Open surfaces of small volume

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ABSTRACT

We construct a surface with log terminal singularities and ample canonical class that has $K_X^2 = 1/48983$ and a log canonical surface pair (X, B) with a nonempty reduced divisor B and ample $K_X + B$ that has $(K_X + B)^2 = 1/462$. Both examples significantly improve known records.

1. Introduction

Let U be a smooth quasiprojective surface, and let S be a smooth compactification such that $D = S \setminus U$ is a normal crossing divisor. The open surface U is said to be of general type if $K_S + D$ is big. This condition and the spaces of pluri log canonical sections $H^0(n(K_S + D))$ for all $n \geq 0$ depend only on U and not on the choice of a particular normal crossing compactification (S, D) .

Since $K_S + D$ is big, the number of its sections grows quadratically: $h^0(n(K_S + D)) \sim cn^2/2$. After passing to the log canonical model $(S_{\text{can}}, D_{\text{can}})$ where $K_{S_{\text{can}}} + D_{\text{can}}$ is ample, one sees that $c = (K_{S_{\text{can}}} + D_{\text{can}})^2$; this is called the *volume* of the surface pair (S, D) , equivalently the volume of U , and is denoted by $\text{vol}(K_S + D) = \text{vol}(U)$.

Vice versa, if (X, B) is a log canonical surface pair with reduced boundary B and ample $K_X + B$, and if $f: S \rightarrow X$ is its log resolution with exceptional divisors $\{E_i\}$, then

$$\text{vol}(S \setminus (f^{-1}(B) \cup E_i)) = \text{vol}\left(K_S + f_*^{-1}B + \sum E_i\right) = \text{vol}(K_X + B) = (K_X + B)^2.$$

Question 1.1. How small could a volume of an open surface U of general type be? Equivalently, how small could $(K_X + B)^2$ be for a log canonical surface pair with reduced boundary B and ample $K_X + B$?

The basic result in this direction is the following quite general theorem.

THEOREM 1.2 (Alexeev, [Ale94]). *Consider the log canonical surface pairs $(X, B = \sum b_i B_i)$ with ample $K_X + B$ and with coefficients b_i belonging to a DCC set $\mathcal{S} \subset [0, 1]$ (that is, a set satisfying the descending chain condition). Then the set of squares $(K_X + B)^2$ is also a DCC set. In particular, it has a minimum, a positive real number, which is rational if $\mathcal{S} \subset \mathbb{Q}$.*

We will denote this minimum by $K^2(\mathcal{S})$. Some interesting DCC sets are $\mathcal{S}_0 = \{0\}$, $\mathcal{S}_1 = \{0, 1\}$, $\mathcal{S}'_2 = \{1 - 1/n, n \in \mathbb{N}\}$, $\mathcal{S}_2 = \mathcal{S}'_2 \cup \{1\}$. The paper [AM04] gives an effective, computable lower

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bound for $K^2(\mathcal{S})$ in terms of the set $\mathcal{S}$, which is however too small to be realistic; see Section 10 where we spell it out for the sets $\mathcal{S}_0$ and $\mathcal{S}_1$.

A version of the above definition is to look at the surface pairs (X, B) with nonempty reduced part of the boundary $\lfloor B \rfloor \neq 0$. We will denote the minimum in this case by $K_1^2(\mathcal{S})$. Clearly, one has

$$K^2(\mathcal{S}_2) \leq K^2(\mathcal{S}_1) \leq K^2(\mathcal{S}_0) \quad \text{and} \quad K^2(\mathcal{S}_1) \leq K_1^2(\mathcal{S}_1).$$

Some published bounds for $K^2(\mathcal{S})$ and $K_1^2(\mathcal{S})$ for the above sets include

- (1) $K^2(\mathcal{S}_0) \leq 3/55$,
- (2) $K_1^2(\mathcal{S}_1) \leq 1/60$,
- (3) $K_1^2(\mathcal{S}_2) = 1/42^2 = 1/1764$, and thus $K_1^2(\mathcal{S}_1) \geq 1/1764$,
- (4) $K^2(\mathcal{S}_2) \leq 1/(42 \cdot 43)^2$, and the bound is conjectured to be sharp.

The first of these bounds follows from an example of Blache [Bla95]. The others are due to Kollár [Kol94, Kol13].

There are also many other examples of log terminal surfaces with ample canonical class appearing in the literature. For example, hypersurfaces $S(a_1, a_2, a_3, a_4)$ of the form $x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}x_4 + x_4^{a_4}x_1 = 0$ in weighted projective spaces $\mathbb{P}(w_1, w_2, w_3, w_4)$ provide such examples under some mild conditions on the a_i . Kollár [Kol08] studied these surfaces and surfaces $X = S^*(a_1, a_2, a_3, a_4)$ with $\rho(X) = 1$ obtained from them by contracting two curves. Hwang and Keum introduced another family of surfaces $X = T(a_1, a_2, a_3, a_4)$ with $\text{rk Pic } X = 1$ depending on four parameters, and Urzúa–Yáñez [UY16] explained the relation between the two series.

These papers are not specifically concerned with the minimal possible value of K_X^2 , but certainly better bounds than $3/55$ can be achieved. José Ignacio Yáñez informed us that the following example appears to achieve the minimum among the surfaces $S(a_1, a_2, a_3, a_4)$ with $\gcd(w_1, w_3) = \gcd(w_2, w_4) = 1$.

Example 1.3 (Urzúa–Yáñez). The hypersurface $X = S(2, 2, 4, 10)$ of degree 159 in the weighted projective space $\mathbb{P}(49, 61, 37, 11)$ has an ample canonical class K_X , cyclic quotient singularities, and

$$K_X^2 = \frac{159 \cdot (159 - 49 - 61 - 37 - 11)^2}{49 \cdot 61 \cdot 37 \cdot 11} = \frac{159}{1216523} \approx \frac{1}{7651}.$$

Knowing the exact bounds is important for many applications. As explained in [Kol94], a stable limit of surfaces of general type of volume d has at most $d/K_1^2(\mathcal{S}_1)$ irreducible components. Thus, as a corollary of Kollár’s bound $K_1^2(\mathcal{S}_1) \geq 1/1764$, the number of irreducible components is at most $1764d$. Other applications include bounds for the automorphism groups of surfaces and surface pairs of general type; see, for example, [Kol94, Ale94] for a broader discussion.

The constant $K^2(\mathcal{S}_1)$ is certainly a very fundamental global invariant in its own right: the smallest volume of a smooth open surface.

The main result of this paper is the following.

