Monodromy map for tropical Dolbeault cohomology

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ABSTRACT

We define monodromy maps for the tropical Dolbeault cohomology of algebraic varieties over non-Archimedean fields. We propose a conjecture of Hodge isomorphisms via monodromy maps and provide some evidence.

1. Introduction

Let K be a complete non-Archimedean field. For a separated scheme X of finite type over K , one has the associated K -analytic space X^{an} in the sense of Berkovich [Ber93]. Using the $\mathbb{Q}$ -sheaf $\mathcal{T}_{X^{\text{an}}}^p$ on X^{an} constructed in [Liu17], we obtain a new cohomology theory $H_{\text{trop}}^{p,q}(X)$, functorial in X . They are rational vector spaces satisfying $H_{\text{trop}}^{p,q}(X) \neq 0$ only when $0 \leq p, q \leq \dim X$. Denote by $h_{\text{trop}}^{p,q}(X)$ the dimension of $H_{\text{trop}}^{p,q}(X)$, which might be infinity.

Using a bicomplex $\mathcal{A}_{X^{\text{an}}}^{\bullet,\bullet}$ of $\mathbb{R}$ -sheaves of real forms on X^{an} , constructed by Chambert-Loir and Ducros [CD12], we can recover $H_{\text{trop}}^{p,q}(X)_{\mathbb{R}} := H_{\text{trop}}^{p,q}(X) \otimes_{\mathbb{Q}} \mathbb{R}$ via certain Dolbeault cohomology groups. In this article, we consider a natural question, motivated by complex geometry, about the Hodge relation: *Do we have the equality $h_{\text{trop}}^{p,q}(X) = h_{\text{trop}}^{q,p}(X)$?*

However, the perspective we take is very different from the one in complex geometry. In fact, we will construct, for $p \geq 1$ and $q \geq 0$, a functorial map

$$N_X: H_{\text{trop}}^{p,q}(X)_{\mathbb{R}} \rightarrow H_{\text{trop}}^{p-1,q+1}(X)_{\mathbb{R}},$$

which we call the *monodromy map*, via constructing a canonical map $\mathcal{A}_{X^{\text{an}}}^{p,q} \rightarrow \mathcal{A}_{X^{\text{an}}}^{p-1,q+1}$ on the level of sheaves. Such a map does not exist in complex geometry. We conjecture (see Conjecture 5.3) that over certain algebraically closed complete non-Archimedean fields K , the iterated map

$$N_X^{p-q}: H_{\text{trop}}^{p,q}(X)_{\mathbb{R}} \rightarrow H_{\text{trop}}^{q,p}(X)_{\mathbb{R}}$$

is an isomorphism for $p \geq q$ if X is proper smooth. We view such an isomorphism as the Hodge isomorphism for tropical Dolbeault cohomology. If this holds, then we have the following numerical relation for the corresponding Hodge numbers:

$$h_{\text{trop}}^{p,0}(X) \leq h_{\text{trop}}^{p-1,1}(X) \leq \dots \geq h_{\text{trop}}^{1,p-1}(X) \geq h_{\text{trop}}^{0,p}(X).$$

This is apparently new from the results in complex geometry. In fact, we believe that the Hodge isomorphism and the above numerical relation should have a connection with the weight spectral

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sequence. We note that the map $N_X^p: H_{\text{trop}}^{p,0}(X)_{\mathbb{R}} \rightarrow H_{\text{trop}}^{0,p}(X)_{\mathbb{R}}$ is simply induced by the “flipping” map J , see [CD12], multiplied by $p!$; see Lemma 3.2(1).

The theorem below provides certain evidence toward our expected Hodge isomorphism. The proof uses arithmetic geometry, especially the weight spectral sequence as we have just mentioned. The method is restrictive in the sense that any further extension would seem to rely on the weight-monodromy conjecture (Conjecture 4.5) in the mixed-characteristic case, among other difficulties. Therefore, it is very interesting and important to find a somewhat analytic proof of the results here. However, the situation is rather subtle since we know that for some algebraically closed, complete, non-Archimedean fields K , such as the field of Puiseux series $\mathbb{C}\{\{t\}\}$, the map $N_X: H_{\text{trop}}^{1,0}(X)_{\mathbb{R}} \rightarrow H_{\text{trop}}^{0,1}(X)_{\mathbb{R}}$ may fail to be an isomorphism (see Remark 5.4).

In this article, by a non-Archimedean field, we mean a complete topological field equipped with a nontrivial, non-Archimedean valuation of rank one. For a non-Archimedean field K , we denote the ring of integers by K° , the residue field by $\tilde{K}$, and a fixed algebraic closure by K^a , with $\widehat{K^a}$ its completion.

THEOREM 1.1. *Let X_0 be a proper, smooth scheme over a non-Archimedean field K_0 . Let K be a closed subfield of $\widehat{K_0^a}$ containing K_0 . Put $X := X_0 \otimes_{K_0} K$.*

- (i) (Corollary 5.12) *Suppose that K_0 is isomorphic to $k((t))$ for k either a finite field or a field of characteristic zero. Then the monodromy map*

$$N_X^p: H_{\text{trop}}^{p,0}(X)_{\mathbb{R}} \rightarrow H_{\text{trop}}^{0,p}(X)_{\mathbb{R}}$$

is injective for every $p \geq 0$. In particular, $h_{\text{trop}}^{p,0}(X)$ is finite.

- (ii) (Theorem 5.13) *Suppose that K_0 is either a finite extension of $\mathbb{Q}_p$ or isomorphic to $k((t))$ for k a finite field, $K = \widehat{K_0^a}$, and X_0 admits a proper, strictly semistable model over K_0° . Then the monodromy map*

$$N_X: H_{\text{trop}}^{1,0}(X)_{\mathbb{R}} \rightarrow H_{\text{trop}}^{0,1}(X)_{\mathbb{R}}$$

is an isomorphism.

Remark 1.2. Let K be an algebraically closed, non-Archimedean field.

- (1) In his thesis, Jell proved that for a proper, smooth scheme X over K of dimension n , the map $N_X^p: H_{\text{trop}}^{p,0}(X)_{\mathbb{R}} \rightarrow H_{\text{trop}}^{0,p}(X)_{\mathbb{R}}$ is injective for $p = 0, 1, n$ [Jel16, Proposition 3.4.11].
- (2) In [JW18], Jell and Wanner proved that for X either $\mathbf{P}_K^1$ or a (proper, smooth) Mumford curve over K , the map $N_X: H_{\text{trop}}^{1,0}(X)_{\mathbb{R}} \rightarrow H_{\text{trop}}^{0,1}(X)_{\mathbb{R}}$ is an isomorphism.

Their methods in both papers are analytic.

In [MZ14], Mikhalkin and Zharkov study the tropical homology groups $H_{p,q}(X)$ of a *tropical space* X . Assuming X compact, they define a map

$$\phi \cap: H_{p,q}(X)_{\mathbb{R}} \rightarrow H_{p+1,q-1}(X)_{\mathbb{R}}$$

using their eigenwave ϕ . In fact, as shown in [JSS18], one can compute the tropical homology $H_{p,q}(X)_{\mathbb{R}}$, or rather the tropical cohomology, via superforms in [CD12] as well. Then our construction in Section 2 would give rise to a map $N_X: H_{p,q}(X)_{\mathbb{R}} \rightarrow H_{p+1,q-1}(X)_{\mathbb{R}}$ (or rather on cohomology) for any tropical space X . We expect that N_X and $\phi \cap$ should coincide when X is compact, possibly up to an elementary factor.

Moreover, in [MZ14], the authors prove that when X is a realizable, smooth, compact tropical space, the iterated map $\phi^{q-p} \cap: H_{p,q}(X)_{\mathbb{R}} \rightarrow H_{q,p}(X)_{\mathbb{R}}$ is an isomorphism. They first realize X as

the tropical limit of a complex, projective, one-parameter, semistable degeneration $\mathcal{X}$ such that all strata of the singular fiber $\mathcal{X}_0$ are blow-ups of projective spaces. By the work [IKMZ16], one knows that $H_{p,q}(X)_{\mathbb{Q}}$ can be identified with E_2 -terms of the Steenbrink–Illusie spectral sequence associated with $\mathcal{X}$. Moreover, Mikhalkin and Zharkov show that under such an identification, the map $\phi \cap$ is simply the monodromy map on E_2 -terms. In our work, X is a proper, smooth¹ *scheme*, and we relate our tropical Dolbeault cohomology $H_{\text{trop}}^{p,q}(X)$ to E_2 -terms of the weight spectral sequence of (various) semistable alterations $\mathcal{X}$ of X (see Definition 5.5 for the precise meaning) for certain p and q , under which N_X is essentially the monodromy map. In our setup, the semistable scheme $\mathcal{X}$ is very general, so that its E_2 -terms cannot be read off solely from the dual complex of $\mathcal{X}$; hence, we do not obtain a strict identification for general p and q . Nevertheless, it would be interesting to compare our approach relating to the weight spectral sequence and their approach relating to the Steenbrink–Illusie spectral sequence in [MZ14, IKMZ16].

Conventions

Let K be a non-Archimedean field. All K -analytic (Berkovich) spaces are assumed to be *good* [Ber93, Remark 1.2.16], *Hausdorff*, and *strictly K -analytic* [Ber93, Proposition 1.2.15].

For a K -analytic space X , we use $H^\bullet(X, -)$ to indicate the cohomology group with respect to the underlying topology of X , and we use $H_{\text{ét}}^\bullet(X, -)$ to indicate the cohomology group with respect to the étale topology of X .

2. Monodromy maps for superforms on vector spaces

In this section, we review the construction of superforms on vector spaces from [CD12] and introduce the corresponding monodromy map.

Let V be an $\mathbb{R}$ -vector space of dimension n . Let T_V be the tangent space of V and T_V^* its dual space. For every open subset U of V and integers $p, q \geq 0$, we have the space of (p, q) -forms on U ,

$$\mathcal{A}_V^{p,q}(U) := \mathcal{A}_V(U) \otimes_{\mathbb{R}} \bigwedge^p T_V^* \otimes_{\mathbb{R}} \bigwedge^q T_V^*,$$

where $\mathcal{A}_V(U)$ is the $\mathbb{R}$ -algebra of smooth functions on U . The direct sum

$$\mathcal{A}_V^{*,*}(U) := \bigoplus \mathcal{A}_V^{p,q}(U)$$

forms a bigraded $\mathbb{R}$ -algebra with the following commutativity law: if ω is a (p, q) -form and ω' is a (p', q') -form, then we have

$$\omega' \wedge \omega = (-1)^{(p+q)(p'+q')} \omega \wedge \omega'.$$

Moreover, $\wedge: \mathcal{A}_V^{p,0}(U) \times \mathcal{A}_V^{0,q}(U) \rightarrow \mathcal{A}_V^{p,q}(U)$ is simply the map induced by the tensor product map.

The real vector spaces $\mathcal{A}_V^{\bullet,\bullet}(U)$ form a bicomplex with two differentials

$$d': \mathcal{A}_V^{p,q}(U) \rightarrow \mathcal{A}_V^{p+1,q}(U), \quad d'': \mathcal{A}_V^{p,q}(U) \rightarrow \mathcal{A}_V^{p,q+1}(U).$$

We also have a convolution map $J: \mathcal{A}_V^{p,q}(U) \rightarrow \mathcal{A}_V^{q,p}(U)$.

¹It is worth noting that the tropicalization of a smooth analytic space, such as X^{an} , under a local tropical chart is in general not a smooth tropical space.

In terms of coordinates, they are described as follows. Let $\{x_1, \dots, x_n\}$ be a system of coordinates of V . Then a (p, q) -form can be written as

$$\omega = \sum_{|I|=p, |J|=q} \omega_{I,J}(x) d'x_I \otimes d''x_J = \sum_{|I|=p, |J|=q} \omega_{I,J}(x) d'x_I \wedge d''x_J,$$

where I and J are subsets of $\{1, \dots, n\}$ and the $\omega_{I,J}(x)$ are smooth functions on U . For such a form ω , we have

$$\begin{aligned} d'\omega &= \sum_{|I|=p, |J|=q} \sum_{i=1}^n \frac{\partial \omega_{I,J}(x)}{\partial x_i} d'x_i \wedge d'x_I \wedge d''x_J, \\ d''\omega &= (-1)^p \sum_{|I|=p, |J|=q} \sum_{j=1}^n \frac{\partial \omega_{I,J}(x)}{\partial x_j} d'x_I \wedge d''x_j \wedge d''x_J, \\ J\omega &= (-1)^{pq} \sum_{|I|=p, |J|=q} \omega_{I,J}(x) d'x_J \wedge d''x_I. \end{aligned}$$

We remind readers of the following relations from [CD12, Section 1.2]:

$$d'' = Jd'J, \quad Jd' = d''J, \quad Jd'' = d'J, \quad d' = Jd''J,$$

and for a (p, q) -form ω and a (p', q') -form ω' ,

$$\begin{aligned} d'(\omega \wedge \omega') &= d'\omega \wedge \omega' + (-1)^{p+q} \omega \wedge d'\omega', \\ d''(\omega \wedge \omega') &= d''\omega \wedge \omega' + (-1)^{p+q} \omega \wedge d''\omega'. \end{aligned}$$

Now we are going to define a map $N: \mathcal{A}_V^{p,q}(U) \rightarrow \mathcal{A}_V^{p-1,q+1}(U)$ for $p \geq 1$. We first recall the notion of coevaluation map. Let W be an arbitrary finite-dimensional $\mathbb{R}$ -vector space with W^* its dual space. We have a canonical evaluation map

$$\text{ev}: W^* \otimes_{\mathbb{R}} W \rightarrow \mathbb{R}.$$

We define the *coevaluation map* to be the unique linear map

$$\text{coev}: \mathbb{R} \rightarrow W \otimes_{\mathbb{R}} W^*$$

such that both composite maps

$$\begin{aligned} W^* &\xrightarrow{1_{W^*} \otimes \text{coev}} W^* \otimes_{\mathbb{R}} (W \otimes_{\mathbb{R}} W^*) \xrightarrow{\sim} (W^* \otimes_{\mathbb{R}} W) \otimes_{\mathbb{R}} W^* \xrightarrow{\text{ev} \otimes 1_{W^*}} W^*, \\ W &\xrightarrow{\text{coev} \otimes 1_W} (W \otimes_{\mathbb{R}} W^*) \otimes_{\mathbb{R}} W \xrightarrow{\sim} W \otimes_{\mathbb{R}} (W^* \otimes_{\mathbb{R}} W) \xrightarrow{1_W \otimes \text{ev}} W \end{aligned}$$

are identity maps. In fact, for an arbitrary basis $\{w_1, \dots, w_n\}$ of W , with the dual basis $\{w_1^*, \dots, w_n^*\}$ of W^* , the map coev sends 1 to $\sum_{i=1}^n w_i \otimes w_i^*$.

Now we apply the coevaluation map to the vector space $W = T_V$.

