)}$. Hence, we have proved the claim. $\square$

6. Vanishing cycles and the middle Picard number

In this section, we apply the theory of Milnor fibers and vanishing cycles to show that there is a smooth cubic fourfold of middle Picard number 1 with an automorphism of order 3; see Proposition 6.5. All varieties in this section are over the complex numbers $\mathbb{C}$. We fix some notation as follows.

Let $F_1(X_1, X_2, \dots, X_5)$ and $F_2(X_1, X_2, \dots, X_5)$ be cubic forms defining smooth cubic threefolds. Let V_1 and V_2 be the smooth cubic threefolds in $\mathbb{P}^4$ defined by $F_1 = 0$ and $F_2 = 0$, respectively. By the theory of Lefschetz pencils, we can assume that F_1 and F_2 are two forms such that the intersection $V_1 \cap V_2$ is smooth and every fiber of the family X of cubic threefolds defined by

$$F_1(X_1, \dots, X_5) + tF_2(X_1, \dots, X_5) = 0, \quad t \in \mathbb{C}$$

has at most one node.

Using this family X of cubic threefolds, we can construct a family $X_{\mathbb{A}^1}$ of cubic fourfolds $f: X_{\mathbb{A}^1} \rightarrow \mathbb{A}^1$ as follows: $X_{\mathbb{A}^1}$ is given by

$$\{(t, [X_0 : \dots : X_5]) \in \mathbb{A}^1 \times \mathbb{P}^5 \mid X_0^3 + F_1(X_1, \dots, X_5) + tF_2(X_1, \dots, X_5) = 0\}. \quad (6.1)$$

In other words, the fiber of $X_{\mathbb{A}^1} \rightarrow \mathbb{A}^1$ over $b \in \mathbb{C}$ is a cubic fourfold which is a 3-cyclic branched covering over $\mathbb{P}^4$ along the cubic threefold $(X)_b$ which is the cubic threefold in X over b . Hence, every fiber of $X_{\mathbb{A}^1} \rightarrow \mathbb{A}^1$ has at most one singularity, and the singularity must be of A_2 -type. Such an A_2 -singularity is locally defined by the equation

$$X_0^3 + X_1^2 + X_2^2 + \dots + X_5^2 = 0.$$

Denote by $x_1, \dots, x_m \in X_{\mathbb{A}^1}$ the A_2 -singularities in $X_{\mathbb{A}^1}$. The Milnor fiber Mil_{f, x_i} corresponding to x_i is homotopic to $S^4 \vee S^4$ since its Milnor number is 2.

The family $X_{\mathbb{A}^1} \rightarrow \mathbb{A}^1$ can be completed to a pencil of cubic fourfolds over $\mathbb{P}^1$ naturally. Denote the pencil by

$$f: X_{\mathbb{P}^1} \rightarrow \mathbb{P}^1.$$

In particular, it is the pencil of cubic fourfolds defined by cubic fourfolds $X_0^3 + F_1 = 0$ and $F_2 = 0$. Over the point $\infty \in \mathbb{P}^1$, the fiber X_∞ is the cubic fourfold defined by $F_2(X_1, \dots, X_5) = 0$. It is clear that the cubic fourfold X_∞ is a cone. An easy calculation shows that the base locus B of this pencil is smooth since $V_1 \cap V_2$ is smooth. Moreover, we have $X_{\mathbb{P}^1} = \text{Bl}_B(\mathbb{P}^5)$.

Let $U = \mathbb{A}^1 - \{f(x_1), \dots, f(x_m)\}$ and $X_b = f^{-1}(b)$ ($b \in U$). It is clear that $X_{\mathbb{P}^1} - X_\infty = X_{\mathbb{A}^1}$. We have the following lemma to describe the vanishing cycles.

LEMMA 6.1. *Let Y be a smooth fiber of $X_{\mathbb{A}^1} \rightarrow \mathbb{A}^1$ over some point $b \in U$. Consider the inclusions $j': Y \subseteq \mathbb{P}^5$ and $j: Y = f^{-1}(b) \subseteq X_{\mathbb{P}^1} = \text{Bl}_B(X)$.*

- (i) $H_{\text{prim}}^4(Y, \mathbb{C}) = H_{\text{van}}^4(Y, \mathbb{C}) := \text{Ker}(j'_*: H^4(Y, \mathbb{C}) \rightarrow H^6(\mathbb{P}^5, \mathbb{C}))$.
- (ii) *By Poincaré duality, the space $H_{\text{van}}^4(Y, \mathbb{C})$ of vanishing cycles is generated by the image of $H_4(\text{Mil}_{f, x_i}, \mathbb{C})$ in*

$$H_4(X_{t_i}, \mathbb{C}) \cong H_4(Y, \mathbb{C}),$$

where t_i is a regular value of f nearby $f(x_i)$ and we identify the homology and cohomology groups of X_{t_i} with those of Y .