THEOREM 1.4. *One has $K_1^2(\mathcal{S}_1) \leq 1/462$ and $K^2(\mathcal{S}_1) = K^2(\mathcal{S}_0) \leq 1/48983$.*

Section 2 explains the method which we used to find the new examples. We restate it in Section 3 as a purely combinatorial game with weights (w_0, w_1, w_2, w_3) . Section 4 contains some easy instances of this game for the simplest weights $(0, 1, 1, 1)$ and $(1, 1, 1, 1)$, giving in particular Kollár’s example with $(K_X + B)^2 = 1/60$. Sections 5 and 6 contain our champion examples for

the winning weights $(1, 2, 3, 5)$: surfaces with a nonempty boundary and $(K_X + B)^2 = 1/462 = 1/(11 \cdot 42)$, and surfaces without boundary and $K_X^2 = 1/48983 = 1/(11 \cdot 61 \cdot 73)$.

In characteristic 0, the champion surfaces have Picard number $\rho(X) = 2$ and they have four (respectively, three) singularities. But in characteristic 2, the rank of the Picard group drops by 1 and there is an additional A_1 -singularity. The surfaces with ample K_X and such configurations of singularities would provide counterexamples to the algebraic Montgomery–Yang problem, were they to exist in characteristic 0. We discuss this connection in Section 7.

Related to this, in Section 8 we list some surfaces of small volume that have Picard rank $\rho(X) = 1$. The smallest volume we were able to find is $K_X^2 = 1/6351$, achieved by $S^*(2, 2, 4, 10) = T(2, 2, 4, 10)$. For the Hwang–Keum surfaces $T(a_1, a_2, a_3, a_4)$ from [HK12], we prove unconditionally that this is the smallest possible volume. DongSeon Hwang informed us that this was independently checked by Jeon–Hwang.

Section 9 explains why we restricted to the case of only four lines in our search. Finally, Section 10 spells out the effective (but very small) bound for K_X^2 which follows from [AM04, Kol94].

Further, we note that Kollár’s lower bound for $K_1^2(\mathcal{S}_2)$ is a combination of two inequalities. One defines two invariants of a DCC set $\mathcal{S}$.

DEFINITION 1.5. (i) Let (X, B) be a log canonical surface with ample $K_X + B$ and $\lrcorner B \lrcorner \neq 0$. Then

$$\epsilon_1(X, B) := \min_{B_0 \subset \lrcorner B \lrcorner} \{(K_X + B)B_0\},$$

and $\epsilon_1(\mathcal{S})$ is the minimum of these numbers as (X, B) goes over all surface pairs with coefficients in $\mathcal{S}$.

(ii) We define $\delta_1(\mathcal{S})$ as the minimum of the numbers $t > 0$ such that there exists a log canonical surface pair $(X, (1 - t)B_0 + \Delta)$ with $K_X + (1 - t)B_0 + \Delta \equiv 0$ such that the coefficients of Δ are in $\mathcal{S}$.

(iii) We also define a closely related invariant of an individual big log canonical divisor: $\delta_1(X, B)$ is the minimum of the numbers t such that $K_X + B - tB_0$ is not big for some $0 \neq B_0 \subset \lrcorner B \lrcorner$, or 1 if this minimum is greater than 1.

Then according to [Kol94], one has $(K_X + B)^2 \geq \epsilon_1(\mathcal{S})\delta_1(\mathcal{S})$ and $\epsilon_1(\mathcal{S}_2) = \delta_1(\mathcal{S}_2) = 1/42$. In Section 5, we give an example that shows that the equality $\epsilon_1(\mathcal{S}_1) = 1/42$ also holds. As for $\delta_1(\mathcal{S}_1)$, we were not able to find better than $1/13$ with the present method, which is the same as in Kollár’s example with $(K_X + B_0)^2 = 1/60$.

All constructions and examples in this paper work over an algebraically closed field of arbitrary characteristic.

2. The method of construction

We begin with four lines L_0, L_1, L_2, L_3 in $\mathbb{P}^2$ in general position. Let $f: \tilde{X} \rightarrow \mathbb{P}^2$ be a sequence of blowups, each at a point of intersection of two curves that appeared so far: exceptional divisors, lines, and their strict preimages. We will call the curves thus obtained on $\tilde{X}$ the *visible curves*. For the examples with coefficients in $\mathcal{S}_0$, we assume that enough blowups were performed so that the strict preimages $\tilde{L}_k$ of the lines L_k satisfy $\tilde{L}_k^2 \leq -2$. Thus, in particular, all visible curves will have negative self-intersections.

Let the E_i with $i \in I$ be the visible curves with $E_i^2 \leq -2$ and the C_j with $j \in J$ be the visible curves with $C_j^2 = -1$, where I and J are two disjoint finite index sets. Note that the strict transforms of the lines L_k for $0 \leq k \leq 3$ on $\tilde{X}$ are among the E_i by our assumption, and we indicate this fact by saying that their indices satisfy $0, 1, 2, 3 \in I$.

Now assume that the curves $\{E_i\}$ form a *log terminal configuration*, that is, a configuration of exceptional curves on the minimal resolution of a surface with log terminal singularities. Each connected component of the dual graph is of type A_n (that is, a chain) with no further restrictions on self-intersections E_i^2 , or one of the graphs of types D_n or E_n , with restrictions on self-intersections; see, for example, [Ale92, Figure 3].

Log terminal configurations are rational and, by Artin [Art62], the curves E_i on $\tilde{X}$ can be contracted to obtain a projective surface X . Let $\pi: \tilde{X} \rightarrow X$ be the contraction morphism. (More generally, one may assume that the E_i form a log canonical configuration. In our examples, only log terminal singularities occur.)

The surface $\tilde{X}$ is then the minimal resolution of singularities of X , and one has

$$K_{\tilde{X}} = \pi^* K_X + \sum_{i \in I} a_i E_i, \quad \pi^* K_X = K_{\tilde{X}} + \Delta, \quad \Delta = \sum_{i \in I} b_i E_i.$$

Here, the a_i are the *discrepancies* and $b_i = -a_i$. The numbers b_i satisfy the following linear system of n equations in n variables:

$$\left(K_{\tilde{X}} + \sum_{i \in I} b_i E_i \right) E_{i_0} = 0 \iff \sum_{i \in I} b_i E_i E_{i_0} = E_{i_0}^2 + 2 \quad \text{for any } i_0 \in I, \quad (2.1)$$

where the equivalence of the equations is because $E_{i_0} \cong \mathbb{P}^1$. By a theorem of Mumford, the matrix $(E_{i_1} E_{i_2})$ is negative-definite, so this system has a unique solution. By Artin [Art62], all entries of the matrix $(E_{i_1} E_{i_2})^{-1}$ are nonpositive. Since the right-hand sides are $E_{i_0}^2 + 2 \leq 0$ for any $i_0 \in I$, it follows that $b_i \geq 0$ for any $i \in I$. The reference [Ale92, Section 3.2] contains some convenient closed-form formulas for the b_i ; see also [Miy01, Section 2.3].