DEFINITION 2.1. Let $p \geq 1$ be an integer. Define the map

$$N: \mathcal{A}_V^{p,q}(U) \rightarrow \mathcal{A}_V^{p-1,q+1}(U)$$

to be the composite map

$$\begin{aligned}
\mathcal{C}^\infty(U) \otimes_{\mathbb{R}} \bigwedge^p T_V^* \otimes_{\mathbb{R}} \bigwedge^q T_V^* &\xrightarrow{\sim} \mathcal{C}^\infty(U) \otimes_{\mathbb{R}} \bigwedge^p T_V^* \otimes_{\mathbb{R}} \mathbb{R} \otimes_{\mathbb{R}} \bigwedge^q T_V^* \\
&\rightarrow \mathcal{C}^\infty(U) \otimes_{\mathbb{R}} \bigwedge^p T_V^* \otimes_{\mathbb{R}} (T_V \otimes_{\mathbb{R}} T_V^*) \otimes_{\mathbb{R}} \bigwedge^q T_V^* \\
&\xrightarrow{\sim} \mathcal{C}^\infty(U) \otimes_{\mathbb{R}} \left(\bigwedge^p T_V^* \otimes_{\mathbb{R}} T_V \right) \otimes_{\mathbb{R}} \left(T_V^* \otimes_{\mathbb{R}} \bigwedge^q T_V^* \right) \\
&\rightarrow \mathcal{C}^\infty(U) \otimes_{\mathbb{R}} \bigwedge^{p-1} T_V^* \otimes_{\mathbb{R}} \bigwedge^{q+1} T_V^*,
\end{aligned}$$

where the second map is given by the coevaluation map for T_V and the last map is given by the contraction map and the wedge product. For $0 \leq r \leq p$, we denote by

$$N^r : \mathcal{A}_V^{p,q}(U) \rightarrow \mathcal{A}_V^{p-r,q+r}(U)$$

the consecutive composition.

In terms of the coordinates $\{x_1, \dots, x_n\}$ of V , for

$$\omega = \sum_{I=\{i_1 < \dots < i_p\}, J=\{j_1 < \dots < j_q\}} \omega_{I,J}(x) d'x_{i_1} \wedge \dots \wedge d'x_{i_p} \wedge d''x_{j_1} \wedge \dots \wedge d''x_{j_q} \quad (2.1)$$

with $p \geq 1$, we have

$$\begin{aligned}
N\omega &= \sum_{k=1}^p \sum_{I,J} (-1)^{p-k} \omega_{I,J}(x) d'x_{i_1} \wedge \dots \wedge \widehat{d'x_{i_k}} \wedge \dots \wedge d'x_{i_p} \wedge d''x_{i_k} \wedge d''x_{j_1} \wedge \dots \wedge d''x_{j_q} \\
&= \sum_{k=1}^p \sum_{I,J} (-1)^{p-k} \omega_{I,J}(x) d'x_{I \setminus \{i_k\}} \wedge d''x_{i_k} \wedge d''x_J.
\end{aligned}$$

LEMMA 2.2. We have $Nd'' = d''N : \mathcal{A}_V^{p,q}(U) \rightarrow \mathcal{A}_V^{p-1,q+2}(U)$ for $p \geq 1$.

Proof. Take a (p, q) -form ω as in (2.1). We have

$$\begin{aligned}
d''N\omega &= d'' \sum_{k=1}^p \sum_{I,J} (-1)^{p-k} \omega_{I,J}(x) d'x_{I \setminus \{i_k\}} \wedge d''x_{i_k} \wedge d''x_J \\
&= - \sum_{I,J} \sum_{j=1}^n \sum_{k=1}^p \frac{(-1)^k \partial \omega_{I,J}(x)}{\partial x_j} d'x_{I \setminus \{i_k\}} \wedge d''x_j \wedge d''x_{i_k} \wedge d''x_J.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
Nd''\omega &= N(-1)^p \sum_{I,J} \sum_{j=1}^n \frac{\partial \omega_{I,J}(x)}{\partial x_j} d'x_I \wedge d''x_j \wedge d''x_J \\
&= \sum_{k=1}^p \sum_{I,J} \sum_{j=1}^n \frac{(-1)^k \partial \omega_{I,J}(x)}{\partial x_j} d'x_{I \setminus \{i_k\}} \wedge d''x_{i_k} \wedge d''x_j \wedge d''x_J \\
&= - \sum_{I,J} \sum_{j=1}^n \sum_{k=1}^p \frac{(-1)^k \partial \omega_{I,J}(x)}{\partial x_j} d'x_{I \setminus \{i_k\}} \wedge d''x_j \wedge d''x_{i_k} \wedge d''x_J.
\end{aligned}$$

The lemma follows. $\square$

LEMMA 2.3. For a (p, q) -form ω and a (p', q') -form ω' with $p, p' \geq 1$ and $p + p' \geq n + 1$, we have

$$N\omega \wedge \omega' = -\omega \wedge N\omega'.$$

Proof. By linearity, we may assume

$$\omega = \omega(x) d'x_I \wedge d''x_J, \quad \omega' = \omega'(x) d'x_{I'} \wedge d''x_{J'}.$$

If $|I \cap I'| \geq 2$, then it is easy to see that $N\omega \wedge \omega' = \omega \wedge N\omega' = 0$. Otherwise, $|I \cap I'| = 1$. Without loss of generality, we may assume $I \cap I' = \{n\}$, hence $d'x_I = d'x_{I \setminus \{n\}} \wedge d'x_n$ and $d'x_{I'} = d'x_{I' \setminus \{n\}} \wedge d'x_n$. Then we have

$$\begin{aligned} N\omega \wedge \omega' &= (\omega(x) d'x_{I \setminus \{n\}} \wedge d''x_n \wedge d''x_J) \wedge (\omega'(x) d'x_{I'} \wedge d''x_{J'}) \\ &= (-1)^{p'(q+1)} \omega(x) \omega'(x) d'x_{I \setminus \{n\}} \wedge d'x_{I'} \wedge d''x_n \wedge d''x_J \wedge d''x_{J'} \\ &= (-1)^{p'(q+1)} \omega(x) \omega'(x) d'x_{I \setminus \{n\}} \wedge d'x_{I' \setminus \{n\}} \wedge d'x_n \wedge d''x_n \wedge d''x_J \wedge d''x_{J'}, \end{aligned}$$

and

$$\begin{aligned} \omega \wedge N\omega' &= (\omega(x) d'x_I \wedge d''x_J) \wedge (\omega'(x) d'x_{I' \setminus \{n\}} \wedge d''x_n \wedge d''x_{J'}) \\ &= (-1)^{p'q} \omega(x) \omega'(x) d'x_I \wedge d'x_{I' \setminus \{n\}} \wedge d''x_n \wedge d''x_J \wedge d''x_{J'} \\ &= (-1)^{p'q} \omega(x) \omega'(x) d'x_{I \setminus \{n\}} \wedge d'x_n \wedge d'x_{I' \setminus \{n\}} \wedge d''x_n \wedge d''x_J \wedge d''x_{J'} \\ &= (-1)^{p'q+(p'-1)} \omega(x) \omega'(x) d'x_{I \setminus \{n\}} \wedge d'x_{I' \setminus \{n\}} \wedge d'x_n \wedge d''x_n \wedge d''x_J \wedge d''x_{J'}. \end{aligned}$$

The lemma follows. $\square$

LEMMA 2.4. Let V' be another $\mathbb{R}$ -vector space of dimension n' and $U' \subset V'$ an open subset. Let $\varphi: V' \rightarrow V$ be an affine map such that $\varphi(U') \subset U$. Then for $p \geq 1$, we have

$$N'\varphi^* = \varphi^*N: \mathcal{A}_V^{p,q}(U) \rightarrow \mathcal{A}_{V'}^{p-1,q+1}(U'),$$

where N' denotes the monodromy map for V' .

Proof. We may assume that φ is a linear map. It suffices to consider cases where φ is injective or surjective.

Suppose that φ is injective. Regard V' as a subspace of V . Choose a basis $\{x_1, \dots, x_n\}$ of V such that V' is spanned by $\{x_1, \dots, x_{n'}\}$ (with $n' \leq n$). Take $\omega = \omega(x) d'x_I \wedge d''x_J \in \mathcal{A}_V^{p,q}(U)$ for $I = \{i_1 < \dots < i_p\}$. If $i_p > n'$, then $N'\varphi^*\omega = \varphi^*N\omega = 0$. If $i_p \leq n'$, then we have

$$N'\varphi^*\omega = \varphi^*N\omega = \left(\omega(x) \sum_{k=1}^p (-1)^{p-k} d'x_{I \setminus \{i_k\}} \wedge d''x_{i_k} \wedge d''x_J \right) \Big|_{U'}.$$

Suppose that φ is surjective. Choose a basis $\{x_1, \dots, x_{n'}\}$ of V' such that $\ker \varphi$ is spanned by $\{x_{n+1}, \dots, x_{n'}\}$ (with $n \leq n'$). We identify V with the subspace of V' spanned by $\{x_1, \dots, x_n\}$. Then, again, we have

$$N'\varphi^*\omega = \varphi^*N\omega = \left(\omega(\varphi(x)) \sum_{k=1}^p (-1)^{p-k} d'x_{I \setminus \{i_k\}} \wedge d''x_{i_k} \wedge d''x_J \right) \Big|_{U'}.$$

The lemma follows. $\square$

LEMMA 2.5. The map $N^p: \mathcal{A}_V^{p,0}(U) \rightarrow \mathcal{A}_V^{0,p}(U)$ coincides with $p! \cdot J: \mathcal{A}_V^{p,0}(U) \rightarrow \mathcal{A}_V^{0,p}(U)$.

Proof. This is elementary. $\square$

Remark 2.6. The presheaf $U \mapsto \mathcal{A}_V^{p,q}(U)$ is already a sheaf on V . The maps d' , d'' , J , $\wedge$, N induce maps of sheaves with the same relations. In particular, we have the map

$$N_V: \mathcal{A}_V^{p,q} \rightarrow \mathcal{A}_V^{p-1,q+1}$$

of sheaves for $p \geq 1$ and the lemmas corresponding to Lemmas 2.2 and 2.3. Lemma 2.4 induces the equality $(\varphi_* N_{V'})\varphi^* = \varphi^* N_V: \mathcal{A}_V^{p,q} \rightarrow \varphi_* \mathcal{A}_{V'}^{p-1,q+1}$ on the level of sheaves.

Let P be a polyhedral complex² in V and $j: P \rightarrow V$ the tautological inclusion map. For every open subset U of P , let $\mathcal{N}_P^{p,q}(U)$ be the subspace of $(j^{-1}\mathcal{A}_V^{p,q})(U)$ of (p, q) -forms ω such that for every polyhedron C of P , the restriction of ω to $\langle C \rangle$ is zero on $\langle C \rangle \cap U$, where $\langle C \rangle$ is the affine subset of V spanned by C . It is clear that $\mathcal{N}_P^{p,q}$ is an $\mathbb{R}$ -subsheaf of $j^{-1}\mathcal{A}_V^{p,q}$. Let $\mathcal{A}_P^{p,q}$ be the quotient sheaf $j^{-1}\mathcal{A}_V^{p,q}/\mathcal{N}_P^{p,q}$.

By Lemma 2.4, the monodromy map $j^{-1}N_V: j^{-1}\mathcal{A}_V^{p,q} \rightarrow j^{-1}\mathcal{A}_V^{p-1,q+1}$ preserves $\mathcal{N}_P^{p,q}$. Thus, we have an induced map

$$N_P: \mathcal{A}_P^{p,q} \rightarrow \mathcal{A}_P^{p-1,q+1}$$

of sheaves for $p \geq 1$.

3. Monodromy maps for real forms on analytic spaces

In this section, we review the construction of (p, q) -forms in [CD12] and tropical Dolbeault cohomology, and we introduce the monodromy map on analytic spaces.

We fix a non-Archimedean field K . Let X be a K -analytic space. Recall that a *tropical chart* of X is a pair $(f: U \rightarrow T, P)$, where $f: X \rightarrow T$ is a map to a torus T over K , usually called a *moment map*, and P is a compact polyhedral complex of T_{trop} that contains $f_{\text{trop}}(X)$. Here, T_{trop} is the tropicalization of T , which is a real vector space of finite dimension, and $f_{\text{trop}}: X \rightarrow T \rightarrow T_{\text{trop}}$ is the composite map.

For every open subset U of X , let $\mathcal{A}_{\text{pre}}^{p,q}(U)$ be the inductive limit of $\mathcal{A}_P^{p,q}(P)$ for all tropical charts $(f: U \rightarrow T, P)$ of U . Again by Lemma 2.4, the monodromy maps N_P are compatible with transition maps. Therefore, we obtain a map $N_{\text{pre}}: \mathcal{A}_{\text{pre}}^{p,q}(U) \rightarrow \mathcal{A}_{\text{pre}}^{p-1,q+1}(U)$ for $p \geq 1$. The sheaf of (p, q) -forms on X is defined as the sheafification of $U \mapsto \mathcal{A}_{\text{pre}}^{p,q}(U)$ and is denoted by $\mathcal{A}_X^{p,q}$.

DEFINITION 3.1. For $p \geq 1$, we define the *monodromy map* for forms on X , denoted by

$$N_X: \mathcal{A}_X^{p,q} \rightarrow \mathcal{A}_X^{p-1,q+1},$$

to be the sheafification of the map $U \mapsto [N_{\text{pre}}: \mathcal{A}_{\text{pre}}^{p,q}(U) \rightarrow \mathcal{A}_{\text{pre}}^{p-1,q+1}(U)]$ of presheaves. For $0 \leq r \leq p$, we denote the iterated composition by $N_X^r: \mathcal{A}_X^{p,q} \rightarrow \mathcal{A}_X^{p-r,q+r}$.

LEMMA 3.2. (i) The map $N_X^p: \mathcal{A}_X^{p,q} \rightarrow \mathcal{A}_X^{0,p}$ coincides with $p! \cdot J$.

(ii) The map N_X commutes with d'' ; that is, $N_X d'' = d'' N_X: \mathcal{A}_X^{p,q} \rightarrow \mathcal{A}_X^{p-1,q+2}$ for $p \geq 1$.

(iii) We have $(N_X \bullet) \wedge - = -(\bullet \wedge (N_X \bullet)): \mathcal{A}_X^{p,q} \times \mathcal{A}_X^{p',q'} \rightarrow \mathcal{A}_X^{p+p'-1,q+q'+1}$ for $p, p' \geq 1$ and $p + p' \geq \dim X + 1$.

(iv) Let $\varphi: X' \rightarrow X$ be a map of K -analytic spaces. Then

$$(\varphi_* N_{X'})\varphi^* = \varphi^* N_X: \mathcal{A}_X^{p,q} \rightarrow \varphi_* \mathcal{A}_{X'}^{p-1,q+1}.$$

Proof. These are consequences of Lemmas 2.5, 2.2, 2.3, and 2.4, respectively. $\square$

²It is called *polytope* in [CD12, Section 1.1].

