Enriques surfaces with finite automorphism group in positive characteristic

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ABSTRACT

We classify Enriques surfaces with smooth K3 cover and finite automorphism group in arbitrary positive characteristic. The classification is the same as over the complex numbers except that some types are missing in small characteristics. Moreover, we give a complete description of the moduli of these surfaces. Finally, we realize all types of Enriques surfaces with finite automorphism group over the prime fields $\mathbb{F}_p$ and $\mathbb{Q}$ whenever they exist.

1. Introduction

Convention 1.1. Unless mentioned otherwise, we will work over an algebraically closed field k of arbitrary characteristic. By Enriques surface, we will mean Enriques surface with a smooth K3 cover throughout this paper. This means that we will not be dealing with classical and supersingular Enriques surfaces in characteristic 2.

Classically known as the first examples of non-rational surfaces with $q = p_g = 0$, Enriques surfaces are one of the building blocks of minimal, smooth and projective surfaces of Kodaira dimension 0. Thus, in the Enriques–Kodaira classification of complex surfaces (see, for example, [BHPVdV04]), they appear next to Abelian, bielliptic and K3 surfaces, to which they are closely related. In positive characteristics, the classification of E. Bombieri and D. Mumford [Mum69, BM77, BM76] shows that the close relation to K3 surfaces persists unless the characteristic of the base field is 2. However, in characteristic 2, three distinct types of Enriques surfaces appear. These types are distinguished by the torsion component of the identity of the Picard scheme Pic^τ , which is one of $\{\mathbb{Z}/2\mathbb{Z}, \mu_2, \alpha_2\}$, and they are called classical, singular and supersingular, respectively. Among these, only the singular Enriques surfaces admit a smooth canonical K3 cover, whereas the canonical cover $\tilde{X}$ of the other types is only “K3-like”, in the sense that $\tilde{X}$ is integral Gorenstein and $\omega_{\tilde{X}} \cong \mathcal{O}_{\tilde{X}}$. However, $\tilde{X}$ might even be non-normal.

Using the period map for complex Enriques surfaces [Hor78a, Hor78b], one can construct “a” moduli space of unpolarized Enriques surfaces over the complex numbers [GH16], which turns out to be 10-dimensional, quasi-affine [Bor96] and rational [Kon94]. There is a codimension 1 subvariety parametrizing Enriques surfaces containing a (-2) -curve, that is, an irreducible

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curve with self-intersection (-2) , and a codimension 1 subvariety of the boundary of the period domain parametrizing Coble surfaces, that is, smooth rational surfaces X with $|-K_X| = \emptyset$ and $|-2K_X| \neq \emptyset$. Both of these codimension 1 subvarieties are rational [DK13]. Thus, one expects that a 1-dimensional family of Enriques surfaces degenerates to a Coble surface at some point. We will also see this kind of behaviour for our examples in positive characteristic.

In positive and mixed characteristic, a similar picture has been established by C. Liedtke in [Lie15] and T. Ekedahl, J. Hyland and N. Shepherd-Barron in [EHS12]: The moduli space of Cossec–Verra polarized Enriques surfaces is a quasi-separated Artin stack of finite type over $\text{Spec } \mathbb{Z}$, which is irreducible, unirational, 10-dimensional, smooth in odd characteristics and consists of two connected components with these properties in characteristic 2. These two connected components parametrize singular and classical Enriques surfaces, respectively. Their 9-dimensional intersection parametrizes supersingular Enriques surfaces. The stack of unpolarized Enriques surfaces is very badly behaved [Lie15, Remark 5.3], because the automorphism group of a generic Enriques surface X is infinite and, moreover, discrete unless X is supersingular or an exceptional [ES04] and classical Enriques surface in characteristic 2.

The automorphism group of a general complex Enriques surfaces was computed by W. Barth and C. Peters [BP83], independently also by V.V. Nikulin [Nik83], and is equal to the 2-congruence subgroup of the group of positive-cone-preserving automorphisms of the E_{10} lattice. However, an Enriques surface may acquire additional (-2) -curves under specializations, causing the automorphism group to become smaller. Therefore, it is a natural question whether this group can degenerate to a finite group.

In 1984, I. Dolgachev [Dol84] found an example (type I) of an Enriques surface with finite automorphism group, and later it was discovered that G. Fano [Fan10] had also found an example (type VII) as early as in 1910, although the automorphism group is not $\mathfrak{S}_3$ as Fano claimed, but $\mathfrak{S}_5$ (see [Kon86, Introduction, p. 191]). The full classification of Enriques surfaces with finite automorphism group over the complex numbers was then carried out by Nikulin [Nik84] in terms of their root invariants and by S. Kondō [Kon86] using elliptic fibrations. There are seven types I, $\dots$, VII of such Enriques surfaces, distinguished by their dual graphs of (-2) -curves; the first two form a 1-dimensional family, and the others are unique [Kon86].

The key observation for Nikulin’s approach to the classification is the fact that for a complex Enriques surface X , the subgroup $W_X \subseteq \text{O}(\text{Num}(X))$ generated by reflections along classes of (-2) -curves has finite index if and only if $\text{Aut}(X)$ is finite. However, while in any characteristic, W_X being of finite index in $\text{O}(\text{Num}(X))$ implies that the automorphism group $\text{Aut}(X)$ is finite [Dol84, Main Theorem], the converse uses the global Torelli theorem proven by E. Horikawa [Hor78a, Hor78b], which is not available in positive characteristic. For this reason, we will not pursue Nikulin’s approach. Nevertheless, it will follow from our explicit classification that $\text{Aut}(X)$ being finite implies that $W_X \subseteq \text{O}(\text{Num}(X))$ has finite index.

Kondō’s approach is based on the observation, due to Dolgachev [Dol84, § 4], that the Mordell–Weil group of the Jacobian of every elliptic fibration of an Enriques surface X acts on X ; hence, it has to be finite if we want X to have finite automorphism group. Using this approach, we will obtain the classification of Enriques surfaces with finite automorphism group and smooth K3 cover in positive characteristic. Recall that the K3 cover of an Enriques surface X is smooth if and only if $\text{char}(k) \neq 2$ or X is a singular Enriques surface, that is, $\text{Pic}^\tau(X) \cong \mu_2$.

Type	Dual graph of (-2) -curves	Aut	Aut _{nt}	char(k)	Moduli
I		D_4	$\mathbb{Z}/2\mathbb{Z}$	any	$\mathbb{A}^1 - \{0, -256\}$
II		$\mathfrak{S}_4$	$\{1\}$	any	$\mathbb{A}^1 - \{0, -64\}$
III		$(\mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2) \rtimes D_4$	$\mathbb{Z}/2\mathbb{Z}$	$\neq 2$	unique
IV		$(\mathbb{Z}/2\mathbb{Z})^4 \rtimes (\mathbb{Z}/5\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z})$	$\{1\}$	$\neq 2$	unique

TABLE 1. Classification

ENRIQUES SURFACES WITH FINITE AUTOMORPHISM GROUP

V		$\mathfrak{S}_4 \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\neq 2, 3$	unique
VI		$\mathfrak{S}_5$	$\{1\}$	$\neq 3, 5$	unique
VII		$\mathfrak{S}_5$	$\{1\}$	$\neq 2, 5$	unique

TABLE 1. (Continued)

MAIN THEOREM (Classification). *Let X be an Enriques surface with smooth $K3$ cover over an algebraically closed field k .*

- (i) *The surface X has finite automorphism group if and only if the dual graph of all (-2) -curves on X is one of the seven dual graphs in Table 1.*
- (ii) *The automorphism groups, the characteristics in which they exist and the moduli of Enriques surfaces of each of the seven types are as in Table 1.*

In Table 1, the group $\mathfrak{S}_n$ is the symmetric group on n letters, D_4 is the dihedral group of

order 8, and for two groups N and H , the group $N \rtimes H$ is a semi-direct product of N and H .

In characteristic 2, the search for Enriques surfaces with finite automorphism group was started recently by T. Katsura and Kondō [KK18]. There, the question of the existence of the seven types in characteristic 2 was settled. Our classification shows that the examples of singular Enriques surfaces with finite automorphism group in [KK18] are in fact all possible examples of such surfaces. For the classification of classical and supersingular Enriques surfaces with finite automorphism group in characteristic 2, we refer the reader to [KKM17]. There are five types of supersingular and eight types of classical Enriques surfaces with finite automorphism group in characteristic 2.

Remark. As an application of our classification, we determine the semi-symplectic parts of the automorphism groups of Enriques surfaces with finite automorphism group. For the precise statement, we refer the reader to Theorem 12.2 and Table 7.

Even though our approach to the classification of possible dual graphs is similar to Kondō's in many aspects, we will encounter several obstacles due to the lack of a Torelli theorem, the existence of different finite-order automorphisms of K3 surfaces [DK09] and a different list of extremal and rational elliptic surfaces in small characteristics [Lan91, Lan94]. We overcome these problems by extending Kondō's universal base change construction [Kon86, Lemma 2.6] of Enriques surfaces with special elliptic fibration π (that is, an elliptic fibration with a (-2) -curve as bisection; see Definition 2.7) to arbitrary characteristic and by using the explicit description of the covering involution of the canonical cover $\tilde{X}$ of X to obtain a map

$$\text{jac}_2: \text{MW}(J(\pi)) \rightarrow \{\text{special bisections of } \pi\}$$

producing new (-2) -curves on X from sections of the Jacobian $J(\pi)$ of π . Moreover, we exhibit “critical” subgraphs, which are dual graphs of singular fibers of a special elliptic fibration π on X together with some special bisection N , for each of Kondō's seven types, and we show that an Enriques surface whose dual graph of all (-2) -curves contains such a diagram is one of the seven types. Therefore, we can use the universal base change construction to construct π and N and hence the Enriques surface itself. Since the base change construction is universal, we can give an explicit description of the moduli of Enriques surfaces with finite automorphism group. Finally, the equations we give can actually be interpreted as integral models of these Enriques surfaces, and some of them were found using the integral models of extremal and rational elliptic surfaces of T. Jarvis, W. Lang and J. Ricks [JLR12].

As we have just mentioned, a closer look at our equations reveals that they do in fact define integral models of these surfaces in the following sense.

THEOREM 11.3 (Integral models). *Let $K \in \{\text{I}, \dots, \text{VII}\}$ and P_K be as in Table 2. There is a family $\varphi_K: \mathcal{X} \rightarrow \text{Spec}(\mathbb{Z}[1/P_K])$ whose fibers are Enriques surfaces of type K with Picard rank 10.*

Note that for $K \neq \text{I}, \text{II}$, the integer P_K is exactly the product over the characteristics where type K does not exist. If $K = \text{I}, \text{II}$, we give two integral models to obtain the following corollary, which solves the existence of the seven types over arbitrary fields.

COROLLARY 11.5. *Suppose that there exists an Enriques surface of type $K \in \{\text{I}, \dots, \text{VII}\}$ in characteristic p . Then, there exists an Enriques surface of type K with Picard rank 10 over $\mathbb{F}_p$ if $p \neq 0$ and over $\mathbb{Q}$ if $p = 0$.*

Type	P_K
I	255, 257
II	63, 65
III	2
IV	2
V	6
VI	15
VII	10

TABLE 2. Integral models

Moreover, we exhibit special generators of the automorphism groups of Enriques surfaces with finite automorphism group, leading to our third result.

THEOREM 11.6. *Let X be an Enriques surface of type $K \in \{\text{I}, \dots, \text{VII}\}$ over a field k such that $\text{Pic}(X) = \text{Pic}(X_{\bar{k}})$.*

- *If $K \neq \text{III}, \text{IV}$, then $\text{Aut}(X)$ is defined over k .*
- *If $K = \text{III}$, then $\text{Aut}(X)$ is defined over $L \supseteq k$ with $[L : k] \leq 2$.*
- *If $K = \text{IV}$, then $\text{Aut}(X)$ is defined over $L \supseteq k$ with $[L : k] \leq 16$.*

Let us explain the structure of the paper. In Section 2, we extend Kondō’s base change construction to positive characteristic after recalling several facts on Enriques surfaces and elliptic fibrations. In Sections 3–9, we construct Enriques surfaces of types I, . . . , VII and compute their automorphism groups as well as their moduli. After that, in Section 10, we classify the dual graphs of Enriques surfaces with finite automorphism group, finishing the proof of our Main Theorem. In Section 11, we explain how to obtain information on the arithmetic of these surfaces, and in Section 12, we give the list of semi-symplectic automorphism groups of Enriques surfaces with finite automorphism group.

2. Preliminaries

2.1 Generalities on Enriques surfaces, dual graphs and elliptic fibrations

Here we recall some basic facts about Enriques surfaces, clarify our terminology and refer the reader to [CD89] for proofs and to [Sil94] for anything related to elliptic curves. In the first ten sections, we will be working over an algebraically closed field k .

DEFINITION 2.1. A *K3 surface* is a smooth and projective surface $\tilde{X}$ over k with $\omega_{\tilde{X}} \cong \mathcal{O}_{\tilde{X}}$ and $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$. An *Enriques surface* X with smooth K3 cover is the quotient of a K3 surface by a fixed-point-free involution σ . We call the K3 surface $\tilde{X}$ with $\tilde{X}/\sigma = X$ the *canonical cover* or *K3 cover* of X .

Convention 2.2. From now on, we will drop the “with smooth K3 cover”, and we will always assume that the Enriques surfaces we talk about have such a cover.

DEFINITION 2.3. An *elliptic fibration* (with base curve $\mathbb{P}^1$) of a smooth surface $\tilde{X}$ is a surjective morphism $\tilde{\pi}: \tilde{X} \rightarrow \mathbb{P}^1$ such that almost all fibers are smooth genus 1 curves, $\tilde{\pi}_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\mathbb{P}^1}$ and no fiber contains a (-1) -curve. We do not require that $\tilde{\pi}$ has a section.

PROPOSITION 2.4 (Bombieri and Mumford [BM76, Theorem 3]). *Every Enriques surface admits an elliptic fibration.*

The reason why we do not assume that elliptic fibrations have a section is that this is never the case for Enriques surfaces.

PROPOSITION 2.5 (Cossec and Dolgachev [CD89, Theorems 5.7.2, 5.7.5 and 5.7.6]). *Let π be an elliptic fibration of an Enriques surface. Then,*

- *if $\text{char}(k) \neq 2$, then π has exactly two tame double fibers, both of which are either of multiplicative type or smooth, and*
- *if $\text{char}(k) = 2$, then π has exactly one wild double fiber, which either is of multiplicative type or is a smooth ordinary elliptic curve.*

Remark 2.6. Since being supersingular is an isogeny-invariant, one can check the type of the double fiber on the K3 cover.

Therefore, the intersection number of any curve with a fiber of an elliptic fibration of an Enriques surface is even. Thus, the best approximation to a section will be a bisection.

DEFINITION 2.7. Let N be an irreducible curve on an Enriques surface X , and let π be an elliptic fibration of X .

- The curve N is a *(−2)-curve* if $N^2 = -2$. Equivalently, $N \cong \mathbb{P}^1$.
- The curve N is a *special bisection* of π if N is a *(−2)-curve* with $F \cdot N = 2$, where F is a general fiber of π .
- If π admits a special bisection, we call π *special*.

In fact, special elliptic fibrations are much more common than one might think. More precisely, we have the following result of F. Cossec, which was shown by W. Lang also to hold in characteristic 2.

PROPOSITION 2.8 (Cossec [Cos85, Theorem 4], Lang [Lan88, Theorem A3]). *An Enriques surface contains a (−2)-curve if and only if it admits a special elliptic fibration.*

Now, we recall some facts on the Jacobian fibrations of elliptic fibrations of Enriques surfaces.

PROPOSITION 2.9 (Cossec and Dolgachev [CD89, Theorem 5.7.1]). *Let π be an elliptic fibration of an Enriques surface. Then, the Jacobian fibration $J(\pi)$ of π is an elliptic fibration of a rational surface.*

Since the group of sections of the Jacobian of an elliptic fibration of an Enriques surface acts on the surface, we will mostly be concerned with extremal and rational elliptic fibrations. The group of sections of an elliptic fibration π is also called the *Mordell–Weil group* of π (see [Sil94, Chapter III, §9]).

DEFINITION 2.10. Let π be an elliptic fibration of an Enriques surface, and let $J(\pi)$ be its Jacobian. We call $J(\pi)$ and π *extremal* if the Mordell–Weil group $\text{MW}(J(\pi))$ is finite.

We will use the Kodaira symbols I_n ($n \geq 1$), I_n^* ($n \geq 0$), II, III, IV, II*, III* and IV* to denote the singular fibers of an elliptic fibration (see, for example, [Sil94, Remark 8.2.3]). The reducible fibers consist of *(−2)-curves*, and their intersection behaviour will play an important role throughout this paper.

DEFINITION 2.11. Let M be a set of (-2) -curves on a smooth surface X .

- The *dual graph* of M is the graph whose vertices are elements of M and in which two vertices $E_i, E_j \in M$ with $i \neq j$ are joined by an n -tuple line if $E_i \cdot E_j = n$.
- If M is the set of all (-2) -curves on X , we will call the corresponding graph the *dual graph of all (-2) -curves on X* .
- If M is the set of all (-2) -curves contained in singular fibers of an elliptic fibration π of X , we call M the *dual graph of the singular fibers of π* .

The dual graphs of the singular fibers of type I_n ($n \geq 2$), I_n^* ($n \geq 0$), III, IV, II^* , III^* and IV^* are $\tilde{A}_{n-1}$, $\tilde{D}_{n+4}$, $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{E}_8$, $\tilde{E}_7$ and $\tilde{E}_6$, respectively [Mir89, I.6]. Conversely, configurations of (-2) -curves whose dual graphs are extended Dynkin diagrams of these types give rise to elliptic fibrations.