THEOREM 2.1. *Let C_j for $j \in J$ be the visible (-1) -curves on $\tilde{X}$. Assume that*

- (i) *for all C_j , one has $K_X \pi_*(C_j) \geq 0$ (respectively, $K_X \pi_*(C_j) > 0$);*
- (ii) $K_X^2 > 0$;
- (iii) *there exist four rational numbers d_0, d_1, d_2, d_3 with $d_0 + d_1 + d_2 + d_3 = 3$ such that the coefficients of C_j in the formula below satisfy $d_j \leq 0$ (respectively, $d_j < 0$) for all $j \in J$ (recall that we assume $0, 1, 2, 3 \in I$):*

$$K_{\tilde{X}} + \tilde{D} := f^* \left(K_{\mathbb{P}^2} + \sum_{0 \leq k \leq 3} d_k L_k \right) = K_{\tilde{X}} + \sum_{i \in I} d_i E_i + \sum_{j \in J} d_j C_j.$$

Then the divisor K_X is big and nef (respectively, ample).

Proof. Of course, conditions (i) and (ii) are necessary for K_X to be big and nef (respectively, ample). Condition (iii) implies that K_X is an effective linear combination of the curves $\pi(C_j)$. Indeed, $K_{\tilde{X}} + \tilde{D} = f^*(0) = 0$, so $K_X = \pi_*(-\tilde{D}) = \sum (-d_j) \pi_*(C_j)$. So, K_X intersects any irreducible curve on X nonnegatively. Thus, K_X is nef.

For ampleness, note that the union of visible curves supports an effective ample divisor. Thus, any curve on X intersects its image, $\cup \pi(C_j)$. Therefore, any irreducible curve on X intersects $\sum (-d_j) \pi_*(C_j)$ positively, and so K_X is ample by the Nakai–Moishezon criterion. $\square$

Remark 2.2. Even if K_X is only big and nef, it is semiample by the Abundance Theorem in dimension 2 (cf. [Fuj12, Theorem 6.1]), so its canonical model has ample $K_{X_{\text{can}}}$ and the same square $K_{X_{\text{can}}}^2 = K_X^2$.

Remark 2.3. We usually assume $0 \leq d_k \leq 1$ (so that $\sum d_k L_k$ is a “boundary” in the standard Minimal Model Program terminology), or at least $d_k \leq 1$ (so that it is a “sub boundary”). But this is not necessary for the above proof.

The coefficients in the divisor $\tilde{D}$ for the visible curves E_i and C_j are readily computable. The formula is especially simple in terms of the quantities $(1 - d_k)$ for $k \in I \cup J$, which are just the *log discrepancies* of $(\mathbb{P}^2, \sum_{k=0}^3 d_k L_k)$: after one blows up the point of intersection of two curves with log discrepancies $1 - d_{k_1}$ and $1 - d_{k_2}$, the new log discrepancy is $1 - d_{k_3} = (1 - d_{k_1}) + (1 - d_{k_2})$. In other words, the log discrepancies add up.

The above will be our essential method for finding new examples in the case of the empty boundary. For the examples with a nonempty boundary, we do not contract the strict preimage of the line L_0 , which we denote by $\tilde{B}_0$. We no longer include $\tilde{B}_0$ in either collection $\{E_i\}$ or $\{C_j\}$. In the formulas below, $0 \notin I \cup J$ and $1, 2, 3 \in I$. We modify the assumption made at the beginning of this section to allow $\tilde{B}_0^2$ to be nonnegative, since it is a “special” curve. Let B_0 be the image of $\tilde{B}_0$ on X . Then we define the discrepancies for the surface pair (X, B_0) via

$$\pi^*(K_X + B_0) = K_{\tilde{X}} + \Delta := K_{\tilde{X}} + \tilde{B}_0 + \sum_{i \in I} b_i E_i.$$

With this modification, Theorem 2.1 readily extends to the following.

THEOREM 2.4. *Let C_j be the visible (-1) -curves on $\tilde{X}$ (not including $\tilde{B}_0$). Assume that*

- (i) *for all C in $\{\pi(C_j), B_0\}$, one has $(K_X + B_0)C \geq 0$ (respectively, $(K_X + B_0)C > 0$);*
- (ii) *$(K_X + B_0)^2 > 0$;*
- (iii) *there exist four rational numbers d_0, d_1, d_2, d_3 with $d_0 + d_1 + d_2 + d_3 = 3$ such that the coefficients of C_j in the formula below satisfy $d_j \leq 0$ (respectively, $d_j < 0$) for all $j \in J$:*

$$K_{\tilde{X}} + \tilde{D} := f^*\left(K_{\mathbb{P}^2} + \sum_{0 \leq k \leq 3} d_k L_k\right) = K_{\tilde{X}} + d_0 \tilde{B}_0 + \sum_{i \in I} d_i E_i + \sum_{j \in J} d_j C_j;$$

in addition, assume $d_0 \leq 1$ (respectively, $d_0 < 1$).

Then the divisor $K_X + B_0$ is big and nef (respectively, ample).

Proof. The same proof as in Theorem 2.1 gives that $K_X + d_0 B_0$ is an effective (respectively, strictly positive) combination of the curves $\pi(C_j)$. But then so is $K_X + B_0 = K_X + d_0 B_0 + (1 - d_0) B_0$. If $d_0 < 1$, then B_0 appears in this sum with a positive coefficient. The rest of the proof is the same. $\square$

We now state without proof some easy formulas.

LEMMA 2.5. *The following hold. (For the surface without a boundary, omit B_0 .)*

- (i) $(K_X + B_0)\pi_*(C_j) = (K_{\tilde{X}} + \Delta)C_j = -1 + \tilde{B}_0 C_j + \sum_{i \in I} b_i E_i C_j$.
- (ii) $(K_X + B_0)^2 = (K_{\tilde{X}} + \Delta)^2 = K_{\tilde{X}}^2 + K_{\tilde{X}} \Delta + (\Delta - \tilde{B}_0)\tilde{B}_0 - 2$. *For the surface without a boundary, $K_X^2 = (K_{\tilde{X}} + \Delta)^2 = K_{\tilde{X}}^2 + K_{\tilde{X}} \Delta$.*
- (iii) $K_{\tilde{X}}^2 = 9 - (\text{the number of blowups in } \tilde{X} \rightarrow \mathbb{P}^2)$.
- (iv) $K_{\tilde{X}} E_i = -E_i^2 - 2$ and $K_{\tilde{X}} \tilde{B}_0 = -\tilde{B}_0^2 - 2$.

3. Combinatorial game

As usual, with a configuration of curves with simple normal crossings on a surface, we associate its dual graph. The vertices are labeled with marks $-E_i^2$ (we call them *marks* because we use *weights* for a different purpose). Thus, the initial configuration of four lines on $\mathbb{P}^2$ in general position corresponds to a complete graph on four vertices with marks $-1, -1, -1, -1$, and we have a graph describing the visible curves on $\tilde{X}$. To simplify the typography, the (-1) -curves C_j are shown in white with no marks. The exceptional curves E_i are shown in black, and the marks 2 are omitted. The reduced boundary $\tilde{B}_0$, if present, is shown as a crossed vertex.