For a fixed integer p , we have the complex

$$(\mathcal{A}_X^{p,\bullet}, d'') : \mathcal{A}_X^{p,0} \xrightarrow{d''} \mathcal{A}_X^{p,1} \xrightarrow{d''} \dots$$

The monodromy map in Definition 3.1 is in fact a map of complexes

$$N_X : \mathcal{A}_X^{p,\bullet} \rightarrow \mathcal{A}_X^{p-1,\bullet}[1]. \quad (3.1)$$

DEFINITION 3.3 (Dolbeault cohomology, [Liu17, Definition 3.1]). Let X be a K -analytic space. We define the *Dolbeault cohomology* of X to be the sheaf cohomology

$$H^{p,q}(X) := H^q(X, \ker(d'' : \mathcal{A}_X^{p,0} \rightarrow \mathcal{A}_X^{p,1})).$$

Let $\mathcal{O}_X$ be the structure sheaf of X . For $p \geq 0$, let $\mathcal{O}_X^{(p)}$ be the sheaf such that for every open subset U of X , the set $\mathcal{O}_X^{(p)}(U)$ is the $\mathbb{Q}$ -vector space spanned by symbols $\{f_1, \dots, f_p\}$ with $f_i \in \mathcal{O}_X^*(U)$. For $p \geq 0$, we have a natural map

$$\tau : \mathcal{O}_X^{(p)} \rightarrow \ker(d'' : \mathcal{A}_X^{p,0} \rightarrow \mathcal{A}_X^{p,1}) \quad (3.2)$$

of $\mathbb{Q}$ -sheaves on X . Let $\mathcal{T}_X^p$ be its image sheaf. We recall the definition of τ . For an open subset U of X and $f_1, \dots, f_p \in \mathcal{O}_X^*(U)$, we have a moment map

$$f = (f_1, \dots, f_p) : U \rightarrow T := (\mathbf{G}_m^{\text{an}})^p.$$

Let $\{x_1, \dots, x_p\}$ be the standard coordinates of $T_{\text{trop}} = \mathbb{R}^p$.³ Then $\tau(\{f_1, \dots, f_p\})$ is defined as $d'x_1 \wedge \dots \wedge d'x_p$, regarded as an element in $\ker(d'' : \mathcal{A}_X^{p,0}(U) \rightarrow \mathcal{A}_X^{p,1}(U))$.

PROPOSITION 3.4. *Let X be a K -analytic space. The canonical map*

$$\mathcal{T}_X^p \otimes_{\mathbb{Q}} \mathbb{R} \rightarrow \ker(d'' : \mathcal{A}_X^{p,0} \rightarrow \mathcal{A}_X^{p,1})$$

is an isomorphism. It induces a canonical isomorphism $H^q(X, \mathcal{T}_X^p) \otimes_{\mathbb{Q}} \mathbb{R} \simeq H^{p,q}(X)$.

Proof. This follows from [Liu17, Proposition 3.6 and Corollary 3.7]. $\square$

REMARK 3.5. If X is paracompact, then by [Liu17, Remark 3.2], we have a canonical isomorphism

$$H^{p,q}(X) \simeq \frac{\ker(d'' : \mathcal{A}_X^{p,q}(X) \rightarrow \mathcal{A}_X^{p,q+1}(X))}{\text{im}(d'' : \mathcal{A}_X^{p,q-1}(X) \rightarrow \mathcal{A}_X^{p,q}(X))},$$

and the monodromy map (3.1) induces a map

$$N_X : H^{p,q}(X) \rightarrow H^{p-1,q+1}(X)$$

of Dolbeault cohomology when $p \geq 1$.

4. Semistable schemes and cohomological monodromy maps

In this section, we introduce some constructions for strictly semistable schemes, and review the (cohomological) monodromy maps coming from the weight spectral sequence.

Let K be a discrete non-Archimedean field, that is, a non-Archimedean field with a discrete valuation. Fix a rational prime ℓ that is invertible in $\tilde{K}$. Denote by $K^{\text{ur}} \subset K^{\text{a}}$ the maximal subextension that is unramified over K , with residue field $\tilde{K}^{\text{s}}$, which is a separable closure of $\tilde{K}$. Let $I_K \subset \text{Gal}(K^{\text{a}}/K)$ be the inertia subgroup so the quotient group $\text{Gal}(K^{\text{a}}/K)/I_K$ can be

³The map $\mathbf{G}_m^{\text{an}} \rightarrow (\mathbf{G}_m^{\text{an}})_{\text{trop}} = \mathbb{R}$ is given by $-\log |\cdot|$.

identified with $\text{Gal}(\tilde{K}^s/\tilde{K})$. Denote by $t_\ell: \mathbf{I}_K \rightarrow \mathbb{Z}_\ell(1)$ the (ℓ -adic) tame quotient homomorphism; that is, the one sending $\sigma \in \mathbf{I}_K$ to $(\sigma(\varpi^{1/\ell^n})/\varpi^{1/\ell^n})_n$ for a uniformizer ϖ of K . We fix an element $T \in \mathbf{I}_K$ such that $t_\ell(T)$ is a topological generator of $\mathbb{Z}_\ell(1)$.

For a separated scheme X of finite type over K , put $X_a := X \otimes_K K^a$, and let X^{an} be the associated K -analytic space as in [Ber93]. Put $X_a^{\text{an}} := (X \otimes_K \widehat{K^a})^{\text{an}}$. For a scheme $\mathcal{X}$ over K° , put $\mathcal{X}_\eta := \mathcal{X} \otimes_{K^\circ} K$. For a scheme Z over $\tilde{K}$, put $Z_s := Z \otimes_{\tilde{K}} \tilde{K}^s$.

We first recall the following definition (see [Sai03], for example).

DEFINITION 4.1 (Strictly semistable scheme). Let $\mathcal{X}$ be a scheme locally of finite presentation over $\text{Spec } K^\circ$. We say that $\mathcal{X}$ is *strictly semistable* if it is Zariski-locally smooth over

$$\text{Spec } K^\circ[t_0, \dots, t_p]/(t_0 \cdots t_p - \varpi)$$

for some integer $p \geq 0$ (which may vary) and a uniformizer ϖ of K .

Let $\mathcal{X}$ be a proper, strictly semistable scheme over K° . The special fiber $Y := \mathcal{X} \otimes_{K^\circ} \tilde{K}$ is a normal crossing divisor of $\mathcal{X}$. Suppose that $\{Y^1, \dots, Y^m\}$ is the set of irreducible components of Y . For a nonempty subset $I \subset \{1, \dots, m\}$, put $Y^I := \bigcap_{i \in I} Y^i$. For $p \geq 0$, put

$$Y^{(p)} := \coprod_{I \subset \{1, \dots, m\}, |I|=p+1} Y^I.$$

Then $Y^{(p)}$ is a finite, disjoint union of smooth, proper subschemes of Y of codimension p over $\tilde{K}$. Denote the image of the canonical morphism $Y^{(p)} \rightarrow Y$ by $Y^{[p]}$.

Notation 4.2. For a subscheme Z of Y , put $I(Z) := \{i \in \{1, \dots, m\} \mid Z \subset Y^i\}$ and $p(Z) := |I(Z)| - 1$.

DEFINITION 4.3. Let y be a point of Y with $p = p(y) \geq 0$. Suppose $I(y) = \{i_0 < \dots < i_p\}$ (Notation 4.2). We define a *semistable chart* at y to be a pair $(\mathcal{V}, \{g_0, \dots, g_p\})$, where $\mathcal{V}$ is a connected, affine, open neighborhood of y in $\mathcal{X}$ and $g_0, \dots, g_p$ are regular functions on $\mathcal{V}$ such that there is a smooth morphism $\mathcal{V} \rightarrow \text{Spec } K^\circ[t_0, \dots, t_p]/(t_0 \cdots t_p - \varpi)$ under which g_j is the pullback of t_j and its associated divisor is $\mathcal{V} \cap Y^{i_j}$ for $0 \leq j \leq p$.

Remark 4.4. For every point y of Y , one can always find a semistable chart at y by Definition 4.1. Moreover, for every irreducible component W of $Y^{[r]}$ with $r \geq 0$, the intersection $\mathcal{V} \cap W$ is either empty or connected since W is smooth and irreducible.

For subsets $J \subset I \subset \{1, \dots, m\}$ such that $|I| = |J| + 1$, let $i_{JI}: \bigcap_{i \in I} Y^i \rightarrow \bigcap_{i \in J} Y^i$ denote the closed immersion. If $I = \{i_0 < \dots < i_p\}$ and $J = I \setminus \{i_j\}$, then we put $\epsilon(J, I) = (-1)^j$. We define the *pullback map*

$$\delta_p^*: H_{\text{ét}}^q(Y_s^{(p)}, \mathbb{Q}_\ell) \rightarrow H_{\text{ét}}^q(Y_s^{(p+1)}, \mathbb{Q}_\ell) \quad (4.1)$$

to be the alternating sum $\sum_{I \supset J, |I|=|J|+1=p+1} \epsilon(I, J) i_{IJ}^*$ of the restriction maps and the *pushforward map*

$$\delta_{p*}: H_{\text{ét}}^q(Y_s^{(p)}, \mathbb{Q}_\ell) \rightarrow H_{\text{ét}}^{q+2}(Y_s^{(p-1)}, \mathbb{Q}_\ell(1)) \quad (4.2)$$

to be the alternating sum $\sum_{I \supset J, |I|=|J|+1=p+1} \epsilon(J, I) i_{JI*}$ of the Gysin maps. For $p \geq 1$, these maps satisfy the formula

$$\delta_{p-1}^* \circ \delta_{p*} + \delta_{p+1*} \circ \delta_p^* = 0.$$

Let us recall the following weight spectral sequence $E_{\mathcal{X},\bullet}^{\bullet,\bullet}$ attached to $\mathcal{X}$,⁴ originally studied in [RZ82]:

$$E_{\mathcal{X},1}^{p,q} = \bigoplus_{i \geq \max(0, -p)} H_{\text{ét}}^{q-2i}(Y_s^{(p+2i)}, \mathbb{Q}_\ell(-i)) \Rightarrow H_{\text{ét}}^{p+q}(\mathcal{X}_{\eta,a}, \mathbb{Q}_\ell).$$

Here, we will follow the convention and discussion in [Sai03]. When $\mathcal{X}$ is fixed, we write $E_{\bullet}^{\bullet,\bullet}$ for $E_{\mathcal{X},\bullet}^{\bullet,\bullet}$ for short. By [Sai03, Corollary 2.8(2)], we have a map $\mu: E_{\bullet}^{-1,\bullet+1} \rightarrow E_{\bullet}^{\bullet+1,\bullet-1}$ of spectral sequences (depending on T). The differential map $d_1^{p,q}$ is an appropriate sum of pullback and pushforward maps. The map $\mu_1^{p,q}: E_1^{p-1,q+1} \rightarrow E_1^{p+1,q-1}$ is the sum of its restrictions to each direct summand $H_{\text{ét}}^{q+1-2i}(Y_s^{(2i+1)}, \mathbb{Q}_\ell(-i))$, and such a restriction is the tensor product map by $t_\ell(T)$ (respectively, the zero map) if $H_{\text{ét}}^{q+1-2i}(Y_s^{(2i+1)}, \mathbb{Q}_\ell(-i+1))$ does (respectively, does not) appear in the target. For integers $p, q, r \geq 0$, the map μ induces a map

$$N_{\mathcal{X}}^r: E_2^{p,q}(r) \rightarrow E_2^{p+2r,q-2r}, \quad (4.3)$$

which depends only on $\mathcal{X}$.

CONJECTURE 4.5 (Weight-monodromy conjecture). Let $\mathcal{X}$ be a proper, strictly semistable scheme over K° . Then for all integers $p, r \geq 0$, the map

$$N_{\mathcal{X}}^r: E_2^{-r,p+r}(r) \rightarrow E_2^{r,p-r}$$

is an isomorphism.

For $p \geq 1$, the map $N_{\mathcal{X}}^p: E_2^{-p,2p}(p) \rightarrow E_2^{p,0}$ is simply the map

$$\begin{aligned} E_2^{-p,2p}(p) &= \ker \left(H_{\text{ét}}^0(Y_s^{(p)}, \mathbb{Q}_\ell) \xrightarrow{(\delta_p^*, \delta_{p*})} H_{\text{ét}}^0(Y_s^{(p+1)}, \mathbb{Q}_\ell) \oplus H_{\text{ét}}^2(Y_s^{(p-1)}, \mathbb{Q}_\ell(1)) \right) \\ &\rightarrow \frac{\ker \left(H_{\text{ét}}^0(Y_s^{(p)}, \mathbb{Q}_\ell) \xrightarrow{\delta_p^*} H_{\text{ét}}^0(Y_s^{(p+1)}, \mathbb{Q}_\ell) \right)}{\text{im} \left(H_{\text{ét}}^0(Y_s^{(p-1)}, \mathbb{Q}_\ell) \xrightarrow{\delta_{p-1}^*} H_{\text{ét}}^0(Y_s^{(p)}, \mathbb{Q}_\ell) \right)} = E_2^{p,0} \end{aligned} \quad (4.4)$$

induced by the identity map on $H_{\text{ét}}^0(Y_s^{(p)}, \mathbb{Q}_\ell)$.

PROPOSITION 4.6. (i) *The spectral sequence $E_{\mathcal{X}}$ degenerates from the second page. In particular, $E_2^{p,0}$ is canonically a subspace of $H_{\text{ét}}^p(\mathcal{X}_{\eta,a}, \mathbb{Q}_\ell)$.*

(ii) *If K has equal characteristic, then Conjecture 4.5 holds.*

(iii) *If $\tilde{K}$ is a purely inseparable extension of a finitely generated extension of a prime field, then $N_{\mathcal{X}}: E_2^{-1,2}(1) \rightarrow E_2^{1,0}$ is an isomorphism.*

Proof. See [Ito05, Theorem 1.1] for parts (i) and (ii). Part (iii) follows from [Ito05, Proposition 2.5 and Remark 2.4]. $\square$

Let X be a separated scheme of finite type over K . From [Ber00, Section 1], we have the map

$$\kappa_X^p: H^p(X^{\text{an}}, \mathbb{Q}_\ell) \rightarrow H_{\text{ét}}^p(X_a, \mathbb{Q}_\ell). \quad (4.5)$$

It is defined as the composition of the restriction map $H^p(X^{\text{an}}, \mathbb{Q}_\ell) \rightarrow H_{\text{ét}}^p(X_a^{\text{an}}, \mathbb{Q}_\ell)$ and the inverse of the comparison isomorphism $H_{\text{ét}}^p(X_a, \mathbb{Q}_\ell) \xrightarrow{\sim} H_{\text{ét}}^p(X_a^{\text{an}}, \mathbb{Q}_\ell)$.

⁴It also depends on the ordering of the set of irreducible components of Y .

