Relative orbifold Donaldson–Thomas theory and the degeneration formula

Zijun Zhou

ABSTRACT

We generalize the notion of expanded degenerations and pairs for a simple degeneration or smooth pair to the case of smooth Deligne–Mumford stacks. We then define stable quotients on the classifying stacks of expanded degenerations and pairs and prove the properness of their moduli. On 3-dimensional smooth projective Deligne–Mumford stacks, this leads to a definition of relative Donaldson–Thomas invariants and the associated degeneration formula.

1. Introduction

1.1 Background and motivation

Let X be a smooth projective 3-fold. Motivated by many as a higher-dimensional gauge theory and introduced by R. Thomas [Tho00] as a holomorphic analog of the Casson invariant, Donaldson–Thomas (DT) theory counts ideal sheaves of curves on X in certain fixed topological classes.

A coherent sheaf $\mathcal{I}$ can be realized as an ideal sheaf of a 1-dimensional subscheme $Z \subset X$ if and only if it is torsion-free of rank 1 with trivial determinant, which means that DT theory can be viewed as either a sheaf-counting theory or a curve-counting theory. The essential connection of DT theory to other curve-counting theories was first established by the work of Maulik, Nekrasov, Okounkov and Pandharipande [MNOP06a, MNOP06b], known as the Gromov–Witten/Donaldson–Thomas correspondence. They conjectured and proved in special cases that the generating functions of Gromov–Witten (GW) and DT theory can be equated to each other after a change of variable. They adopted the localization technique in the toric setting and developed the theory of DT topological vertex.

The GW/DT correspondence has also been proved for local curves [BP08, OP10]. This is the first nontoric case one can actually do computations on. The crucial tool is the DT degeneration formula, motivated by the degeneration formula in GW theory and developed by J. Li and B. Wu [LW15]. With the degeneration formula in the simple degeneration case, one can “split” X into two simpler spaces Y_- and Y_+ and express the DT invariant of X in terms of the relative theory of $Y_{\pm}$ with respect to the divisor D .

The goal of this paper is to generalize the relative Donaldson–Thomas theory and the degen-

Received 27 July 2015, accepted in final form 6 May 2017.

2010 Mathematics Subject Classification 14N35 (primary), 14C05, 14D23 (secondary).

Keywords: relative Donaldson–Thomas invariants, Deligne–Mumford stacks, degeneration formula.

This journal is © Foundation Compositio Mathematica 2018. This article is distributed with Open Access under the terms of the Creative Commons Attribution Non-Commercial License, which permits non-commercial reuse, distribution, and reproduction in any medium, provided that the original work is properly cited. For commercial re-use, please contact the Foundation Compositio Mathematica.

eration formula to 3-dimensional smooth projective *orbifolds*, which in algebraic settings refers to Deligne–Mumford stacks. One important reason why people care about orbifolds is Ruan’s crepant resolution conjecture [Rua06, BG09]: the GW or DT invariants of a 3-orbifold satisfying the hard Lefschetz condition should be equivalent to those of its crepant resolution (if it exists), up to change of variables and analytic continuation. Orbifold GW theory and its degeneration formula have already appeared in [AGV08, AF16].

On the DT side, the orbifold topological vertex technique was developed in [BCY12] for toric Calabi–Yau (CY) 3-orbifold. Orbifold DT theory is also treated in [GT13], for projective CY 3-orbifolds. In both cases, the CY condition gives a symmetric obstruction theory and defines a Behrend’s function ν , and the DT invariants are defined as weighted Euler characteristics with respect to ν . The GW/DT correspondence for the orbifold topological vertex with transversal A_n -singularities was proved in [Zon15, RZ13, RZ15, Ros15], and the crepant resolution conjecture for DT vertex with transversal A_n -singularities was proved in [Ros17].

We are particularly interested in the following picture, which indicates the relationship between various theories involved and will be pursued in future work:

The upper triangle in the diagram is established in [Mau09, MO09a, MO09b], and the vertical lines stand for crepant transformation correspondences. We hope to establish the lower triangle to complete the whole diagram.