We call the dual graph of the visible curves on $\tilde{X}$ the *visible graph*. It can be obtained by performing a series of insertions in the initial graph on four vertices with given marks $-1, -1, -1, -1$. Each instance is an insertion of a new vertex v_3 with mark 1 between two vertices v_1 and v_2 , at the same time increasing the marks of v_1 and v_2 by 1.

Now we attach a weight to each vertex of the visible graph. First we choose rational numbers w_0, w_1, w_2, w_3 for the four vertices of the initial graph, called the *initial weights*. We define the weights for the other vertices inductively in the process of inserting vertices, as follows. As a new vertex v_3 is inserted between two vertices v_1 and v_2 with already assigned weights $w(v_1)$ and $w(v_2)$, we define the weight of v_3 to be $w(v_3) = w(v_1) + w(v_2)$.

Indeed, our weights are just the suitably normalized log discrepancies for the pair $(\mathbb{P}^2, \sum_{k=0}^3 d_k L_k)$: we have $w_s = n(1 - d_s)$ for some positive rational number n , for all the visible curves. We can always replace n with a sufficiently divisible multiple to make the w_s integers, if we like.

LEMMA 3.1. *In terms of the weights, the conditions on the coefficients in Theorems 2.1 and 2.4 translate to the following:*

- (i) We have $d_0 + d_1 + d_2 + d_3 = 3$; equivalently, $n = w_0 + w_1 + w_2 + w_3$.
- (ii) We have $d_j \leq 0$ (respectively, $d_j < 0$) for a visible (-1) -curve C_j ; equivalently, the weight of the corresponding white vertex satisfies $w_j \geq n$ (respectively, $w_j > n$).
- (iii) In the case with a nonempty boundary, we have $d_0 \leq 1$ (respectively, $d_0 < 1$) for the curve $\tilde{B}_0$; equivalently, we have $w_0 \geq 0$ (respectively, $w_0 > 0$) for the corresponding crossed vertex.

For as long as the weights satisfy these conditions, all we have to do is this:

- (1) Make sure that the configuration of the black vertices $\{E_i\}$ is log terminal.
- (2) Compute the negatives of the discrepancies b_i from the linear system (2.1), or using the formulas from [Ale92], or by any other method.
- (3) Make sure that $K_X \pi_*(C_j) \geq 0$, $K_X \tilde{B}_0 > 0$ (if $\tilde{B}_0$ is present), and $K_X^2 > 0$ (respectively, $(K_X + B_0)^2 > 0$) using the formulas in Lemma 2.5.

If all of these are satisfied, then we get ourselves an example of a log canonical surface (which is either X or X_{can}) with ample (log) canonical divisor.

The visible graph is homeomorphic to a complete graph on four vertices, but the edges between the corners may contain many intermediate vertices.

DEFINITION 3.2. We call an edge a *Calabi–Yau* (or *CY*) *edge* if all the white vertices on this edge have weights n , that is, they have discrepancies $d_i = 0$.

The picture below shows two examples of edges that we use:

In general, if the end vertices have weights w_1 and w_2 , then vertices in the interior of this chain have weights of the form $m_1 w_1 + m_2 w_2$ for some coprime positive integers m_1 and m_2 , and the way in which these integers are produced in the sequence of blowups is equivalent to the, well-known in number theory, Stern–Brocot tree; see, for example, [GKP94, Section 4.5]. Thus, every edge is encoded by a sequence of positive rational numbers $\{m_{1,i}/m_{2,i}\}$ for the white vertices. For the two examples above, the sequences are $\{m/1\}$ and $\{m/1, 1/m'\}$. For a CY edge, these numbers must satisfy the condition $m_{1,i} w_1 + m_{2,i} w_2 = n$.

The following two lemmas are very easy and are given without proof.

LEMMA 3.3. (i) Suppose that all edges in the visible graph are CY edges. If $B_0 \neq 0$, then in addition suppose that the weight w_0 is 0. Then one has $K_X \equiv 0$ (respectively, $K_X + B_0 \equiv 0$).

(ii) Suppose that all edges are CY, except for one where there is a unique white vertex of weight $n + 1$. If $B_0 \neq 0$, then, in addition, suppose that the weight w_0 is 0. Then $K_X \equiv (1/n)C$, where C is an image of a (-1) -curve corresponding to that special white vertex (respectively, $K_X + B_0 \equiv (1/n)C$).

(iii) In the case with the nonempty boundary, suppose that all edges are CY and that the weight of B_0 is 1. Then $K_X + B_0 \equiv (1/n)B_0$ and $K_X \equiv -((n - 1)/n)B_0$.

LEMMA 3.4. In the last case (iii) of Lemma 3.3, the invariant δ_1 defined in (1.5) equals $\delta_1(X, B_0) = 1/n$. In all cases with nonempty boundary, one has $\epsilon_1(X, B_0) = -2 + \sum_{i \in I} b_i E_i \tilde{B}_0$.

Our main strategy for finding interesting examples is this: start with a CY situation, that is, case (i) of Lemma 3.3, and then go just one step up, to get in the situation of cases (ii) or (iii). The (log) canonical divisor is then guaranteed to be quite small.

Remark 3.5. The surfaces X appearing in case (i) of Lemma 3.3 with empty boundary satisfy $K_X \equiv 0$ and $H^1(\mathcal{O}_X) = 0$ and are sometimes called log Enriques surfaces. Our construction with weights provides a huge supply of such surfaces.

Remark 3.6. Another convenient way to compute $(K_X + B)^2$, respectively K_X^2 , is the following. If $K_X + B = (1/n)C$, then certainly $(K_X + B)^2 = (1/n)(K_X + B)C$. So one has to maximize the sum of the weights n and to minimize $(K_X + B)C$.

4. Simplest weights $(0, 1, 1, 1)$ and $(1, 1, 1, 1)$

Figure 1 gives the smallest volumes that can be achieved playing the combinatorial game with $B \neq 0$ and weights $(0, 1, 1, 1)$, and with $B = 0$ and weights $(1, 1, 1, 1)$.

Notation 4.1. The large numbers are marks when not equal to 1 or 2. The small (blue) integers underneath (or to the right of) the vertices are the weights, and the (red) fractional numbers above (or to the left of) the vertices are the b_i , the negatives of discrepancies.