LEMMA 4.7. *The map $\kappa_{\mathcal{X}_\eta}^p$ in (4.5) is injective with image contained in $E_2^{p,0}$. Moreover, if every irreducible component of $Y^{(r)}$ ($r \geq 0$) is geometrically irreducible, then $\kappa_{\mathcal{X}_\eta}^p$ induces an isomorphism $H^p(\mathcal{X}_\eta^{\text{an}}, \mathbb{Q}_\ell) \xrightarrow{\sim} E_2^{p,0}$.*

Proof. From [Ber00, Section 4], we have the diagram

$$\begin{array}{ccc} H_{\text{Zar}}^p(Y, \mathbb{Q}_\ell) & \xrightarrow{\sim} & H^p(\mathcal{X}_\eta^{\text{an}}, \mathbb{Q}_\ell) \\ \downarrow & & \downarrow \\ H_{\text{Zar}}^p(Y_s, \mathbb{Q}_\ell) & \xrightarrow{\sim} & H^p(\mathcal{X}_{\eta,a}^{\text{an}}, \mathbb{Q}_\ell) \\ \downarrow & & \downarrow \\ H_{\text{ét}}^p(Y_s, \mathbb{Q}_\ell) & \longrightarrow & H_{\text{ét}}^p(\mathcal{X}_{\eta,a}, \mathbb{Q}_\ell), \end{array}$$

in which the composition of the two right vertical maps is just $\kappa_{\mathcal{X}_\eta}^p$. By [Ber00, Lemma 4.1], the upper and middle horizontal maps are both isomorphisms. By the discussion after [Ber00, Theorem 1.1], the map $H^p(\mathcal{X}_\eta^{\text{an}}, \mathbb{Q}_\ell) \rightarrow H^p(\mathcal{X}_{\eta,a}^{\text{an}}, \mathbb{Q}_\ell)$ and therefore the map $H_{\text{Zar}}^p(Y, \mathbb{Q}_\ell) \rightarrow H_{\text{Zar}}^p(Y_s, \mathbb{Q}_\ell)$ are injective. Now we compute the composite map

$$H_{\text{Zar}}^p(Y_s, \mathbb{Q}_\ell) \rightarrow H_{\text{ét}}^p(Y_s, \mathbb{Q}_\ell) \rightarrow H_{\text{ét}}^p(\mathcal{X}_{\eta,a}, \mathbb{Q}_\ell). \quad (4.6)$$

To compute the term $H_{\text{Zar}}^p(Y_s, \mathbb{Q}_\ell)$, we apply the topological proper descent theorem [SGA4-2, Exposé vbis, Corollaire 4.1.6] to the covering $\coprod_{i=1}^m Y_s^i \rightarrow Y_s$. Note that for every smooth scheme Z over $\tilde{K}^s$, the cohomology $H_{\text{Zar}}^i(Z, \mathbb{Q}_\ell)$ vanishes for $i > 0$ as constant sheaves are flasque over Z_{Zar} . By the descent spectral sequence, we have a canonical isomorphism

$$H_{\text{Zar}}^p(Y_s, \mathbb{Q}_\ell) \simeq \frac{\ker(H_{\text{Zar}}^0(Y_s^{(p)}, \mathbb{Q}_\ell) \xrightarrow{\delta_p^*} H_{\text{Zar}}^0(Y_s^{(p+1)}, \mathbb{Q}_\ell))}{\text{im}(H_{\text{Zar}}^0(Y_s^{(p-1)}, \mathbb{Q}_\ell) \xrightarrow{\delta_{p-1}^*} H_{\text{Zar}}^0(Y_s^{(p)}, \mathbb{Q}_\ell))}.$$

Thus, the map (4.6) is an isomorphism onto its image, which is $E_2^{p,0}$. Therefore, the map $\kappa_{\mathcal{X}_\eta}^p$ from (4.5) is injective with image contained in $E_2^{p,0}$.

Moreover, if every irreducible component of $Y^{(r)}$ is geometrically irreducible, then the restriction map $H_{\text{Zar}}^0(Y^{(p)}, \mathbb{Q}_\ell) \rightarrow H_{\text{Zar}}^0(Y_s^{(p)}, \mathbb{Q}_\ell)$ is an isomorphism for every $p \geq 0$. Thus, again by topological proper descent, the map $H_{\text{Zar}}^p(Y, \mathbb{Q}_\ell) \rightarrow H_{\text{Zar}}^p(Y_s, \mathbb{Q}_\ell)$ is an isomorphism. The lemma follows. $\square$

LEMMA 4.8. *Suppose that one of following conditions holds:*

- (i) *The field K is local non-Archimedean.*
- (ii) *There is a finite extension K' of K such that $X \otimes_K K'$ is the generic fiber of a proper strictly semistable scheme over K'° .*

Then the map κ_X^p is injective for all $p \geq 0$.

Proof. Case (i) follows from [Ber00, Corollary 1.2]. Case (ii) follows from Lemma 4.7 and the fact that the map $H^p(X^{\text{an}}, \mathbb{Q}_\ell) \rightarrow H^p((X \otimes_K K')^{\text{an}}, \mathbb{Q}_\ell)$ is injective. $\square$

5. Monodromy maps for tropical Dolbeault cohomology

In this section, we propose the conjecture on the isomorphism of tropical Dolbeault cohomology groups under monodromy maps. Then, we prove our main results.

Let K be a non-Archimedean field. Let X be a separated scheme of finite type over K . Note that X^{an} is paracompact.

DEFINITION 5.1 (Tropical Dolbeault cohomology, [Liu17]). We define the *tropical Dolbeault cohomology* of X to be

$$H_{\text{trop}}^{p,q}(X) := H^q(X^{\text{an}}, \mathcal{T}_{X^{\text{an}}}^p).$$

Then $H_{\text{trop}}^{p,q}(X)_{\mathbb{R}}$ is canonically isomorphic to $H^{p,q}(X^{\text{an}})$ by Proposition 3.4. We define the corresponding *tropical Hodge number* of X to be

$$h_{\text{trop}}^{p,q}(X) := \dim_{\mathbb{Q}} H_{\text{trop}}^{p,q}(X).$$

We have the monodromy map

$$N_X := N_{X^{\text{an}}}: H_{\text{trop}}^{p,q}(X)_{\mathbb{R}} \rightarrow H_{\text{trop}}^{p-1,q+1}(X)_{\mathbb{R}}$$

for $p \geq 1$ by Remark 3.5.

Remark 5.2. Since $\mathcal{T}_{X^{\text{an}}}^0$ is the constant sheaf $\mathbb{Q}$, we have $H_{\text{trop}}^{0,q}(X) = H^q(X^{\text{an}}, \mathbb{Q})$.

CONJECTURE 5.3. Suppose that K is an algebraically closed, non-Archimedean field such that $\tilde{K}$ is algebraic over a finite field. Let X be a proper, smooth scheme over K . Then for $p \geq q \geq 0$, the (iterated) monodromy map

$$N_X^{p-q}: H_{\text{trop}}^{p,q}(X)_{\mathbb{R}} \rightarrow H_{\text{trop}}^{q,p}(X)_{\mathbb{R}}$$

is an isomorphism.

Remark 5.4. Conjecture 5.3 does not hold for arbitrary algebraically closed non-Archimedean fields. In fact, let X_0 be the generic fiber of the scheme $\mathfrak{X}$ in [BGS95, Section (6.1)], which is a geometrically connected, projective, smooth curve over $K_0 := \mathbb{C}((t))$. Now take $K = \widehat{K_0^a}$ and $X = X_0 \otimes_{K_0} K$. One can show that $h_{\text{trop}}^{1,0}(X) = 0$ but $h_{\text{trop}}^{0,1}(X) = 2$. This also implies that the canonical pairing

$$H_{\text{trop}}^{1,0}(X)_{\mathbb{R}} \times H_{\text{trop}}^{0,1}(X)_{\mathbb{R}} \rightarrow H_{\text{trop}}^{1,1}(X)_{\mathbb{R}} \xrightarrow{\text{Tr}} \mathbb{R}$$

(where Tr is the integration map) is not a perfect pairing; in other words, the Poincaré duality fails.

DEFINITION 5.5. Let X be a proper, smooth scheme over a discrete non-Archimedean field K . A *strict semistable alteration* of X is a proper, strictly semistable scheme $\mathcal{X}$ over $K_{\mathcal{X}}^{\circ}$ for some finite extension $K_{\mathcal{X}}/K$ contained in K^a , together with a proper, generically étale morphism $\phi: \mathcal{X}_{\eta} \rightarrow X$ over K , such that every irreducible component of $Y^{(r)}$ for $r \geq 0$ is geometrically irreducible.

THEOREM 5.6. Let X be a proper, smooth scheme over a discrete non-Archimedean field K and $p \geq 0$ an integer. Suppose that the map κ_X^p from (4.5) is injective.

(i) If $\log |x| \in \mathbb{Q}$ for every $x \in K^{\times}$, then the monodromy map

$$N_X^p: H_{\text{trop}}^{p,0}(X)_{\mathbb{R}} \rightarrow H_{\text{trop}}^{0,p}(X)_{\mathbb{R}}$$

preserves the rational structure; that is, it sends $H_{\text{trop}}^{p,0}(X)$ into $H_{\text{trop}}^{0,p}(X)$.

(ii) Suppose that for every strict semistable alteration $\mathcal{X}$ of X (Definition 5.5), the map

$$N_{\mathcal{X}}^p: E_2^{-p,2p}(p) \rightarrow E_2^{p,0}$$

from (4.3) is injective. Then $N_{\mathcal{X}}^p: H_{\text{trop}}^{p,0}(X)_{\mathbb{R}} \rightarrow H_{\text{trop}}^{0,p}(X)_{\mathbb{R}}$ is injective.

We need some preparation before the proof. The case $p = 0$ is trivial. So we assume $p \geq 1$. The following notation will be used later.

Notation 5.7. Let $h: T' \rightarrow T$ be a homomorphism of K -analytic torus. Then we denote by $h^b: T'_{\text{trop}} \rightarrow T_{\text{trop}}$ the induced linear map under tropicalization.

For a proper, strictly semistable scheme $\mathcal{X}$ over $K_{\mathcal{X}}^{\circ}$, recall that we have the reduction map

$$\pi_{\mathcal{X}}: \mathcal{X}_{\eta}^{\text{an}} \rightarrow Y,$$

where Y is the special fiber of $\mathcal{X}$ as before.

Let ω be an element in $H_{\text{trop}}^{p,0}(X) = H^0(X^{\text{an}}, \mathcal{T}_{X^{\text{an}}}^p)$. We say that a strict semistable alteration $\mathcal{X}$ of X (with a morphism $\phi: \mathcal{X}_{\eta} \rightarrow X$) *presents* ω if for every irreducible component Y^i of Y , there exist

- an open subset U of $\mathcal{X}_{\eta}^{\text{an}}$ containing $\pi_{\mathcal{X}}^{-1}Y^i$,
- $c_l \in \mathbb{Q}$ for $1 \leq l \leq M$ with some integer $M = M_U \geq 1$,
- $f_{lk} \in \mathcal{O}_{\mathcal{X}_{\eta}^{\text{an}}}^*(U)$ for $1 \leq l \leq M$ and $1 \leq k \leq p$ satisfying $|f_{lk}| = 1$ on $\pi_{\mathcal{X}}^{-1}(Y^i \setminus Y^{[1]})$

such that

$$\phi^{\text{an}*}\omega|_U = \tau \left(\sum_{l=1}^M c_l \{f_{l1}, \dots, f_{lp}\} \right),$$

where τ is the map (3.2). We call such data $(U, \{c_l\}, \{f_{lk}\})$ a *presentation* of ω on Y^i .

Now let Z be an irreducible component of Y^I for some $I = \{i_0 < \dots < i_p\}$ with $p \geq 1$. Choose a presentation $(U, \{c_l\}, \{f_{lk}\})$ of ω on Y^{i_0} . For $j = 1, \dots, p$, let $a_{lkj} \in \mathbb{Z}$ be the order of zero (or the negative order of pole) of f_{lk} along the connected component of $Y^{i_0} \cap Y^{i_j}$ containing Z .

LEMMA 5.8. *Let the notation be as above. The rational number $\sum_{l=1}^M c_l \det(a_{lkj})_{k,j=1}^p$ depends only on $\phi^{\text{an}*}\omega$ and Z .*

Proof. Let η_Z be the generic point of Z . Then $I(\eta_Z) = \{i_0 < \dots < i_p\}$ (Notation 4.2). We choose a semistable chart $(\mathcal{V}, \{g_0, \dots, g_p\})$ of η_Z (Definition 4.3). Put $V := \mathcal{V} \cap Y^{i_0}$. Since U contains $\pi_{\mathcal{X}}^{-1}Y^{i_0}$, since $\mathcal{V}_{\eta}^{\text{an}}$ contains $\pi_{\mathcal{X}}^{-1}(\mathcal{V} \cap Y)$, and since both U and $\mathcal{V}_{\eta}^{\text{an}}$ are open in $\mathcal{X}_{\eta}^{\text{an}}$, we know that $U \cap \mathcal{V}_{\eta}^{\text{an}}$ is an open neighborhood of $\pi_{\mathcal{X}}^{-1}V$. Now we replace U by $U \cap \mathcal{V}_{\eta}^{\text{an}}$ and regard $g_1, \dots, g_p$ as elements in $\mathcal{O}_{\mathcal{X}_{\eta}^{\text{an}}}^*(U)$. Then for $1 \leq j \leq p$, we have $|g_j| = 1$ on $\pi_{\mathcal{X}}^{-1}(V \setminus Y^{[1]})$, and the divisor associated with the reduction $\tilde{g}_j$ on V is $V \cap Y^{i_j}$.

Note that the divisor associated with $\widetilde{f'_{lk}}|_V$ is supported on $V \cap (\bigcup_{j=1}^p Y^{i_j})$. Put

$$f'_{lk} = f_{lk} \cdot \prod_{j=1}^p g_j^{-a_{lkj}}.$$

Then the reduction $\widetilde{f'_{lk}}$ is invertible on V . In other words, $|f'_{lk}| = 1$ on $\pi_{\mathcal{X}}^{-1}V$. Thus, we have

$\tau(\{f'_{lk}\}) = 0 \in \mathbb{R}$ on $\pi_{\mathcal{X}}^{-1}V$. An elementary calculation shows that

$$\phi^{\text{an}*}\omega|_{\pi_{\mathcal{X}}^{-1}V} = \tau\left(\sum_{l=1}^M c_l \{f_{l1}, \dots, f_{lp}\}\right) = \left(\sum_{l=1}^M c_l \det(a_{lkj})_{k,j=1}^p\right) \tau(\{g_1, \dots, g_p\}). \quad (5.1)$$

It is clear that $\tau(\{g_1, \dots, g_p\})$ is nonzero as V contains η_Z . Since the choice of the semistable chart is independent of the presentation ω on Y^{i_0} , the lemma then follows. $\square$

We denote the rational number in Lemma 5.8 by $\text{ord}_{\omega}(Z)$. The assignment $Z \mapsto \text{ord}_{\omega}(Z)$ gives rise to an element $\text{ord}_{\omega} \in H_{\text{ét}}^0(Y_s^{(p)}, \mathbb{Q}_{\ell})$. Denote by $H_{\text{trop}}^{p,0}(X)_{\mathcal{X}}$ the subset of $H_{\text{trop}}^{p,0}(X)$ consisting of the ω which $\mathcal{X}$ presents. Then it is easy to see that $H_{\text{trop}}^{p,0}(X)_{\mathcal{X}}$ is a $\mathbb{Q}$ -subspace of $H_{\text{trop}}^{p,0}(X)$.