1.2 Outline

For simplicity, we work over the field of complex numbers $\mathbb{C}$. We always use A_* and A^* to denote appropriate cohomology and homology theory over $\mathbb{Q}$, which could be Chow groups, Borel–Moore homology groups, etc. K-theory will be either topological or algebraic K-theory, over $\mathbb{Q}$.

In this paper, we consider a smooth projective Deligne–Mumford stack W and define the absolute DT invariants with descendents and insertions, generalizing the DT invariants in the scheme case. For a smooth divisor $D \subset W$, we also define the relative DT invariants of W with respect to D . We follow the approach of introducing a perfect obstruction theory in each case, *without* the CY assumption.

For simplicity, we assume that the divisor D is *connected*. The theory can be generalized easily for disconnected D .

We treat the case of the simple degeneration in this paper. Let $\pi: X \rightarrow C$ be a projective family of smooth Deligne–Mumford stacks of relative dimension 3. By a simple degeneration, we roughly mean that π is in the form of a simple normal crossing near some point $0 \in C$, with singular fiber $X_0 = Y_- \cup_D Y_+$ splitting into two pieces. Let X_c be a smooth fiber.

Let $F_1 K(X)$ be the subgroup of $K(X)_{\mathbb{Q}}$ generated by the structure sheaves of 1-dimensional closed substacks. Given $P \in K(X)$, let $\mathcal{M}^P$ be the stack parameterizing 1-dimensional closed substacks on expanded degenerations

$$X_0[k] := Y_- \cup_D \Delta_1 \cup_D \cdots \cup_D \Delta_k \cup_D Y_+,$$

with topological datum P . Similarly, given $\theta_{\pm} \in K(Y_{\pm})$ and $\theta_0 \in K(\text{Hilb}(D))$, let $\mathcal{N}_{\pm}^{\theta_{\pm}, \theta_0}$ be the stack parameterizing 1-dimensional closed substacks on expanded pairs

$$Y[l] := Y \cup_D \Delta_1 \cup_D \cdots \cup_D \Delta_l,$$

with topological datum $(\theta_{\pm}, \theta_0)$. Let $\mathcal{M}^{\theta} \subset \mathcal{M}^P$ be the open and closed substacks associated with the relative data θ , respectively. We will construct perfect obstruction theories on these stacks.

Let $i_c : \{c\} \hookrightarrow C$ be the inclusion of a point. One has the following two Cartesian diagrams:

$$\begin{array}{ccc} \text{Hilb}_{X_c}^P & \longrightarrow & \mathcal{M}^P \\ \downarrow & & \downarrow \\ \{c\} & \hookrightarrow & C, \end{array} \quad \begin{array}{ccc} \mathcal{M}^{\theta} & \xrightarrow{\iota_{\theta}} & \mathcal{Q}\text{uot}_{\mathfrak{X}_0/\mathfrak{C}_0}^{\mathcal{O}_{X,P}} \hookrightarrow \mathcal{M}^P \\ \downarrow & & \downarrow \\ \{0\} & \hookrightarrow & C, \end{array}$$

where the map $\mathcal{M}^P \rightarrow C$ is the composition $\mathcal{M}^P \rightarrow \mathfrak{C}^P \rightarrow C$. Our main theorem is the following.

THEOREM 1.1 (Degeneration formula: cycle version). *We have*

$$\begin{aligned} i_c^! [\mathcal{M}^P]^{\text{vir}} &= [\text{Hilb}_{X_c}^P]^{\text{vir}}, \\ i_0^! [\mathcal{M}^P]^{\text{vir}} &= \sum_{\theta \in \Lambda_P^{\text{spl}}} \iota_{\theta*} \Delta^! ([\mathcal{N}_{-}^{\theta_{-}, \theta_0}]^{\text{vir}} \times [\mathcal{N}_{+}^{\theta_{+}, \theta_0}]^{\text{vir}}), \end{aligned}$$

where the classes in the second row are viewed in $0 \times_{\mathbb{A}^1} \mathcal{M}^P$.