PROPOSITION 2.12 (Kodaira [Kod63a, Kod63b], Mumford [Mum69]). *A connected, reduced divisor D on an Enriques surface X is equal to the support of a fiber of an elliptic fibration if and only if D is an irreducible genus 1 curve or the irreducible components of D are (-2) -curves whose dual graph is an extended Dynkin diagram of type $\tilde{A}\tilde{D}\tilde{E}$.*

Note that one cannot always reconstruct the fiber type from the graph. Using this notation, we can give the list of extremal and rational elliptic fibrations in every characteristic, due to R. Miranda, U. Persson and W. E. Lang.

PROPOSITION 2.13 (Miranda and Persson [MP86], Lang [Lan91, Lan94]). *Let π be an extremal fibration of a rational surface. Then, the singular fibers of π are given in Table 3.*

The extremal and rational elliptic surfaces with singular fibers (I_0^, I_0^*) in characteristic different from 2 and the ones with singular fiber (I_4^*) in characteristic 2 form 1-dimensional families, and all other fibrations are unique.*

Remark 2.14. From Table 3 we see that the fibrations in small characteristics differ from the characteristic 0 cases only if either a II^* fiber is involved, or the characteristic divides the number of simple components of some fiber of the fibration.

In fact, the Shioda–Tate formula implies that the dual graph of the (-2) -curves contained in singular fibers of an elliptic fibration π determines whether π is extremal or not.

LEMMA 2.15 (Shioda, [Shi72, Corollary 1.5]). *Let π be an elliptic fibration of a rational surface or of an Enriques surface. Then, π is extremal if and only if the lattice spanned by the fiber components of π has rank 9.*

Extremal elliptic fibrations of Enriques surfaces over the complex numbers were studied by the author in his master’s thesis (München, 2016), where he classified those extremal fibrations with at least one reducible double fiber.

2.2 Base change construction

Notation 2.16. Let $\pi: X \rightarrow \mathbb{P}^1$ be an elliptic fibration with section of a rational surface or of a K3 surface. We denote the composition in $MW(\pi)$ with respect to some fixed zero section by $\oplus$, the inverse of a section P is denoted by $\ominus P$, and the translation by a section P is denoted by t_P . By abuse of notation, we will also use t_P for the induced automorphism of X .

$\text{char}(k) \neq 2, 3, 5$	$\text{char}(k) = 5$	$\text{char}(k) = 3$	$\text{char}(k) = 2$
(II^*, II)	(II^*, II)	(II^*)	(II^*)
$(\text{III}^*, \text{III})$	$(\text{III}^*, \text{III})$	$(\text{III}^*, \text{III})$	–
(IV^*, IV)	(IV^*, IV)	–	(IV^*, IV)
$(\text{I}_0^*, \text{I}_0^*)$	$(\text{I}_0^*, \text{I}_0^*)$	$(\text{I}_0^*, \text{I}_0^*)$	–
$(\text{II}^*, \text{I}_1, \text{I}_1)$	$(\text{II}^*, \text{I}_1, \text{I}_1)$	$(\text{II}^*, \text{I}_1)$	$(\text{II}^*, \text{I}_1)$
$(\text{III}^*, \text{I}_2, \text{I}_1)$	$(\text{III}^*, \text{I}_2, \text{I}_1)$	$(\text{III}^*, \text{I}_2, \text{I}_1)$	$(\text{III}^*, \text{I}_2)$
$(\text{IV}^*, \text{I}_3, \text{I}_1)$	$(\text{IV}^*, \text{I}_3, \text{I}_1)$	$(\text{IV}^*, \text{I}_3)$	$(\text{IV}^*, \text{I}_3, \text{I}_1)$
$(\text{I}_4^*, \text{I}_1, \text{I}_1)$	$(\text{I}_4^*, \text{I}_1, \text{I}_1)$	$(\text{I}_4^*, \text{I}_1, \text{I}_1)$	(I_4^*)
$(\text{I}_2^*, \text{I}_2, \text{I}_2)$	$(\text{I}_2^*, \text{I}_2, \text{I}_2)$	$(\text{I}_2^*, \text{I}_2, \text{I}_2)$	–
$(\text{I}_1^*, \text{I}_4, \text{I}_1)$	$(\text{I}_1^*, \text{I}_4, \text{I}_1)$	$(\text{I}_1^*, \text{I}_4, \text{I}_1)$	$(\text{I}_1^*, \text{I}_4)$
$(\text{I}_9, \text{I}_1, \text{I}_1, \text{I}_1)$	$(\text{I}_9, \text{I}_1, \text{I}_1, \text{I}_1)$	(I_9, II)	$(\text{I}_9, \text{I}_1, \text{I}_1, \text{I}_1)$
$(\text{I}_8, \text{I}_2, \text{I}_1, \text{I}_1)$	$(\text{I}_8, \text{I}_2, \text{I}_1, \text{I}_1)$	$(\text{I}_8, \text{I}_2, \text{I}_1, \text{I}_1)$	(I_8, III)
$(\text{I}_5, \text{I}_5, \text{I}_1, \text{I}_1)$	$(\text{I}_5, \text{I}_5, \text{II})$	$(\text{I}_5, \text{I}_5, \text{I}_1, \text{I}_1)$	$(\text{I}_5, \text{I}_5, \text{I}_1, \text{I}_1)$
$(\text{I}_6, \text{I}_3, \text{I}_2, \text{I}_1)$	$(\text{I}_6, \text{I}_3, \text{I}_2, \text{I}_1)$	$(\text{I}_6, \text{I}_3, \text{III})$	$(\text{I}_6, \text{IV}, \text{I}_2)$
$(\text{I}_4, \text{I}_4, \text{I}_2, \text{I}_2)$	$(\text{I}_4, \text{I}_4, \text{I}_2, \text{I}_2)$	$(\text{I}_4, \text{I}_4, \text{I}_2, \text{I}_2)$	–
$(\text{I}_3, \text{I}_3, \text{I}_3, \text{I}_3)$	$(\text{I}_3, \text{I}_3, \text{I}_3, \text{I}_3)$	–	$(\text{I}_3, \text{I}_3, \text{I}_3, \text{I}_3)$

TABLE 3. Extremal and rational elliptic fibrations

Over the complex numbers, the following is due to Kondō [Kon86, Lemma 2.6]. There are generalizations of this result in [HS11] and [Sch16]. Since we need this construction for our classification, we will extend it to arbitrary characteristic.

LEMMA 2.17. *Let $f: \tilde{X} \rightarrow X$ be the canonical cover of an Enriques surface X , and let σ be the covering involution. Let $\pi: X \rightarrow \mathbb{P}^1$ be a special elliptic fibration of X with a special bisection N , let F be a double fiber of π , and let $J(\pi): J(X) \rightarrow \mathbb{P}^1$ be the Jacobian fibration associated with π . Let $\tilde{\pi}$ be the fibration of $\tilde{X}$ induced by $|f^{-1}F|$, and denote by $\varphi: |f^{-1}F| = \mathbb{P}^1 \rightarrow \mathbb{P}^1 = |2F|$ the induced morphism on the base curve.*

- (i) *The bisection N splits into two sections N^+ and N^- of $\tilde{\pi}$. In particular, the minimal proper smooth models of the base changes of $J(\pi)$ and π along φ are isomorphic.*
- (ii) *Choose N^+ as the zero section of $\tilde{\pi}$. Then, $J(\sigma) = t_{\ominus N^-} \circ \sigma$ is an involution whose quotient, after minimalizing the obtained fibration, is $J(\pi)$.*
- (iii) *The section N^- satisfies $N^- \cdot N^+ = 0$ and $J(\sigma)(N^-) = \ominus N^-$, and it does not meet the preimage of a singular double fiber of π in the identity component.*

The main tool to establish this result in arbitrary characteristic is the following lemma, which is a close study of how automorphisms of the generic fiber of an elliptic fibration with section extend to special fibers. For lack of a reference, we will give a proof.

LEMMA 2.18. *Let R be a discrete valuation ring with residue field k , and let $K = \text{Quot}(R)$. Let (E, O) be an elliptic curve over K , and let $\mathcal{E}$ be the Néron model of E over R . Let E_0 be the identity component of the special fiber of $\mathcal{E}$. Let $\rho: \text{Aut}(E, O) \rightarrow \text{Aut}(E_0, \overline{O}|_{E_0})$ be the natural map obtained from the Néron mapping property and restriction. Then, ρ is injective if and only if*

- $\text{char}(k) \notin \{2, 3\}$, or
- $\text{char}(k) \in \{2, 3\}$ and E_0 is not additive.

If E_0 is additive, then $\ker(\rho)$ consists of all elements of order p^n in $\text{Aut}(E, O)$, where $p = \text{char}(k)$.

Proof. We will compute the reduction of the automorphisms explicitly using Weierstrass equations and the description of automorphisms in [Sil09, Appendix A, proof of Proposition 1.2, p. 411] (see also [Sil94, Chapter IV, § 9] for an exposition of Tate's algorithm). Throughout, we denote by π a uniformizer of R .

If $\text{char}(k) \geq 5$, then we use a minimal and integral Weierstrass equation

$$y^2 = x^3 + a_4x + a_6.$$

Since all $g \in \text{Aut}(E, O)$ are of the form $g: (x, y) \mapsto (\zeta^2x, \zeta^3y)$ for some 12th root of unity ζ , they induce non-trivial automorphisms of E_0 independently of a_4 and a_6 .

If $\text{char}(k) = 3$, then we use a minimal and integral Weierstrass equation

$$y^2 = x^3 + a_2x^2 + a_4x + a_6.$$

If $a_2 \neq 0$, then the same argument as before works, so we may assume $a_2 = 0$. Then, an automorphism $g \in \text{Aut}(E, O)$ is given by $g: (x, y) \mapsto (\zeta^2x + r, \zeta^3y)$, where $\zeta^4 = 1$ and $r^3 + a_4r + (1 - \zeta^2)a_6 = 0$. If $\zeta \neq 1$, then $\rho(g) \neq \text{id}$ since ζ does not depend on a_4 and a_6 . But if $\zeta = 1$ and $r = \pm\sqrt{-a_4}$, then $\rho(g)$ is trivial if and only if $\pi \mid a_4$, that is, if and only if E_0 is of additive type. Note that the automorphisms with $\zeta = 1$ are exactly the automorphisms of order 3^n in $\text{Aut}(E, O)$. This yields the last claim in the case $p = 3$.

If $\text{char}(k) = 2$, then we use a minimal and integral Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

The inversion involution $g \in \text{Aut}(E, O)$ is given by $(x, y) \mapsto (x, y + a_1x + a_3)$. Thus, $\rho(g)$ is trivial if and only if $\pi \mid a_1, a_3$, that is, if and only if E_0 is of additive type. Now if $j(E_0) = 0$, then we can assume $a_1 = a_2 = 0$. An automorphism $g \in \text{Aut}(E, O)$ is given by $g: (x, y) \mapsto (\zeta^2x + s^2, \zeta^3y + \zeta^2sx + t)$, where $\zeta^3 = 1$, $s^4 + a_3s + (1 - \zeta)a_4 = 0$ and $t^2 + a_3t + s^6 + a_4s^2 = 0$. If $\zeta \neq 1$, then we have $\rho(g) \neq \text{id}$. Therefore, assume $\zeta = 1$ and $s^3 + a_3 = 0$. Now, $\rho(g) = \text{id}$ if and only if $\pi \mid a_3$, that is, if and only if E_0 is additive. Again, note that the automorphisms with $\zeta = 1$ are exactly those of order 2^n in $\text{Aut}(E, O)$. This is the last statement of the lemma in the case $p = 2$. $\square$

Proof of Lemma 2.17. Since $\tilde{X} \rightarrow X$ is étale of degree 2, every (-2) -curve on X splits into two disjoint (-2) -curves on $\tilde{X}$. In particular, N splits into two (-2) -curves N^+ and N^- . We claim that a general fiber of π also splits into two components. Indeed, if not, then the degree 2 morphism $\varphi: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ induced by f is ramified everywhere and therefore inseparable. Hence, $\text{char}(k) = 2$ and σ acts on every fiber of $\tilde{\pi}$. Since σ is fixed-point-free and additive and supersingular fibers do not admit fixed-point-free involutions, every fiber of $\tilde{\pi}$ is multiplicative or ordinary. This implies that the j -invariant of $\tilde{\pi}$ has no zeroes at all; hence, it is a non-zero constant. But then $\tilde{\pi}$ has no multiplicative fibers, hence no singular fibers at all, which is impossible for surfaces with non-zero topological Euler characteristic, by [Dol72]. Both N^+ and N^- have to be sections of the fibration $\tilde{\pi}$ since a general fiber of π splits into two components F_1 and F_2 , both of which are fibers of $\tilde{\pi}$, and therefore $2 = N.F = 2N^+.F_1 = 2N^-.F_1$.

Next, we show that $J(\sigma)$ is indeed an involution. Let F_0 be the identity component of a fiber of $\tilde{\pi}$ which is fixed (not necessarily pointwise) by σ . Note that F_0 is either multiplicative or smooth by Proposition 2.5. Since σ is fixed-point free, it induces a translation on F_0 if F_0 is smooth. Moreover, because $J(\sigma)(N^+) = t_{\ominus N^-} \circ \sigma(N^+) = N^+$, the restriction $J(\sigma)|_{F_0}$ is the identity if F_0 is smooth, and it can have at most order 2 if F_0 is multiplicative. We obtain $J(\sigma)^2|_{F_0} = \text{id}$ in any

case. Now, $J(\sigma)^2$ fixes $\tilde{\pi}$, and hence it is an automorphism of the generic fiber of $\tilde{\pi}$ fixing the zero section N^+ . By Lemma 2.18, we have $J(\sigma)^2 = \text{id}$ because it restricts to the identity on F_0 . Since $J(\sigma)(N^+) = N^+$, this section descends to the quotient. Thus, after possibly contracting (-1) -curves in the fibers, we obtain an elliptic fibration with section $\pi': X' \rightarrow \mathbb{P}^1$ on this quotient. Since π' has a section by construction and since it coincides with π after passing to the common cover $\tilde{X}$, it is the Jacobian fibration of π , and we have $\pi' = J(\pi)$ as well as $X' = J(X)$.

Finally, if π has a singular double fiber F of type I_n , the preimage of F in $\tilde{X}$ is a fiber F' of $\tilde{\pi}$ of type I_{2n} since this happens with the corresponding fiber on the Jacobian. Now, σ has to act without fixed points; hence, it acts as a rotation of order 2 on the corresponding $\tilde{A}_{2n-1}$ diagram, while $J(\sigma)$ fixes the diagram. In particular, the preimage of N meets two opposite curves of the diagram; that is, N^- does not meet the identity component of F' if we choose N^+ to be the zero section of $\tilde{\pi}$. $\square$

In particular, we obtain a distinguished non-zero section of $\tilde{\pi}$ if $\tilde{\pi}$ arises as the base change of a special elliptic fibration π of an Enriques surface with a given special bisection. Conversely, we will see that we can reconstruct π from $J(\pi)$ by exhibiting a suitable section on a degree 2 base change of $J(\pi)$. This has been studied by K. Hulek and M. Schütt in [HS11] using quadratic twists. Since in our case $J(\pi)$ is an extremal and rational elliptic fibration and extremal and rational elliptic surfaces are classified, we can approach the classification problem in a very explicit way. First, let us clarify what we mean by a “suitable section”.

DEFINITION 2.19. Let $J(\pi): J \rightarrow \mathbb{P}^1$ be an elliptic fibration of a rational surface J with zero section N^+ . Let $\varphi: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be a separable degree 2 morphism such that no branch point of φ is a point of additive reduction of $J(\pi)$. If $\text{char}(k) = 2$, assume further that the branch point is not a point of good supersingular reduction of $J(\pi)$. Then, a minimal proper smooth model of the base change $\tilde{\pi}$ of $J(\pi)$ along φ is an elliptic fibration of a K3 surface $\tilde{X}$. Also denote the zero section of $\tilde{\pi}$ by N^+ , and let $J(\sigma)$ be a covering involution of $\tilde{X} \rightarrow J$ such that $J(\sigma)(N^+) = N^+$. A section N^- of $\tilde{\pi}$ is called a $J(\pi)$ -Enriques section of $\tilde{\pi}$ if

- (i) $N^- \cdot N^+ = 0$,
- (ii) $J(\sigma)(N^-) = \ominus N^-$, and
- (iii) N^- does not meet the identity component of the fiber over $\varphi^{-1}(x)$ if φ is branched over a point x with $J(\pi)^{-1}(x)$ singular.

Remark 2.20. There are several ways of proving that a minimal proper smooth model of $\tilde{\pi}$ is an elliptic fibration of a K3 surface. One of them is that under our assumptions, a given minimal integral Weierstrass equation for $J(\pi)$ stays minimal after base changing via φ , and the type of the surface can be read off from the degrees of the coefficients in a minimal integral Weierstrass equation (see, for example, [SS10, § 4.8]).

Remark 2.21. Observe that the conditions in Definition 2.19 are exactly the properties satisfied by N^- in Lemma 2.17(iii). We will encounter several examples of such $J(\pi)$ -Enriques sections throughout this paper. The quickest way to achieve conditions (i) and (ii) is to take for N^- an everywhere integral (that is, $N^- \cdot N^+ = 0$) 2-torsion section of $\tilde{\pi}$ since such a section will be a base change of a 2-torsion section of $J(\pi)$. However, this does not guarantee condition (iii) to hold, as we will see later.

The following is the main ingredient in our approach to the classification. Over the complex numbers, this is implicitly contained in [Kon86] (for a variation of this, see [HS11]).