LEMMA 5.9. *Let $\text{ord}: H_{\text{trop}}^{p,0}(X)_{\mathcal{X}} \rightarrow H_{\text{ét}}^0(Y_s^{(p)}, \mathbb{Q}_{\ell})$ be the map sending ω to ord_{ω} .*

(i) *The map ord is injective.*

(ii) *The image of ord is contained in the subspace $E_2^{-p,2p}(p) \subset H_{\text{ét}}^0(Y_s^{(p)}, \mathbb{Q}_{\ell})$ defined in (4.4).*

Proof. For statement (i), take an element $\omega \in H_{\text{trop}}^{p,0}(X)_{\mathcal{X}}$. Let $x \in \mathcal{X}_{\eta}^{\text{an}}$ be a point, and put $y := \pi_{\mathcal{X}}(x) \in Y$. Suppose $I(y) = \{i_0 < \dots < i_r\}$ (Notation 4.2) for some $r \geq 0$, and choose a semistable chart $(\mathcal{V}, \{g_0, \dots, g_r\})$ of y (Definition 4.3). Put $V := \mathcal{V} \cap Y^{i_0}$.

Choose a presentation $(U, \{c_l\}, \{f_{lk}\})$ of ω on Y^{i_0} . As in the proof of Lemma 5.8, we replace U with $U \cap \mathcal{V}_{\eta}^{\text{an}}$ and view $g_1, \dots, g_r$ as being in $\mathcal{O}_{\mathcal{X}_{\eta}^{\text{an}}}(U)$. Then there exist unique integers a_{lkj} such that if we put $f'_{lk} := \prod_{j=1}^r g_j^{a_{lkj}}$, then $|f'_{lk}| = |f_{lk}|$ on $\pi_{\mathcal{X}}^{-1}V$. Let f (respectively, f') be the moment map $\pi_{\mathcal{X}}^{-1}V \rightarrow (\mathbf{G}_m^{\text{an}})^{Mp}$ induced by $\{f_{lk}\}$ (respectively, $\{f'_{lk}\}$). Then $f_{\text{trop}} = f'_{\text{trop}}$. In particular,

$$\phi^{\text{an}*}\omega|_{\pi_{\mathcal{X}}^{-1}V} = \tau\left(\sum_{l=1}^M c_l \{f'_{l1}, \dots, f'_{lp}\}\right).$$

Let $g: \pi_{\mathcal{X}}^{-1}V \rightarrow (\mathbf{G}_m^{\text{an}})^r$ be the moment map induced by $\{g_1, \dots, g_r\}$. Then there exists a unique homomorphism $h: (\mathbf{G}_m^{\text{an}})^r \rightarrow (\mathbf{G}_m^{\text{an}})^{Mp}$, determined by integers $\{a_{lkj}\}$ such that $f_{\text{trop}} = (h \circ g)_{\text{trop}}$. Now if $\text{ord}_{\omega}(Z) = 0$ for every irreducible component Z of $Y^{[p]}$, then the pullback of $\tau(\sum_{l=1}^M c_l \{f'_{l1}, \dots, f'_{lp}\})$ under h^b (Notation 5.7) is zero. In other words, we have $\phi^{\text{an}*}\omega|_{\pi_{\mathcal{X}}^{-1}V} = 0$ by the equality (5.1). Thus, the map $\text{ord}: H_{\text{trop}}^{p,0}(X)_{\mathcal{X}} \rightarrow H_{\text{ét}}^0(Y_s^{(p)}, \mathbb{Q}_{\ell})$ is injective since $\pi_{\mathcal{X}}^{-1}V$ contains x and x is arbitrary; statement (i) follows.

For statement (ii), we need to show that $\delta_{p*} \text{ord}_{\omega} = \delta_p^* \text{ord}_{\omega} = 0$. Suppose that Z is an irreducible component of Y^I for some $I = \{i_0 < \dots < i_p\}$. For every permutation σ of the set $\{0, \dots, p\}$, we may define in the same way a rational number $\text{ord}_{\omega}^{\sigma}(Z)$ by replacing i_j by $i_{\sigma(j)}$. If $\sigma(0) = 0$, then we let $\epsilon(\sigma) \in \{\pm 1\}$ be the sign of the permutation $\sigma|_{\{1, \dots, p\}}$. If $\sigma(0) \neq 0$, then we let $\epsilon(\sigma) \in \{\pm 1\}$ be the negative of the sign of the permutation from $\{1, \dots, 0, \dots, p\}$ (where $\sigma(0)$ is replaced by 0) to $\{\sigma(1), \dots, \sigma(p)\}$. Then a calculation similar to that in the proof of Lemma 5.8 implies

$$\text{ord}_{\omega}^{\sigma}(Z) = \epsilon(\sigma) \cdot \text{ord}_{\omega}(Z). \quad (5.2)$$

We start from $\delta_{p*} \text{ord}_{\omega}$. Fix an irreducible component W of Y^J for some $J = \{i_0 < \dots < i_{p-1}\}$. For $1 \leq j \leq p-1$, let W_j be the unique irreducible component of $Y^{i_0} \cap Y^{i_j}$ that contains W . Choose a presentation $(U, \{c_l\}, \{f_{lk}\})$ of ω on Y^{i_0} . By linear algebra, it is easy to see that we may choose f_{lk} such that $\text{ord}_{W_j}(f_{lk}) = 0$ if $j < k$. In particular, the restriction of $\widetilde{f_{lp}}$ on W is a nonzero

rational function, which we denote by f_{lp}^W . Put $b_{l,j} = \text{ord}_{W_j}(\widetilde{f_{lj}})$ for $1 \leq j \leq p-1$. Let Z be an irreducible component of $W \cap Y^{[p]}$. Then there is a unique element $i_p \in \{1, \dots, m\} \setminus \{i_0, \dots, i_{p-1}\}$ such that $Z \subset W \cap Y^{i_p}$. Let $0 \leq \epsilon_Z \leq p$ be the integer such that there are exactly ϵ_Z elements in $\{i_0, \dots, i_{p-1}\}$ that are greater than i_p . Then by (5.2), we have

$$\text{ord}_\omega(Z) = (-1)^{\epsilon_Z} \sum_{l=1}^M c_l b_{l,1} \cdots b_{l,p-1} \text{ord}_Z(f_{lp}^W).$$

Therefore, we have the equality

$$(-1)^{p-\epsilon_Z} \sum_{Z \subset W} \text{ord}_\omega(Z) Z = (-1)^p \sum_{l=1}^M c_l b_{l,1} \cdots b_{l,p-1} \text{div}(f_{lp}^W) \quad (5.3)$$

of divisors on W . By the definition (4.2) of δ_{p*} , the value of $\delta_{p*} \text{ord}_\omega$ on W , which is an element in $H_{\text{et}}^2(W_s, \mathbb{Q}_\ell(1))$, is equal to the cycle class of the left-hand side of (5.3). Since $\text{div}(f_{lp}^W) = 0$ in $H_{\text{et}}^2(W_s, \mathbb{Q}_\ell(1))$, we have $\delta_{p*} \text{ord}_\omega = 0$ by (5.3).

Now we consider $\delta_p^* \text{ord}_\omega$. Let W be an irreducible component of Y^J for some $J = \{i_0 < \dots < i_{p+1}\}$. For $0 \leq j \leq p+1$, let Z_j be the unique irreducible component of $Y^{J \setminus \{i_j\}}$ containing W . Let $(\mathcal{V}, \{g_0, \dots, g_{p+1}\})$ be a semistable chart of the generic point of W (Definition 4.3). Put $V = \mathcal{V} \cap Y$ and $V_j = V \setminus Y^{i_j}$. From the proof of part (i), we know that

$$\phi^{\text{an}*} \omega|_{\pi_X^{-1}V} = \sum_{\alpha} c_{\alpha} \tau(\{g_{\alpha(1)}, \dots, g_{\alpha(p)}\})$$

for some $c_{\alpha} \in \mathbb{Q}$, where the sum is taken over all strictly increasing maps $\alpha: \{1, \dots, p\} \rightarrow \{0, \dots, p+1\}$. We take such a map α . Now let $\alpha_0 < \alpha_1$ be the two integers in $\{0, \dots, p+1\}$ not in the image of α . We have three cases:

- If $j \in \text{im}(\alpha)$, then $\tau(\{g_{\alpha(1)}, \dots, g_{\alpha(p)}\})|_{\pi_X^{-1}V_j} = 0$.
- If $j = \alpha_0$, then $\tau(\{g_{\alpha(1)}, \dots, g_{\alpha(p)}\})|_{\pi_X^{-1}V_j} = (-1)^{\alpha_1-1} \tau(\{g_{\alpha'(1)}, \dots, g_{\alpha'(p)}\})|_{\pi_X^{-1}V_j}$, where α' is the unique map whose image does not include α_0 and $\min(\{0, \dots, p+1\} \setminus \{\alpha_0\})$.
- If $j = \alpha_1$, then $\tau(\{g_{\alpha(1)}, \dots, g_{\alpha(p)}\})|_{\pi_X^{-1}V_j} = (-1)^{\alpha_0} \tau(\{g_{\alpha'(1)}, \dots, g_{\alpha'(p)}\})|_{\pi_X^{-1}V_j}$, where α' is the unique map whose image does not include $\{0, \alpha_1\}$.

Now we may choose an open affine subscheme $\mathcal{V}_j$ of $\mathcal{V}$ with $\mathcal{V}_j \cap Y = V_j$ such that $(\mathcal{V}_j, \{g_0, \dots, \widehat{g_j}, \dots, g_{p+1}\})$ is a semistable chart of the generic point of Z_j . Then the above three cases imply that

$$\phi^{\text{an}*} \omega|_{\pi_X^{-1}V_j} = \left(\sum_{\alpha, \alpha_0=j} c_{\alpha} (-1)^{\alpha_1-1} + \sum_{\alpha, \alpha_1=j} c_{\alpha} (-1)^{\alpha_0} \right) \tau(\{g_1, \dots, \widehat{g_j}, \dots, g_{p+1}\}).$$

Comparing with the formula (5.1), we have

$$\text{ord}_\omega(Z_j) = \sum_{\alpha, \alpha_0=j} c_{\alpha} (-1)^{\alpha_1-1} + \sum_{\alpha, \alpha_1=j} c_{\alpha} (-1)^{\alpha_0}.$$

However, by (4.1), we have

$$\delta_p^* \omega|_W = \sum_{j=0}^{p+1} (-1)^j \text{ord}_\omega(Z_j) = \sum_{\alpha} c_{\alpha} (-1)^{\alpha_0+\alpha_1-1} + c_{\alpha} (-1)^{\alpha_0+\alpha_1} = 0.$$

Thus, part (ii) follows. $\square$

The map ord in Lemma 5.9 is inspired by and closely related to the construction in [Liu17, Section 4]. The following lemma is the key step to the proof of Theorem 5.6. It also justifies the terminology *monodromy map* for N_X in Definitions 3.1 and 5.1.

LEMMA 5.10. *Let $\mathcal{X}$ be a strict semistable alteration of X , with $\phi: \mathcal{X}_\eta \rightarrow X$. Suppose that we have $\log |\varpi| = -1$ for a uniformizer ϖ of K . Then the image of the restriction of $N_X^p: H_{\text{trop}}^{p,0}(X)_{\mathbb{R}} \rightarrow H_{\text{trop}}^{0,p}(X)_{\mathbb{R}}$ to $H_{\text{trop}}^{p,0}(X)_{\mathcal{X}}$ is contained in $H_{\text{trop}}^{0,p}(X) = H^p(X^{\text{an}}, \mathbb{Q})$ (Remark 5.2). Moreover, the diagram*

$$\begin{array}{ccc} H_{\text{trop}}^{p,0}(X)_{\mathcal{X}} & \xrightarrow{\text{ord}} & E_2^{-p,2p}(p) \\ N_X^p \downarrow & & \downarrow N_{\mathcal{X}}^p \\ H^p(X^{\text{an}}, \mathbb{Q}) & \longrightarrow & E_2^{p,0} \end{array} \quad (5.4)$$

commutes. Here, the bottom map is the composition of

- the multiplication map $(-1)^{p(p+1)/2}: H^p(X^{\text{an}}, \mathbb{Q}) \rightarrow H^p(X^{\text{an}}, \mathbb{Q}_\ell)$,
- the map $\kappa_X^p: H^p(X^{\text{an}}, \mathbb{Q}_\ell) \rightarrow H_{\text{ét}}^p(X_a, \mathbb{Q}_\ell)$ defined in (4.5) (which is assumed to be injective), and
- the pullback map $\phi^*: H_{\text{ét}}^p(X_a, \mathbb{Q}_\ell) \rightarrow H_{\text{ét}}^p(\mathcal{X}_{\eta,a}, \mathbb{Q}_\ell)$.

The image of the bottom map is contained in $E_2^{p,0}$ by Lemma 4.7.

Proof. We first claim that $\phi^*: H_{\text{ét}}^p(X_a, \mathbb{Q}_\ell) \rightarrow H_{\text{ét}}^p(\mathcal{X}_{\eta,a}, \mathbb{Q}_\ell)$ is injective. In fact, for every $\alpha \in H_{\text{ét}}^p(X_a, \mathbb{Q}_\ell)$ and $\beta \in H_{\text{ét}}^{2\dim X - p}(X_a, \mathbb{Q}_\ell)$, we have $\phi^*\alpha \cup \phi^*\beta = \phi^*(\alpha \cup \beta)$. If $\phi^*\alpha = 0$, then $\phi^*(\alpha \cup \beta) = 0$. However, since $\phi^*: H_{\text{ét}}^{2\dim X}(X_a, \mathbb{Q}_\ell) \rightarrow H_{\text{ét}}^{2\dim X}(\mathcal{X}_{\eta,a}, \mathbb{Q}_\ell)$ is injective as ϕ is proper generically étale, we must have $\alpha \cup \beta = 0$ for every β . Thus, $\alpha = 0$ by Poincaré duality [Ber93, Theorem 7.3.1].

Now for every embedding $\iota: \mathbb{R} \hookrightarrow \mathbb{Q}_\ell^a$, we have an induced map $\kappa_\iota: H^p(X^{\text{an}}, \mathbb{R}) \rightarrow E_2^{p,0} \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell^a$ from the bottom map of (5.4), which is injective by the previous claim. If we can show that the diagram

$$\begin{array}{ccc} H_{\text{trop}}^{p,0}(X)_{\mathcal{X}} & \xrightarrow{\text{ord}} & E_2^{-p,2p}(p) \\ N_X^p \downarrow & & \downarrow N_{\mathcal{X}}^p \\ H^p(X^{\text{an}}, \mathbb{R}) & \xrightarrow{\kappa_\iota} & E_2^{p,0} \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell^a \end{array} \quad (5.5)$$

commutes for every ι , then the composite map $\kappa_\iota \circ N_X^p$ does not depend on the choice of ι . An easy argument from linear algebra implies that the image of N_X^p must be contained in $H^p(X^{\text{an}}, \mathbb{Q})$ and that the original diagram (5.4) commutes.