For the first pair (X, B) , one has $\epsilon_1 = 1 - 1/3 - 1/4 - 1/5 = 13/60$. There is an alternative choice of weights $(1, 3, 4, 5)$ in this case, for which this pair fits into the case (iii) of Lemma 3.3; that is, all edges are CY. Then $\delta_1 = 1/13$ by Lemma 3.4, and $(K + B)^2 = (13/60) \cdot (1/13) = 1/60$. In fact,

 FIGURE 1. Surfaces with $(K_X + B)^2 = 1/60$ and $K_X^2 = 1/462$

another description for this pair is $(X, B) = (\mathbb{P}(3, 4, 5), D_{13})$, where D_{13} is a degree 13 weighted hypersurface, and this is exactly Kollár's example from [Kol13]. One has $\rho(X) = \text{rank Pic } X = 1$, and $K_X + B$, B , and $-K_X$ are ample.

For the second pair, we can also choose the weights $(1 - 2\epsilon, 1, 1 + \epsilon, 1 + \epsilon)$ with $0 < \epsilon < 1/2$, for which Theorem 2.1 and Lemma 3.1 give that K_X is ample and not just big and nef.

This and other examples were found by playing the combinatorial game explained in Section 3 using a custom Sage [S⁺17] script that one of us wrote. In more detail, in the case $B = \emptyset$, the script does the following:

- (1) Fix the positive weights $w_0 + w_1 + w_2 + w_3 = n$, for example the weights $1 + 1 + 1 + 1 = 4$ in Figure 1.
- (2) For each pair (w, w') , find all the possible CY edges as in Definition 3.2, encoded by pairs of coprime positive integers (m, m') such that $mw + m'w' = n$. There are only finitely many solutions.
- (3) Similarly, find all the CY+1 edges, solutions to the equation $mw + m'w' = n + 1$. There are only finitely many of those as well.
- (4) Now look at all the graphs obtained from the initial graph of four lines on $\mathbb{P}^2$ by inserting exactly one CY+1 edge, some CY edges, and leaving the other edges intact.
- (5) For each of the graphs, compute the intersection numbers $\pi^*K_X \cdot C_j$ with the visible curves using the formulas of Lemma 2.5. Leave only those where the intersection numbers are positive.
- (6) For each of the cases that passed the above filter, compute K_X^2 using (2.5) or (3.3) and pick the smallest.
- (7) The final result represents a surface with big and nef K_X whose canonical model $\overline{X}$ has ample $K_{\overline{X}}$ (no matter how it was found.) We easily check it by hand using Theorem 2.4 and Lemma 3.1.

For each quadruple $w_0 + w_1 + w_2 + w_3 = n$, this is a finite algorithm ending in a finite list. For the record examples given in the next sections, we then picked four-partitions of larger and larger integers n to run this script. It turns out that for n large enough, there are very few interesting CY edges and very few interesting surfaces. The records are achieved for relatively small values

of n for which there are many interesting solutions to the CY edge equation $mw + m'w' = n$.

We also wrote and ran versions of this algorithm with more than one CY+1 edge, and with CY+2 edges, and so on. The values of K_X^2 in those cases turned out to be significantly larger.

With some (quite significant) effort, this ad hoc algorithm perhaps can be turned into a proof in the case when the minimal resolution $\tilde{X}$ of X is obtained by the blowup of the four-line configuration in the plane.

5. Example with nonempty boundary: $1/462$

Figure 2 shows two nonisomorphic visible graphs producing surface pairs (X, B) of the smallest volume with nonempty boundary that we were able to find by our method.

FIGURE 2. Surfaces with $(K_X + B)^2 = 1/462 = 1/(11 \cdot 42)$ and $B \neq 0$

The winning weights are $(1, 2, 3, 5)$. Both pairs achieve the minimal possible value of $\epsilon_1 = 1 - 1/2 - 1/3 - 1/7 = 1/42$. On the other hand, one has $K_X + B \equiv (1/11)B$ and $\delta_1 = 1/11$ by Lemma 3.4. Thus, one has $(K_X + B)^2 = \epsilon_1 \delta_1$. The divisor $K_X + B$ is big and nef, but there are no weights for which Theorem 2.1, or Lemma 3.1, or a variation of them, shows that $K_X + B$ is ample. The canonical model has ample $K_{X_{\text{can}}} + B_{\text{can}}$ and $(K_{X_{\text{can}}} + B_{\text{can}})^2 = (K_X + B)^2 = 1/462$.

Notation 5.1. We use Figure 2 for an alternative labeling of the curves, as follows. We denote the strict preimages of the four lines by $\tilde{L}^w$, where the superscript $w = 1, 2, 3, 5$ is the (blue) weight of the corresponding vertex, given below or to the right of the vertex in the figure. Similarly, we denote one of the remaining curves by $\tilde{F}_w^{i,j}$ if its vertex lies on the edge between $\tilde{L}^i$ and $\tilde{L}^j$ and has weight w . In the same vein, we call the initial lines in the plane L^w and the intersection points $P^{i,j} = L^i \cap L^j$.

Finally, we use this labeling for the standard orthogonal basis of $\text{Pic } \tilde{X}$ consisting of the pullback H of a line from $\mathbb{P}^2$ and the full preimages of the (-1) -curves from the intermediate blowups $\tilde{X} \rightarrow \mathbb{P}^2$. Thus, in $\text{Pic } \tilde{X}$ we have $\tilde{L}^1 = H - F_4^{1,3} - F_6^{1,5}$, $\tilde{F}_4^{1,3} = F_4^{1,3} - F_7^{1,3} - F_{11}^{1,3}$, $\tilde{F}_7^{1,3} = F_7^{1,3} - F_{11}^{1,3}$, $\tilde{F}_{11}^{1,3} = F_{11}^{1,3}$, etc.

THEOREM 5.2. *The two visible graphs of Figure 2 describe the same surface X . If $\text{char } k \neq 2$, then $K_X + B$ is ample, $\rho(X) = 2$, and $X = X_{\text{can}}$ has four singularities. If $\text{char } k = 2$, then $K_X + B$ is big, nef, but not ample, and it contracts a (-2) -curve. One has $\rho(X_{\text{can}}) = 1$, and X_{can} has five singularities, the last one a simple A_1 .*

Proof. The second graph has an extra (-1) -curve $F_{11}^{1,5}$ not present in the first graph. It is easy to see that with respect to the first graph, it is simply the strict preimage of the line in $\mathbb{P}^2$ connecting the points $P^{1,5}$ and $P^{2,3}$. (We are using Notation 5.1 here.) Thus, the surfaces are the same, but different curves are illuminated as being visible.