PROPOSITION 2.22. *With notation as in Definition 2.19, let N^- be a section of $\tilde{\pi}$ such that $J(\sigma)(N^-) = \ominus N^-$ and $N^+.N^- = 0$. Then, the quotient of $\tilde{X}$ by the involution $\sigma := t_{N^-} \circ J(\sigma)$ is an Enriques surface X with a special elliptic fibration π induced by $\tilde{\pi}$ if and only if N^- is a $J(\pi)$ -Enriques section. The Jacobian of π is $J(\pi)$, and the double fibers of π occur over the branch points of φ .*

Proof. Let us first show that σ is an involution. Denote by F_0 a fiber which is fixed (not necessarily pointwise) by $J(\sigma)$. We have $\sigma^2|_{F_0} = t_{N^-}|_{F_0} \circ J(\sigma)|_{F_0} \circ t_{N^-}|_{F_0} \circ J(\sigma)|_{F_0} = t_{N^-}|_{F_0} \circ t_{\ominus N^-}|_{F_0} = \text{id}|_{F_0}$ and since F_0 is either multiplicative or smooth and σ^2 fixes $\tilde{\pi}$ and N^+ , we obtain $\sigma^2 = \text{id}$ by Lemma 2.18.

Since translation by a section fixes all fibers and $J(\sigma)$ fixes at most two fibers F_0 and F_1 , we have $\text{Fix}(\sigma) \subseteq F_0 \cup F_1$. If $F \in \{F_0, F_1\}$ is smooth, we claim that $J(\sigma)$ acts trivially on F . In characteristic different from 2, this follows because $J(\sigma)$ acts non-trivially on a global 2-form. Indeed, $J(\sigma)$ has 1-dimensional fixed locus, as can be seen by choosing local coordinates u_1, u_2 with $J(\sigma)(u_1) = u_1$ and $J(\sigma)(u_2) = -u_2$ at a fixed point of $J(\sigma)$. Since $J(\sigma)$ fixes $N^+ \cap F$, it has to fix the whole fiber F pointwise. In characteristic 2, the restriction $J(\sigma)|_F$ is either the identity or a hyperelliptic involution since it fixes N^+ and F is ordinary. The latter case is impossible by [DK01, Theorem 1]. Since $J(\sigma)$ acts trivially on a smooth fiber $F \in \{F_0, F_1\}$ and $N^+.N^- = 0$, the restriction $\sigma|_F = t_{N^-}|_F$ will have no fixed points on F . As for a multiplicative fiber $F \in \{F_0, F_1\}$, the involution $J(\sigma)$ fixes the components of F (not necessarily pointwise); hence, σ has fixed points if and only if N^- meets the identity component of this fiber, that is, if and only if N^- is not a $J(\pi)$ -Enriques section.

Now, if N^- is a $J(\pi)$ -Enriques section, this means that the quotient of $\tilde{X}$ by σ is an Enriques surface X . Moreover, the divisors F and $N^+ + N^-$ are fixed by σ and thus descend to X , giving a special elliptic fibration π on X . Additionally, F_0 and F_1 descend to the two double fibers of π and $J(\pi)$ is the Jacobian of π by construction. $\square$

Remark 2.23. If σ has fixed points, we claim that it actually has a fixed locus of dimension 1. To see this, note that σ fixing two points on a (-2) -curve in characteristic 2 means that the whole curve is fixed (see also [DK01]). For the other characteristics, we refer the reader to [Zha98]. After contracting the fixed locus, the quotient by σ is nothing but a rational log Enriques surface of index 2 (see [Zha91]) and its minimal resolution is a Coble surface (see [DZ01]). We will not study these surfaces here, but the attentive reader will see them occur naturally as degenerations of the models we give for the surfaces in our Main Theorem.

Remark 2.24. We see from the proof that one can also obtain an Enriques surface as a quotient by σ if one weakens the assumption $N^+.N^- = 0$ to $N^+ \cap N^- \cap F_0 = N^+ \cap N^- \cap F_1 = \emptyset$. However, in general, this will not produce a *smooth* bisection. For more on this, see [HS11].

With this explicit and universal construction at our disposal, we can have a look at the relation between special bisections of an elliptic fibration of an Enriques surface and sections of its Jacobian.

COROLLARY 2.25. *Let π be a special elliptic fibration of an Enriques surface X with a special bisection N splitting into N^+ and N^- on the K3 cover $\tilde{X}$ of X . There is a map*

$$\text{jac}_2: \text{MW}(J(\pi)) \rightarrow \{\text{special bisections of } \pi\}$$

which is

- *injective if N^- is not 2-torsion after fixing N^+ as the zero section and*

– 2-to-1 onto its image otherwise.

Moreover, $\text{MW}(J(\pi))$ acts transitively on the image of jac_2 via its action on X .

Proof. We use the notation of Lemma 2.17. There is a natural injection $\text{MW}(J(\pi)) \rightarrow \text{MW}(\tilde{\pi})$; using this, we will consider sections of $J(\pi)$ as sections of $\tilde{\pi}$ by abuse of notation. Let $P \in \text{MW}(J(\pi))$. Since P comes from $J(\pi)$, it is fixed by $J(\sigma)$. Now, we compute

$$P.\sigma(P) = P.(t_{N^-} \circ J(\sigma))(P) = P.(P \oplus N^-) = N^+.N^- = 0.$$

Therefore, the divisor $P + \sigma(P)$ descends to a (-2) curve $\text{jac}_2(P)$ on X , which is necessarily a bisection of π since $2 = (P + \sigma(P)).\tilde{F} = \text{jac}_2(P).F$, where $\tilde{F}$ (respectively, F) is a general fiber of $\tilde{\pi}$ (respectively, π). For the injectivity, observe that $\sigma(P) \in \text{MW}(J(\pi))$ if and only if $J(\sigma)(\sigma(P)) = \sigma(P)$, that is, if and only if

$$P \oplus N^- = (t_{N^-} \circ J(\sigma))(P) = \sigma(P) = J(\sigma)(\sigma(P)) = P \ominus N^-,$$

which happens if and only if N^- is 2-torsion. The statement about the action of $\text{MW}(J(\pi))$ is clear by the construction of jac_2 . $\square$

To compute the intersection behaviour of the special bisections obtained via jac_2 , we will use the height pairing on $\text{MW}(\tilde{\pi})$.

PROPOSITION 2.26 (Shioda [Shi90]). *Let $\tilde{\pi}$ be an elliptic fibration of a K3 surface with zero section N^+ . The pairing*

$$\begin{aligned} \text{MW}(\tilde{\pi}) \times \text{MW}(\tilde{\pi}) &\rightarrow \mathbb{Q}, \\ (P, Q) &\mapsto \langle P, Q \rangle = 2 + P.N^+ + Q.N^+ - P.Q - \sum_{\nu \in \mathbb{P}^1} \text{contr}_{\nu}(P, Q), \end{aligned}$$

where the $\text{contr}_{\nu}(P, Q)$ are local correction terms depending on the intersection of P and Q with the fiber over ν , is a symmetric, bilinear pairing on $\text{MW}(\tilde{\pi})$ which induces the structure of a positive-definite lattice on $\text{MW}(\tilde{\pi})/\text{MW}(\tilde{\pi})_{\text{tors}}$. It is called the height pairing on $\text{MW}(\tilde{\pi})$. We write $h(P)$ for $\langle P, P \rangle$.

Remark 2.27. Note that this implies immediately that $h(P) = 0$ if and only if $P \in \text{MW}(\tilde{\pi})_{\text{tors}}$. Moreover, $\langle P, Q \rangle = 0$ as soon as P or Q is in $\text{MW}(\tilde{\pi})_{\text{tors}}$.

For the reader's convenience, we recall the correction terms of the height pairing following [SS10, § 11.8]. First, we have to fix a numbering of the simple components of a reducible fiber F_{ν} of an elliptic fibration π with zero section N^+ depending on the dual graph Γ of F_{ν} . In any case, denote the component of F_{ν} which meets N^+ by E_0 .

- If $\Gamma = \tilde{A}_{n-1}$, denote the components of F_{ν} by $E_0, \dots, E_{n-1}$ such that $E_i.E_j = 1$ if and only if $i - j = \pm 1 \pmod n$.
- If $\Gamma = \tilde{D}_{n+4}$, denote the simple components of F_{ν} by E_0, E_1, E_2 and E_3 such that E_1 is a simple component with minimal distance to E_0 .

Now, let $P, Q \in \text{MW}(\pi)$ be such that P meets E_i and Q meets E_j , and assume $i \leq j$. If $i = 0$, the correction term is 0. Otherwise, the value of $\text{contr}_{\nu}(P, Q)$ is given in Table 4.

2.3 Example

We keep the notation introduced in the previous subsection. Since we know how sections coming from $J(\pi)$ intersect the fibers of $\tilde{\pi}$, we can compute the intersection behaviour of the corresponding bisections on X once we know how N^- intersects the fibers of $\tilde{\pi}$. But this is already

Γ	$\tilde{E}_7$	$\tilde{E}_6$	$\tilde{D}_{n+4}$	$\tilde{A}_{n-1}$
Case $i = j$	$3/2$	$4/3$	$\begin{cases} 1 & \text{if } i = 1 \\ 1 + n/4 & \text{else} \end{cases}$	$i(n-i)/n$
Case $i < j$	$-$	$2/3$	$\begin{cases} 1/2 & \text{if } i = 1 \\ 1/2 + n/4 & \text{else} \end{cases}$	$i(n-j)/n$

TABLE 4. Correction terms for the height pairing

determined by the intersection behaviour of the special bisection N on X with the fibers of π . We will leave these computations to the reader but give a detailed description of the procedure in the following example.

Suppose that an Enriques surface contains the following dual graph of (-2) -curves with N as indicated:

This is the dual graph of a special elliptic fibration with a singular fiber of type I_8 and a double fiber of type I_2 . Note that the I_2 fiber has to be double since N meets its components only once and N is a bisection. On the K3 cover, this yields the following configuration:

On the other hand, we know that the Jacobian of π together with its four sections $P_1, P_2, P_3,$ and P_4 has the following dual graph:

One can explicitly compute the dual graph of a degree 2 base change of $J(\pi)$ ramified over the I_2 fiber (and not ramified over I_8):

To put this picture together with the second one, we set $N^+ = P_1$ as the zero section, add the sections $N^- \oplus P_i$ for all i to the diagram and calculate the intersection of N^- with P_i using the height pairing and the equality $0 = \langle P_i, N^- \rangle = 2 - N^- \cdot P_i - \sum_\nu \text{contr}_\nu(P_i, N^-)$ which follows from Remark 2.27. By using translations, we obtain the remaining intersection numbers and the following graph, where we denote P_i and $P_i \oplus N^-$ by P_i^+ and P_i^- , respectively:

$$P_2 \cdot N^- = P_4 \cdot N^- = 2 - \left(\frac{6}{8} + \frac{2}{8} + 1\right) = 0, \quad P_3 \cdot N^- = 2 - \left(\frac{4}{8} + \frac{4}{8}\right) = 1.$$

This yields the following configuration on the quotient Enriques surface, where we denote the special bisection corresponding to P_i again by P_i :

In fact, we can produce six more (-2) -curves using different fibrations with a double I_3 fiber to obtain the dual graph of type VII. For example, one may look at the following subgraph:

By Proposition 2.12, the (-2) -curve N_1 is a special bisection of a fibration with fibers I_6 and I_3 (not IV since it is double) and another reducible fiber. By Lemma 2.15, the corresponding fibration is extremal and by Table 3, the last reducible fiber is of type I_2 (respectively, III in characteristic 3) and it is simple since N_1 meets its reduced components twice. Hence, we can add the missing component of the I_2 (respectively, III) fiber to the graph. Similarly, one finds five more (-2) -curves and finally obtains the dual graph of type VII. The configuration we started with is what we will later call the “critical subgraph of type VII” since we have shown that any Enriques surface containing this graph is of type VII.

Remark 2.28. Note that the crucial point in all examples is the computation of the intersection numbers of the bisections using the height pairing. The intersection of the bisections obtained via jac_2 with the fibers is just a “translation” of the intersection of N with the fibers. In particular, the process is much easier if N^- is a 2-torsion section since the bisections arising via jac_2 are disjoint.

2.4 Vinberg’s criterion and numerically trivial automorphisms

In order to check that the (-2) -curves in the graphs for types I, $\dots$, VII are all (-2) -curves on the Enriques surface, one uses Vinberg’s criterion.

PROPOSITION 2.29 (Vinberg [Vin75, Theorem 2.6]). *Let Γ be a dual graph of finitely many (-2) -curves on an Enriques surface X . Suppose that Γ contains no m -tuple lines with $m \geq 3$, and suppose that the cone $K = \{C \in \text{Num}(X)_{\mathbb{R}} \mid C.E \geq 0 \text{ for all } E \in \Gamma\}$ is strictly convex. Then, the group W_{Γ} generated by reflections along (-2) -curves in Γ has finite index in $O(\text{Num}(X))$ if and only if the fibration π induced by every subgraph F of Γ of type $\tilde{A}-\tilde{D}-\tilde{E}$ is extremal and Γ contains the dual graph of the singular fibers of π . In this case, Γ is the dual graph of all (-2) -curves on X .*

Remark 2.30. This is a reformulation of the version of Vinberg’s criterion presented by Kondō [Kon86, Theorem 1.9]. The last statement is due to Y. Namikawa [Nam85, Proposition 6.9]. The strict convexity of K can be achieved, for example, if Γ contains the dual graph of the singular fibers of an elliptic fibration π and also contains another (-2) -curve which is not contained in a fiber of π .

The following corollary is a straightforward application of Vinberg’s criterion.

COROLLARY 2.31. *Let X be an Enriques surface whose dual graph of all (-2) -curves contains a graph Γ which is one of the seven dual graphs in the Main Theorem. Then, the (-2) -curves in Γ are all (-2) -curves on X .*

Therefore, we can check the action of $\text{Aut}(X)$ on $\text{Num}(X)$ directly on the dual graph of all (-2) -curves on X .

DEFINITION 2.32. An automorphism of an Enriques surface X is called *numerically trivial* if it acts trivially on $\text{Num}(X)$. It is called *cohomologically trivial* if it acts trivially on $\text{Pic}(X)$. We denote the respective groups by $\text{Aut}_{\text{nt}}(X)$ and $\text{Aut}_{\text{ct}}(X)$.

Recall that $\text{Num}(X)$ is a quotient of $\text{Pic}(X)$; hence, $\text{Aut}_{\text{ct}}(X)$ is a normal subgroup of $\text{Aut}_{\text{nt}}(X)$. Over the complex numbers, a complete classification of such automorphisms is available [MN84, Muk10]. There are three types of Enriques surfaces X with numerically trivial automorphisms; they satisfy $\text{Aut}_{\text{nt}}(X) \in \{\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}\}$. In positive characteristics, however, we only have bounds on the size of these groups.

PROPOSITION 2.33 (Dolgachev [Dol13]). *Let X be an Enriques surface. Then,*

$$|\text{Aut}_{\text{ct}}(X)| \leq 2 \quad \text{and} \quad |\text{Aut}_{\text{nt}}(X)/\text{Aut}_{\text{ct}}(X)| \leq 2.$$

However, we will not use this result since we are interested in the precise shape of the automorphism group. Therefore, we give explicit arguments in every case.

3. Enriques surfaces of type I

3.1 Main theorem for type I

THEOREM 3.1. *Let X be an Enriques surface. The following are equivalent:*

- (i) *The surface X is of type I.*
- (ii) *The dual graph of all (-2) -curves on X contains the graph in Figure 1.*

FIGURE 1. Critical subgraph for type I

- (iii) *The canonical cover $\tilde{X}$ of X admits an elliptic fibration with a Weierstrass equation of the form*

$$y^2 + \beta(s^2 + s)xy = x^3 + \beta^3(s^2 + s)^3x$$

such that the covering morphism $\rho: \tilde{X} \rightarrow X$ is given as quotient by the involution $\sigma = t_{N^-} \circ J(\sigma)$, where $J(\sigma): s \mapsto -s - 1$ and t_{N^-} is translation by $N^- = (0, 0)$.

Proof. First, observe that the dual graph of type I (see Table 1) contains the graph in Figure 1. This subgraph can be interpreted as the dual graph of a special elliptic fibration π with singular fibers of types III^* and I_2 (not III since this fiber is a double fiber) and special bisection N as follows, where the dotted rectangles mark the fibers:

As explained in Lemma 2.17, the bisection N splits into two sections N^+ and N^- of the elliptic fibration $\tilde{\pi}$ induced by π on the K3 cover $\tilde{X}$. Fixing N^+ as the zero section, we can compute $h(N^-) = 0$, and we see that N^- is a 2-torsion section of $\tilde{\pi}$. Starting from the subgraph in Figure 1, we get the last missing (-2) -curve from the elliptic fibration with a double fiber of type I_8 , which is induced by the $\tilde{A}_7$ diagram, as follows: The fibration is extremal by Lemma 2.15, the second reducible fiber is of type I_2 (except in characteristic 2, where it is of type III) by Table 3, and the intersection behaviour can be determined from the dual graph. These are all (-2) -curves on X by Corollary 2.31.

Now, we pursue the converse process dictated by Proposition 2.22 and calculate all elliptic fibrations of K3 surfaces obtained as separable quadratic base changes of $J(\pi)$ together with a section having the same intersection behaviour as N^- with curves obtained from (-2) -curves on X .

By [JLR12] we have the following equation for the unique rational elliptic surface with singular fibers of types III^* and I_2 :

$$y^2 + txy = x^3 + t^3x,$$

where t is a coordinate on $\mathbb{P}^1$. The I_2 fiber is at $t = \infty$, while the III^* fiber is at $t = 0$. Moreover, if $\text{char}(k) \neq 2$, there is an I_1 fiber at $t = 64$ and all other fibers are smooth. The non-trivial 2-torsion section is $S = (0, 0)$.