Now we show the commutativity of (5.5). To simplify the notation, put

$$\mathcal{A}_?^{p,q,\text{cl}} := \ker(d'': \mathcal{A}_?^{p,q} \rightarrow \mathcal{A}_?^{p,q+1}).$$

Take an element $\omega \in H_{\text{trop}}^{p,0}(X)_{\mathcal{X}}$. By Lemma 3.2(i), the map $N_X^p(\omega)$ is represented by the Dolbeault representative $p!J\omega \in H^0(X^{\text{an}}, \mathcal{A}_{X^{\text{an}}}^{0,p,\text{cl}})$.

For $1 \leq i \leq m$, put $U^i := \pi_{\mathcal{X}}^{-1}Y^i$, and choose a presentation $(U^i, \{c_l^i\}, \{f_{lk}^i\} \mid 1 \leq l \leq M_i, 1 \leq k \leq p)$ of ω on Y^i . For every subset $I \subset \{1, \dots, m\}$, put $U^I := \bigcap_{i \in I} U^i = \pi_{\mathcal{X}}^{-1}Y^I$. Since $\pi_0(U^I) = \pi_0(Y^I)$, the assignment $Z \mapsto \text{ord}_\omega(Z)$ gives rise to a Čech p -cocycle θ_ω for the sheaf $\mathbb{Q}$

with respect to the ordered open covering $\underline{U} = \{U^1, \dots, U^m\}$ of $\mathcal{X}_\eta^{\text{an}}$. Since $\delta_p^* \text{ord}_\omega = 0$ by Lemma 5.9(ii), the cocycle θ_ω is closed hence gives rise to a class in $H^p(\underline{U}, \mathbb{Q})$, whose image $[\theta_\omega] \in H^p(\mathcal{X}_\eta^{\text{an}}, \mathbb{Q})$ coincides with $(\kappa_{\mathcal{X}_\eta}^p)^{-1}(N_{\mathcal{X}}^p(\text{ord}_\omega))$ by (4.4), where $\kappa_{\mathcal{X}_\eta}^p: H^p(\mathcal{X}_\eta^{\text{an}}, \mathbb{Q}_\ell) \xrightarrow{\sim} E_2^{p,0}$ is the isomorphism in Lemma 4.7. Therefore by Lemma 3.2(i), the commutativity of (5.5) is equivalent to the statement that $(-1)^{p(p+1)/2} p! J\omega$ is a Dolbeault representative of $[\theta_\omega]$.

Let us recall the construction of Dolbeault representatives, following the similar procedure in the proof of [GH94, Section 5.1, first lemma, p. 651]. For an abelian sheaf $\mathcal{F}$ on $\mathcal{X}_\eta^{\text{an}}$ and $r \geq 0$, denote by

$$C^r(\underline{U}, \mathcal{F}) = \bigoplus_{|I|=r+1} \Gamma(U^I, \mathcal{F})$$

the abelian group of Čech r -cocycles for $\mathcal{F}$ with respect to $\underline{U}$. We have a coboundary map $\delta: C^r(\underline{U}, \mathcal{F}) \rightarrow C^{r+1}(\underline{U}, \mathcal{F})$. If we have elements $\theta_r \in C^r(\underline{U}, \mathcal{A}_{\mathcal{X}_\eta^{\text{an}}}^{0,p-r-1})$ for $0 \leq r \leq p-1$ satisfying

$$d''\theta_0 = J\omega, \quad d''\theta_1 = \delta\theta_0, \dots, \quad d''\theta_{p-1} = \delta\theta_{p-2}, \quad \frac{(-1)^{p(p+1)/2}}{p!} \theta_\omega = \delta\theta_{p-1}, \quad (5.6)$$

then $(-1)^{p(p+1)/2} p! J\omega$ is a Dolbeault representative of $[\theta_\omega]$ and (5.5) commutes. Here, we regard $J\omega$ as an element in $C^0(\underline{U}, \mathcal{A}_{\mathcal{X}_\eta^{\text{an}}}^{0,p})$.

The remainder of the proof will be dedicated to the construction of θ_r . For $r \geq 0$, denote the set of irreducible components of $Y^{[r]}$ by $\mathcal{D}^r$.

Step 1. For $Z \in \mathcal{D}^r$, put $U_Z := \pi_{\mathcal{X}}^{-1}Z$ and $M_Z := \sum_{i \in I(Z)} M_i$, and let $f_Z: U_Z \rightarrow (\mathbf{G}_m^{\text{an}})^{M_Z p}$ be the moment map given by invertible functions $\{f_{lk}^i \mid i \in I(Z), 1 \leq l \leq M_i, 1 \leq k \leq p\}$ on U . Put $U_Z^\circ := \pi_{\mathcal{X}}^{-1}(Z \setminus Y^{[r+1]})$. The image of the tropicalization map $f_{Z, \text{trop}}: U_Z \rightarrow \mathbb{R}^{M_Z p}$, denoted by C_Z , is canonically a simplicial complex, induced by the reduced, normal crossing divisor $Z \cap Y^{[r+1]}$ of Z , with the unique minimal simplex $\Delta_Z := f_{Z, \text{trop}}(U_Z^\circ)$. We now define the barycenter P_Z of Δ_Z as follows.

Let η_Z be the generic point of Z . Let $(\mathcal{V}, \{g_0, \dots, g_r\})$ be a semistable chart at η_Z (Definition 4.3). We let $g_Z: U_Z \cap \mathcal{V}_\eta^{\text{an}} \rightarrow (\mathbf{G}_m^{\text{an}})^{r+1}$ be the moment map induced by $\{g_0, \dots, g_r\}$. Then the image of $g_{Z, \text{trop}}$ is the standard (open) simplex Δ° in $\mathbb{R}^{r+1}$, which is $\{(x_0, \dots, x_r) \in \mathbb{R}^{r+1} \mid x_0 + \dots + x_r = 1, x_i > 0\}$. As in the proof of Lemma 5.9, we have unique integers a_{lkj}^i such that

$$f_{lk}^i \cdot \prod_{j=0, j \neq i}^r g_j^{-a_{lkj}^i}$$

has norm 1 on $\pi_{\mathcal{X}}^{-1}(Z \cap \mathcal{V}) \subset U_Z^\circ$. Then there exists a unique homomorphism

$$h_Z: (\mathbf{G}_m^{\text{an}})^{r+1} \rightarrow (\mathbf{G}_m^{\text{an}})^{M_Z p} \quad (5.7)$$

determined by integers $\{a_{lkj}^i\}$ such that $f_{Z, \text{trop}}|_{\pi_{\mathcal{X}}^{-1}(Z \cap \mathcal{V})} = (h_Z \circ g_Z)_{\text{trop}}$. Then we have

$$h_Z^b(\Delta^\circ) = f_{Z, \text{trop}}(\pi_{\mathcal{X}}^{-1}(Z \cap \mathcal{V})) \subset \Delta_Z,$$

where $h_Z^b: \mathbb{R}^{r+1} \rightarrow \mathbb{R}^{M_Z p}$ is the map induced by (5.7) (Notation 5.7). We now define P_Z to be the image of the barycenter of Δ° , which has coordinates $(1/(r+1), \dots, 1/(r+1))$, under h_Z^b . It is easy to see that the point P_Z does not depend on the choice of the semistable chart of η_Z (and in fact, $h_Z^b(\Delta^\circ) = \Delta_Z$). To summarize, we have $P_Z \in \Delta_Z \subset C_Z \subset \mathbb{R}^{M_Z p}$.

Step 2. Take another irreducible component $Z' \in \mathcal{D}^{r'}$ such that $Z \subset Z'$. In particular, we have $r' \leq r$ and $I(Z') \subset I(Z)$. Let

$$h_{Z,Z'} : (\mathbf{G}_m^{\text{an}})^{M_Z p} \rightarrow (\mathbf{G}_m^{\text{an}})^{M_{Z'} p} \quad (5.8)$$

be the canonical projection by forgetting components with $i \in I(Z) \setminus I(Z')$. Then the diagram

$$\begin{array}{ccc} U_Z & \xrightarrow{f_Z} & (\mathbf{G}_m^{\text{an}})^{M_Z p} \\ i_{Z,Z'} \downarrow & & \downarrow h_{Z,Z'} \\ U_{Z'} & \xrightarrow{f_{Z'}} & (\mathbf{G}_m^{\text{an}})^{M_{Z'} p} \end{array}$$

commutes, where $i_{Z,Z'}$ is the natural inclusion map. It induces, for every $q \geq 0$, the commutative diagram

$$\begin{array}{ccc} \mathcal{A}_{C_{Z'}}^{0,q} & \xrightarrow{\iota_{Z'}} & \mathcal{A}_{\mathcal{X}_\eta^{\text{an}}}^{0,q}(U_{Z'}) \\ (h_{Z,Z'}^\flat)^* \downarrow & & \downarrow i_{Z,Z'}^* \\ \mathcal{A}_{C_Z}^{0,q} & \xrightarrow{\iota_Z} & \mathcal{A}_{\mathcal{X}_\eta^{\text{an}}}^{0,q}(U_Z), \end{array}$$

where ι_Z and $\iota_{Z'}$ are natural maps from the definition of $\mathcal{A}_{\mathcal{X}_\eta^{\text{an}}}^{0,q}$.

On the other hand, we have a canonical isomorphism $C^r(\underline{U}, \mathcal{A}_{\mathcal{X}_\eta^{\text{an}}}^{0,q}) \simeq \bigoplus_{Z \in \mathcal{D}^r} \mathcal{A}_{\mathcal{X}_\eta^{\text{an}}}^{0,q}(U_Z)$, which induces a canonical map

$$\iota^r : \bigoplus_{Z \in \mathcal{D}^r} \mathcal{A}_{C_Z}^{0,q} \rightarrow C^r(\underline{U}, \mathcal{A}_{\mathcal{X}_\eta^{\text{an}}}^{0,q})$$

for $r \geq 0$, and there is a similarly defined coboundary map

$$\delta : \bigoplus_{Z \in \mathcal{D}^r} \mathcal{A}_{C_Z}^{0,q} \rightarrow \bigoplus_{Z' \in \mathcal{D}^{r+1}} \mathcal{A}_{C_{Z'}}^{0,q}, \quad (5.9)$$

using $(h_{Z,Z'}^\flat)^*$ as restriction maps, that is compatible with the coboundary map for $C^\bullet(\underline{U}, \mathcal{A}_{\mathcal{X}_\eta^{\text{an}}}^{0,q})$ under $\iota^\bullet$.

Moreover, since C_Z is star-shaped with respect to P_Z , we have the star-shape integration map

$$\mathcal{I}_{P_Z}'' : \mathcal{A}_{C_Z}^{0,q}(C_Z) \rightarrow \mathcal{A}_{C_Z}^{0,q-1}(C_Z)$$

for $q \geq 1$ (see the appendix for a formula on the standard simplex). We put

$$\mathcal{I}'' := \bigoplus_{Z \in \mathcal{D}^r} \mathcal{I}_{P_Z}'' : \bigoplus_{Z \in \mathcal{D}^r} \mathcal{A}_{C_Z}^{0,q}(C_Z) \rightarrow \bigoplus_{Z \in \mathcal{D}^r} \mathcal{A}_{C_Z}^{0,q-1}(C_Z).$$

Step 3. Note that $\mathcal{D}^0 = \{Y^i \mid 1 \leq i \leq m\}$. Define $\vartheta_0 \in \bigoplus_{Z \in \mathcal{D}^0} \mathcal{A}_{C_Z}^{0,p-1}(C_Z)$ by the formula

$$\vartheta_0(Y^i) = \mathcal{I}_{P_{Y^i}}'' \left(\sum_{l=1}^{M_i} c_l^i d'' x_{l1}^i \wedge \cdots \wedge d'' x_{lp}^i \right), \quad (5.10)$$

where $\{x_{lk}^i\}$ are the standard coordinates on $\mathbb{R}^{M_i p}$. For $1 \leq r \leq p-1$, define

$$\vartheta_r = \mathcal{I}''(\delta \vartheta_{r-1}) \in \bigoplus_{Z \in \mathcal{D}^r} \mathcal{A}_{C_Z}^{0,p-r-1}(C_Z).$$

We claim that

$$d'(\delta\vartheta_{r-1}) = d''(\delta\vartheta_{r-1}) = 0 \quad (5.11)$$

for $1 \leq r \leq p-1$ and

$$\delta\vartheta_{p-1} = \frac{(-1)^{p(p+1)/2}}{p!} \theta_w. \quad (5.12)$$

We leave the proof to the next step. Assuming these, we have $d''\vartheta_r = \delta\vartheta_{r-1}$ as $d''(\delta\vartheta_{r-1}) = 0$. Finally, we put $\theta_r = \iota^r \vartheta_r \in C^r(\underline{U}, \mathcal{A}_{\chi_\eta^{\text{an}}}^{0,p-r-1})$ for $0 \leq r \leq p-1$. The θ_r satisfy (5.6), and the lemma follows.

Step 4. We have to verify (5.11) and (5.12) for $(\delta\vartheta_{r-1})(Z) \in \mathcal{A}_{C_Z}^{0,p-r}(C_Z)$ for every $Z \in \mathcal{D}^r$. Without loss of generality, we assume that X has dimension n . For each fixed $Z \in \mathcal{D}^r$, it suffices to consider the restriction of $(\delta\vartheta_{r-1})(Z)$ to maximal (open) cells of C_Z , all of which are of the form $f_{Z,\text{trop}}(U_z)$, where $z \in Z$ is a closed point that belongs to $\mathcal{D}^n$.