Let us work with the first representation. Suppose that there exists another, not visible, curve $\tilde{D}$ such that $\tilde{D} \cdot \pi^* K_X = 0$, which is then contracted by a linear system $|N\pi^* K_X|$ for $N \gg 0$. Since $\rho(X) = 2$, there could only be one such irreducible curve. We write $\tilde{D} = dH - \sum m_w^{ij} F_w^{ij}$ in the basis H, F_w^{ij} of $\text{Pic } \tilde{X}$. The divisor $\tilde{D}$ has zero intersection with the curves

$$\tilde{L}^1 = H - F_4^{1,3} - F_6^{1,5}, \quad \tilde{L}^2 = H - F_5^{2,3} - F_7^{2,3} - F_9^{2,3} - F_{11}^{2,3}, \quad \tilde{F}_4^{1,3}, \tilde{F}_6^{1,5}, \tilde{F}_5^{2,3}, \tilde{F}_8^{2,3}$$

since the full pullback $\pi^* K_X = (1/11)\pi^* B_0$ is a strictly positive combination of these curves. Also, $\tilde{D}$ intersects nonnegatively all the other visible curves. This gives an explicit set of identities and inequalities. By hand or using linear programming software, one checks that the only solution is

$$\tilde{D} = c[3H - (2F_4^{1,3} + F_7^{1,3} + F_{11}^{1,3}) - (F_6^{1,5} + F_{11}^{1,5}) - (F_7^{2,5} + F_9^{2,5} + F_{11}^{2,5})].$$

Then $\tilde{D}K_{\tilde{X}} = 0$ and $\tilde{D}^2 = -2c^2$. It follows from the genus formula that $c = 1$, and $\tilde{D}$ is a smooth rational (-2) -curve. It must be a strict preimage of a cubic curve D in $\mathbb{P}^2$ which has

- (1) a cusp at $P^{1,3}$ with the tangent direction L^3 ,
- (2) a flex at $P^{2,5}$ with the tangent direction L^2 ,
- (3) a tangent at $P^{1,5}$ to the line L^5 .

(We are using Notation 5.1 here.) Thus, D is a cuspidal cubic, and the set of points of $D \setminus P^{1,3}$ has the structure of the additive group $\mathbb{G}_a$. We see that the cubic D with the above properties exists if and only if the system of equations $3P^{2,5} = P^{2,5} + 2P^{1,5} = 0$ has a solution in the base field k with $P^{2,5} \neq P^{1,5}$. This is possible if and only if $\text{char } k = 2$; then $P^{2,5} = 0$ and $P^{1,5}$ is any other smooth point. This completes the proof.

A second proof using the alternative presentation of surface X works in characteristics 2 and 0, and by extension in all but finitely many other positive characteristics. The second visible graph of Figure 2 leads to a smooth rational (-2) -curve

$$\tilde{D} = 4H - (2F_4^{1,3} + F_7^{1,3} + F_{11}^{1,3}) - (2F_6^{1,5} + 2F_{11}^{1,5}) - (F_5^{2,3} + F_7^{2,3} + F_9^{2,3} + F_{11}^{2,3}).$$

It must then be a strict preimage of a quartic curve D on $\mathbb{P}^2$ that has

- (1) an A_2 -singularity (a cusp) at $P^{1,3}$ with the tangent line L^3 ,
- (2) an A_3 -singularity (a tacnode) or A_4 -singularity at $P^{1,5}$ with the tangent line L^5 ,
- (3) a hyperflex at $P^{2,3}$ with the tangent direction L^2 ; that is, $L^2 \cap D$ is a smooth point of D and the intersection is of multiplicity 4.

If $\text{char } k = 0$, then such quartic curves do not exist. Indeed, [Wal95a, Table 2] shows that irreducible quartic curves in characteristic 0 with A_2A_3 - or A_2A_4 -singularities do not have any hyperflexes. Since the property of $K_X + B$ being ample is open, the same is true in all but finitely many prime characteristics. On the other hand, if $\text{char } k = 2$, then there exists a unique such curve (with A_2A_3), which can be concluded from the normal forms of quartics given in [Wal95b]. Explicitly, the equation of D can be taken to be $f = x^4 + x^2yz + x^2y^2 + y^2z^2$, and the lines are $L^1 = (x)$, $L^2 = (y + z)$, $L^3 = (x + z)$, $L^5 = (y)$. $\square$

Remark 5.3. Combining the two presentations of the surface X in the proof, we see that a quartic

with the above configuration of singularities and tangent lines does not exist in prime characteristics $p \neq 2$.

Remark 5.4. The intersections of $\tilde{D}$ with the visible curves are zero except for the following:

- (1) For the first graph, $\tilde{D}\tilde{F}_{11}^{1,3} = \tilde{D}\tilde{F}_{11}^{1,5} = \tilde{D}\tilde{F}_{11}^{2,3} = 1$.
- (2) For the second graph, $\tilde{D}\tilde{F}_{11}^{1,5} = 2$ and $\tilde{D}\tilde{F}_{11}^{1,3} = \tilde{D}\tilde{F}_{11}^{2,3} = 1$.

6. Example with empty boundary: $1/48983$

There exist at least four visible graphs that produce surfaces X without a boundary and $K_X^2 = 1/48983 = 1/(11 \cdot 61 \cdot 73)$. Three of them share the same list of singularities; the list is different for the fourth graph. The first two graphs can be obtained directly by inserting ten vertices into the edge from $\tilde{L}^1$ to $\tilde{L}^2$, that is, by blowing up the surfaces $\tilde{X}$ of Figure 2 ten times above point $P^{1,2}$. We show one of these surfaces in Figure 3. It leads to a surface with $\rho(X) = 2$ and three singularities.

The fourth graph describes a surface X' with $\rho(X') = 3$ which has four singularities: the singularity with the minimal resolution $[2, 4, 2, 2, 2]$ of Figure 3 is replaced by two singularities $[2, 6]$ and $[2, 3, 2, 2]$.

FIGURE 3. Surface with $K_X^2 = 1/48983 = 1/(11 \cdot 61 \cdot 73)$

The set of winning weights in these cases is again $(1, 2, 3, 5)$. Since $48983 > 42^2$, these examples show that $K^2(\mathcal{S}_1)$ is achieved when the boundary is empty, that is, $K_1^2(\mathcal{S}_1) > K^2(\mathcal{S}_1) = K^2(\mathcal{S}_0)$.

Similarly to the previous case, the divisor K_X is big and nef, but there are no weights for which Theorem 2.1, or Lemma 3.1, or a variation of them, shows that K_X is ample. But the canonical model X_{can} has ample canonical class and $K_{X_{\text{can}}}^2 = K_X^2 = 1/48983$.

THEOREM 6.1. *The three distinct visible graphs describe the same surface X . The fourth graph describes a different surface X' which, however, has the same canonical model since there is a crepant blowdown $X' \rightarrow X$ contracting the image of a (-1) -curve from $\tilde{X}'$. If $\text{char } k \neq 2$, then K_X is ample, $\rho(X) = 2$, and $X = X_{\text{can}}$ has three singularities. If $\text{char } k = 2$, then K_X is big,*

nef, but not ample, and it contracts a (-2) -curve; $\rho(X_{\text{can}}) = 1$; and X_{can} has four singularities, the last one a simple A_1 .