Here, Δ is the diagonal map in the following diagram:

$$\begin{array}{ccc} \mathcal{M}^{\theta} \xleftarrow[\Phi_{\theta}]{\cong} \mathcal{N}_{-}^{\theta_{-}, \theta_0} \times_{\text{Hilb}_D^{\theta_0}} \mathcal{N}_{+}^{\theta_{+}, \theta_0} & \longrightarrow & \mathcal{N}_{-}^{\theta_{-}, \theta_0} \times \mathcal{N}_{+}^{\theta_{+}, \theta_0} \\ \downarrow & & \downarrow \\ \text{Hilb}_D^{\theta_0} & \xrightarrow{\Delta} & \text{Hilb}_D^{\theta_0} \times \text{Hilb}_D^{\theta_0}. \end{array}$$

Using virtual classes, one can define the corresponding numerical DT invariants $\langle \prod \tau_{k_i}(\gamma_i) \rangle_{X_c}^P$ and $\langle \prod \tau_{k_i}(\gamma_{i,\pm}) | C \rangle_{Y_{\pm}, D}^{\theta_{\pm}}$, where the γ_i are descendent insertions and C is the relative insertion. Let $\{C_k\}$ be a basis for the cohomology of $\text{Hilb}(D)$, and let g^{kl} be the inverse matrix for the Poincaré pairing under this basis. We have the following numerical version of the degeneration formula.

THEOREM 1.2 (Degeneration formula: numerical version). *Given $P \in F_1 K(X_c)$, assume that $\gamma_{i,\pm}$ are disjoint with D . We have*

$$\left\langle \prod_{i=1}^r \tau_{k_i}(\gamma_i) \right\rangle_{X_c}^P = \sum_{\substack{\theta_{-} + \theta_{+} - P_0 = P, \\ S \subset \{1, \dots, r\}, k, l}} \left\langle \prod_{i \in S} \tau_{k_i}(\gamma_{i,-}) \middle| C_k \right\rangle_{Y_-, D}^{\theta_{-}} g^{kl} \left\langle \prod_{i \notin S} \tau_{k_i}(\gamma_{i,+}) \middle| C_l \right\rangle_{Y_+, D}^{\theta_{+}},$$

where the $\theta_{\pm} \in F_1 K(Y_{\pm})$ range over all configurations that satisfy $\theta_{-} + \theta_{+} - P_0 = P$.

Let $F_1^{\text{mr}}K(X)$ be the *multi-regular* subgroup, generated by multi-regular stacky curves, that is, those whose associated representation of the stabilizer group at the generic point is a multiple of the regular representation. In the multi-regular case, consider the image (β, ε) of P in the quotient $F_1^{\text{mr}}K(X_c)/F_0(X_c) \oplus K_0(X_c)$ as the topological datum.

THEOREM 1.3 (Degeneration formula: numerical version for multi-regular case). *Given $\beta \in F_1^{\text{mr}}K(X_c)/F_0K(X_c)$, assume that the $\gamma_{i,\pm}$ are disjoint with D . We have*

$$\left\langle \prod_{i=1}^r \tau_{k_i}(\gamma_i) \right\rangle_{X_c}^{\beta, \varepsilon} = \sum_{\substack{\beta_- + \beta_+ = \beta, \\ \varepsilon_- + \varepsilon_+ = \varepsilon + m, \\ S \subset \{1, \dots, r\}, k, l}} \left\langle \prod_{i \in S} \tau_{k_i}(\gamma_{i,-}) \middle| C_k \right\rangle_{Y_-, D}^{\beta_-, \varepsilon_-} g^{kl} \left\langle \prod_{i \notin S} \tau_{k_i}(\gamma_{i,+}) \middle| C_l \right\rangle_{Y_+, D}^{\beta_+, \varepsilon_+},$$

where $\beta_- \in F_1^{\text{mr}}K(Y_-)/F_0K(Y_-)$ and $\beta_+ \in F_1^{\text{mr}}K(Y_+)/F_0K(Y_+)$ range over all curve classes that coincide on D and satisfy $\beta_- + \beta_+ = \beta$.