Proof. The first statement follows from [Voi02, Proposition 2.27]. For the second statement, we

claim that

$$\mathrm{Ker}(j'_* : H_4(Y, \mathbb{C}) \rightarrow H_4(\mathbb{P}^5, \mathbb{C})) = \mathrm{Ker}(j_* : H_4(Y, \mathbb{C}) \rightarrow H_4(X_{\mathbb{P}^1} - X_\infty, \mathbb{C})),$$

where the left-hand side is $H_{\mathrm{van}}^4(Y, \mathbb{C})$ by Poincaré duality. In fact, $X_{\mathbb{P}^1} = \mathrm{Bl}_B(\mathbb{P}^5)$ and $j' = \mathrm{Bl}|_{X_{\mathbb{A}^1}} \circ j$ where Bl is the blow-up morphism

$$\mathrm{Bl} : X_{\mathbb{P}^1} = \mathrm{Bl}_B(\mathbb{P}^5) \rightarrow \mathbb{P}^5.$$

So we have a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Ker}(j_*) & \longrightarrow & H_4(Y) & \xrightarrow{j_*} & H_4(X_{\mathbb{A}^1}) \\ & & \downarrow h & & \parallel & & \downarrow (Bl|_{X_{\mathbb{A}^1}})_* \\ 0 & \longrightarrow & \mathrm{Ker}(j'_*) & \longrightarrow & H_4(Y) & \xrightarrow{j'_*} & H_4(\mathbb{P}^5), \end{array}$$

where $X_{\mathbb{A}^1} = X_{\mathbb{P}^1} - X_\infty$. If j_* is surjective, then h is an isomorphism by the snake lemma. To show the surjectivity of j_* , we consider the long exact sequence of homology groups associated with $(X_{\mathbb{A}^1}, Y)$,

$$H_5(X_{\mathbb{A}^1}, Y) \longrightarrow H_4(Y) \xrightarrow{j_*} H_4(X_{\mathbb{A}^1}) \longrightarrow H_4(X_{\mathbb{A}^1}, Y).$$

By Lemma 6.2 below, we have $H_4(X_{\mathbb{A}^1}, Y) = 0$. Therefore, we have shown that j_* is surjective. $\square$

LEMMA 6.2. *With the notation as above, we have*

$$H_j(X_{\mathbb{A}^1}, Y) \simeq \oplus_{i=1}^m H_j(X_{\Delta_i}, Y) \simeq \oplus_{i=1}^m \tilde{H}_{j-1}(\mathrm{Mil}_{f, x_i}),$$

where Δ_i is a small disk containing $f(x_i)$ and we identify Y with any smooth fiber of f , so that the relative homology groups make sense. Moreover, we have $H_4(X_{\mathbb{A}^1}, Y) = 0$ and $H_5(X_{\mathbb{A}^1}, Y) = \oplus_{i=1}^m H_4(\mathrm{Mil}_{f, x_i})$.

Proof. In fact, we can choose a good cover $\{U_1, \dots, U_{s+1}\}$ of $\mathbb{P}^1$ such that

- the finite intersections of $\{U_i\}$ are contractible,
- the intersection $\cap_{i \in I} U_i$ does not contain any point of $\{x_1, \dots, x_m, \infty\}$ if $\#(I)$ (where $I \subseteq \{1, 2, \dots, s+1\}$) is greater than 1,
- $\mathbb{P}^1 - (U_1 \cup \dots \cup U_s)$ contains ∞ , and it is contractible to ∞ ,
- U_i contains x_i , and U_j does not contain any x_i if $j \geq m+1$.

It follows from a deformation retraction that the inclusion $(X_{U_1 \cup \dots \cup U_s}, Y) \subseteq (X_{\mathbb{A}^1}, Y)$ is a homotopy equivalence. It follows that $H_j(X_{\mathbb{A}^1}, Y) \simeq H_j(X_{U_1 \cup \dots \cup U_s}, Y)$. By the Mayer–Vietoris sequence and induction on the number of the coverings $\{U_i\}$, we have $H_j(X_{U_1 \cup \dots \cup U_s}, Y) = \oplus_{i=1}^m H_j(X_{U_i}, Y)$. Since we can shrink U_i to a small disk Δ_i , we have shown that $H_j(X_{\mathbb{A}^1}, Y) \simeq \oplus_{i=1}^m H_j(X_{\Delta_i}, Y)$. By the excision theorem (see [PS08, Proof of Lemma C.13]), we have

$$H_*(X_{\Delta_i}, Y) \simeq H_*(B, \mathrm{Mil}_{f, x_i}) \simeq \tilde{H}_{*-1}(\mathrm{Mil}_{f, x_i}),$$

where B is a small ball that is centered at x_i . We have proved the lemma. $\square$

From now on, we denote $H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3))$ by V . Suppose that V has a basis $\{F_1, F_2, \dots, F_{N+1}\}$. Using this basis, we identify $\mathbb{P}(V)$ with $\mathbb{P}^N$. Let $[t_1 : \dots : t_{N+1}]$ be the coordinates of $\mathbb{P}^N$. So we have a natural inclusion

$$\mathbb{A}^1 \hookrightarrow \mathbb{A}^{N-1} = \{t_1 \neq 0\}, \quad (6.2)$$

which maps $t \in \mathbb{A}^1$ to $[F_1 + tF_2]$.