Proof. The proof of the equivalence for the first three graphs is the same as in Theorem 5.2. Since the surface in the end is unique, we do not draw the other two graphs but indicate the two invisible curves that have to be added to Figure 3 to obtain them. These are the strict preimages of a line in $\mathbb{P}^2$ joining $P^{1,5}$ and $P^{2,3}$, and of a conic passing through $P^{1,2}$ generically, through $P^{1,5}$ with the tangent L^5 , and through $P^{2,3}$ with the tangent L^3 .

One way to see surface X' with $\rho(X') = 3$ and four singularities is to start with the graph in Figure 3 and blow up twice above point $P^{3,5}$. This splits the chain $[2, 4, 2, 2, 2]$ into two chains $[2, 6]$ and $[2, 3, 2, 2]$. However, for the new (-1) -curve C , one has $C \cdot \pi^*K_X = 0$, so the canonical model is the same.

This operation adds a *visible* (-1) -curve with $C \cdot \pi^*K_X = 0$, which is not allowed. However, if one uses the presentation of X given by the third visible graph (which we do not draw), this operation adds an *invisible* (-1) -curve. This way, one obtains the fourth legal visible graph that gives surface X' .

From now on, we work with the surface X described by Figure 3. Again, if there exists an invisible curve $\tilde{D}$ with $\tilde{D} \cdot \pi^*K_X = 0$, then it must have zero intersection with the curves effectively supporting π^*K_X , which include the ten newly inserted curves $\tilde{F}_w^{12}$. Thus, the inequalities in this case are reduced to those in Theorem 5.2, and the rest of the proof is the same. $\square$

As in Remark 5.4, the intersections of D with the visible curves are zero except for those listed there.

7. Connection with the algebraic Montgomery–Yang problem

The algebraic Montgomery–Yang problem [Kol08, Conjecture 30] asks whether there exists a surface with $\rho(S) = 1$ and $\pi_1(S \setminus \text{Sing } S) = 1$ that has *four* quotient singularities. Conjecturally, the answer is no. All the possibilities for such surfaces were ruled out except when K_S is ample; see [HK12].

It is amusing to note that if the characteristic 2 surface $S = X_{\text{can}}$ with four singularities which we constructed in Theorem 6.1 existed in characteristic 0, then it would provide a counterexample to the above conjecture.

Let $U_s \ni s$ be a small neighborhood of a singular point s and $L_s = U_s \setminus s$. Then the three singularities whose determinants $m = 11, 61, 73$ are coprime to 2 are quotient singularities, and one has $\pi_1^{\text{alg}}(L_s) = \mathbb{Z}_m$. For the fourth singularity obtained by contracting a (-2) -curve, one has $\pi_1^{\text{alg}}(L_s) = 1$ in characteristic 2. One can prove that $\pi_1^{\text{alg}}(S \setminus \text{Sing } S) = 1$ by the usual methods, by considering the images of the (-1) -curves C_j connecting the singularities and using the van Kampen theorem, which still holds for the étale fundamental group in positive characteristic by [Gro63, Chapter IX, Theorem 5.1]; cf. [Mor12].

In any case, this surface also violates the orbifold Bogomolov–Miyaoka–Yau inequality $c_1^2(S) \leq 3e_{\text{orb}}(S)$, for which one may see the discussion in [Kol08, Section 1]. Namely, it violates its corollary, the inequality

$$\sum_{s \in \text{Sing } S} \left(1 - \frac{1}{|\pi_1(U_s \setminus s)|} \right) \leq 3,$$

if one literally replaces π_1 with π_1^{alg} , or if one replaces $|\pi_1(U_s \setminus s)|$ with m_i . Thus, this configuration of singularities cannot appear in characteristic 0 if $\rho(X) = 1$.

8. The case of Picard rank 1

In part because of the connection with the algebraic Montgomery–Yang problem, it is of interest to know the minimal volume for surfaces with the additional condition $\rho(X) = 1$ in characteristic 0. For surfaces without the boundary, the best we were able to find by our method is $K_X^2 = 1/6351 = 1/(3 \cdot 29 \cdot 73)$.

There are three possible graphs for the visible curves in this case. One of these graphs describes a surface $T(2, 2, 4, 10)$ from [HK12]. In that paper, Hwang and Keum construct, for any $a_1, a_2, a_3, a_4 \geq 2$, a surface $T = T(a_1, a_2, a_3, a_4)$ with $\rho(T) = 1$ obtained by blowing up the four-line configuration; it has two cyclic singularities corresponding to the chains $[2 * (a_4 - 1), a_3, a_1, 2 * (a_2 - 1)]$ and $[2 * (a_3 - 1), a_2, a_4, 2 * (a_1 - 1)]$. By [UY16], these surfaces include all the Kollár's surfaces $S^*(a_1, a_2, a_3, a_4)$ with $\gcd(w_1, w_3) = \gcd(w_2, w_4) = 1$ and $w^* = 1$. In particular, $T(2, 2, 4, 10)$ is the same as Kollár's surface $S^*(2, 2, 4, 10)$.

The other two visible graphs are not of this type, and we give one of them in Figure 4. However, all three graphs share the same list of singularities. Indeed, the argument we gave in the proofs of Theorems 5.2 and 6.1 shows that the three visible graphs describe the same surface $X = T(2, 2, 4, 10) = S^*(2, 2, 4, 10)$.

FIGURE 4. Surface with $K_X^2 = 1/6351 = 1/(3 \cdot 29 \cdot 73)$ and $\rho(X) = 1$

Remark 8.1. Comparing Figures 3 and 4, one can see that our surface with the minimal volume can be obtained from the surface $S^*(2, 2, 4, 10)$ by a weighted blowup at one point.

THEOREM 8.2. *Let $a_1, a_2, a_3, a_4 \geq 2$ be the parameters for the surfaces $T(a_1, a_2, a_3, a_4)$ defined immediately above. We set*

$$A = a_1a_2a_3a_4 - a_2a_3a_4 - a_1a_3a_4 - a_1a_2a_4 - a_1a_2a_3 \\ + a_1a_2 + a_2a_3 + a_3a_4 + a_1a_4 - a_1 - a_2 - a_3 - a_4 + 3,$$

$$\begin{aligned} B_1 &= a_1 a_2 a_3 a_4 - a_1 a_3 a_4 - a_1 a_2 a_3 + a_2 a_3 + a_1 a_4 - a_1 - a_3 + 1, \\ B_2 &= a_1 a_2 a_3 a_4 - a_2 a_3 a_4 - a_1 a_2 a_4 + a_1 a_2 + a_3 a_4 - a_2 - a_4 + 1. \end{aligned}$$

Then the following are true:

- (i) The surface $T(a_1, a_2, a_3, a_4)$ has ample canonical class K_T if and only if $A > 0$.
- (ii) The determinants of the two singularities are B_1 and B_2 .
- (iii) $K_T^2 = A^2/B_1 B_2$.
- (iv) The minimum $K_T^2 = 1/6351$ is achieved for $(a_i) = (2, 2, 4, 10)$, up to a cyclic rotation.