The theorem can be restated in the form of generating functions, obtained by summing over ε .

COROLLARY 1.4. *Given $\beta \in F_1^{\text{mr}}K(X_c)/F_0K(X_c)$, assume that the $\gamma_{i,\pm}$ are disjoint with D . Then,*

$$\begin{aligned} Z_\beta \left(X_c; q \middle| \prod_{i=1}^r \tau_{k_i}(\gamma_i) \right) &= \sum_{\substack{\beta_- + \beta_+ = \beta \\ S \subset \{1, \dots, r\}, k, l}} \frac{g^{kl}}{q^m} Z_{\beta_-, C_k} \left(Y_-, D; q \middle| \prod_{i \in S} \tau_{k_i}(\gamma_{i,-}) \right) \\ &\quad \times Z_{\beta_+, C_l} \left(Y_+, D; q \middle| \prod_{i \notin S} \tau_{k_i}(\gamma_{i,+}) \right). \end{aligned}$$

The paper is organized as follows. In Section 2, we generalize J. Li's construction of expanded degenerations and pairs to the orbifold case, and in Section 3, we discuss the concept of stable quotients. Sections 3, 4 and 5 aim to define and prove the properness of the moduli stacks of stable quotients on the classifying stacks of expanded degenerations and pairs. Finally, in Sections 6, 7 and 8, we specialize to the case of 3-orbifolds, construct perfect obstruction theories on such stacks and prove the degeneration formula, for both the cycle version and the numerical version.

2. Stacks of expanded degenerations and pairs

In this section, we construct the stacks of expanded degenerations and expanded pairs, which serve as the target spaces for the relative curve-counting theories and the degeneration of curve-counting theories. There are several approaches to this construction. The first algebraic-geometric approach is due to J. Li [Li01], who explicitly constructs standard families of expanded degenerations and pairs, and then forms the stacks as limits of them. Various equivalent approaches are well summarized in the paper [ACFW13], which works with algebraic stacks and proves the independence of the resulting stacks on the original target.

We will mainly adopt J. Li's explicit process. In fact, the following are just direct generalizations of his method to the case of Deligne–Mumford stacks. We will define the notions of expanded degenerations and pairs, build standard families of expanded degenerations and pairs, and then construct the general stacks using the standard models as coverings. Our basic setting is as follows.

DEFINITION 2.1. Let Y be a separated Deligne–Mumford stack of finite type over $\mathbb{C}$, and let $D \subset Y$ be a locally smooth connected effective divisor. By locally smooth, we mean that étale locally near a point on D , the pair (Y, D) is a pair of a smooth stack and a smooth divisor. We call such a pair (Y, D) *locally smooth*. We call it a *smooth pair* if both Y and D are smooth.

DEFINITION 2.2. Let X be a separated Deligne–Mumford stack of finite type over $\mathbb{C}$, and let $(C, 0)$ be a smooth pointed curve over $\mathbb{C}$. A *locally simple degeneration* is a flat morphism $\pi: X \rightarrow C$, with central fiber $X_0 = Y_- \cup_D Y_+$, where $Y_{\pm} \subset X_0$ are closed substacks, $(Y_{\pm}, D)$ are locally smooth pairs and $Y_{\pm}$ intersect transversally as $Y_- \cap Y_+ = D$.

By $Y_- \cup_D Y_+$, we mean the pushout in the 2-category of algebraic stacks in the sense of [AGV08]. We denote the divisor D in $Y_{\pm}$ by $D_{\pm}$.