By the theory of Lefschetz pencils, we know that there is an irreducible discriminant hypersurface H' in $\mathbb{P}(V)$ parametrizing singular cubic threefolds; see [Voi02, (2.2) and Section 3.2.2]. Denote the universal smooth hypersurface by

$$\varphi: \mathcal{X} \rightarrow \mathbb{P}(V) - H'.$$

Let H'_o be the open subset of H' parametrizing cubic threefolds with at most one ordinary double point. The subvariety H'_o is smooth; see [Voi02, Lemma 2.7 and Proposition 2.9]. Let H be $H' \cap \mathbb{A}^{N-1}$ and H_o be $H'_o \cap \mathbb{A}^{N-1}$, where $\mathbb{A}^{N-1} = \{t_1 \neq 0\}$; see (6.2). In particular, we have $H_o \cap \mathbb{A}^1 = \{x_1, \dots, x_m\}$. Assume $w = \exp(2\pi i/3)$.

LEMMA 6.3. *Let r_i be the local monodromy operator of $\pi_1(\Delta_i^*)$ on the homology $H_4(\text{Mil}_{f,x_i})$. We have the following decomposition:*

$$H_4(\text{Mil}_{f,x_i}, \mathbb{C}) = H_w^i \oplus H_{w^2}^i,$$

where H_w^i (respectively, $H_{w^2}^i$) is the w (respectively, w^2)-eigenspace of r_i , which is of dimension 1. Denote the corresponding eigenvector by δ_w^i (respectively, $\delta_{w^2}^i$). Under the monodromy action of $\pi_1(U)$, all the image of δ_w^i (respectively, $\delta_{w^2}^i$) in $H_{\text{prim}}^4(X_b, \mathbb{C}) = H_{\text{van}}^4(X_b, \mathbb{C})$ are conjugate up to a nonzero scalar, where b is a point in U .

Proof. Since the singularity x_i is of A_2 -type, that is, is locally as

$$X_0^3 + X_1^2 + \dots + X_5^2 = 0,$$

the first statement follows from the Brieskorn–Pham theorem and the fact that $\text{Mil}_{f,x_i} \simeq S^4 \vee S^4$; see [Mil68, Theorem 9.1]. For the second statement, we notice that $H = H' \cap \mathbb{A}^{N-1}$ is irreducible, so its smooth open subset H_o is connected. For points x_i and x_j in H_o , we can choose a path l from x_i to x_j in H_o . We lift l to a path l' from x'_i to x'_j in the boundary of a tubular neighborhood of H_o in $\mathbb{A}^{N-1}$, where x'_i (respectively, x'_j) is a point very closed to x_i (respectively, x_j). Obviously, a trivialization of φ over l' transports the vanishing cycle classes generated by the image of $H_4(\text{Mil}_{f,x_i})$ to the vanishing cycle classes generated by the image of $H_4(\text{Mil}_{f,x_j})$; see Lemma 6.1. Under this transportation, the monodromy operators r_i and r_j are conjugate by the path l' . So we identify H_w^i with H_w^j and $H_{w^2}^i$ with $H_{w^2}^j$ by the monodromy action. We have proved the lemma. $\square$

Consider $f|_{X_U}: X_U \rightarrow U$; see (6.1). There is a natural $\mathbb{Z}/3\mathbb{Z}$ -action on X_U fiberwise. More precisely, the action of the generator of $\mathbb{Z}/3\mathbb{Z}$ is given by

$$(t, [X_0 : X_1 : \dots : X_5]) \mapsto (t, [wX_0 : X_1 : \dots : X_5]),$$

where w is $\exp(2\pi i/3)$. This action induces an action on the local system $(R^4 f_*(\mathbb{C}))_0$ (over U) whose stalk over $b \in U$ is $H_{\text{prim}}^4(X_b, \mathbb{C})$. So we have

$$H_{\text{prim}}^4(X_b, \mathbb{C}) = H_w^4 \oplus H_{w^2}^4,$$

where H_w^4 (respectively, $H_{w^2}^4$) is the w (respectively, w^2)-eigenspace of the $\mathbb{Z}/3\mathbb{Z}$ -action. The eigenspace H_w^4 (respectively, $H_{w^2}^4$) contains $H^{3,1}$ (respectively, $H^{1,3}$) by the Griffiths residues; see (7.1). Since the $\mathbb{Z}/3\mathbb{Z}$ -action commutes with the monodromy action of $\pi_1(U)$, the summands H_w^4 and $H_{w^2}^4$ are $\pi_1(U)$ -modules. In particular, the summand H_w^4 (respectively, $H_{w^2}^4$) is generated by the vanishing cycles $\{\delta_w^i\}$ (respectively, $\{\delta_{w^2}^i\}$) for $i = 1, 2, \dots, m$; see Lemma 6.1 and [Mil68, Theorem 9.1].