The idea is to reverse the consideration. We fix a closed point $z \in \mathcal{D}^n$ and consider the restriction of $(\delta\vartheta_{r-1})(Z)$ to $f_{Z,\text{trop}}(U_z)$ for all $1 \leq r \leq p$ and all $Z \in \mathcal{D}^r$ such that $z \in Z$. We fix a semistable chart $(\mathcal{V}, \{g_0, \dots, g_n\})$ at z . Then $\mathcal{V}_\eta^{\text{an}}$ contains U_z . We have the moment map $g: \mathcal{V}_\eta^{\text{an}} \rightarrow (\mathbf{G}_m^{\text{an}})^{n+1}$ and $g_{\text{trop}}: \mathcal{V}_\eta^{\text{an}} \rightarrow \mathbb{R}^{n+1}$ such that $g_{\text{trop}}(U_z)$ is the standard open simplex Δ° in $\mathbb{R}^{n+1}$. Note that for every subset $I \subset I(z)$ of cardinality $r+1$, there is a unique element $Z \in \mathcal{D}^r$ that contains z , which we denote by Z_I . For every I as above, put

$$h_I := h_{z,Z_I} \circ h_z: (\mathbf{G}_m^{\text{an}})^{n+1} \rightarrow (\mathbf{G}_m^{\text{an}})^{M_{Z_I}p},$$

where h_z and h_{z,Z_I} are defined in (5.7) and (5.8), respectively. Then $f_{Z,\text{trop}}(U_z)$ is the image of Δ° under $h_I^\flat$. Consider the composite map

$$h_I^\dagger: \mathcal{A}_{C_{Z_I}}^{0,q}(C_{Z_I}) \xrightarrow{(h_I^\flat)^*} \mathcal{A}_\Delta^{0,q}(\Delta^\circ) \rightarrow \Omega^q(\Delta),$$

where the second map is simply regarding a $(0,q)$ -superform as an ordinary q -form (see the appendix for notation concerning Δ). Since $h_I^\flat$ is affine and $h_I(P^I) = P_{Z_I}$, the diagram

$$\begin{array}{ccc} \mathcal{A}_{C_{Z_I}}^{0,q}(C_{Z_I}) & \xrightarrow{h_I^\dagger} & \Omega^q(\Delta) \\ \mathcal{I}_{P_{Z_I}}'' \downarrow & & \downarrow \mathcal{I}_{P^I} \\ \mathcal{A}_{C_{Z_I}}^{0,q-1}(C_{Z_I}) & \xrightarrow{h_I^\dagger} & \Omega^{q-1}(\Delta) \end{array}$$

commutes. Moreover, we have the commutative diagram

$$\begin{array}{ccc} \bigoplus_{Z \in \mathcal{D}^r} \mathcal{A}_{C_Z}^{0,q}(C_Z) & \xrightarrow{h_r} & C^r(\Omega^q(\Delta)) \\ \delta \downarrow & & \downarrow \delta \\ \bigoplus_{Z' \in \mathcal{D}^{r+1}} \mathcal{A}_{C_{Z'}}^{0,q}(C_{Z'}) & \xrightarrow{h_{r+1}} & C^{r+1}(\Omega^q(\Delta)) \end{array}$$

concerning coboundary maps. Here, δ on the left-hand side is (5.9) and δ on the right-hand side

is defined in the appendix. The map h_r is the composition of the projection

$$\bigoplus_{Z \in \mathcal{D}^r} \mathcal{A}_{C_Z}^{0,q}(C_Z) \rightarrow \bigoplus_{I \subset I(z), |I|=r+1} \mathcal{A}_{C_{Z_I}}^{0,q}(C_{Z_I})$$

and $\bigoplus_{I \subset I(z), |I|=r+1} h_I^\dagger$.

Recall that we have the element $\vartheta_0 \in \bigoplus_{Z \in \mathcal{D}^0} \mathcal{A}_{C_Z}^{0,p-1}(C_Z)$ defined in (5.10). Put $\beta_0 := h_0(d''\vartheta_0) \in C^0(\Omega^p(\Delta))$. Since

$$(d''\vartheta_0)(Y^i) = \sum_{l=1}^{M_i} c_l^i d''x_{l1}^i \wedge \cdots \wedge d''x_{lp}^i$$

(Remark A.1), the element $h_0(d''\vartheta_0)(i)$ represents the form ω over U_z , for every $i \in I(z)$. In particular, $h_0(d''\vartheta_0)$ is of constant value $\beta \in \Omega^p(\Delta)$. Now we are in the situation of Proposition A.2. In particular, we have elements $\beta_r \in C^r(\Delta^{p-r}(\Delta))$ satisfying $\beta_r = \delta(\mathcal{I}\beta_{r-1})$ for $1 \leq r \leq p$. We prove by induction that

$$\beta_r = h_r(\delta\vartheta_{r-1}).$$

When $r = 1$, we have

$$\beta_1 = \delta(\mathcal{I}\beta_0) = \delta(\mathcal{I}(h_0(d''\vartheta_0))) = \delta(h_0(\mathcal{I}''(d''\vartheta_0))) = \delta(h_0(\vartheta_0)) = h_1(\delta\vartheta_0).$$

Suppose $\beta_r = h_r(\delta\vartheta_{r-1})$. Then we have

$$\beta_{r+1} = \delta(\mathcal{I}\beta_r) = \delta(\mathcal{I}(h_r(\delta\vartheta_{r-1}))) = \delta(h_r(\mathcal{I}''(\delta\vartheta_{r-1}))) = \delta(h_r(\vartheta_r)) = h_{r+1}(\delta\vartheta_r),$$

where we recall that $\vartheta_r = \mathcal{I}''(\delta\vartheta_{r-1})$ by definition.

Now Proposition A.2(i) (respectively, (ii)) implies that (5.11) (respectively, (5.12)) holds when restricted to $f_{Z,\text{trop}}(U_z)$. Since z is arbitrary, (5.11) and (5.12) hold. $\square$

LEMMA 5.11. We have $H_{\text{trop}}^{p,0}(X) = \bigcup_{\mathcal{X}} H_{\text{trop}}^{p,0}(X)_{\mathcal{X}}$, where the union is taken over all strict semistable alterations $\mathcal{X}$ of X .

Proof. Let ω be an element in $H_{\text{trop}}^{p,0}(X)$. Since X is proper, we may choose a finite open covering $\underline{U}$ of X^{an} such that for every $U \in \underline{U}$, there are $c_l \in \mathbb{Q}$ for $1 \leq l \leq M$ with some integer $M \geq 1$, and $f_{lk} \in \mathcal{O}_{X^{\text{an}}}^*(U)$ for $1 \leq l \leq M$ and $1 \leq k \leq p$, such that $\omega|_U = \tau(\sum_{l=1}^M c_l \{f_{l1}, \dots, f_{lp}\})$.

By [Liu17, Lemma 7.8], after taking a finite extension K'/K inside K^a , we have a (proper, flat) integral model $\mathcal{X}_0$ of $X \otimes_K K'$ such that if $Y_0^1, \dots, Y_0^{m_0}$ are all reduced irreducible components of $\mathcal{X}_0 \otimes_{K^{\circ}} \widehat{K}'$, then the covering $\{\pi_{\mathcal{X}_0}^{-1} Y_0^i | i = 1, \dots, m_0\}$ refines $\underline{U}$. By [dJo96, Theorem 8.2], we have a proper, strictly semistable scheme $\mathcal{X}$ over $K_{\mathcal{X}}^{\circ}$, where $K_{\mathcal{X}}/K'$ is a finite extension inside K^a , with a proper, generically étale morphism $\mathcal{X} \rightarrow \mathcal{X}_0$. Replacing $K_{\mathcal{X}}$ by a finite unramified extension inside K^a , we may assume that every irreducible component of $Y^{(p)}$ for $p \geq 0$ is geometrically irreducible. Therefore, $(\mathcal{X}, \phi: \mathcal{X}_{\eta} \rightarrow X)$ is a strict semistable alteration of X such that $\{\pi_{\mathcal{X}}^{-1} Y^i | i = 1, \dots, m\}$ refines $(\phi^{\text{an}})^{-1} \underline{U}$.

Now for an arbitrary irreducible component Y^i of Y , take an open subset $U \in \underline{U}$ such that $\pi_{\mathcal{X}}^{-1} Y^i \subset (\phi^{\text{an}})^{-1} U$. For every f_{lk} as above, we may choose an element $a_{lk} \in K^*$ such that $|a_{lk} \phi^{\text{an}*} f_{lk}| = 1$ on $\pi_{\mathcal{X}}^{-1}(Y^i \setminus Y^{[1]})$. Thus, $((\phi^{\text{an}})^{-1} U, \{c_l\}, \{a_{lk} \phi^{\text{an}*} f_{lk}\})$ is a presentation of ω on Y^i . Therefore, ω belongs to $H_{\text{trop}}^{p,0}(X)_{\mathcal{X}}$. $\square$

Proof of Theorem 5.6. Part (i) is a consequence of Lemmas 5.10 and 5.11.

For part (ii), it is equivalent to check that the map $N_X^p : H_{\text{trop}}^{p,0}(X) \rightarrow H^p(X^{\text{an}}, \mathbb{Q})$ is injective. Let ω be an element of $H_{\text{trop}}^{p,0}(X)$. By Lemma 5.11, we may assume $\omega \in H_{\text{trop}}^{p,0}(X)_{\mathcal{X}}$ for a strict semistable alteration (Definition 5.5) $\mathcal{X}$ of X . Then the injectivity follows from Lemmas 5.9 and 5.10 as we assume that $N_{\mathcal{X}}^p$ is injective. $\square$

COROLLARY 5.12. *Let X_0 be a proper smooth scheme over a non-Archimedean field K_0 that is isomorphic to $k((t))$ for k either a finite field or a field of characteristic zero. Let K be a closed subfield of $\widehat{K_0^a}$ containing K_0 . Then the monodromy map*

$$N_X^p : H_{\text{trop}}^{p,0}(X)_{\mathbb{R}} \rightarrow H_{\text{trop}}^{0,p}(X)_{\mathbb{R}}$$

is injective, where $X = X_0 \otimes_{K_0} K$. In particular, we have $h_{\text{trop}}^{p,0}(X) < \infty$ and $h_{\text{trop}}^{p-q,q}(X) \geq h_{\text{trop}}^{p,0}(X)$ for $0 \leq q \leq p$.

Proof. First, suppose that K/K_0 is a finite extension. By Lemma 4.8 and resolution of singularity when $\text{char } k = 0$, the map κ_X^p is injective. Then the corollary follows from Theorem 5.6 and Proposition 4.6(ii).

In general, since X_0 is proper, we have

$$H_{\text{trop}}^{p,0}(X) = \bigcup_{K_0 \subset K' \subset K} H_{\text{trop}}^{p,0}(X_0 \otimes_{K_0} K'),$$

where the union is taken over all finite extensions K'/K_0 contained in K . By [Ber99, Theorem 10.1], we know that $H_{\text{trop}}^{0,p}(X_0 \otimes_{K_0} K')$, which is isomorphic to $H^p(X_0^{\text{an}} \widehat{\otimes}_{K_0} K', \mathbb{Q})$ by Remark 5.2, stabilizes when K' increases. Therefore, the injectivity of N_X^p is reduced to the case where K/K_0 is finite. $\square$

In the case where $p = 1$, we have the following stronger result over certain non-Archimedean fields.

THEOREM 5.13. *Let K_0 be a local, non-Archimedean field and X_0 a proper, smooth scheme over K_0 that admits a proper, strictly semistable model over K_0° . Then the monodromy map*

$$N_X : H_{\text{trop}}^{1,0}(X)_{\mathbb{R}} \rightarrow H_{\text{trop}}^{0,1}(X)_{\mathbb{R}}$$

is an isomorphism, where $X = X_0 \otimes_{K_0} \widehat{K_0^a}$.

Proof. Note that every form in $H_{\text{trop}}^{1,0}(X)_{\mathbb{R}}$ and $H_{\text{trop}}^{0,1}(X)_{\mathbb{R}}$ is defined over $X_0 \otimes_{K_0} K$ for some finite extension K of K_0 contained in $\widehat{K_0^a}$. We need some particular strict semistable alteration of $X_0 \otimes_{K_0} K$ for sufficiently large K . Since X_0 admits a proper, strictly semistable model over K_0° , it is well known (see, for example, [Sai03, Lemma 1.11]) that $X_0 \otimes_{K_0} K$ admits a proper, strictly semistable model over K° for any finite extension of K of K_0 . Replacing K by a finite unramified extension, we may assume that X_0 admits a strict semistable alteration $\mathcal{X}$ over K° such that $\phi : \mathcal{X}_\eta \rightarrow X_0$ induces an isomorphism $\mathcal{X}_\eta \xrightarrow{\sim} X_0 \otimes_{K_0} K$. We then have an infinite sequence of successive finite field extensions $K_0 \subset K_1 \subset \cdots$ contained in K_0^a such that for $i \geq 1$, the scheme X_0 admits a strict semistable alteration $\mathcal{X}_i$ over K_i° with $\phi : \mathcal{X}_{i,\eta} \xrightarrow{\sim} X_i$ an isomorphism, where $X_i := X_0 \otimes_{K_0} K_i$.

By the similar argument in the proof of Corollary 5.12, it suffices to show that

$$N_{X_i} : H_{\text{trop}}^{1,0}(X_i)_{\mathbb{R}} \rightarrow H_{\text{trop}}^{0,1}(X_i)_{\mathbb{R}} \quad (5.13)$$

is an isomorphism for every $i \geq 1$. By Theorem 5.6(ii), Lemma 4.8, and Proposition 4.6(iii), we know that (5.13) is injective. Thus, if we can show $h_{\text{trop}}^{1,0}(X_i) \geq h_{\text{trop}}^{0,1}(X_i)$, then (5.13) is an isomorphism.

We fix such an index i and suppress it from the notation. In particular, we now have $X = X_0 \otimes_{K_0} K$ with K a finite extension of K_0 . Let $\mathcal{X}$ be a strict semistable alteration of X with an isomorphism $\phi: \mathcal{X}_\eta \xrightarrow{\sim} X$. We identify X with $\mathcal{X}_\eta$. By Lemma 4.7 and Proposition 4.6(iii), the composite map

$$N_{\mathcal{X}}^{-1} \circ \kappa_X^1: H^1(X^{\text{an}}, \mathbb{Q}_\ell) \rightarrow E_2^{-1,2}(1)$$

is an isomorphism, under which the image of $H^1(X^{\text{an}}, \mathbb{Q})$ is contained in $E_2^{-1,2}(1) \cap H_{\text{ét}}^0(Y_s^{(1)}, \mathbb{Q})$, which we denote by $E_{\mathcal{X}}$. The theorem will follow if we can construct an injective (linear) map

$$T_{\mathcal{X}}: E_{\mathcal{X}} \rightarrow H^0(X^{\text{an}}, \mathcal{T}_{X^{\text{an}}}^1).$$

For every point $y \in Y$, we fix a semistable chart $(\mathcal{V}_y, \{g_0^y, \dots, g_{p(y)}^y\})$ at y (Definition 4.3). We can write an element in $E_{\mathcal{X}}$ in the form

$$D = \frac{1}{q} \sum_{Z \in \mathcal{D}^1} a(Z)Z$$

for some integers $a(Z)$ and $q > 0$, where the sum is taken over $\mathcal{D}^1$, the set of all irreducible components of $Y^{[1]}$. As $\delta_{1*}D = 0$ by the description of the map $N_{\mathcal{X}}^1$ via (4.4), the divisor $(qD) \cap Y^i$ on Y^i is cohomologically trivial for every irreducible component Y^i of Y ($1 \leq i \leq m$). Thus, there exists some integer $q_i > 0$ such that $(q_i q D) \cap Y^i$ is algebraically equivalent to zero on Y^i ; in particular, $\mathcal{O}_{Y^i}((q_i q D) \cap Y^i)$ is an element in $\text{Pic}_{Y^i/\tilde{K}}^0(\tilde{K})$. Since $\text{Pic}_{Y^i/\tilde{K}}^0$ is a projective scheme over the *finite* field $\tilde{K}$, one may replace q_i by some integer multiple such that $\mathcal{O}_{Y^i}((q_i q D) \cap Y^i)$ is a trivial line bundle. Replacing q by $q \prod_{i=1}^m q_i$, we may assume that for every i , there is a rational function $f_i \in \tilde{K}(Y^i)$ such that its divisor is exactly $(qD) \cap Y^i$.