In every characteristic, we can write a degree 2 morphism $\mathbb{P}^1 \rightarrow \mathbb{P}^1$ branched over $t = \infty$ and not branched over $t = 0$ in the form $t \mapsto \beta(s^2 + s)$, where s is the new parameter on $\mathbb{P}^1$ and $\beta \in k - \{0\}$. We are allowed to assume that $t = 0$ is not a branch point since the III^* fiber is not multiple. The covering involution is given by $J(\sigma): s \mapsto -s - 1$. The second branch point of this degree 2 cover in characteristic different from 2 is at $t = -\beta/4$, which corresponds to $s = -1/2$. Now, we get the equation

$$y^2 + \beta(s^2 + s)xy = x^3 + \beta^3(s^2 + s)^3x$$

together with the 2-torsion section $S' = (0, 0)$ obtained by pulling back S . This equation defines an elliptic fibration $\tilde{\pi}$ on a K3 surface. As explained in Section 2.2, if $\tilde{\pi}$ is obtained as base change of a fibration of an Enriques surface, then $S' = N^-$ and σ is the covering involution. $\square$

Remark 3.2. Note that we have not yet claimed the existence of Enriques surfaces of type I. However, we have reduced this problem to the question of whether N^- is a $J(\pi)$ -Enriques section. We answer this question in Section 3.3.

3.2 Automorphisms

The automorphism groups of Enriques surfaces with finite automorphism group over the complex numbers have been computed in [Dol84, Kon86]. However, since some of these computations use the Torelli theorem for K3 surfaces and the classification of Enriques surfaces with numerically trivial automorphisms, which are not yet available in positive characteristic, we have to be careful when trying to extend these results. Therefore, we will give new calculations of these automorphism groups while pointing out where the old arguments have to be modified in order to work in arbitrary characteristic.

PROPOSITION 3.3. *Let X be an Enriques surface of type I. Then, $\text{Aut}(X) \cong D_4$ and this group is generated by automorphisms induced by 2-torsion sections of the Jacobian fibrations of elliptic fibrations of X . Moreover, $\text{Aut}_{\text{nt}}(X) \cong \mathbb{Z}/2\mathbb{Z}$ and $\text{Aut}(X)/\text{Aut}_{\text{nt}}(X) = (\mathbb{Z}/2\mathbb{Z})^2$.*

Proof. Recall that the dual graph of type I is as follows [Kon86]:

As has already been explained by Kondō [Kon86, Proof of Theorem 3.1.1, p. 205] and Dolgachev [Dol84, § 4.4, p. 175], the symmetry group of the dual graph of all (-2) -curves is $(\mathbb{Z}/2\mathbb{Z})^2$, and the 2-torsion section of the fibration π induced by the linear system $|2(F_9 + F_{11})|$ acts as a reflection along the horizontal axis, while the 2-torsion section of the fibration induced by $|F_{11} + F_{12}|$ acts trivially on the graph. A non-trivial numerically trivial automorphism g fixes F_3 and F_7 pointwise; hence, g fixes the fibration π and at least one geometric point on the generic fiber of π . Since π is non-isotrivial, g is the unique hyperelliptic involution of the generic fiber of π fixing the geometric points defined by F_3 and F_7 . Since $\text{Aut}(X)$ contains a translation by a 4-torsion section of the Jacobian of $|F_{11} + F_{12}|$, to show that $\text{Aut}(X) \cong D_4$, it suffices to observe that the 2-torsion section of a fibration with I_4^* fiber acts as a reflection along the vertical axis. This follows from Corollary 2.25. $\square$

3.3 Degenerations and moduli

PROPOSITION 3.4. *Let $\beta \neq 0$, and let*

$$y^2 + \beta(s^2 + s)xy = x^3 + \beta^3(s^2 + s)^3x$$

be the Weierstrass equation of an elliptic fibration $\tilde{\pi}_\beta$ with section on a K3 surface $\tilde{X}$. Define the involution $\sigma = t_{N^-} \circ J(\sigma)$, where $J(\sigma): s \mapsto -s - 1$ and t_{N^-} is translation by the section $N^- = (0, 0)$. Then, the following statements are true:

- (i) *The involution σ is fixed-point free if and only if $\beta \neq -256$. If $\beta = -256$, then the fixed locus of σ is one (-2) -curve.*
- (ii) *Two fibrations $\tilde{\pi}_\beta$ and $\tilde{\pi}_{\beta'}$ are isomorphic up to automorphisms of $\mathbb{P}^1$ if and only if $\beta = \beta'$.*

Proof. For the first claim, by Lemma 2.22, we have to check whether N^- is a $J(\pi)$ -Enriques section. First, observe that $N^-.N^+ = 0$ and $J(\sigma)(N^-) = N^- = \ominus N^-$ and that N^- does not meet the I_4 fiber in the identity component. Therefore, we are done if the second fiber fixed by $J(\sigma)$, namely $F_{-1/2}$, is smooth. This happens if and only if $\beta \neq -256$, and otherwise $F_{-1/2}$ is an I_2 fiber. In the latter case, N^- does not meet the singular point $(-2^9, 2^{14})$ of the Weierstrass equation at $s = -1/2$, and therefore it meets the identity component of $F_{-1/2}$. Hence, N^- is not a $J(\pi)$ -Enriques section in this case, and σ is not fixed-point free by Proposition 2.22.

The second claim follows immediately from a comparison of j -invariants since in any characteristic and independently of β , the locations of the III^* and I_4 fibers are at $s = -1, 0, \infty$. $\square$

We have seen in the proof of Proposition 3.3 that the two elliptic fibrations with singular fiber III^* on an Enriques surface of type I are isomorphic since they are interchanged by a reflection along the vertical axis. Therefore, we can describe the moduli space of these Enriques surfaces using Proposition 3.4.

COROLLARY 3.5. *Enriques surfaces of type I are parametrized by $\mathbb{A}^1 - \{0, -256\}$ in every characteristic.*

While $\beta \in \{0, \infty\}$ leads to very degenerate surfaces, we still get an involution if $\beta = -256$, while the K3 surface acquires an additional rational double point. The minimal resolution of the quotient is a Coble surface (see also Remark 2.23).

4. Enriques surfaces of type II

4.1 Main theorem for type II

THEOREM 4.1. *Let X be an Enriques surface. The following are equivalent:*

- (i) *The surface X is of type II.*
- (ii) *The dual graph of all (-2) -curves on X contains the graph in Figure 2.*

FIGURE 2. Critical subgraph for type II

- (iii) *The canonical cover $\tilde{X}$ of X admits an elliptic fibration with a Weierstrass equation of the form*

$$y^2 + \beta(s^2 + s)xy + \beta^2(s^2 + s)^2y = x^3 + \beta(s^2 + s)x^2$$

such that the covering morphism $\rho: \tilde{X} \rightarrow X$ is given as quotient by the involution $\sigma = t_{N^-} \circ J(\sigma)$, where $J(\sigma): s \mapsto -s - 1$ and t_{N^-} is translation by $N^- = (0, 0)$.

Proof. First, observe that the dual graph of type II (see Table 1) contains the graph in Figure 2. This subgraph can be interpreted as the dual graph of a special elliptic fibration π with singular fibers I_1^* and I_4 and special bisection N as follows, where the dotted rectangles mark the fibers:

Note that the I_4 fiber is a double fiber. Similarly to the case of type II, we compute $h(N^-) = 0$ and find the last missing (-2) -curves via jac_2 .

We found the following equation for the unique rational elliptic surface with singular fibers of type I_1^* and I_4 in arbitrary characteristic:

$$y^2 + txy + t^2y = x^3 + tx^2,$$

where t is a coordinate on $\mathbb{P}^1$. The I_4 fiber is at $t = \infty$, while the I_1^* fiber is at $t = 0$. Moreover, if $\text{char}(k) \neq 2$, then there is an I_1 fiber at $t = 16$ and all other fibers are smooth. The non-trivial 2-torsion section is $S = (0, 0)$.

In every characteristic, we can write every degree 2 morphism $\mathbb{P}^1 \rightarrow \mathbb{P}^1$ branched over $t = \infty$ and not branched over $t = 0$ in the form $t \mapsto \beta(s^2 + s)$, where s is the new parameter on $\mathbb{P}^1$ and

$\beta \in k - \{0\}$. The covering involution is given by $s \mapsto -s - 1$. The second branch point of this degree 2 cover in characteristic different from 2 is at $t = -\beta/4$. Now, we get the equation

$$y^2 + \beta(s^2 + s)xy + \beta^2(s^2 + s)^2y = x^3 + \beta(s^2 + s)x^2$$

together with the 2-torsion section $S' = (0, 0)$ obtained by pulling back S . This equation defines an elliptic fibration $\tilde{\pi}$ on a K3 surface. As explained in Section 2.2, if $\tilde{\pi}$ is obtained as base change of a fibration of an Enriques surface, then $S' = N^-$ and σ is the covering involution. $\square$

4.2 Automorphisms

PROPOSITION 4.2. *Let X be an Enriques surface of type II. Then, $\text{Aut}(X) \cong \mathfrak{S}_4$ and this group is generated by automorphisms induced by 2-torsion sections of the Jacobian fibrations of elliptic fibrations of X . Moreover, $\text{Aut}_{\text{nt}}(X) \cong \{1\}$.*

Proof. Kondō's proof works in arbitrary characteristic [Kon86, Proof of Theorem 3.2.1, p. 208] once we show that the surface has no numerically trivial automorphisms. Recall that the dual graph of all (-2) -curves for type II is as follows [Kon86]:

A numerically trivial automorphism g fixes the two bisections F_1 and F_7 of the non-isotrivial fibration π induced by the linear system $|2(F_9 + F_{10} + F_{11} + F_{12})|$ pointwise. Both F_1 and F_7 are separable (that is, the projection to the base curve is separable) bisections of π since they meet distinct points on the I_1^* fiber; hence, g fixes at least four geometric points on the generic fiber of π . If $\text{char}(k) = 2$, then g is trivial. If $\text{char}(k) \neq 2$, then we may assume that g is non-trivial. Then, g is a hyperelliptic involution of π , and the four geometric points on the generic fiber are 2-torsion points relative to one another. But in characteristic different from 2, the fibration π has an I_1 fiber which has only two 2-torsion points. Therefore, F_1 and F_7 would have to meet, but they do not. Hence, g is trivial. $\square$

4.3 Degenerations and moduli

As in the case of type I, one proves the following.

PROPOSITION 4.3. *Let $\beta \neq 0$, and let*

$$y^2 + \beta(s^2 + s)xy + \beta^2(s^2 + s)^2y = x^3 + \beta(s^2 + s)x^2$$

be the Weierstrass equation of an elliptic fibration $\tilde{\pi}_\beta$ with section on a K3 surface $\tilde{X}$. Define the involution $\sigma = t_{N^-} \circ J(\sigma)$, where $J(\sigma): s \mapsto -s - 1$ and t_{N^-} is translation by the section $N^- = (0, 0)$. Then, the following statements are true:

- (i) *The involution σ is fixed-point free if and only if $\beta \neq -64$. If $\beta = -64$, the fixed locus of σ is one (-2) -curve.*
- (ii) *Two fibrations $\tilde{\pi}_\beta$ and $\tilde{\pi}_{\beta'}$ are isomorphic up to automorphisms of $\mathbb{P}^1$ if and only if $\beta = \beta'$.*

COROLLARY 4.4. *Enriques surfaces of type II are parametrized by $\mathbb{A}^1 - \{0, -64\}$ in every characteristic.*

As in the case of type I, the cases where $\beta \in \{0, \infty\}$ are very degenerate surfaces, and $\beta = -64$ leads to a Coble surface.

5. Enriques surfaces of type III

5.1 Main theorem for type III

THEOREM 5.1. *Let X be an Enriques surface. The following are equivalent:*

- (i) *The surface X is of type III.*
- (ii) *The dual graph of all (-2) -curves on X contains the graph in Figure 3.*

FIGURE 3. Critical subgraph for type III

- (iii) *The canonical cover $\tilde{X}$ of X admits an elliptic fibration with a Weierstrass equation of the form*

$$y^2 + xy = x^3 + 4s^4x^2 + s^4x$$

such that the covering morphism $\rho: \tilde{X} \rightarrow X$ is given as quotient by the involution $\sigma = t_{N^-} \circ J(\sigma)$, where $J(\sigma): s \mapsto -s$ and t_{N^-} is translation by $N^- = (0, 0)$.

Moreover, Enriques surfaces of type III do not exist in characteristic 2.

Proof. Note that the dual graph of type III (see Table 1) contains the graph in Figure 3. This subgraph can be interpreted as the dual graph of a special elliptic fibration π with singular fibers (I_4, I_4, I_2, I_2) and special bisection N as follows, where the dotted rectangles mark the fibers:

As before, the bisection N splits into two sections N^+ and N^- of the elliptic fibration $\tilde{\pi}$ induced by π on the K3 cover $\tilde{X}$. Fixing N^+ as the zero section, we compute $h(N^-) = 0$, and we see that N^- is a 2-torsion section of $\tilde{\pi}$ meeting the I_8 fibers in a non-identity component.

Note that the existence of this fibration already implies the non-existence of this type of Enriques surfaces in characteristic 2 since a fibration with singular fibers (I_4, I_4, I_2, I_2) does not exist on rational surfaces in characteristic 2, as can be seen in Table 3.

Now, Corollary 2.25 gives three more (-2) -curves, resulting in the following graph:

We find a graph of an elliptic fibration with singular fibers (I_0^*, I_0^*) and special bisection N :

With the usual notation, we compute $h(N^-) = 2$ and add two bisections coming from Corollary 2.25. In the following figure, we added only one of these bisections to maintain readability:

Note that one of the bisections arising via jac_2 has already been part of the graph to begin with. Hence, it remains to produce two more (-2) -curves using another fibration. We leave the details to the reader.

By [JLR12], we have the following equation for the unique rational elliptic surface with singular fibers of type (I_4, I_4, I_2, I_2) in characteristic different from 2 (the equation can be simplified over $\mathbb{Z}$):

$$y^2 + xy = x^3 + 4t^2x^2 + t^2x,$$

where t is a coordinate on $\mathbb{P}^1$. The I_4 fibers are at $t = 0, \infty$, while the I_2 fibers are at $t = \pm \frac{1}{4}$. The non-trivial 2-torsion sections are $S_1 = (-4t^2, 2t^2)$, $S_2 = (0, 0)$ and $S_3 = (-\frac{1}{4}, \frac{1}{8})$.

In characteristic different from 2, we can write a degree 2 morphism $\mathbb{P}^1 \rightarrow \mathbb{P}^1$ with $t = 0, \infty$ as branch points in the form $t \mapsto s^2$, where s is the new parameter on $\mathbb{P}^1$. The covering involution

$J(\sigma)$ is given by $s \mapsto -s$. Now, we get the equation

$$y^2 + xy = x^3 + 4s^4x^2 + s^4x \quad (5.1)$$

together with the 2-torsion sections $S'_1 = (-4s^4, 2s^4)$, $S'_2 = (0, 0)$ and $S'_3 = (-\frac{1}{4}, \frac{1}{8})$ obtained by pulling back S_1 , S_2 and S_3 . All of them are $J(\sigma)$ -(anti-)invariant. However, S'_1 (respectively, S'_3) meets the identity component of the fiber at $s = \infty$ (respectively, $s = 0$). Therefore, S'_2 is the section we are looking for. $\square$

Remark 5.2. Note that equation (5.1) has an automorphism $\iota: s \mapsto \sqrt{-1}s$ which commutes with σ . Therefore, ι induces an automorphism of the Enriques surface, which we will also denote by ι . Moreover, ι fixes the 2-torsion sections of (5.1). Note also that this automorphism acts as $\sqrt{-1}$ on a non-zero global 2-form of the K3 surface.

5.2 Automorphisms

PROPOSITION 5.3. *Let X be an Enriques surface of type III. Then, $\text{Aut}(X) \cong (\mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2) \rtimes D_4$, and this group is generated by automorphisms induced by 2-torsion sections of the Jacobian fibrations of non-isotrivial elliptic fibrations of X and the automorphism exhibited in Remark 5.2. Moreover, $\text{Aut}_{\text{nt}}(X) \cong \mathbb{Z}/2\mathbb{Z}$ and $\text{Aut}(X)/\text{Aut}_{\text{nt}}(X) = (\mathbb{Z}/2\mathbb{Z})^3 \rtimes D_4$.*

Proof. Recall that the dual graph of all (-2) -curves for type III is as follows [Kon86]:

Let us first show that $|\text{Aut}_{\text{nt}}(X)| \leq 2$. Consider the elliptic fibration π induced by the linear system $|2(E_3 + E_4 + E_5 + E_{11})|$, and let $g \in \text{Aut}_{\text{nt}}(X)$ be a non-trivial automorphism. If g fixes one of the bisections E_2 , E_9 , E_6 and E_{12} pointwise, then it is the hyperelliptic involution of the generic fiber of π fixing the geometric points defined by the bisection. Moreover, g induces a unique involution on such a bisection if it acts non-trivially on it. In any case, $\text{ord}(g) = 2^n$ for some $n \in \mathbb{N}$ and, since $\text{char}(k) \neq 2$, the automorphism g is tame. The fixed locus of a tame automorphism is smooth by the Lefschetz fixed-point formula [Ive72]. Since g fixes E_1 , E_3 , E_5 and E_7 pointwise, it has to act non-trivially on E_2 , E_9 , E_6 and E_{12} . In particular, g is unique.

As explained by Kondō [Kon86, Proof of Theorem 3.3.4, p. 214], the automorphism group of the graph is the same as the automorphism group of the subgraph Σ generated by $\{E_i\}_{i \in \{1, \dots, 12\}}$,

which is $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes D_4$. Moreover, since the intersection behaviour of the curves is the same in any characteristic, it is still true that only an index 2 subgroup of $\text{Aut}(\Sigma)$ may be realized.