We use the Picard–Lefschetz transformation formula to show the following proposition.

PROPOSITION 6.4. *With the notation as above, the monodromy representation $\pi_1(U)$ of $H_{\text{prim}}^4(X_b, \mathbb{C})$ is complete reducible. The irreducible decomposition is*

$$H_{\text{prim}}^4(X_b, \mathbb{C}) = H_w^4 \oplus H_{w^2}^4.$$

Proof. Let W be a nonzero subrepresentation of $H_{\text{prim}}^4(X_b)$. We claim that W contains H_w^4 or $H_{w^2}^4$. In fact, it follows from Lemma 6.1 that $H_{\text{prim}}^4(X_b)$ is generated by δ_w^i and $\delta_{w^2}^i$. For a nonzero element $y \in W$, it follows that $y = \sum_{i=1}^m y_i$, where

$$y_i \in \text{Im}(H^4(\text{Mil}_{f,x_i})) = \text{Span}(\delta_w^i, \delta_{w^2}^i).$$

It is clear that the local monodromy operator r_j fixes $\text{Im}(H^4(\text{Mil}_{f,x_i}, \mathbb{C}))$ if $i \neq j$.

We use the notation of [Dim92, Chapter 3]. On the space $H_4(\text{Mil}_{f,x_i})$, there is an intersection form $\langle, \rangle_i$. It is nondegenerate on $H_4(\text{Mil}_{f,x_i})$ since the singularity is of A_2 -type; see [Dim92, Chapter 3, Proposition 4.7(A) and Theorem 4.10].

Recall the Picard–Lefschetz transformation [Dim92, Chapter 3, (3.11)]: Suppose that T_k is an endomorphism of $H_4(\text{Mil}_{f,x_i})$ defined by

$$T_k(z) = y \pm \langle z, \Delta_k^i \rangle_i \Delta_k^i,$$

where $k = 1, 2$ and $\{\Delta_k^i\}$ is a distinguished basis of vanishing cycles $H_4(\text{Mil}_{f,x_i})$, see [Dim92, Chapter 3, Definition 3.4]. The monodromy operator r_i on $H_4(\text{Mil}_{f,x_i})$ is the composition of T_1 and T_2 ; see [Dim92, Chapter 3, (3.10)]. Therefore, we have

$$r_j(y_j) = y_j \pm \langle y_j, \Delta_1^j \rangle_j \Delta_1^j \pm (\langle y_j, \Delta_2^j \rangle_j \pm \langle y_j, \Delta_1^j \rangle_j \langle \Delta_1^j, \Delta_2^j \rangle_j) \Delta_2^j.$$

If $y \neq 0$, then $y_j \neq 0$ for some j . Since $r_j(y_i) = y_i$ ($i \neq j$), we have

$$r_j(y) = \left(\sum_{i \neq j}^m y_i \right) + r_j(y_j) = y \pm \langle y_j, \Delta_1^j \rangle_j \Delta_1^j \pm (\langle y_j, \Delta_2^j \rangle_j \pm \langle y_j, \Delta_1^j \rangle_j \langle \Delta_1^j, \Delta_2^j \rangle_j) \Delta_2^j.$$

Since the intersection form $\langle, \rangle_j$ is nondegenerate, it follows that

$$r_j(y) = y + A\Delta_1^j + B\Delta_2^j = y + A'\delta_w^j + B'\delta_{w^2}^j,$$

where A or $B \neq 0$, and A' or $B' \neq 0$. This implies that

- $c := A'\delta_w^j + B'\delta_{w^2}^j = r_j(y) - y \in W$ and
- $r_j(c) = A'w\delta_w^j + B'w^2\delta_{w^2}^j \in W$.

In particular, we conclude that δ_w^j or $\delta_{w^2}^j$ is in the subrepresentation W . It follows from Lemma 6.3 that W contains H_w^4 or $H_{w^2}^4$. $\square$

PROPOSITION 6.5. *Suppose that X_3 is a smooth cubic fourfold defined by a cubic equation*

$$X_0^3 + G(X_1, \dots, X_5) = 0.$$

If X_3 is very general, then the middle Picard number of X_3 is 1.

Proof. Recall that Hodge loci are unions of countably many proper analytic subsets; see [Voi13]. To prove the proposition, it suffices to show that there exists a smooth cubic fourfold defined by

$$X_0^3 + G(X_1, \dots, X_5) = 0,$$

whose middle Picard number is 1. Consider the family $X_{A^1} \rightarrow \mathbb{A}^1$ of cubic fourfolds which we describe at the beginning of this section. With the notation as before, if $b \in U$ is a very general

point which is out of the Hodge locus of $X_{\mathbb{A}^1} \rightarrow \mathbb{A}^1$, then $\pi_1(U)$ preserves the rational Hodge classes $\text{Hdg}(X_b)_{\mathbb{Q}}$; see [Voi13]. Therefore, the vector space