For every point y of Y , we are going to construct an open neighborhood $U_y \subset X^{\text{an}}$ of $\pi_{\mathcal{X}}^{-1}y$ and an element $\omega_y \in \mathcal{T}_{X^{\text{an}}}^1(U_y)$. For every nonempty subset I of $I(y)$ (Notation 4.2), let Z_y^I be the unique irreducible component of Y^I that contains y . Put $U_y := \mathcal{V}_{y,\eta}^{\text{an}} \cap \pi_{\mathcal{X}}^{-1}Z_y^{I(y)}$. For ω_y , there are two cases:

- (1) If $p(y) = 0$, that is, $Z_y^{I(y)} = Y^i$ for some i , then we set

$$\omega_y = \tau \left(\frac{1}{q} \{f_i^\#\} \right) \Big|_{U_y},$$

where $f_i^\#$ is a lift of $f_i|_{\mathcal{V}_y \cap Y^i}$ to an invertible regular function on $\mathcal{V}_{y,\eta}$.

- (2) If $p(y) > 0$, then we set

$$\omega_y = \tau \left(\frac{1}{q} \prod_{j=1}^{p(y)} (g_j^y)^{a(Z_y^{\{i_0, i_j\}})} \right) \Big|_{U_y},$$

where $I(y) = \{i_0 < \dots < i_{p(y)}\}$. Note that $Z_y^{\{i_0, i_j\}} \in \mathcal{D}^1$.

We claim that the (U_y, ω_y) patch together to give an element in $H^0(X^{\text{an}}, \mathcal{T}_{X^{\text{an}}}^1) = H_{\text{trop}}^{1,0}(X)$, which we define as $T_{\mathcal{X}}D$. Given two points y and y' of Y , this amounts to showing that $\omega_y = \omega_{y'}$ on $U_y \cap U_{y'}$. If one of y and y' is in Case (1), then the compatibility can be shown in the same way as in the proof of Lemma 5.8. In particular, ω_y does not depend on the choice of the lift $f_i^\#$

in Case (1). If both y and y' are in Case (2), then it is a straightforward consequence of the fact that $\delta_1^* D = 0$ by the description of the map $N_{\mathcal{X}}^1$ via (4.4). We leave the details to the reader.

By construction, $T_{\mathcal{X}}$ is apparently linear and injective. The theorem is proved. $\square$

Remark 5.14. Suppose that we are in the situation of Theorem 5.13. In particular, we have a sequence of finite extensions $K_0 \subset K_1 \subset \cdots$ contained in $\widehat{K}_0^a$ such that $\bigcup_{i \geq 0} K_i$ is an algebraic closure of K_0 and a sequence of schemes $\cdots \rightarrow \mathcal{X}_i \rightarrow \mathcal{X}_{i-1} \rightarrow \cdots \rightarrow \mathcal{X}_0$, where each $\mathcal{X}_i$ is a strict semistable alteration of $X_0 \otimes_{K_0} K_i$ such that $\mathcal{X}_{i,\eta} \simeq X_0 \otimes_{K_0} K_i$. It follows from Lemma 4.7 that there is a canonical isomorphism $\varinjlim_i E_{\mathcal{X}_{i,2}}^{p,0} \simeq H_{\text{trop}}^{0,p}(X) \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell}$ for $p \geq 0$. Thus, Conjectures 4.5 and 5.3, together with Theorem 5.6(i), imply that there should be an isomorphism $\varinjlim_i E_{\mathcal{X}_{i,2}}^{-p,2p}(p) \simeq H_{\text{trop}}^{p,0}(X) \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell}$ for $p \geq 0$. This is trivial when $p = 0$. The case $p = 1$ is essentially Theorem 5.13. For $p \geq 2$, we do not know how to prove this at the moment. Moreover, for general (p, q) , we expect some relation between $H_{\text{trop}}^{p,q}(X)$ and $\varinjlim_i E_{\mathcal{X}_{i,2}}^{\bullet,\bullet}$ but do not know if there is one.

Appendix. Star-shaped integration on simplex

In this appendix, we prove a technical result for the local computation used in the proof of Lemma 5.10.

Let $n \geq 1$ be an integer. Let $\Delta \subset \mathbb{R}^{n+1}$ be the standard (closed) simplex of dimension n . In other words, if we denote by $x_0, \dots, x_n$ the standard coordinates on $\mathbb{R}^{n+1}$, then

$$\Delta = \{(x_0, \dots, x_n) \mid x_0 + \cdots + x_n = 1, x_i \geq 0\}.$$

For $p \geq 0$, denote by $\Omega^p(\Delta)$ the space of (smooth) p -differential forms on Δ and by $\Omega_{\text{cl}}^p(\Delta)$ the subspace of closed forms. Let ω be an element of $\Omega^p(\Delta)$; then for every $0 \leq i \leq n$, the form ω can be uniquely written as

$$\omega = \sum_{I \subset \{0, \dots, n\} \setminus \{i\}, |I|=p} f_I(x) dx_I,$$

where f_I is a continuous function on Δ , smooth in the interior of Δ . We say that ω has *constant coefficients* if f_I is a constant function for every such I . Note that this property is independent of the choice of i . Let $\Omega_{\text{cons}}^p(\Delta) \subset \Omega_{\text{cl}}^p(\Delta)$ be the subspace of forms with constant coefficients.

For every nonempty subset $J \subset \{0, \dots, n\}$, let Δ^J be the face of Δ generated by $\{P^j \mid j \in J\}$, where P^j is the point with coordinates $x_i = \delta_{ij}$, and let P^J be the barycenter of Δ^J . For every point $P \in \Delta$, the simplex Δ is star-shaped with respect to P . Thus, one has the *star-shaped integration map*

$$\mathcal{I}_P: \Omega^p(\Delta) \rightarrow \Omega^{p-1}(\Delta)$$

for $p \geq 1$. We briefly recall the definition. Consider the map $\gamma_P: \Delta \times [0, 1] \rightarrow \Delta$ sending (x, t) to $(1-t)P + tx$. Then

$$\mathcal{I}_P(\alpha) = \int_0^1 \left\langle \frac{\partial}{\partial t}, \gamma_P^* \alpha \right\rangle dt,$$

where $\langle \cdot, \cdot \rangle$ denotes the contraction.

Remark A.1. By the Poincaré lemma for a star-shaped region (see, for example, [Spi65, Theorem 4-11], we know that if $d\alpha = 0$, then $d(\mathcal{I}_P(\alpha)) = \alpha$.

Let F be an abelian group. For every $r \geq 0$, put

$$C^r(F) := \text{Map}(\{J \subset \{0, \dots, n\} \mid |J| = r+1\}, F).$$

We have a coboundary map $\delta: C^r(F) \rightarrow C^{r+1}(F)$ similar to the one for Čech cocycles. Namely, for $\beta \in C^r(F)$, we set

$$(\delta\beta)(J) = \sum_{k=0}^{r+1} (-1)^k \beta(\{j_0, \dots, \widehat{j_k}, \dots, j_{r+1}\})$$

for $J = \{j_0, \dots, j_{r+1}\}$ with $0 \leq j_0 < \dots < j_{r+1} \leq n$.

For $p - r \geq 1$, we define the map

$$\mathcal{I}: C^r(\Omega^{p-r}(\Delta)) \rightarrow C^r(\Omega^{p-r-1}(\Delta)) \quad (\text{A.1})$$

sending ω_r in the source to $\mathcal{I}\omega_r$ in the target such that $(\mathcal{I}\omega_r)(J) = \mathcal{I}_{PJ}(\omega_r(J))$.

PROPOSITION A.2. *Let $1 \leq p \leq n$ be an integer. Let β be an element in $\Omega_{\text{cons}}^p(\Delta)$. We define $\beta_0 \in C^0(\Omega^p(\Delta))$ by $\beta_0(\{j\}) = \beta$ for every $0 \leq j \leq n$. For $1 \leq r \leq p$, we inductively define $\beta_r \in C^r(\Omega^{p-r}(\Delta))$ by the formula $\beta_r = \delta(\mathcal{I}\beta_{r-1})$. Then*

- (i) β_r belongs to $C^r(\Omega_{\text{cons}}^{p-r}(\Delta))$ for $0 \leq r \leq p$, and in particular $\beta_p \in C^p(\mathbb{R})$; and
- (ii) we have the formula

$$\beta|_{\Delta^I} = (-1)^{p(p+1)/2} p! \cdot \beta_p(I) \, dx_{i_1} \wedge \dots \wedge dx_{i_p}$$

for every subset $I = \{i_0 < \dots < i_p\} \subset \{1, \dots, n\}$.

Proof. We fix some $i \in \{0, \dots, n\}$ and write

$$\beta = \sum_{I \subset \{0, \dots, n\} \setminus \{i\}, |I|=p} f_I dx_I \quad (\text{A.2})$$

for some $f_I \in \mathbb{R}$. Regard β as a p -form on $\mathbb{R}^{n+1}$ with constant coefficients, which we denote by $\beta' \in \Omega_{\text{cons}}^p(\mathbb{R}^{n+1})$. For a point $Q \in \mathbb{R}^{n+1}$ and $r \geq 1$, denote by

$$\mathcal{I}'_Q: \Omega^r(\mathbb{R}^{n+1}) \rightarrow \Omega^{r-1}(\mathbb{R}^{n+1})$$

the corresponding star-shaped integral on $\mathbb{R}^{n+1}$ with respect to Q . If Q belongs to Δ , then for $\alpha \in \Omega^r(\mathbb{R}^{n+1})$, we have

$$\mathcal{I}'_Q(\alpha)|_{\Delta} = \mathcal{I}_Q(\alpha|_{\Delta}). \quad (\text{A.3})$$

Suppose $Q = (q_0, \dots, q_n)$; we define a map $\mathcal{C}_Q: \Omega_{\text{cons}}^r(\mathbb{R}^{n+1}) \rightarrow \Omega_{\text{cons}}^{r-1}(\mathbb{R}^{n+1})$ sending α to the contraction of α by $\sum_{j=0}^n q_j (\partial/\partial x_j)$ (from the left). For example,

$$\mathcal{C}_{P^j}(dx_0 \wedge \dots \wedge dx_{r-1}) = \begin{cases} (-1)^j dx_0 \wedge \dots \wedge \widehat{dx_j} \wedge \dots \wedge dx_{r-1}, & 0 \leq j \leq r-1, \\ 0, & r \leq j \leq n. \end{cases}$$

Denote the origin of $\mathbb{R}^{n+1}$ by O . It is an elementary exercise to see that for $\alpha \in \Omega_{\text{cons}}^r(\mathbb{R}^{n+1})$, we have

$$\mathcal{I}'_Q(\alpha) - \mathcal{I}'_O(\alpha) = -\frac{1}{r} \mathcal{C}_Q(\alpha). \quad (\text{A.4})$$

For $0 \leq r \leq p$, we define an element $\beta'_r \in C^r(\Omega_{\text{cons}}^{p-r}(\mathbb{R}^{n+1}))$ by the formula

$$\beta'_r(I) = \frac{(-1)^r}{p(p-1) \dots (p-r+1)} \sum_{j=0}^r (-1)^j (\mathcal{C}_{P^{i_0}} \circ \dots \circ \widehat{\mathcal{C}_{P^{i_j}}} \circ \dots \circ \mathcal{C}_{P^{i_r}})(\beta'). \quad (\text{A.5})$$

for $I = \{i_0 < \dots < i_r\}$. We claim that

$$\beta'_r = \delta(\mathcal{I}' \beta'_{r-1}), \quad (\text{A.6})$$

where $\mathcal{I}': C^r(\Omega^{p-r}(\mathbb{R}^{n+1})) \rightarrow C^r(\Omega^{p-r-1}(\mathbb{R}^{n+1}))$ is defined similarly to (A.1). In fact, by the definition of δ , for $r \geq 1$, we have

$$\delta(\mathcal{I}' \beta'_{r-1}) = \sum_{j=0}^r (-1)^j \mathcal{I}'_{P^{I \setminus \{i_j\}}}(\beta'_{r-1}(I \setminus \{i_j\})). \quad (\text{A.7})$$

As β'_{r-1} is a closed cocycle, we have $\sum_{j=0}^r (-1)^j \beta'_{r-1}(I \setminus \{i_j\}) = 0$. Thus,

$$\begin{aligned} (\text{A.7}) &= \sum_{j=0}^r (-1)^j (\mathcal{I}'_{P^{I \setminus \{i_j\}}} - \mathcal{I}'_O)(\beta'_{r-1}(I \setminus \{i_j\})) \\ &= \frac{-1}{p-r+1} \sum_{j=0}^r (-1)^j \mathcal{C}_{P^{I \setminus \{i_j\}}}(\beta'_{r-1}(I \setminus \{i_j\})) \end{aligned}$$

by (A.4), which equals

$$\frac{(-1)^r}{p(p-1) \cdots (p-r+1)} \sum_{j=0}^r (-1)^j \mathcal{C}_{P^{I \setminus \{i_j\}}} \left(\sum_{k=0}^r \epsilon(j, k) \mathcal{C}(j, k)(\beta') \right). \quad (\text{A.8})$$

Here $\epsilon(j, k)$ is set to be $(-1)^k$ (respectively, 0, $(-1)^{k+1}$) if $k < j$ (respectively, $k = j$, $k > j$), and $\mathcal{C}(j, k)$ is the map $\mathcal{C}_{P^{i_0}} \circ \dots \circ \mathcal{C}_{P^{i_r}}$ with $\mathcal{C}_{P^{i_j}}$ and $\mathcal{C}_{P^{i_k}}$ removed. Since

$$\mathcal{C}_{P^{I \setminus \{i_j\}}} = \frac{1}{r} (\mathcal{C}_{P^{i_0}} + \dots + \widehat{\mathcal{C}_{P^{i_j}}} + \dots + \mathcal{C}_{P^{i_r}}),$$

it is easy to see that

$$(\text{A.8}) = \frac{(-1)^r}{p(p-1) \cdots (p-r+1)} \sum_{j=0}^r (-1)^j \mathcal{C}(j, j)(\beta) = \beta'_r(I).$$

Thus, (A.6) holds. By (A.3), we have $\beta_r = \beta'_r|_{\Delta}$ which belongs to $C^r(\Omega_{\text{cons}}^{p-r}(\Delta))$. Thus, statement (i) follows.

Now we compute the value of $\beta_p(I) = \beta'_p(I)$ for $I = \{i_0 < \dots < i_p\}$. Note that the representative β defined in (A.2) can be chosen with respect to an arbitrary $i \in \{0, \dots, n\}$. In particular, we may take $i = i_0$. By (A.5), we have

$$\beta_p(I) = \frac{(-1)^p}{p!} (\mathcal{C}_{P^{i_1}} \circ \dots \circ \mathcal{C}_{P^{i_p}})(\beta) = \frac{(-1)^{p(p+1)/2}}{p!} (\mathcal{C}_{P^{i_p}} \circ \dots \circ \mathcal{C}_{P^{i_1}})(\beta).$$

Thus, statement (ii) follows as $\beta|_{\Delta I} = (\mathcal{C}_{P^{i_p}} \circ \dots \circ \mathcal{C}_{P^{i_1}})(\beta) \, dx_{i_1} \wedge \dots \wedge dx_{i_p}$. $\square$

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