As for the realization of the automorphisms, note the following:

- A reflection r_d along a diagonal axis is realized by a 2-torsion section of the Jacobian of $|E_2 + E_9 + E_6 + E_{12} + 2(E_1 + E_7 + E_8)|$.
- A reflection r_v along the vertical axis is realized by the 2-torsion section of the Jacobian of $|E_2 + E_9 + E_8 + E_{10} + 2(E_3 + E_4 + E_5 + E_6 + E_7)|$.
- There is a 2-torsion section of the Jacobian of the fibration $|2(E_3 + E_4 + E_5 + E_{11})|$ which interchanges E_2 and E_9 as well as E_6 and E_{12} while fixing E_4, E_{11}, E_8 and E_{10} . Another 2-torsion section of the same fibration induces the numerically trivial involution.
- After fixing E_6 as a special bisection N of $|2(E_3 + E_4 + E_5 + E_{11})|$, the automorphism ι of Remark 5.2 fixes E_6 and E_{12} and interchanges E_2 and E_9 . Moreover, it acts non-trivially on exactly one of the pairs (E_3, E_{10}) and (E_4, E_{11}) .

These facts are checked by using Corollary 2.25 and following the construction of jac_2 . Now, note that we can compute the pointwise stabilizer G of the set $\{E_1, E_3, E_5, E_7\}$ using equation (5.1). It is generated by t_{S_1}, t_{S_3} and t_{S_2} as well as ι and the inversion involution. All these automorphisms commute with one another and $\iota^2 = t_{S_2}$; hence, $G \cong \mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2$. Therefore, we have a short exact sequence

$$0 \longrightarrow G \longrightarrow \text{Aut}(X) \longrightarrow D_4 \longrightarrow 0.$$

We claim that this sequence splits. Indeed, by [MO14, Corollary 4.7 and Section 7.1], a tame semi-symplectic automorphism (that is, an automorphism acting trivially on $H^0(X, \omega_X^{\otimes 2})$) has order at most 6. We have realized all reflections using translations by 2-torsion sections, which are semi-symplectic, since they fix the base of an elliptic fibration and act as translation on the fibers, and tame since we are working in characteristic different from 2. Therefore, $r_d \circ r_v$ has order 4, and the group generated by r_d and r_v is a subgroup of $\text{Aut}(X)$ isomorphic to D_4 . Hence, the sequence splits, and the proof is finished. $\square$

Remark 5.4. In particular, note that $\text{Aut}(X)$ is not a semi-direct product $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes D_4$. This was already observed by H. Ohashi in [Oha15] and corrects a small mistake in [Kon86].

5.3 Degenerations and moduli

This is similar to the first two cases. However, the involution is always fixed-point free since the branch points of the degree 2 morphism $\mathbb{P}^1 \rightarrow \mathbb{P}^1$ do not move.

PROPOSITION 5.5. *Assume $\text{char}(k) \neq 2$. Let*

$$y^2 + xy = x^3 + 4s^4x^2 + s^4x$$

be the Weierstrass equation of an elliptic fibration $\tilde{\pi}$ with section on a K3 surface $\tilde{X}$. Define the involution $\sigma = t_{N^-} \circ J(\sigma)$, where $J(\sigma): s \mapsto -s$ and t_{N^-} is translation by the section $N^- = (0, 0)$. Then, σ is fixed-point free.

COROLLARY 5.6. *Enriques surfaces of type III exist if and only if $\text{char}(k) \neq 2$. Moreover, they are unique if they exist.*

Remark 5.7. The equation we took from [JLR12] for $J(\pi)$ makes sense in characteristic 2, where it defines a rational elliptic surface with singular fibers I_4 at $t = 0$ and I_1^* at $t = \infty$. The degree 2

cover $t \mapsto s^2$ given in Proposition 5.5 is the Frobenius morphism, and the base change along this morphism defines a rational elliptic surface with singular fibers (I_8, III) . This surface is the minimal resolution of singularities of a surface covering a 1-dimensional family of classical Enriques surfaces with finite automorphism group of “type VIII”, as is shown by Katsura, Kondō and the author in [KKM17].

6. Enriques surfaces of type IV

6.1 Main theorem for type IV

THEOREM 6.1. *Let X be an Enriques surface. The following are equivalent:*

- (i) *The surface X is of type IV.*
- (ii) *The dual graph of all (-2) -curves on X contains the graph in Figure 4.*

FIGURE 4. Critical subgraph for type IV

- (iii) *The canonical cover $\tilde{X}$ of X admits an elliptic fibration with a Weierstrass equation of the form*

$$y^2 = x^3 + 2(s^4 + 1)x^2 + (s^4 - 1)^2x$$

such that the covering morphism $\rho: \tilde{X} \rightarrow X$ is given as quotient by the involution $\sigma = t_{N^-} \circ J(\sigma)$, where $J(\sigma): s \mapsto -s$ and t_{N^-} is translation by $N^- = (-(s^2 - 1)^2, 0)$.

Moreover, Enriques surfaces of type IV do not exist in characteristic 2.

Proof. First, we observe that the dual graph of type IV (see Table 1) contains the graph in Figure 4. This subgraph can be interpreted as the dual graph of a special elliptic fibration π with singular fibers (I_4, I_4, I_2, I_2) and special bisection N as follows:

The bisection N splits into two sections N^+ and N^- of the elliptic fibration $\tilde{\pi}$ induced by π on the K3 cover $\tilde{X}$. Fixing N^+ as the zero section, we compute $h(N^-) = 0$, and we see that N^- is a 2-torsion section of $\tilde{\pi}$ meeting the I_4 fibers coming from the I_2 fibers of π in a non-identity component. The same argument as for type III shows that this type cannot exist in characteristic 2.

Now, Corollary 2.25 gives three more (-2) -curves, resulting in the following graph:

Again, to produce additional (-2) -curves, we find a different special fibration with special bisection N on this surface as follows:

This special fibration has one I_0^* fiber and four disjoint (-2) -curves contained in some other fibers. Such a fibration will be extremal in any case by Lemma 2.15, so by Table 3 the fibers are (I_0^*, I_0^*) . Hence, we obtain one more (-2) -curve. We leave it to the reader to find three more such diagrams and to check that the resulting graph is the one of type IV.

We use the same equation as for surfaces of type III,

$$y^2 + xy = x^3 + 4t^2x^2 + t^2x,$$

where t is a coordinate on $\mathbb{P}^1$. Recall that the I_4 fibers are at $t = 0, \infty$, while the I_2 fibers are at $t = \pm \frac{1}{4}$. The non-trivial 2-torsion sections are $S_1 = (-4t^2, 2t^2)$, $S_2 = (0, 0)$ and $S_3 = (-\frac{1}{4}, \frac{1}{8})$.

In characteristic different from 2, we can write a degree 2 morphism $\mathbb{P}^1 \rightarrow \mathbb{P}^1$ with $t = \pm \frac{1}{4}$ as branch points in the form $t \mapsto \frac{1}{4}(s^2 - 1)/(s^2 + 1)$, where s is the new parameter on $\mathbb{P}^1$. The covering involution $J(\sigma)$ is given by $s \mapsto -s$. After scaling x and y and simplifying, we get the equation

$$y^2 = x^3 + 2(s^4 + 1)x^2 + (s^4 - 1)^2x \quad (6.1)$$

together with the 2-torsion sections $S'_1 = (-(s^2 - 1)^2, 0)$, $S'_2 = (0, 0)$ and $S'_3 = (-(s^2 + 1)^2, 0)$ obtained by pulling back S_1 , S_2 and S_3 . All of them are $J(\sigma)$ -anti-invariant. However, S'_2 meets the identity component of the fiber at $s = 0$. Moreover, the surface defined by equation (6.1) has an automorphism ι interchanging S'_1 and S'_3 given by $\iota: s \mapsto \sqrt{-1}s$.

Therefore, we can choose S'_1 as N^- . □

Remark 6.2. It is important to observe that the fibration $\tilde{\pi}$ defined by equation (6.1) has more torsion sections than the ones coming from the rational surface. For example, one can check that $P = (-(s - \sqrt{-1})^2(s^2 - 1), -2s(s - \sqrt{-1})^2(s^2 - 1))$ is a section satisfying $P \oplus P = N^-$. Since $t_P \circ \iota$ commutes with σ , it will induce an automorphism of the Enriques surface, which we will

also denote by $t_P \circ \iota$. Moreover, $(t_P \circ \iota)^2 = t_Q \circ J(\sigma)$ for a 4-torsion section Q of $\tilde{\pi}$. Again, note that $t_P \circ \iota$ acts as $\sqrt{-1}$ on a non-zero global 2-form of the K3 surface.

6.2 Automorphisms

PROPOSITION 6.3. *Let X be an Enriques surface of type IV. Then, $\text{Aut}(X) \cong (\mathbb{Z}/2\mathbb{Z})^4 \rtimes (\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z})$, and this group is generated by automorphisms induced by sections of the Jacobian fibrations of elliptic fibrations of X and an automorphism exhibited in Remark 6.2. More precisely, we can choose the sections in such a way that at most one of them is not 2-torsion and that none of them is a section of an isotrivial fibration. Moreover, $\text{Aut}_{\text{nt}}(X) \cong \{1\}$.*

Proof. Recall that the dual graph of all (-2) -curves for type IV is as follows [Kon86]:

We claim that $\text{Aut}_{\text{nt}}(X)$ is trivial. Indeed, a numerically trivial automorphism g acts trivially on the base of the fibration $|2(E_1 + E_{11})|$ since this fibration has four reducible fibers and g fixes the four bisections E_2, E_4, E_{13} and E_{14} pointwise; hence, it is trivial.

Following [Kon86, Proof of Theorem 3.4.1, p. 217], we look at the action of $\text{Aut}(X)$ on the set of five fibrations $\{\Delta_i \mid i = 1, \dots, 5\}$ with $\Delta_1 = |2(E_1 + E_{11})|$, $\Delta_2 = |2(E_2 + E_{10})|$, $\Delta_3 = |2(E_5 + E_{15})|$, $\Delta_4 = |2(E_6 + E_{13})|$ and $\Delta_5 = |2(E_{17} + E_{19})|$. The kernel of the induced homomorphism $\psi: \text{Aut}(X) \rightarrow \mathfrak{S}_5$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$, and it is generated by translations by 2-torsion sections of the Jacobians of the Δ_i [Kon86, Proof of Theorem 3.4.1, p. 218]. From the dual graph, we see that an automorphism of X cannot act as a permutation of order 3 or as a transposition on $\{\Delta_1, \dots, \Delta_5\}$. Now, we show that the image of ψ is the group G generated by the permutations $\varphi_1 = (13425)$ and $\varphi_2 = (1324)$. Using Corollary 2.25, these permutations are realized as follows:

- The Jacobian of $|E_5 + E_6 + E_{10} + E_{18} + E_{11}|$ has a 5-torsion section which realizes φ_1 .
- If we fix E_{11} as a special bisection of Δ_5 , by Remark 6.2, we obtain a section P such that φ_2 is realized by the automorphism $t_P \circ \iota$. To see this, note that a 4-torsion section of the Jacobian of Δ_5 acts as $\Delta_1 \mapsto \Delta_2, \Delta_3 \mapsto \Delta_4$.

We have $G \cong \mathbb{Z}/5\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$ and, since $[\mathfrak{S}_5 : G] \geq 6$, this yields the claim on the image of ψ . Now, note that we can compose $t_P \circ \iota$ with an involution interchanging the two I_2 fibers of the Δ_5 fibration to obtain an automorphism of order 4 realizing φ_2 . Hence, we obtain a splitting of

$$0 \longrightarrow (\mathbb{Z}/2\mathbb{Z})^4 \longrightarrow \text{Aut}(X) \longrightarrow \mathbb{Z}/5\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z} \longrightarrow 0.$$

This finishes the proof. $\square$

6.3 Degenerations and moduli

As in the previous case, we obtain information about degenerations and moduli by direct calculation.

PROPOSITION 6.4. *Assume $\text{char}(k) \neq 2$. Let*

$$y^2 = x^3 + 2(s^4 + 1)x^2 + (s^4 - 1)^2 x$$

be the Weierstrass equation of an elliptic fibration $\tilde{\pi}$ with section on a K3 surface $\tilde{X}$. Define the involution $\sigma = t_{N^-} \circ J(\sigma)$, where $J(\sigma): s \mapsto -s$ and t_{N^-} is translation by the section $N^- = (-(s^2 - 1)^2, 0)$. Then, σ is fixed-point free.

COROLLARY 6.5. *Enriques surfaces of type IV exist if and only if $\text{char}(k) \neq 2$. Moreover, they are unique if they exist.*

7. Enriques surfaces of type V

7.1 Main theorem for type V

THEOREM 7.1. *Let X be an Enriques surface. The following are equivalent:*

- (i) *The surface X is of type V.*
- (ii) *The dual graph of all (-2) -curves on X contains the graph in Figure 5.*

FIGURE 5. Critical subgraph for type V.

- (iii) *The canonical cover $\tilde{X}$ of X admits an elliptic fibration with a Weierstrass equation of the form*

$$y^2 + (s^2 + 1)xy + (s^2 + 1)y = x^3 + (s^2 + 2)x^2 + (s^2 + 1)x$$

such that the covering morphism $\rho: \tilde{X} \rightarrow X$ is given as quotient by the involution $\sigma = t_{N^-} \circ J(\sigma)$, where $J(\sigma): s \mapsto -s$ and t_{N^-} is translation by $N^- = (-1, 0)$.

Moreover, Enriques surfaces of type V do not exist in characteristics 2 and 3.

Proof. First, we observe that the dual graph of type V (see Table 1) contains the graph in Figure 5. This subgraph can be interpreted as the dual graph of a special elliptic fibration π with singular fibers I_6 , I_2 (not III since it is double) and I_3 (or IV) and special bisection N as follows, where the dotted rectangles mark the fibers:

As before, the bisection N splits into two sections N^+ and N^- of the elliptic fibration $\tilde{\pi}$ induced by π on the K3 cover $\tilde{X}$. Fixing N^+ as the zero section, we can compute $h(N^-) = 0$, and we see that N^- is a 2-torsion section of $\tilde{\pi}$ meeting the I_6 and I_2 fibers in a non-identity component.

Note that the existence of this fibration already gives the non-existence of this type of Enriques surfaces in characteristics 2 and 3 since an extremal fibration with singular fibers I_6 and I_2 does not exist on rational surfaces in characteristic 3 (see Table 3) and a fibration with two double fibers cannot exist in characteristic 2. Therefore, we will assume $\text{char}(k) \neq 2, 3$ from now on.

Now, Corollary 2.25 gives two more (-2) -curves, resulting in the following graph:

For this example, one can use a fibration with an I_2^* fiber to produce another (-2) -curve:

As usual, the remaining curves can be found similarly.

By [JLR12], we have, after simplifying, the following equation for the unique extremal and

rational elliptic surface with singular fibers (I_6, I_3, I_2, I_1) :

$$y^2 + txy + ty = x^3 + (1+t)x^2 + tx,$$

where t is a coordinate on $\mathbb{P}^1$. The I_6 fiber is at $t = \infty$, the I_3 fiber is at $t = 0$, the I_2 fiber is at $t = 1$, and the I_1 fiber is at $t = -8$. The non-trivial 2-torsion section is $S = (-1, 0)$.

In characteristic different from 2, we can write a degree 2 morphism $\mathbb{P}^1 \rightarrow \mathbb{P}^1$ with $t = 1, \infty$ as branch points in the form $t \mapsto s^2 + 1$, where s is the new parameter on $\mathbb{P}^1$. The covering involution $J(\sigma)$ is given by $s \mapsto -s$. Now, we have the equation

$$y^2 + (s^2 + 1)xy + (s^2 + 1)y = x^3 + (s^2 + 2)x^2 + (s^2 + 1)x \quad (7.1)$$

together with the 2-torsion sections $S' = (-1, 0)$ obtained by pulling back S . Since S' is $J(\sigma)$ -(anti-)invariant and meets the fibers in the correct components, it is the section we are looking for. $\square$

7.2 Automorphisms

PROPOSITION 7.2. *Let X be an Enriques surface of type V. Then, $\text{Aut}(X) \cong \mathfrak{S}_4 \times \mathbb{Z}/2\mathbb{Z}$ and this group is generated by automorphisms induced by 2-torsion sections of the Jacobian fibrations of elliptic fibrations of X . Moreover, $\text{Aut}_{\text{nt}}(X) \cong \mathbb{Z}/2\mathbb{Z}$ and $\text{Aut}(X)/\text{Aut}_{\text{nt}}(X) \cong \mathfrak{S}_4$.*

Proof. Recall that the dual graph of (-2) -curves for type V is as follows [Kon86]:

We claim that $|\text{Aut}_{\text{nt}}(X)| = 2$. Indeed, a numerically trivial automorphism g acts trivially on the base of the fibration $|2(E_1 + E_2 + E_3 + E_4 + E_5 + E_6)|$ since this fibration has at least three singular fibers and g acts trivially or induces a unique involution on the three bisections E_{10} , E_7 and E_9 . By the same argument as for type III, there is at most one such g . Now, note that the 2-torsion section of the Jacobian of this fibration acts identically on the graph of (-2) -curves.

The automorphism group of the graph is $\mathfrak{S}_4$ (see [Kon86, Proof of Theorem 3.5.1, p. 223]). It suffices to look at the action of $\text{Aut}(X)$ on the set $\{E_1, E_3, E_5, E_8\}$.

- Transpositions of E_5 with another curve of the set are induced by 2-torsion sections of fibrations with a singular fiber of type I_2^* . For example, there is a 2-torsion section of the

Jacobian of $|E_2 + E_6 + E_7 + E_9 + 2(E_1 + E_8 + E_{10})|$ which interchanges E_3 and E_5 , by Corollary 2.25.

- All transpositions of two curves different from E_5 are induced by 2-torsion sections of fibrations with a singular fiber of type III*; for example, the 2-torsion section of the Jacobian of $|E_{10} + E_9 + 2E_1 + 2E_3 + 2E_7 + 3E_6 + 3E_4 + 4E_5|$ interchanges E_{10} and E_9 .