$$\text{Hdg}(X_b)_{\mathbb{C}} \cap H_{\text{prim}}^4(X_b, \mathbb{C})$$

is a subrepresentation of the $\pi_1(U)$ -representation $H_{\text{prim}}^4(X_b)$. It follows from Proposition 6.4 that $\text{Hdg}(X_b)_{\mathbb{C}} \cap H_{\text{prim}}^4(X_b, \mathbb{C})$ is either zero or one of $H_{\text{prim}}^4(X_b)$, H_w^4 , and $H_{w^2}^4$. However, the last three cases are impossible since they contain either $H^{3,1}$ or $H^{1,3}$. So $\text{Hdg}(X_b)_{\mathbb{C}} \cap H_{\text{prim}}^4(X_b, \mathbb{C}) = 0$. In other words, the middle Picard number of X_b is 1. $\square$

7. Lattices and automorphisms

In this section, we will use the theory of lattices to classify the automorphism groups of cubic fourfolds. For the theory of lattices, we refer to [Nik80] and [Huy16, Chapter 14]. We use the notation of [Huy16, Chapter 14]. Suppose that F is a cubic form defining a smooth projective cubic fourfold X_3 . Denote by σ the $(3, 1)$ -form

$$\text{Res}_{X_3} \left(\frac{\sum_{i=0}^5 (-1)^i X_i dX_0 \wedge \cdots \wedge \widehat{dX_i} \wedge \cdots \wedge dX_5}{F^2} \right) \in H^{3,1}(X_3).$$

It follows from the Griffiths residues that

$$H^{3,1}(X_3) = \mathbb{C}\langle \sigma \rangle. \quad (7.1)$$

Let $T(X_3)$ be the smallest primitive integral sub-Hodge structure of $H^4(X_3, \mathbb{Z})$ containing $H^{3,1}(X_3)$. It is called the *transcendental part* of the Hodge structure on $H^4(X_3, \mathbb{Z})$. We have a natural action of the automorphism group $\text{Aut}(X_3)$ on the 1-dimensional vector space $H^{3,1}(X_3)$. Since the automorphism group $\text{Aut}(X_3)$ is finite, we have the exact sequence

$$1 \rightarrow \text{Aut}_s(X_3) \rightarrow \text{Aut}(X_3) \xrightarrow{p} \mu_m \rightarrow 1, \quad (7.2)$$

where μ_m is the group of m th roots of unity acting as multiplication on $H^{3,1}(X_3) = \mathbb{C}$ and the kernel $\text{Aut}_s(X_3)$ is the subgroup of automorphisms preserving σ .

It follows from Poincaré duality and Lemma 2.3 that $(H^4(X_3, \mathbb{Z}), \langle, \rangle)$ is a unimodular lattice. Denote this lattice by Λ . We define the middle Picard group $\text{MP}(X_3)$ as the sublattice of $H^4(X_3, \mathbb{Z})$ which is generated by algebraic cycles

$$\text{MP}(X_3) := \text{Im}(\text{cl}: \text{CH}^2(X_3) \rightarrow H^4(X_3, \mathbb{Z})) = H^4(X_3, \mathbb{Z}) \cap H^{2,2}(X),$$

where the second equality follows from the fact that the integral Hodge conjecture holds for smooth cubic fourfolds; see [Voi07, Theorem 18].

LEMMA 7.1. *If $g \in \text{Aut}_s(X_3)$, then $g^* = \text{Id}$ on $T(X_3)$.*

Proof. Consider a map $\psi: T(X_3) \rightarrow \mathbb{C}$ that sends $x \in T(X_3)$ to the intersection product $\langle x, \sigma \rangle$ of x and σ . Since g^* preserves the intersection product and $g \in \text{Aut}_s(X_3)$, we have

$$\psi(g^*(x)) = \langle g^*(x), \sigma \rangle = \langle g^*(x), g^*\sigma \rangle = \psi(x).$$

It follows that $g^*(x) - x \in T(X_3) \cap \text{MP}(X_3)$. Since we know that $\langle \sigma, \bar{\sigma} \rangle \neq 0$, we conclude that $\text{Ker}(\psi) \subseteq T(X_3) \cap \text{MP}(X_3)$.

We claim that $T^\perp(X_3) = \text{MP}(X_3)$. In fact, since $H^{3,1}(X_3)$ is orthogonal to $\text{MP}(X_3)$, we conclude that $T(X_3) \subseteq \text{MP}(X_3)^\perp$ by the minimality of $T(X_3)$. Therefore, we have $\text{MP}(X_3)^{\perp\perp} \subseteq T(X_3)^\perp$. On the other hand, an integral class x which is orthogonal to $T(X_3)$ is orthogonal to $H^{3,1}(X_3)$. Therefore, we have $x \in H^{2,2}(X_3)$. This implies that $T(X_3)^\perp \subseteq \text{MP}(X_3)$. Obviously, we have $\text{MP}(X_3) \subseteq \text{MP}(X_3)^{\perp\perp}$. In summary, we get the inclusions

$$\text{MP}(X_3) \subseteq \text{MP}(X_3)^{\perp\perp} \subseteq T(X_3)^\perp \subseteq \text{MP}(X_3).$$

It follows that $\text{MP}(X_3) = T(X_3)^\perp$. We have proved the claim.