By *transversal* intersection, we mean the following. For any point p in the divisor D , there are étale neighborhoods V of $0 \in C$ and U of $p \in X$, such that the following diagram is commutative, with all horizontal maps étale:

$$\begin{array}{ccccc} \mathrm{Spec}(\mathbb{C}[x, y, t]/(xy - t)) \times \mathbb{A}^n & \xleftarrow{\text{ét}} & U & \xrightarrow{\text{ét}} & X \\ \downarrow \pi & & \downarrow \pi|_U & & \downarrow \pi \\ \mathrm{Spec} \mathbb{C}[t] & \xleftarrow{\text{ét}} & V & \xrightarrow{\text{ét}} & C. \end{array}$$

In other words, there exist a common étale neighborhood V of $\mathrm{Spec} \mathbb{C}[t]$ and C and a common étale neighborhood U of $\mathrm{Spec}(\mathbb{C}[x, y, t]/(xy - t)) \times \mathbb{A}^n$ and X .

We call such data a *simple degeneration* if, furthermore, $(Y_{\pm}, D_{\pm})$ are smooth pairs and for any $c \in C$ with $c \neq 0$, the fiber $X_c := \pi^{-1}(c)$ is smooth.

We call such a U together the restricted family π_U a *standard local model*.

Let $N_- := N_{D/Y_-}$ and $N_+ := N_{D/Y_+}$. Checking on local coordinates, one can easily find that $N_- \otimes N_+ \cong \mathcal{O}_D$.

Remark 2.3. By Artin’s algebraic approximation theorem, our definition of transversality is equivalent to that in the language of formal neighborhoods, as in [Li01].

Remark 2.4. In the study of degenerations, we are only interested in local behavior around the singular divisor. For convenience, we can always replace the curve C by an étale neighborhood around $0 \in C$ and just assume $C = \mathbb{A}^1$.

2.1 Expanded degenerations and pairs

For the central fiber $X_0 = Y_- \cup_D Y_+$, consider the $\mathbb{P}^1$ -bundle over D

$$\Delta := \mathbb{P}_D(\mathcal{O}_D \oplus N_+) \cong \mathbb{P}_D(N_- \oplus \mathcal{O}_D),$$

with two sections D_- and D_+ corresponding to the 0-sections of the two expressions, where $N_{D_-/\Delta} \cong N_+$ and $N_{D_+/\Delta} \cong N_-$. The $\mathbb{P}^1$ -bundle Δ will often be called a “bubble component.”

For an integer $k \geq 0$, take k copies of Δ , indexed by $\Delta_1, \dots, \Delta_k$, and insert them between $\Delta_0 := Y_-$ and $\Delta_{k+1} := Y_+$. We have

$$X_0[k] := Y_- \cup_D \Delta_1 \cup_D \dots \cup_D \Delta_k \cup_D Y_+,$$

where $D_- \subset \Delta_i$ is glued to $D_+ \subset \Delta_{i+1}$ for $i = 0, \dots, k$.

The stack $X_0[k]$ is a Deligne–Mumford stack of nodal singularities, with $k + 2$ irreducible components. Following [AF16], we call $X_0[k]$ an *expanded degeneration of length k* with respect to X_0 . We index the divisors in $X_0[k]$ in order as $D_0, \dots, D_k$.

For the expanded pair, consider a locally smooth pair (Y, D) . In practice, the pair we mostly care about is $(Y_\pm, D_\pm)$. Denote by N the normal bundle $N_{D/Y}$. In the same manner, consider the $\mathbb{P}^1$ -bundle

$$\Delta := \mathbb{P}_D(\mathcal{O}_D \oplus N),$$

and for an integer $l \geq 0$, let

$$Y[l] := Y \cup_D \Delta_1 \cup_D \dots \cup_D \Delta_l,$$

where the gluing is carried out in the same manner.

Denote by $D[l]$ the last divisor $D_- \subset \Delta_l$ (in cases where l is implicit, we might also use $D[\infty]$). The pair $(Y[l], D[l])$ forms a new locally smooth pair. We call $(Y[l], D[l])$ the *expanded pair of length l* with respect to (Y, D) . We index the divisors in $(Y[l], D[l])$ as $D_0, \dots, D_{l-1}, D_l = D[l]$.