Finally, we claim that these transpositions generate a subgroup of $\text{Aut}(X)$ which is isomorphic to $\mathfrak{S}_4$. Indeed, this can be checked by using equation (7.1) to compute the stabilizer G of E_1 (which is D_6) and by using the fact that the maximal order of a tame semi-symplectic automorphism is 6 [MO14]. This finishes the proof. $\square$

7.3 Degenerations and moduli

As in the previous cases, we prove the existence of this type by explicit calculation.

PROPOSITION 7.3. Assume $\text{char}(k) \neq 2, 3$. Let

$$y^2 + (s^2 + 1)xy + (s^2 + 1)y = x^3 + (s^2 + 2)x^2 + (s^2 + 1)x$$

be the Weierstrass equation of an elliptic fibration $\tilde{\pi}$ with section on a K3 surface $\tilde{X}$. Define the involution $\sigma = t_{N^-} \circ J(\sigma)$, where $J(\sigma): s \mapsto -s$ and t_{N^-} is translation by the section $N^- = (-1, 0)$. Then, σ is fixed-point free.

COROLLARY 7.4. Enriques surfaces of type V exist if and only if $\text{char}(k) \neq 2, 3$. Moreover, they are unique if they exist.

Remark 7.5. Again, the equation makes sense in characteristic 2, where it defines a K3 surface covering a 1-dimensional family of classical and supersingular Enriques surfaces of type VII (see [KK18]).

8. Enriques surfaces of type VI

8.1 Main theorem for type VI

Remark 8.1. In the first five cases, every base change with the correct ramification points produced an elliptic fibration of a K3 surface with $J(\pi)$ -Enriques section N^- . This happened because the section N^- was a 2-torsion section. In the last two cases, however, we do not get this section for free.

LEMMA 8.2. Let $\text{char}(k) \neq 3$ and $J(\sigma): s \mapsto -s - \beta$, and let

$$y^2 - 3(3(s^2 + \beta s) + 1)xy + (3(s^2 + \beta s) + 1)^2 y = x^3 \quad (8.1)$$

with $\beta \in k - \{\pm 2/\sqrt{3}\}$ be the Weierstrass equation of an elliptic fibration of a K3 surface. Then, an everywhere integral, $J(\sigma)$ -anti-invariant section N^- meeting the fiber at $s = \infty$ in a non-identity component exists if and only if $\beta = \pm 1$. Moreover, it is unique up to the sign if it exists. Both cases are isomorphic, and if $\beta = 1$, the section is given by $N^- = (s + s^2, s^3)$.

Proof. By [Shi10, Lemma 1.2], an everywhere integral section N^- is given by $(x(s), y(s))$, where $x(s)$ and $y(s)$ are polynomials in s with $\deg_s(x) \leq 4$ and $\deg_s(y) \leq 6$. Since the singular point of equation (8.1) at $s = \infty$ is $(0, 0)$, we even need $\deg_s(x) \leq 3$ and $\deg_s(y) \leq 5$ if we want N to meet a non-identity component of the fiber at $s = \infty$.

Since the x -coordinate of $\ominus N$ is $x(s)$, we need $x(s)$ to be a $J(\sigma)$ -invariant polynomial of degree at most 3 if we want $J(\sigma)(N) = \ominus N$ to hold. In every characteristic, such a polynomial

has at most degree 2 and is of the form $x(s) = a_0 + a_1\beta s + a_1s^2$ with $a_0, a_1 \in k$. Now, we set $y(s) = b_0 + b_1s + b_2s^2 + b_3s^3 + b_4s^4 + b_5s^5$ with $b_i \in k$ and plug $x(s)$ and $y(s)$ into equation (8.1). A straightforward comparison of coefficients taking into account $J(\sigma)(N) = \ominus N$ gives the desired result $\beta = \pm 1$ and, up to the sign, $(x(s), y(s)) = (\beta s + s^2, \beta s^3)$.

Finally, note that the automorphism $s \mapsto -s$ interchanges the two cases. $\square$

THEOREM 8.3. *Let X be an Enriques surface. The following are equivalent:*

- (i) *The surface X is of type VI.*
- (ii) *The dual graph of all (-2) -curves on X contains the graph in Figure 6.*

FIGURE 6. Critical subgraph for type VI.

- (iii) *The canonical cover $\tilde{X}$ of X admits an elliptic fibration with a Weierstrass equation of the form*

$$y^2 - 3(3s^2 + 3s + 1)xy + (3s^2 + 3s + 1)^2y = x^3$$

such that the covering morphism $\rho: \tilde{X} \rightarrow X$ is given as quotient by the involution $\sigma = t_{N^-} \circ J(\sigma)$, where $J(\sigma): s \mapsto -s - 1$ and t_{N^-} is translation by $N^- = (s + s^2, s^3)$.

Moreover, Enriques surfaces of type VI do not exist in characteristic 3.

Proof. First, observe that the dual graph of type VI (see Table 1) contains the graph in Figure 6. This subgraph can be interpreted as the dual graph of a special elliptic fibration π with singular fibers IV^* , I_3 (not III since it is double) and special 2-section N . With the same notation as in the previous cases, we can compute $h(N^-) \neq 0$, and from Corollary 2.25 we obtain two more (-2) -curves, as follows:

There are three subgraphs of type $\tilde{A}_1$ such that the graph of (-2) -curves disjoint from this diagram together with a special bisection has the following form:

The only rational elliptic fibration with a singular fiber of type I_2 and some other singular fibers whose dual graphs contain an A_5 and an A_2 diagram is the one with singular fibers (I_6, I_3, I_2, I_1) (respectively, (I_6, IV, I_2) in characteristic 2). Using the other (-2) -curves in the graph, one deduces that the I_6 and I_3 (respectively, IV) fibers are simple. These fibrations give the seven remaining (-2) -curves for the dual graph of type VI. Observe that the existence of such a fibration excludes this case in characteristic 3 since the I_2 fiber is double.

We have found the following equation for the unique rational and extremal elliptic surface with singular fibers IV^* and I_3 in any characteristic:

$$y^2 + txy + t^2y = x^3,$$

where t is a coordinate on $\mathbb{P}^1$. By a change of coordinates (valid away from characteristic 3), we obtain

$$y^2 - 3(3t + 1)xy + (3t + 1)^2y = x^3.$$

The IV^* fiber is at $t = -\frac{1}{3}$, the I_3 fiber is at $t = \infty$, and there is an I_1 fiber at $t = -\frac{2}{3}$.

In characteristic different from 3, we can write a degree 2 morphism $\mathbb{P}^1 \rightarrow \mathbb{P}^1$ branched over $t = \infty$ and not branched over $t = -\frac{1}{3}$ as $t \mapsto s^2 + \beta s$, where s is the new parameter on $\mathbb{P}^1$ and $\beta \neq \pm 2/\sqrt{3}$. The covering involution $J(\sigma)$ is given by $s \mapsto -s - \beta$. We obtain the equation

$$y^2 - 3(3(s^2 + \beta s) + 1)xy + (3(s^2 + \beta s) + 1)^2y = x^3.$$

By Lemma 8.2, if a suitable section N^- exists, we can assume $\beta = 1$ and $N^- = (s + s^2, s^3)$. $\square$

8.2 Automorphisms

PROPOSITION 8.4. *Let X be an Enriques surface of type VI. Then, $\text{Aut}(X) \cong \mathfrak{S}_5$ and this group is generated by automorphisms induced by 2-torsion sections of the Jacobian fibrations of elliptic fibrations of X . Moreover, $\text{Aut}_{\text{nt}}(X) \cong \{1\}$.*

Proof. Recall that the dual graph of all (-2) -curves for type VI is as follows [Kon86]:

Let us first show that $\text{Aut}_{\text{nt}}(X)$ is trivial. Indeed, the three separable bisections E_7 , E_9 and E_{10} of $|E_1 + E_2 + E_3 + E_4 + E_5 + E_6|$ are fixed pointwise by any numerically trivial automorphism, which therefore has to be the identity.

The automorphism group of the graph is $\mathfrak{S}_5$ (see [Kon86, Proof of Theorem 3.6.1, p. 227]). We look at the induced action of $\text{Aut}(X)$ on the set $\Sigma = \{E_1, \dots, E_{10}\}$ and note the following points:

- The pointwise stabilizer of the set $\Gamma_1 = \{E_4, E_5, E_6, E_7\}$ is $\mathbb{Z}/2\mathbb{Z}$. It is realized by the 2-torsion section of the Jacobian of $|2(E_5 + E_{13})|$.
- The stabilizer of E_5 under the action of the automorphism group of the graph is $\mathfrak{S}_3 \times \mathbb{Z}/2\mathbb{Z}$. It is realized by the stabilizer of Γ_1 and the 2-torsion sections of the Jacobian fibrations with a fiber of type I_1^* . For example, the Jacobian of $|E_6 + E_7 + E_3 + E_{10} + 2(E_4 + E_5)|$ has a 2-torsion section which interchanges E_6 and E_7 .
- Since the stabilizer of E_5 has order 12, it suffices to show that the group generated by 2-torsion sections acts transitively on Σ . We show that we can map E_5 to E_{10} , E_3 and E_6 . The rest can be done similarly.
- Indeed, the 2-torsion sections of the Jacobians of $|2(E_3 + E_{17})|$, $|2(E_{10} + E_{16})|$ and $|2(E_8 + E_{11})|$ interchange E_5 and E_{10} , E_5 and E_3 , and E_3 and E_6 , respectively. $\square$

8.3 Degenerations and moduli

PROPOSITION 8.5. Assume $\text{char}(k) \neq 3$. Let

$$y^2 - 3(3(s^2 + s) + 1)xy + (3(s^2 + s) + 1)^2 y = x^3$$

be the Weierstrass equation of an elliptic fibration $\tilde{\pi}$ with section on a K3 surface $\tilde{X}$. Define the involution $\sigma = t_{N^-} \circ J(\sigma)$, where $J(\sigma): s \mapsto -s - 1$ and t_{N^-} is translation by the section $N^- = (s + s^2, s^3)$. Then, σ is fixed-point free if and only if $\text{char}(k) \neq 5$. If $\text{char}(k) = 5$, then σ has exactly one (-2) -curve as fixed locus.

Proof. The only possibility for σ to have fixed points is the case where $\varphi: t \mapsto s^2 + s$ is branched over the point lying under the nodal fiber. Hence, we may assume $\text{char}(k) \neq 2$. The branch points of φ are $t = \infty$ and $t = -\frac{1}{4}$, while the I_1 fiber of π lies over $t = -\frac{2}{3}$. Hence, φ is branched over the point lying under the nodal fiber if and only if $-\frac{2}{3} = -\frac{1}{4}$, that is, if and only if $5 = 0$.

Now, if $\text{char}(k) = 5$, the location of the I_2 fiber of $\tilde{\pi}$ is $s = -\frac{1}{2} = 2$. The singular point of the Weierstrass equation at $s = 2$ is $(-1, 1)$, while N^- passes through $(1, 3)$. Hence, N^- meets the identity component of the I_2 fiber, and therefore it is not a $J(\pi)$ -Enriques section, and σ fixes a (-2) -curve. $\square$

COROLLARY 8.6. *Enriques surfaces of type VI exist if and only if $\text{char}(k) \neq 3, 5$. Moreover, they are unique if they exist.*

As in the cases of type I and II, one obtains a Coble surface if σ has a fixed curve, that is, if $\text{char}(k) = 5$.

9. Enriques surfaces of type VII

9.1 Main theorem for type VII

LEMMA 9.1. *Let $\text{char}(k) \neq 2$ and $J(\sigma): s \mapsto -s$, and let*

$$y^2 = x^3 - (s_\beta^2 + s_\beta)x^2 + (2s_\beta^3 - 3s_\beta^2 + 4s_\beta - 2)x + (-s_\beta^3 + 2s_\beta^2 - 2s_\beta + 1),$$

where $s_\beta = s^2 + \beta$ with $\beta \in k - \{1\}$, be the Weierstrass equation of an elliptic fibration of a K3 surface. Then, an everywhere integral, $J(\sigma)$ -anti-invariant section N^- meeting the fibers at $s = \infty$ and $s = \pm\sqrt{1-\beta}$ in a non-identity component exists if and only if $\beta \in \{0, 2\}$. Moreover, it is unique up to sign if it exists. Both cases are isomorphic, and if $\beta = 0$, the section is $N^- = (1, s - s^3)$.

Proof. As in the previous case, one obtains conditions on β by direct calculation. Let us show the existence of the automorphism. The Weierstrass equation for the rational elliptic fibration

$$y^2 = x^3 - (t^2 + t)x^2 + (2t^3 - 3t^2 + 4t - 2)x + (-t^3 + 2t^2 - 2t + 1)$$

has an automorphism defined by $t \mapsto 2 - t$ and $x \mapsto x - 2 + 2t$. This automorphism induces the desired isomorphism. $\square$

THEOREM 9.2. *Let X be an Enriques surface. The following are equivalent:*

- (i) *The surface X is of type VII.*
- (ii) *The dual graph of all (-2) -curves on X contains the graph in Figure 7.*

FIGURE 7. Critical subgraph for type VII

- (iii) *The canonical cover $\tilde{X}$ of X admits an elliptic fibration with a Weierstrass equation of the form*

$$y^2 = x^3 - (s^4 + s^2)x^2 + (2s^6 - 3s^4 + 4s^2 - 2)x + (-s^6 + 2s^4 - 2s^2 + 1)$$

such that the covering morphism $\rho: \tilde{X} \rightarrow X$ is given as quotient by the involution $\sigma = t_{N^-} \circ J(\sigma)$, where $J(\sigma): s \mapsto -s$ and t_{N^-} is translation by $N^- = (1, s - s^3)$.

Moreover, singular Enriques surfaces of type VII do not exist in characteristic 2.

Proof. First, observe that the dual graph of type VII (see Table 1) contains the graph in Figure 7.

Conversely, we have shown in Example 2.3 that we recover type VII from the critical subgraph and, since an elliptic fibration with singular fibers I_8 and I_2 (not III since it is a double fiber) does not exist in characteristic 2, this type cannot exist in characteristic 2.

We have found the following equation for the unique rational and extremal elliptic surface with singular fibers (I_8, I_2, I_1, I_1) in characteristic different from 2:

$$y^2 = x^3 - (t^2 + t)x^2 + (2t^3 - 3t^2 + 4t - 2)x + (-t^3 + 2t^2 - 2t + 1),$$

where t is a coordinate on $\mathbb{P}^1$. The I_8 fiber is at $t = 1$, the I_2 fiber is at $t = \infty$, and there are two I_1 fibers at $t = 1 \pm \sqrt{-1}/2$.

In characteristic different from 2, we can write a degree 2 morphism $\mathbb{P}^1 \rightarrow \mathbb{P}^1$ branched over $t = \infty$ and not branched over $t = 0$ as $t \mapsto s^2 + \beta$, where s is the new parameter on $\mathbb{P}^1$ and $\beta \neq 0$. The covering involution $J(\sigma)$ is given by $s \mapsto -s$. Now, note that we are looking for a section N^- which meets the I_4 and I_8 fibers in non-identity components. By Lemma 9.1, if a suitable section N^- exists, we can assume $\beta = 0$ and $N^- = (1, s - s^3)$. Moreover, one can check that N^- has the correct intersection behaviour with the I_8 fibers. $\square$

9.2 Automorphisms

PROPOSITION 9.3. *Let X be an Enriques surface of type VII. Then, $\text{Aut}(X) \cong \mathfrak{S}_5$ and this group is generated by automorphisms induced by 2-torsion sections of the Jacobian fibrations of elliptic fibrations of X . Moreover, $\text{Aut}_{\text{nt}}(X) \cong \{1\}$.*

Proof. Recall that the dual graph of all (-2) -curves for type VII is as follows [Kon86]:

We claim that $\text{Aut}_{\text{nt}}(X)$ is trivial. Indeed, one can check that the bisections E_2, E_3, E_5, E_6, E_8 and E_9 of $|K_4 + K_5|$ are fixed pointwise by any numerically trivial automorphism, which therefore has to be trivial.

The automorphism group of the graph is $\mathfrak{S}_5$. Following [Kon86, Proof of Theorem 3.7.5, p. 233], we look at the induced action on the set $\Sigma = \{K_1, \dots, K_5\}$ and observe that the

pointwise stabilizer of Σ is trivial. Now, each K_i , for $i \in \{1, \dots, 5\}$, meets exactly three E_j , where $j \in \{1, \dots, 15\}$, twice. The 2-torsion sections of the Jacobians of the elliptic fibrations $|2(K_i + E_j)|$ act as permutations of cycle type $(2, 2)$ on $\Sigma - K_i$. Note that the 2-torsion section of the Jacobian of $|K_4 + K_5|$ interchanges K_4 and K_5 while fixing K_1 , K_2 and K_3 . Together, these involutions generate the full automorphism group. $\square$

9.3 Degenerations and moduli

PROPOSITION 9.4. *Assume $\text{char}(k) \neq 2$. Let*

$$y^2 = x^3 - (s^4 + s^2)x^2 + (2s^6 - 3s^4 + 4s^2 - 2)x + (-s^6 + 2s^4 - 2s^2 + 1)$$

be the Weierstrass equation of an elliptic fibration $\tilde{\pi}$ with section on a K3 surface $\tilde{X}$. Define the involution $\sigma = t_{N^-} \circ J(\sigma)$, where $J(\sigma): s \mapsto -s$ and t_{N^-} is translation by the section $N^- = (1, s - s^3)$. Then, σ is fixed-point free if and only if $\text{char}(k) \neq 5$. If $\text{char}(k) = 5$, then σ has exactly one (-2) -curve as fixed locus.

Proof. The branch points of φ are $t = \infty$ and $t = 0$, while the I_1 fibers of π lie over $t = 1 \pm \sqrt{-1}/2$. Hence, φ is branched over a point lying under a nodal fiber if and only if $1 \pm \sqrt{-1}/2 = 0$, that is, if and only if $5 = 0$.