By the claim, if $y \in T(X_3) \cap \text{MP}(X_3)$, then $\langle y, y \rangle = 0$ and $\langle y, H^2 \rangle = 0$, where H is $c_1(\mathcal{O}_{X_3}(1))$. It follows that $y = 0$ by the Hodge index theorem for $H^4(X_3)$. Hence, we have $g^*(x) - x = 0$ for all $x \in T(X_3)$. We have proved the lemma. $\square$

Remark 7.2. A similar argument shows that $T(X_3)_\mathbb{Q} \cap \text{MP}(X_3)_\mathbb{Q} = 0$.

LEMMA 7.3. *Suppose that $(\Lambda, \langle \cdot, \cdot \rangle)$ is a unimodular lattice. Let Λ_1 be a primitive sublattice of Λ . Denote by Λ_2 the sublattice $\Lambda_1^\perp$. Define A_{Λ_i} to be Λ_i^*/Λ_i , where the inclusion $\Lambda_i \subseteq \Lambda_i^*$ is induced by the intersection form $\langle \cdot, \cdot \rangle$; see [Huy16, Chapter 14]. If $\Lambda_1 \cap \Lambda_2 = 0$, then we have $A_{\Lambda_1} \simeq A_{\Lambda_2}$ via a natural identification.*

Proof. We mimic the proof of [Huy16, Chapter 14, Proposition 0.2]. Suppose that ψ is the composition of i and π :

$$\psi: \Lambda \xhookrightarrow{i} \Lambda_1^* \xrightarrow{\pi} \frac{\Lambda_1^*}{\Lambda_1},$$

where i sends x to the linear map $\langle x, - \rangle$ on Λ_1 and π is the natural quotient. The kernel $\text{Ker}(\psi)$ of ψ is given by

$$\{x \in \Lambda \mid \text{there is an } x' \in \Lambda_1 \text{ such that } \langle x, y \rangle = \langle x', y \rangle \text{ for all } y \in \Lambda_1\}.$$

Suppose $x \in \text{Ker}(\psi)$. There is an $x' \in \Lambda_1$ such that $x - x' \in \Lambda_2$. It follows that $x \in \Lambda_1 \oplus \Lambda_2$. Therefore, we have a natural inclusion

$$\overline{\psi}: \frac{\Lambda}{\Lambda_1 \oplus \Lambda_2} \hookrightarrow \frac{\Lambda_1^*}{\Lambda_1} = A_{\Lambda_1}.$$

Since $\Lambda_1 \hookrightarrow \Lambda$ is primitive and Λ is unimodular, the map $\Lambda \simeq \Lambda^* \rightarrow \Lambda_1^*$ is surjective. Therefore, the natural map $\overline{\psi}$ is surjective. We conclude that $\overline{\psi}$ is an isomorphism. A similar argument shows the same result for A_{Λ_2} . So we have a natural identification

$$A_{\Lambda_1} \simeq \frac{\Lambda}{\Lambda_1 \oplus \Lambda_2} \simeq A_{\Lambda_2}. \quad \square$$

LEMMA 7.4. *We have $\text{Aut}_s(X_3) = \{\text{Id}\}$ if the rank of $\text{MP}(X_3)$ is at most 2.*

Proof. By Proposition 4.4, it suffices to show that $g^* = \text{Id}$ on $H^4(X_3, \mathbb{Q})$ for all $g \in \text{Aut}_s(X_3)$ if the rank of $\text{MP}(X_3)$ is at most 2. By Lemma 7.1 and the fact that $\text{MP}(X_3) + T(X_3)$ has the same rank as $H^4(X, \mathbb{Z})$, we just need to prove that $g^* = \text{Id}$ on $\text{MP}(X_3)$ since $g^* = \text{Id}$ on $T(X_3)$ by definition. We consider two cases.

(1) If the rank of $\text{MP}(X_3)$ is 1, then $\text{MP}(X_3)$ is generated by $H \cup H$, where $H = c_1(\mathcal{O}_{X_3}(1))$. We have $g^*(H^2) = H^2$ since g is linear by Lemma 2.1. It follows that $g^* = \text{Id}$ on $\text{MP}(X_3)$.

(2) If the rank of $\text{MP}(X_3)$ is 2, then we can assume $\text{MP}(X_3) = \langle H^2, S \rangle$ by the fact that the subgroup $\langle H^2 \rangle$ generated by H^2 is primitive. As in Case 1, we have $g^*(H^2) = H^2$. Suppose

$g^*(S) = aH^2 + bS$, where a and b are integers. It follows from $\det(g^*) = 1$ or -1 that $b = 1$ or -1 , respectively. On the other hand, we have

$$H^2 \cdot S = g^*(H^2) \cdot g^*(S) = 3a + bH^2 \cdot S. \quad (7.3)$$

If $\det(g^*) = 1$, then $b = 1$. It follows from (7.3) that $a = 0$. Therefore, we have $g^*(S) = S$.