Proof. (i) Using Lemma 2.5(i) and a formula in [Ale92, Section 3.2] for discrepancies of a log terminal singularity, we compute that for the four visible (-1) -curves C_i , one has $\pi^* K_T \cdot C_i = A w_i / B_1 B_2$, where the $w_i > 0$ are the weights. Then statement (i) follows.

(ii) This is [HK12, Lemma 2.4].

(iii) This follows by a direct computation, applying Lemma 2.5(ii) and a formula for discrepancies of a log terminal singularity in [Ale92, Section 3.2].

(iv) It is somewhat more convenient to use the variables $x_i = a_i - 1$. Then

$$\frac{A}{x_1 x_2 x_3 x_4} = 1 - \frac{1}{x_1} \frac{1}{x_3} \left(1 + \frac{1}{x_2} + \frac{1}{x_4} \right) - \frac{1}{x_2} \frac{1}{x_4} \left(1 + \frac{1}{x_1} + \frac{1}{x_3} \right).$$

One easily checks that for $j = 1, 2$, the partial derivatives satisfy $\partial(A/B_j)/\partial x_i > 0$ when $A > 0$ and $x_i \geq 1$. Thus, it is sufficient to check the minimal collections (a_i) for which $A > 0$. Here, $(a'_i) > (a_i)$ if all $a'_i \geq a_i$ and at least one inequality is strict.

We first find the “critical” collections, for which $A = 0$. These are $(3, 3, 3, 3)$, $(2, 8, 3, 3)$, $(2, 3, 11/2, 3)$, $(2, 3, 3, 8)$, $(2, 2, 4, 9)$, $(2, 2, 5, 6)$, $(2, 2, 6, 5)$, $(2, 2, 9, 4)$, ordered by the number of 2’s and 3’s in them. Note that $A(2, a_2, 2, a_4) < 0$.

Then, modulo rotational symmetry, the smallest collections (a_i) for which $A > 0$ are $(4, 3, 3, 3)$, $(2, 9, 3, 3)$, $(2, 8, 4, 3)$, $(2, 8, 3, 4)$, $(2, 3, 6, 3)$, $(2, 4, 3, 8)$, $(2, 3, 4, 8)$, $(2, 3, 3, 9)$, $(3, 2, 4, 9)$, $(2, 3, 4, 9)$, $(2, 2, 5, 9)$, $(2, 2, 4, 10)$, $(3, 2, 5, 6)$, $(2, 3, 5, 6)$, $(2, 2, 6, 6)$, $(2, 2, 5, 7)$, $(3, 2, 6, 5)$, $(2, 3, 6, 5)$, $(2, 2, 7, 5)$, $(2, 2, 6, 6)$, $(2, 3, 9, 4)$, $(2, 2, 10, 4)$, $(2, 2, 9, 5)$. Among these, the minimal value $A^2/B_1 B_2 = 1/6351$ is achieved only for $(a_1, a_2, a_3, a_4) = (2, 2, 4, 10)$. $\square$

For surfaces with boundary, we found a pair with $(K_X + B_0)^2 = 1/78$. The marks of the corners are 1, 3, 2', 2'' (B_0 goes first, we use the notation 2' and 2'' to distinguish the two vertices with the same marks), and the curves along the edges have marks 1–3, 1–2–2–1–2', 1–2–1–2'', 3–2', 3–1–2–2–2–2'', 2'–2''. The weights that work for Lemma 3.3(iii) are $(1, 1, 2, 3)$, $n = 7$, $\delta_1 = 1/7$, and $\epsilon_1 = 1 - 1/2 - 1/3 - 1/13 = 7/78$.

9. Why only four lines?

It may seem naive and insufficient in search of examples to reduce oneself only to the simplest of line arrangements: four lines in the plane. Why not consider some more interesting configurations, for example, the Fano configuration of seven lines, which is defined over the field $\mathbb{F}_2$, or the Hesse (respectively, dual Hesse) configuration of twelve (respectively, nine) lines? And why lines and not conics or curves of higher degree? In fact, there are good ad hoc reasons for this:

- (1) For all examples of log surfaces arising from four lines, one *a priori* has $K_X^2 \leq (K_{\mathbb{P}^2} + \sum_{i=0}^3 L_i)^2 = 1$. Similarly, for d lines in general position, an upper bound is $(d - 3)^2$. For

special line arrangements, the upper bound is smaller, but it starts with 2 for a special configuration of five lines. Although this is an *upper* and not a lower bound, it shows how hard one has to work to achieve a minimum. Indeed, for $d \geq 7$ lines in general position, it is easy to show that

$$K_X^2 \geq 9 - \frac{d(d-1)}{2} + \frac{d-4}{d-2}(d-4)d = \frac{d^2}{2} - \frac{11d}{2} + 13 + \frac{8}{d-2} \geq \frac{3}{5},$$

with the minimum achieved by blowing up all of the $d(d-1)/2$ intersection points of the d lines.

- (2) The combinatorial game similar to the one we described in Section 3 becomes very hard to play for more than four lines. For example, for five lines, the condition on the weights becomes $w_0 + \cdots + w_4 = 2n$, and there are very few interesting examples. A similar thing happens if one works with conics instead of lines.
- (3) We also note that constructions of many interesting log surfaces with ample K_X can be reduced to blowups of the same four-line configuration in $\mathbb{P}^2$, even when the initial definition is different; see, for example, [Kol08, HK11, HK12, UY16]. The surfaces of [HK12] that use conics and cubics all have bigger volumes.

10. Lower bound for $K^2(\mathcal{S}_1) = K^2(\mathcal{S}_0)$

In this section, we spell out the explicit effective lower bound for $K^2(\mathcal{S}_0)$ provided by [AM04, Theorem 4.8]. In our present notation, it says the following:

$$K_X^2 \geq \frac{1}{\ell \cdot (2\ell)^N}, \quad \text{where } N = 128\ell^5 + 4\ell \quad \text{and} \quad \ell = \lceil 1/\delta_1(\mathcal{S}_1) \rceil.$$

Together with Kollár's bound $\delta_1(\mathcal{S}_1) \geq 1/42$, this gives

$$K_X^2 \geq \frac{1}{42 \cdot 84^{128 \cdot 42^5 + 4 \cdot 42}} \approx 10^{-3 \cdot 10^{10}}.$$

Certainly this is not a realistic bound. Many improvements can be made to the estimates in [AM04], but they would not cardinally change the estimate without introducing some cardinally new methods. The true lower bound for $K^2(\mathcal{S}_0)$ may be closer to Kollár's conjectural bound for $K^2(\mathcal{S}_2) = 1/(42 \cdot 43)^2$. Indeed, we dare to think that it could be close, or equal to the value $1/48983 \approx 67/(42 \cdot 43)^2$ that we give here.

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