Now, if $\text{char}(k) = 5$, the location of the I_2 fiber of $\tilde{\pi}$ is $s = 0$. The singular point of the Weierstrass equation at $s = 0$ is $(2, 0)$, while N^- passes through $(1, 0)$. Hence, N^- meets the identity component of the I_2 fiber, and therefore it is not a $J(\pi)$ -Enriques section, and σ fixes a (-2) -curve. $\square$

COROLLARY 9.5. *Enriques surfaces of type VII with smooth K3 cover exist if and only if $\text{char}(k) \neq 2, 5$. Moreover, they are unique if they exist.*

REMARK 9.6. Here, it is important to recall our convention on Enriques surfaces in characteristic 2. In fact, by [KK18], there is a 1-dimensional family of classical and supersingular Enriques surfaces of type VII in characteristic 2. Note also that the involution σ produces a Coble surface in characteristic 5.

10. The classification theorem

Now that we have completed the construction of the seven types of Enriques surfaces with finite automorphism group, it remains to show that these seven types are indeed all possible Enriques surfaces with finite automorphism group. Hence, the goal of this section is to prove the following classification theorem, finishing the proof of the Main Theorem. Recall that all our Enriques surfaces are assumed to have a smooth canonical cover.

THEOREM 10.1. *Let X be an Enriques surface. The following are equivalent:*

- (i) *The surface X has finite automorphism group.*
- (ii) *Every elliptic fibration of X is extremal.*
- (iii) *Every special elliptic fibration of X is extremal, and X contains a (-2) -curve.*
- (iv) *The dual graph of all (-2) -curves on X contains a critical subgraph for one of the types I, $\dots$, VII.*
- (v) *The dual graph of all (-2) -curves on X is one of the seven types I, $\dots$, VII.*
- (vi) *The surface X contains only finitely many, but at least one, (-2) -curve.*

Before giving the proof of Theorem 10.1, we need to introduce the tools for the classification of dual graphs.

10.1 Preparations for the proof of the classification theorem

Corollary 2.25 and the height pairing of sections of elliptic fibrations of the K3 cover will play an important role. More precisely, we have the following lemma.

LEMMA 10.2. *Let $\pi: X \rightarrow \mathbb{P}^1$ be a special and extremal elliptic fibration of an Enriques surface X with special bisection N . Let $\tilde{\pi}$ be the corresponding elliptic fibration of the K3 cover $\tilde{X}$ of X . Denote the irreducible curves on $\tilde{X}$ mapping surjectively onto N by N^+ and N^- . Let $J(\pi)$ be the Jacobian of π . We choose N^+ as the zero section of $\tilde{\pi}$.*

Then,

- either $h(N^-) = 0$ and N^- is a 2-torsion section in $\text{MW}(J(\pi)) \subseteq \text{MW}(\tilde{\pi})$,
- or N^- satisfies

$$\sum_{\nu} \text{contr}_{\nu}(N^-) < 4 \quad \text{and} \quad \sum_{\nu} \text{contr}_{\nu}(N^-, P) \in \{0, 1, 2\}$$

for all $P \in \text{MW}(J(\pi)) \subseteq \text{MW}(\tilde{\pi})$ with $P \neq N^-$.

Proof. Since

$$0 \leq h(N^-) = 4 + 2N^- \cdot N^+ - \sum_{\nu} \text{contr}_{\nu}(N^-) = 4 - \sum_{\nu} \text{contr}_{\nu}(N^-)$$

and N^- restricts to a 2-torsion section on a fiber F of $\tilde{\pi}$ lying over a double fiber of π , either we have $h(N^-) = 0$ and we claim that N^- is 2-torsion, or we have $h(N^-) > 0$ and therefore $\sum_{\nu} \text{contr}_{\nu}(N^-) < 4$.

Indeed, suppose $h(N^-) = 0$ and that N^- is not 2-torsion. Then, $N^- \oplus N^-$ meets the zero section in F ; hence, its order is divisible by $\text{char}(k) = 2$ by [IL13, Proposition 2.4]. But if $\text{char}(k) = 2$, the fiber F is either multiplicative or ordinary by Proposition 2.5, contradicting [IL13, Proposition 2.1].

Since every $P \in \text{MW}(J(\pi)) \subseteq \text{MW}(\tilde{\pi})$ is disjoint from N^+ , we have

$$\begin{aligned} 0 &= \langle P, N^- \rangle = 2 + P \cdot N^+ + N^- \cdot N^+ - P \cdot N^- - \sum_{\nu} \text{contr}_{\nu}(N^-, P) \\ &= 2 - P \cdot N^- - \sum_{\nu} \text{contr}_{\nu}(N^-, P), \end{aligned}$$

which yields the second claim. □

Remark 10.3. By Table 4, the local contributions to the height pairing can be read off almost completely from the dual graph of the singular fibers. However, a remark about the cases where π has a double fiber of type I_1 is in order. Since sections $P \in \text{MW}(J(\pi))$ meet the corresponding I_2 fiber of $\tilde{\pi}$ in the identity component, N^- cannot be 2-torsion. Moreover, $\sum_{\nu} \text{contr}_{\nu}(N^-)$ will decrease by $\frac{1}{2}$, while $\sum_{\nu} \text{contr}_{\nu}(N^-, P)$ stays the same; hence, N^- can only satisfy the conditions of the lemma if it does so when we ignore the double I_1 fiber. We will do this from now on.

DEFINITION 10.4. Let Γ_1 be the dual graph of the singular fibers of a rational and extremal elliptic fibration. A graph $\Gamma \supseteq \Gamma_1$ is called a *fiber-bisection configuration* for Γ_1 if the following two conditions hold:

- (i) The graph $\Gamma - \Gamma_1$ consists of one vertex N , called the special bisection.
- (ii) The special bisection N meets every connected component of Γ_1 of type $\tilde{D}$ and $\tilde{E}$ exactly twice and every component of type $\tilde{A}$ at least once and at most twice. Moreover, N meets at most two connected components of Γ_1 exactly once.

Given a fiber-bisection configuration Γ , we can check whether it could be the dual graph of a special elliptic fibration π on an Enriques surface as follows: Suppose that it is the dual graph of π . Then, we can pass to the canonical cover, add the sections coming from the Jacobian $J(\pi)$ of π and check the conditions of Lemma 10.2. By Remark 10.3, it makes sense to say that a fiber-bisection configuration satisfies the conditions of Lemma 10.2.

DEFINITION 10.5. A fiber-bisection configuration is called *admissible* if it satisfies the conditions of Lemma 10.2

10.2 Outline of proof

In this section, we outline the proof of the following lemma, which is the main ingredient in the proof of Theorem 10.1.

LEMMA 10.6. *Let X be an Enriques surface such that every special elliptic fibration of X is extremal and X contains a (-2) -curve. Then, the dual graph of all (-2) -curves on X contains a critical subgraph (see Figures 1, ..., 7) for one of the types I, ..., VII.*

Proof of Theorem 10.1 (assuming Lemma 10.6). As observed by Dolgachev [Dol84, § 4], if X has finite automorphism group, then every elliptic fibration π on X is extremal since the Mordell–Weil group of $J(\pi)$ acts faithfully on X . In particular, since X admits an elliptic fibration by Proposition 2.4, the surface X contains a (-2) -curve by Lemma 2.15, and every special elliptic fibration of X is extremal. From Lemma 10.6, we deduce that X contains a critical subgraph, which, by the earlier sections, implies that the dual graph of all (-2) -curves on X is one of the types I, ..., VII.

The seven dual graphs in Table 1 consist of 12 (respectively, 20) vertices; hence, X contains finitely many, and at least one, (-2) -curve. Moreover, we have computed the automorphism groups of these surfaces. They are finite. Finally, by Corollary 2.25, the only special elliptic fibrations of Enriques surfaces with finitely many, but at least one, (-2) -curve are the extremal ones. $\square$

Since we have constructed all seven types in the previous sections, Theorem 10.1 will finish the classification. The strategy for the proof of Lemma 10.6 can be summarized as follows:

- (1) Let X be an Enriques surface with a (-2) -curve such that every special elliptic fibration of X is extremal. By Proposition 2.8, the surface X admits such a special elliptic fibration π .
- (2) Pick a dual graph Γ_1 of singular fibers of a rational and extremal elliptic fibration and some admissible fiber-bisection configuration $\Gamma \supseteq \Gamma_1$. Suppose that Γ is the dual graph of the fibers and a special bisection of π .
- (3) Apply Corollary 2.25 to find additional (-2) -curves, and obtain a bigger graph Γ_2 .
- (4) If Γ_2 contains one of the critical subgraphs, we have shown in the previous sections that X is of one of the seven types.

- (5) If not, find a different subgraph Γ_3 of Γ_2 of type $\tilde{A}_n$ together with a vertex N meeting Γ_3 exactly once. By Proposition 2.12, the subgraph Γ_3 is the dual graph of a singular fiber of a special elliptic fibration π_1 , and N is a special bisection of π_1 . By the assumption on X , the fibration π_1 is extremal; that is, we can extend Γ_3 to a dual graph Γ_4 of singular fibers of an extremal elliptic fibration such that $\Gamma_4 \cup N$ is an admissible fiber-bisection configuration for Γ_4 . Now, go back to Step (3).

We will show that the above process will terminate at some point for every choice of Γ_1 , either with a contradiction or with Step (4).

10.3 Proof of the classification theorem

The following lemma shows that the number of admissible fiber-bisection configurations we have to check is “not too big”.

LEMMA 10.7. *Let X be an Enriques surface with a special and extremal elliptic fibration π . Then, X admits a special elliptic fibration with a double fiber of type I_n with $n \geq 2$. Moreover, if π has double fibers of types I_{n_1} and I_{n_2} , then $n_1 + n_2 \leq 8$.*

Proof. For the first claim, let π be a special and extremal elliptic fibration of X , and let N be a special bisection of π . If π has a fiber of type IV^* , III^* , II^* , I_n^* or I_n with $n \geq 5$, then N and fiber components form a fiber of type I_n and a component of the fiber takes the role of a special bisection. The remaining possibilities for π are the one with fibers (I_4, I_4, I_2, I_2) and the one with fibers (I_3, I_3, I_3, I_3) . These are checked similarly, using more than one fiber.

For the second claim, let π be a special elliptic fibration of X with double fibers of types I_{n_1} and I_{n_2} . Denote a special bisection by N and the corresponding curves on the K3 cover by N^+ and N^- , as usual. Then, we compute $\sum_{\nu} \text{contr}_{\nu}(N^-) \geq (n_1 + n_2)/2$ using Table 4. Since $\sum_{\nu} \text{contr}_{\nu}(N^-) \leq 4$, this gives the second claim. $\square$

It is straightforward to give a complete list of admissible fiber-bisection configurations for dual graphs of singular fibers of extremal elliptic fibrations. We leave the details to the reader. Note that it follows from the classification of extremal and rational elliptic surfaces (see Table 3) that we do not have to take special care of small characteristics.

LEMMA 10.8. *Let Adm_p be the set of admissible fiber-bisection configurations for dual graphs of extremal elliptic fibrations over an algebraically closed field of characteristic p . Then $\text{Adm}_p \subseteq \text{Adm}_0$.*

LEMMA 10.9. *Table 5 shows the list of all admissible fiber-bisection configurations for dual graphs of singular fibers of extremal elliptic fibrations, where the special bisection meets at least one $\tilde{A}$ subgraph (marked with a 2 in front) only once.*

Remark 10.10. In fact, many of these admissible fiber-bisection configurations are realizable over the complex numbers; see the author’s master’s thesis (München, 2016).

From these tables, we can deduce the following improvement of Lemma 10.7.

COROLLARY 10.11. *If an Enriques surface X admits a special and extremal elliptic fibration, then either X is of type II, or it admits a special elliptic fibration with a double fiber of type I_2 .*

Dual graph of fibers	Admissible fiber-bisection configurations
$\tilde{E}_7 \oplus 2\tilde{A}_1$	
$\tilde{E}_6 \oplus 2\tilde{A}_2$	
$\tilde{D}_5 \oplus 2\tilde{A}_3$	
$\tilde{D}_6 \oplus 2\tilde{A}_1 \oplus 2\tilde{A}_1$	
$\tilde{D}_6 \oplus 2\tilde{A}_1 \oplus \tilde{A}_1$	
$2\tilde{A}_7 \oplus \tilde{A}_1$	
$\tilde{A}_7 \oplus 2\tilde{A}_1$	
$2\tilde{A}_4 \oplus \tilde{A}_4$	
$2\tilde{A}_5 \oplus \tilde{A}_2 \oplus 2\tilde{A}_1$	
$\tilde{A}_5 \oplus 2\tilde{A}_2 \oplus 2\tilde{A}_1$	
$2\tilde{A}_5 \oplus \tilde{A}_2 \oplus \tilde{A}_1$	
$\tilde{A}_5 \oplus 2\tilde{A}_2 \oplus \tilde{A}_1$	

TABLE 5. Admissible fiber-bisection configurations for extremal fibrations

Proof. By Lemma 10.7, we know that X admits an elliptic fibration with a double fiber of type I_n for some n . Almost every graph in Lemma 10.9 admits an $\tilde{A}_1$ subgraph and a vertex meeting this subgraph exactly once; the only exception is the critical subgraph for type II. Hence, the claim follows. $\square$

$\tilde{A}_5 \oplus \tilde{A}_2 \oplus 2\tilde{A}_1$	
$2\tilde{A}_3 \oplus 2\tilde{A}_3 \oplus \tilde{A}_1 \oplus \tilde{A}_1$	
$\tilde{A}_3 \oplus \tilde{A}_3 \oplus 2\tilde{A}_1 \oplus 2\tilde{A}_1$	
$2\tilde{A}_3 \oplus \tilde{A}_3 \oplus 2\tilde{A}_1 \oplus \tilde{A}_1$	
$2\tilde{A}_3 \oplus \tilde{A}_3 \oplus \tilde{A}_1 \oplus \tilde{A}_1$	
$\tilde{A}_3 \oplus \tilde{A}_3 \oplus 2\tilde{A}_1 \oplus \tilde{A}_1$	
$2\tilde{A}_2 \oplus 2\tilde{A}_2 \oplus \tilde{A}_2 \oplus \tilde{A}_2$	
$2\tilde{A}_2 \oplus \tilde{A}_2 \oplus \tilde{A}_2 \oplus \tilde{A}_2$	

TABLE 5. (Continued)

Before we start with the proof of Lemma 10.6, we need the following auxiliary result.

LEMMA 10.12. *There is no Enriques surface with a special elliptic fibration with singular fibers*

- (I_3, I_3, I_3, I_3) such that two of the I_3 fibers are multiple or
- (I_6, I_3, I_2, I_1) such that the I_3 and I_2 fibers are multiple.

Proof. We will only show the first claim; the proof of the second one is similar. The claim is true if $\text{char}(k) \in \{2, 3\}$ since there is no rational elliptic surface with singular fibers (I_3, I_3, I_3, I_3) in characteristic 3 and an elliptic fibration of an Enriques surface in characteristic 2 cannot have two multiplicative double fibers.

Let us assume $\text{char}(k) \notin \{2, 3\}$. The rational elliptic surface $J(\pi)$ with singular fibers (I_3, I_3, I_3, I_3) has the Weierstrass equation

$$y^2 = x^3 + (-3t^4 + 24t)x + 2t^6 + 40t^3 - 16. \quad (10.1)$$

If an Enriques surface with this Jacobian and two double I_3 fibers exists, it is covered by the base change of (10.1) via $t \mapsto s^2 - 1$.

A $J(\pi)$ -Enriques section $N^- = (x(s), y(s))$ meets the fibers of $J(\pi)$ at $s = 0$ and at $s = \infty$ in a non-identity component and is $J(\sigma)$ -anti-invariant, where $J(\sigma): s \mapsto -s$. Since the singular point of the fiber at $s = 0$ (respectively, $s = \infty$) is $(-3, 0)$ (respectively, $(1, 0)$), the section N^- has the form

$$x = -3 + x_2 s^2 + s^4, \quad y = y_1 s + y_3 s^3 + y_5 s^5.$$

Plugging this into the base change of equation (10.1), we additionally obtain $y_1 = y_5 = 0$, $y_3 = \pm 8$, $x_2 = -2$ and finally $144 = 0$, which is not allowed, since $\text{char}(k) \neq 2, 3$. $\square$

Proof of Lemma 10.6. (For a detailed explanation of how to add (-2) -curves using jac_2 , see Section 2.3.) By Corollary 10.11, it suffices to check the admissible fiber-bisection configurations with a $2\tilde{A}_1$ component. We will treat them in the following order:

Γ_1	Number of admissible fiber-bisection configurations
$\tilde{D}_6 \oplus 2\tilde{A}_1 \oplus 2\tilde{A}_1$	2
$\tilde{D}_6 \oplus 2\tilde{A}_1 \oplus \tilde{A}_1$	4
$\tilde{E}_7 \oplus 2\tilde{A}_1$	2
$\tilde{A}_3 \oplus \tilde{A}_3 \oplus 2\tilde{A}_1 \oplus 2\tilde{A}_1$	2
$2\tilde{A}_3 \oplus \tilde{A}_3 \oplus 2\tilde{A}_1 \oplus \tilde{A}_1$	1
$\tilde{A}_3 \oplus \tilde{A}_3 \oplus 2\tilde{A}_1 \oplus \tilde{A}_1$	4
$2\tilde{A}_5 \oplus \tilde{A}_2 \oplus 2\tilde{A}_1$	1
$\tilde{A}_5 \oplus \tilde{A}_2 \oplus 2\tilde{A}_1$	4
$\tilde{A}_7 \oplus 2\tilde{A}_1$	2

- $\Gamma_1 = \tilde{D}_6 \oplus 2\tilde{A}_1 \oplus 2\tilde{A}_1$
 - a) Fiber-bisection configuration:

After adding a bisection with jac_2 , we find another special fibration with two double I_2 fibers and bisection N as follows, where the dotted rectangles mark the fibers:

There is a D_4 diagram which is disjoint from the two $\tilde{A}_1$ subgraphs. By Table 3, the only extremal fibration with two singular fibers of type I_2 and one singular fiber whose dual graph contains a D_4 is the one with singular fibers (I_2^*, I_2, I_2) . However, the bisection N cannot meet the I_2^* fiber in an admissible way; hence, this fiber-bisection configuration does not occur.

b) Fiber-bisection configuration:

After adding a bisection with jac_2 , we find another special fibration with two double I_4 fibers and bisection N as follows:

By Table 3, the only extremal fibration with two singular fibers of type I_4 is the one with singular fibers (I_4, I_4, I_2, I_2) and the only admissible fiber-bisection configuration with $\Gamma_1 = 2\tilde{A}_3 \oplus 2\tilde{A}_3 \oplus \tilde{A}_1 \oplus \tilde{A}_1$ is the critical subgraph for type III.

$$- \Gamma_1 = \tilde{D}_6 \oplus 2\tilde{A}_1 \oplus \tilde{A}_1$$

a) Fiber-bisection configuration:

After adding a bisection corresponding to a 2-torsion section via jac_2 , we obtain another special fibration with double singular fibers I_6 and I_2 and bisection N as follows:

The only admissible fiber-bisection configuration for such a fibration is the critical subgraph for type V.

b) Fiber-bisection configuration:

Adding another bisection corresponding to a 2-torsion section via jac_2 , we obtain another special fibration with two singular double fibers of type I_4 , giving the critical subgraph for type III:

c) Fiber-bisection configuration:

Adding another bisection corresponding to a 2-torsion section via jac_2 , we obtain another special fibration with two singular double fibers of type I_2 , bisection N and some fiber whose dual graph contains a D_4 . The only extremal fibration satisfying this is the one with fibers (I_2^*, I_2, I_2) , and we have already treated the cases where both I_2 fibers are double.

d) Fiber-bisection configuration:

There is another special elliptic fibration with double fiber of type I_2 as in the following figure:

There are a D_4 diagram and three disjoint vertices, which are disjoint from the marked subgraph. The only extremal fibration whose dual graph of the singular fibers contains these diagrams is the one with singular fibers (I_2^*, I_2, I_2) . But the bisection N meets the fibers in such a way that the fiber-bisection configuration will be one of the configurations we have already treated.