We now prove the lemma.

Suppose $\det(g) = -1$. Then $b = -1$. By equation (7.3), we have

$$a = \frac{2S \cdot H^2}{3}, \quad \text{that is,} \quad g^*(S) = \frac{2}{3}(S \cdot H^2)H^2 - S.$$

In particular, it follows that $S \cdot H^2$ is divisible by 3. Therefore, the lattice $(\text{MP}(X_3), \langle \cdot, \cdot \rangle)$ is equivalent to the lattice

$$(H^2) \oplus_{\perp} (S') = (3) \oplus_{\perp} (2n) \quad \text{with} \quad \langle H^2, H^2 \rangle = 3 \quad \text{and} \quad \langle H^2, S' \rangle = 0,$$

where $S' = S - \frac{1}{3}(S \cdot H^2)H^2$. Note that the discriminant of $\text{MP}(X_3)$ is $6n$. It follows from [Has00, Theorem 1.0.1] that $n \geq 2$. It is clear that

$$g^*(S') = -S'.$$

It follows that $g^* = (\text{Id}, -\text{Id})$ on $A_{\text{MP}(X_3)} \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/2n\mathbb{Z}$. Since $n \geq 2$, we know that $g^* \neq \text{Id}$. On the other hand, by Lemma 7.1, we know that $g^* = \text{Id}$ on $A_{T(X_3)}$. By the claim in the proof of Lemma 7.1, we have $\text{MP}(X_3) = T(X_3)^{\perp}$. Under the natural identification $A_{\text{MP}(X_3)} = A_{T(X_3)}$ (see Lemma 7.3), we have $\text{Id} = g^* \neq \text{Id}$, which is absurd. This gives a contradiction. So $\det(g)$ cannot be -1 . $\square$

COROLLARY 7.5. *The automorphism group $\text{Aut}(X_3)$ is the cyclic group μ_m if the middle Picard number of X_3 is at most 2.*

Proof. This follows from Lemma 7.4 and the exact sequence (7.2). $\square$

The rest of this section is used to figure out what the number m is.

LEMMA 7.6. *Let Λ be a nondegenerate lattice of rank at most 2. For A_{Λ} , if one of the following assumptions holds*

- (i) $\Lambda = \mathbb{Z}\langle H \rangle$ and $\langle H, H \rangle$ is at least 3,
- (ii) $\text{rk}(\Lambda) = 2$ and $\text{disc}(\Lambda) > 4$,

then $\text{Id}_{A_{\Lambda}} \neq -\text{Id}_{A_{\Lambda}}$.

Proof. Under assumption (i), we have $A_{\Lambda} = \Lambda^*/\Lambda = \mathbb{Z}/(H \cdot H)\mathbb{Z} \simeq \mathbb{Z}/n\mathbb{Z}$, where $n \geq 3$. The lemma follows.

Under assumption (ii), we identify Λ^* with $\{x \in \Lambda_{\mathbb{Q}} \mid \langle x, \Lambda \rangle \subseteq \mathbb{Z}\}$. So we can pick a basis $\{H, S\}$ of Λ such that $\Lambda^* = \mathbb{Z}\langle H/N_1, S/N_2 \rangle$ for some integers N_1 and N_2 . Therefore, we have $A_{\Lambda} = \Lambda^*/\Lambda = \mathbb{Z}/N_1\mathbb{Z} \oplus \mathbb{Z}/N_2\mathbb{Z}$. If $\text{Id}_{A_{\Lambda}} = -\text{Id}_{A_{\Lambda}}$, then $N_1, N_2 \in \{1, 2\}$. This contradicts the hypothesis $\text{disc}(\Lambda) (= |\Lambda^*/\Lambda| = N_1 \cdot N_2) > 4$. $\square$

LEMMA 7.7. *The number $\varphi(m)$ divides the rank of $T(X_3)$, where $\varphi(-)$ is Euler's function. Suppose that the middle Picard number of X_3 is at most 2. Then*

- $\varphi(m) \in \{1, 2, 22\}$ if the rank of $\text{MP}(X_3)$ is 1,
- $\varphi(m) = 1$ if the rank of $\text{MP}(X_3)$ is 2.

Proof. We mimic the proof of [Huy16, Chapter 15, Corollary 1.14]. Let T be an integral Hodge structure of K3-type that does not contain any proper primitive sub-Hodge structure T' of K3-type.

Fact: For $a, a' \in \text{End}_{\text{H.S.}}(T)$, if $a = a'$ on $T^{2,0}$, then $a = a'$.

Note that the action of $\text{Aut}(X_3)$ on $T(X_3)_{\mathbb{C}}$ is induced by $p: \text{Aut}(X_3) \rightarrow \mu_m$, see (7.2), where μ_m acts on $T(X_3)_{\mathbb{C}}$ faithfully. Let $f \in \text{Aut}(X_3)$ be the automorphism such that

$$p(f) = \xi,$$

where ξ is a primitive m th root of unity. Therefore, the minimal polynomial $\Phi_m(x)$ of ξ divides the characteristic polynomial $G(x)$ of the linear map f^* on $T(X_3)_{\mathbb{Q}}$. In order to show that $\phi(m)$ divides $\text{rk}(T(X_3))$, we simply remark that all nontrivial irreducible subrepresentations of $\mu_m = \langle p(f) \rangle$ on $T(X_3)_{\mathbb{Q}}$ are of rank $\phi(m)$.

Indeed, otherwise, for some $n < m$, there exists a nonzero $b \in T(X_3)$ with $(f^n)^*(b) = b$. Suppose $b = b_{1,3} + b_{2,2} + b_{3,1}$. It follows from Remark 7.2 that $b_{3,1} \neq 0$. By the fact we mention at the beginning of the proof, we conclude that $f^n(b_{3,1}) = (\xi)^n b_{3,1} \neq b_{3,1}$. This is absurd. In other words, as a representation of μ_m , we have

$$T(X_3) \simeq \mathbb{Z}[\xi]^{\oplus r}$$

with $r = \text{rk}(T(X_3))/\phi(m)$. It follows that $\phi(m) \mid \text{rk}(T(X_3)) = 23 - \text{rk}(\text{MP}(X_3))$. So we conclude that

- $\phi(m) \in \{1, 2, 11, 22\}$ if $\text{rk}(\text{MP}(X_3)) = 1$ and
- $\phi(m) \in \{1, 3, 7, 21\}$ if $\text{rk}(\text{MP}(X_3)) = 2$.

By the elementary facts that there is no m such that $\phi(m) = 11$ and that $\phi(m)$ is even if $m > 2$, we have proved the lemma. $\square$

Proof of Theorem 1.5. Suppose $\text{rk}(\text{MP}(X_3)) = 1$. It follows from Corollary 7.5 that $\text{Aut}(X_3) = \mu_m$. Using Lemma 7.7 and solving the equations $\phi(m) = 1, 2$ or 22 , we conclude that $m \in \{1, 2, 3, 23, 46\}$. By [GL11, Theorem 1.6], we know that $m \neq 23$ or 46 . We claim that $|\text{Aut}(X_3)|$ is odd. This implies that $m \neq 2$ or 46 .

We suppose that $g \in \text{Aut}(X_3)$ is of order 2. Recall that the action of g^* on $T(X_3)$ is determined by the action on $T^{3,1}(X_3)$ (see the fact in the proof of Lemma 7.7). It follows from this fact that $g^* = -\text{Id}$ on $T(X_3)$. Therefore, we have $g^* = -\text{Id}$ on $A_{T(X_3)} \cong A_{\text{MP}(X_3)}$ (Lemma 7.3). On the other hand, we have $g^* = \text{Id}$ on $\text{MP}(X_3)$ since g is linear. It follows that $g^* = \text{Id}$ on $A_{\text{MP}(X_3)} \cong A_{T(X_3)}$. However, this is absurd by Lemma 7.6.

We conclude that $m = 1$ or 3 . According to a theorem of Deligne [DK73, Exposé XIX], we know that a very general cubic fourfold is of middle Picard number 1. By [MM63], the automorphism group of a general cubic fourfold is trivial. Therefore, we show that there is a cubic fourfold of middle Picard number 1 with trivial automorphism group. For $m = 3$, it follows from Proposition 6.5 that there is a cubic fourfold X of middle Picard number 1 with an automorphism of order 3 defined by

$$[X_0 : X_1 : \dots : X_5] \mapsto [\xi_3 X_0 : X_1 : \dots : X_5],$$

where $\xi_3 = \exp(2\pi i/3)$.

For $\text{rk}(\text{MP}(X_3)) = 2$, it follows from Corollary 7.5 that $\text{Aut}(X_3) = \mu_m$. By Lemma 7.7 and the equation $\phi(m) = 1$, we conclude that $m = 1$ or 2 . We claim that $m \neq 2$. In fact, if $g \in \text{Aut}(X_3)$

is of order 2, then $g^* = -\text{Id}$ on $T(X_3)$ (the argument is similar to that above). It follows that $g^* = -\text{Id}$ on $A_{T(X_3)} \cong A_{\text{MP}(X_3)}$. From the proof of Lemma 7.4, we know that there are only two possible cases for g^* :

- (1) $g^* = \text{Id}$ on $\text{MP}(X_3)$ (that is, $\det g^*|_{\text{MP}(X_3)} = 1$),
- (2) $g^* = (\text{Id}, -\text{Id})$ on $\text{MP}(X_3) = (3) \oplus_{\perp} (2n)$, where $n \geq 2$.

However, by Lemmas 7.3 and 7.6, both cases are absurd. Therefore, we have $m \neq 2$. In summary, we have proved the theorem. $\square$

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