– $\Gamma_1 = \tilde{E}_7 \oplus 2\tilde{A}_1$

a) Fiber-bisection configuration:

This is the critical subgraph for type I.

b) Fiber-bisection configuration:

There is another special elliptic fibration with a double fiber of type I_2 and a bisection N , as follows:

There are a D_6 diagram and an isolated vertex which are disjoint from the marked subgraph. Moreover, from the intersection behaviour of N , we can exclude the case that the new fibration has a singular fiber of type III^* . The only extremal fibration satisfying these conditions is the one with singular fibers (I_2^*, I_2, I_2) . We have already treated all fiber-bisection configurations for this fibration.

– $\Gamma_1 = \tilde{A}_3 \oplus \tilde{A}_3 \oplus 2\tilde{A}_1 \oplus 2\tilde{A}_1$

a) Fiber-bisection configuration:

This is the critical subgraph for type IV.

b) Fiber-bisection configuration:

After adding bisections coming from 2-torsion sections via jac_2 , we obtain another (maybe non-special) fibration with two double I_2 fibers, as follows:

There are six disjoint vertices which are disjoint from the two I_2 fibers. There is no extremal elliptic fibration whose dual graph of the singular fibers contains two $\tilde{A}_1$ diagrams and six disjoint vertices.

$$- \Gamma_1 = 2\tilde{A}_3 \oplus \tilde{A}_3 \oplus 2\tilde{A}_1 \oplus \tilde{A}_1$$

Fiber-bisection configuration:

Adding a bisection corresponding to a 2-torsion section via jac_2 , we find another special fibration with two double I_2 fibers and special bisection N .

Since we have already treated all cases with two double I_2 fibers, we are done with this case.

$$- \Gamma_1 = \tilde{A}_3 \oplus \tilde{A}_3 \oplus 2\tilde{A}_1 \oplus \tilde{A}_1$$

a) Fiber-bisection configuration:

After adding a bisection corresponding to a different 2-torsion section via jac_2 , we obtain another special elliptic fibration with two double fibers of type I_2 and a special bisection N , as follows:

Since we have treated all fibrations with two double I_2 fibers, we are done.

- b) The other fiber-bisection configuration where the bisection meets both components of the simple I_2 fiber is treated similarly to Case a).
- c) Fiber-bisection configuration:

We add another bisection arising via jac_2 and find a special elliptic fibration with double fibers of type I_3 and I_2 and bisection N as follows:

The only extremal fibration with these fibers is the one with singular fibers (I_6, I_3, I_2, I_1) . But the I_3 and I_2 fibers cannot both be double, by Lemma 10.12. Therefore, this fiber-bisection configuration does not occur.

d) Fiber-bisection configuration:

There is another special elliptic fibration with a double singular fiber of type I_2 and special bisection N as in the following figure:

There are an A_3 diagram and three disjoint vertices which are disjoint from the I_2 fiber. The extremal fibrations whose dual graphs of singular fibers satisfy these conditions are the ones with singular fibers (I_2^*, I_2, I_2) and (I_4, I_4, I_2, I_2) . Since we have already treated the first fibration, we can assume that the second one occurs. But the bisection N and the fibers form a fiber-bisection configuration which we have already treated; hence, this case is settled.

$$- \Gamma_1 = 2\tilde{A}_5 \oplus \tilde{A}_2 \oplus 2\tilde{A}_1$$

Fiber-bisection configuration:

This is the critical subgraph for type V.

$$- \Gamma_1 = \tilde{A}_5 \oplus \tilde{A}_2 \oplus 2\tilde{A}_1$$

a) Fiber-bisection configuration:

There is another special elliptic fibration with a double fiber of type I_5 and bisection N . We leave it to the reader to check that one obtains the critical subgraph for type VI from a fibration with singular fibers I_5 and I_5 , where one of the I_5 fibers is double.

b) Fiber-bisection configuration:

Adding another special bisection corresponding to the 2-torsion section via jac_2 , we obtain another special elliptic fibration with two double singular fibers of type I_3 and bisection N as follows:

The only extremal and rational elliptic fibration with two fibers of type I_3 is the fibration with fibers (I_3, I_3, I_3, I_3) . By Lemma 10.12, there is no such fibration with two double I_3 fibers.

c) Fiber-bisection configuration:

Adding another special bisection corresponding to a 6-torsion section via jac_2 , we obtain another special fibration with a double fiber of type I_2 and a special bisection N as follows:

There are diagrams of types A_3 , A_2 and A_1 which are disjoint from the double I_2 fiber. Therefore, the fibration cannot have an I_8 fiber. Since we have treated all the other cases with a double I_2 fiber, we can assume that the fibration has singular fibers of types I_6 , I_3 (or IV) and I_2 such that the I_6 fiber is simple. But then, the fibers together with the bisection N form the admissible fiber-bisection configuration of Case a) or b) since N meets distinct components of the I_6 fiber. Therefore, this case is settled.

d) Fiber-bisection configuration:

Here, we can use the same (-2) -curves as in the previous case and the same argument right away without adding additional bisections.

$$- \Gamma_1 = \tilde{A}_7 \oplus 2\tilde{A}_1$$

a) Fiber-bisection configuration:

This is the critical subgraph for type VII.

b) Fiber-bisection configuration:

There is another special elliptic fibration with a double fiber of type I_2 and special bisection N as follows:

This fiber-bisection configuration is not the same as the one we started with, and since this is the last case, we have already treated this. $\square$

11. Arithmetic of Enriques surfaces with finite automorphism group

In this section, we explain how to derive the results on the arithmetic of Enriques surfaces with finite automorphism group which we mentioned in the introduction from the equations we gave in Sections 3–9. In particular, we establish explicit models of Enriques surfaces of every type over the prime fields $\mathbb{F}_p$ and $\mathbb{Q}$.

LEMMA 11.1. *The integral Weierstrass models of the elliptic K3 surfaces in Table 6 admit a resolution of singularities over the ring R , where R is as in the table.*

Equation	R	Type
$y^2 + (s^2 + s)xy = x^3 + (s^2 + s)^3 x$	$\mathbb{Z}[1/257]$	I
$y^2 - (s^2 + s)xy = x^3 - (s^2 + s)^3 x$	$\mathbb{Z}[1/255]$	I
$y^2 + (s^2 + s)xy + (s^2 + s)^2 y = x^3 + (s^2 + s)x^2$	$\mathbb{Z}[1/65]$	II
$y^2 - (s^2 + s)xy + (s^2 + s)^2 y = x^3 - (s^2 + s)x^2$	$\mathbb{Z}[1/63]$	II
$y^2 + xy = x^3 + 4s^4 x^2 + s^4 x$	$\mathbb{Z}[1/2]$	III
$y^2 = x^3 + 2(s^4 + 1)x^2 + (s^4 - 1)^2 x$	$\mathbb{Z}[1/2]$	IV
$y^2 + (s^2 + 1)xy + (s^2 + 1)y = x^3 + (s^2 + 2)x^2 + (s^2 + 1)x$	$\mathbb{Z}[1/6]$	V
$y^2 - 3(3s^2 + 3s + 1)xy + (3s^2 + 3s + 1)^2 y = x^3$	$\mathbb{Z}[1/15]$	VI
$y^2 = x^3 - (s^4 + s^2)x^2 + (2s^6 - 3s^4 + 4s^2 - 2)x + (-s^6 + 2s^4 - 2s^2 + 1)$	$\mathbb{Z}[1/10]$	VII

TABLE 6. Weierstrass equations for the K3 covers of integral models of Enriques surfaces with finite automorphism group

Proof. Let $f: \mathcal{X} \rightarrow \text{Spec}(\mathbb{R})$ be one of the families defined by the above equations. Since the non-smooth locus of f is closed and f is proper, the non-smooth locus of f is proper. Hence, every singular point of the generic fiber X_η of f is the generic point of a subscheme Z of $\mathcal{X}$ which is completely contained in the singular locus of f and flat over $\text{Spec}(\mathbb{R})$.

Since Z is flat over $\text{Spec}(\mathbb{R})$, a local computation shows that blowing up along Z commutes with taking fibers of f . Moreover, we know that every fiber of f has the same types of rational double points; hence, we can repeat the above argument and deduce that the minimal resolution of singularities of the generic fiber extends uniquely to a minimal resolution of the whole family. $\square$

Remark 11.2. The reason why we have to exclude some seemingly arbitrary characteristics is that the surface defined by the Weierstrass equation acquires additional singularities in these characteristics, because the degree 2 morphism to a rational elliptic surface we used to find the equations branches over a multiplicative fiber. This happens for the first four equations and for the last two, where the double cover branches over a nodal fiber, producing an additional A_1 singularity in some fibers. This singularity cannot be resolved in families without a base change to an algebraic space [Art74].

THEOREM 11.3. *Let $K \in \{\text{I}, \dots, \text{VII}\}$. There is a morphism $\varphi_K: \mathcal{X} \rightarrow \text{Spec}(\mathbb{Z}[1/P_K])$ whose fibers are Enriques surfaces of type K with full Picard rank, that is, $\text{Pic}(\mathcal{X}_{\mathbb{F}_p}) = \text{Pic}(\mathcal{X}_{\mathbb{F}_p})$. The numbers P_K are given in Table 2.*

Proof. By Lemma 11.1, we have a family of K3 surfaces over $\mathbb{Z}[1/P_K]$. Now, observe that the Enriques involution is also defined over this ring. Hence, the only remaining claim is the one that the fibers of the family have full Picard rank.

Let X_p be the fiber over p of one of the families of Enriques surfaces over R , and let $\tilde{X}_p$ be its canonical cover. By Vinberg's criterion (Proposition 2.29), the geometric Picard group of X_p is generated by (-2) -curves; hence, it suffices to check that all these curves are defined over $\mathbb{F}_p$ (respectively, over $\mathbb{Q}$ if $p = 0$). Then, one uses our explicit equations to check that the Galois action preserves the preimages of these curves in $\tilde{X}_p$, and therefore all (-2) -curves on X_p are defined over $\mathbb{F}_p$ (respectively, over $\mathbb{Q}$ if $p = 0$). Note that it suffices to check that the fiber

components and special bisections of the fibration we used to construct the surfaces are fixed since this will imply that the Galois action is trivial on the whole graph. $\square$

Remark 11.4. In particular, note that there are Enriques surfaces of types VI and VII with full Picard rank over $\mathbb{Q}$, while this is not possible for their canonical cover due to a result of N. D. Elkies [Sch10].

Moreover, Theorem 11.3 proves the existence of a model for every type of Enriques surfaces with finite automorphism group together with its dual graph of all (-2) -curves over the prime fields.

COROLLARY 11.5. *Suppose that there exists an Enriques surface of type $K \in \{I, \dots, VII\}$ in characteristic p . Then, there exists an Enriques surface of type K with Picard rank 10 over $\mathbb{F}_p$ if $p \neq 0$ and over $\mathbb{Q}$ if $p = 0$.*

THEOREM 11.6. *Let X be an Enriques surface of type $K \in \{I, \dots, VII\}$ over a field k such that $\text{Pic}(X) = \text{Pic}(X_{\bar{k}})$.*

- *If $K \neq III, IV$, then $\text{Aut}(X)$ is defined over k .*
- *If $K = III$, then $\text{Aut}(X)$ is defined over $L \supseteq k$ with $[L : k] \leq 2$.*
- *If $K = IV$, then $\text{Aut}(X)$ is defined over $L \supseteq k$ with $[L : k] \leq 16$.*

Proof. Let X be an Enriques surface over k such that $|\text{Aut}(X_{\bar{k}})| < \infty$ and $\text{rk}(\text{Pic}(X)) = 10$. Since $\text{rk}(\text{Pic}(X)) = 10$, every elliptic fibration of X is defined over k . Therefore, all Jacobian fibrations of elliptic fibrations of X are defined over k . Now, if X is of type I, II, V, VI or VII, the generic fiber of an elliptic fibration of X whose Jacobian has non-trivial sections has j -invariant different from 0, 1728. Therefore, the Jacobian is unique up to quadratic twisting with elements in $\bar{k}$. We have shown in Propositions 3.3, 4.2, 7.2, 8.4 and 9.3 that $\text{Aut}(X_{\bar{k}})$ is generated by the actions of 2-torsion sections of the Jacobian fibrations of elliptic fibrations of X . Since quadratic twisting preserves 2-torsion sections and all extremal and rational elliptic fibrations have a model over k such that their 2-torsion is already defined over k , all such sections, and hence $\text{Aut}(X)$, are defined over k .

If X is of type III, we need to realize the additional automorphism of Remark 5.2. For this, a quadratic extension is sufficient.

If X is of type IV, we need the automorphism of Remark 6.2 and one non-2-torsion section (see Proposition 6.3). As before, we need a field extension of degree at most 2 per non-2-torsion section. To define the automorphism of Remark 6.2, we need a field extension of degree at most 8 since we found a model of the corresponding fibration which acquires the required section after a quadratic extension and we need a quadratic extension to define ι (see Remark 6.2). $\square$

Remark 11.7. Over finite fields (and for our model), the proof shows that an extension of degree 4 suffices to realize all automorphisms for type IV.

12. Semi-symplectic automorphisms

As an application of our explicit classification of Enriques surfaces with finite automorphism group, we determine the semi-symplectic automorphism groups of these surfaces.

DEFINITION 12.1. Let X be an Enriques surface. An automorphism of X is called *semi-symplectic* if it acts trivially on $H^0(X, \omega_X^{\otimes 2})$. We denote the group of all semi-symplectic automorphisms of X by $\text{Aut}_{\text{ss}}(X)$.

These automorphisms are studied in [MO14]. There, the semi-symplectic automorphism groups of Enriques surfaces of types VI and VII have already been computed. See [Oha15] for a study of finite and non-semi-symplectic automorphisms.

THEOREM 12.2. *Let X be an Enriques surface of type $K \in \{I, \dots, VII\}$. Then, $\text{Aut}_{\text{ss}}(X)$ is as given in Table 7.*

Type	$\text{Aut}_{\text{ss}}(X)$
I	D_4
II	$\mathfrak{S}_4$
III	$(\mathbb{Z}/2\mathbb{Z})^3 \rtimes D_4$
IV	$(\mathbb{Z}/2\mathbb{Z})^4 \rtimes (\mathbb{Z}/5\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z})$
V	$\mathfrak{S}_4 \times \mathbb{Z}/2\mathbb{Z}$
VI	$\mathfrak{S}_5$
VII	$\mathfrak{S}_5$

TABLE 7. Semi-symplectic automorphism groups

Proof. Note that an automorphism induced by a section of the Jacobian of an elliptic fibration of X is semi-symplectic since it fixes the base of the fibration and acts as translation on the fibers. For all K , the group generated by such automorphisms is equal to the group given in Table 7. If $K \neq \text{III}, \text{IV}$, these are all automorphisms, and if $K \in \{\text{III}, \text{IV}\}$, we have exhibited non-semi-symplectic automorphisms in Remarks 5.2 and 6.2. Since the groups in Table 7 have index 2 in $\text{Aut}(X)$ for $K \in \{\text{III}, \text{IV}\}$, this finishes the proof. $\square$

Remark 12.3. The fact that surfaces of types III and IV admit non-semi-symplectic automorphisms is the reason why, in general, we need a field extension to realize all automorphisms of these surfaces. These non-semi-symplectic automorphisms act as $\sqrt{-1}$ on a non-zero global 2-form of the K3 cover; hence, it is necessary to adjoin at least $\sqrt{-1}$ to k to realize all automorphisms of these surfaces. Since the K3 cover of Enriques surfaces of type III and IV is the Kummer surface associated with the self-product of an elliptic curve with j -invariant 1728 (see [Kon86, Main Theorem, Table 1, p. 193]), it is likely that this field extension always suffices.

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