Remark 2.5. For $X_0[k]$ (respectively, $Y[l]$) as above, we can introduce a natural $(\mathbb{C}^*)^k$ (respectively, $(\mathbb{C}^*)^l$)-action: the i th factor of $(\mathbb{C}^*)^k$ (respectively, $(\mathbb{C}^*)^l$) acts on Δ_i fiberwise, and trivially on $Y_\pm$ (respectively, Y).

2.2 Expanded degeneration of the standard local model

Now, we describe the process of building the standard family of expanded degenerations. First, look at the baby case

$$\pi: X = \operatorname{Spec} \frac{\mathbb{C}[x, y, t]}{(xy - t)} \rightarrow \mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[t],$$

with central fiber $X_0 = \operatorname{Spec} (\mathbb{C}[x, y]/(xy))$ and singular divisor $D = \operatorname{pt}$, where Y_- is given by the equation $y = 0$. Consider the base change

$$m: \mathbb{A}^2 \rightarrow \mathbb{A}^1, \quad (t_0, t_1) \mapsto t_0 t_1.$$

The family becomes

$$\pi: X \times_{\mathbb{A}^1} \mathbb{A}^2 = \operatorname{Spec} \frac{\mathbb{C}[x, y, t_0, t_1]}{(xy - t_0 t_1)} \rightarrow \mathbb{A}^2 = \operatorname{Spec} \mathbb{C}[t_0, t_1],$$

with smooth generic fibers, and fibers over the axes $t_0 t_1 = 0$ are isomorphic to X_0 .

Blow up $X \times_{\mathbb{A}^1} \mathbb{A}^2$ along the singular divisor $D \times_{\mathbb{A}^1} 0$, and denote the result by $\widetilde{X}(1)$. It is the proper transform of the degree 2 hypersurface $(xy - t_0 t_1 = 0) \subset \mathbb{A}^4$ in $\operatorname{Bl}_0 \mathbb{A}^4 = \mathcal{O}_{\mathbb{P}^3}(-1) \subset \mathbb{A}^4 \times \mathbb{P}^3$. We have

$$\widetilde{X}(1) \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1),$$

where $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ is given by the Segre embedding.

The next step is to contract one factor of the $\mathbb{P}^1 \times \mathbb{P}^1$. For details, one can look in [Li01], where calculations are given in explicit coordinates. The contraction map and resulting space are

$$p: \widetilde{X}(1) \rightarrow X(1), \quad X(1) := \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2},$$

with the associated projection

$$\pi: X(1) \rightarrow \mathbb{A}^2.$$

The space $X(1)$ obtained this way is a resolution of singularities of $X \times_{\mathbb{A}^1} \mathbb{A}^2$. This is the first standard expanded degeneration family for the local model $X \rightarrow \mathbb{A}^1$. The properties of this

family will be summarized in the next subsection.

Remark 2.6. In the contraction $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1) \rightarrow \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$, the choice of the $\mathbb{P}^1$ -factor to contract is not canonical. Here, we adopt the convention that x is the coordinate on Y_- and y is the coordinate on Y_+ , and we choose to contract the first $\mathbb{P}^1$ -factor. The global criterion for this choice is: the restriction of resulting family $\pi: X(1) \rightarrow \mathbb{A}^2$ to the coordinate line $L_0 = (t_1 = 0)$ is a smoothing of the divisor $D_0 \subset X(1) \times_{\mathbb{A}^2} 0 \cong X[1]$.

Different choices of contracted factors give different resolutions. Our baby case is actually the simplest example of Atiyah's flop.

Remark 2.7. For those familiar with toric geometry, the relationship between $X \times_{\mathbb{A}^1} \mathbb{A}^2$, $\widetilde{X(1)}$ and $X(1)$ can be understood as follows. Let e_i , for $1 \leq i \leq 3$, be the standard basis in $\mathbb{R}^3$. The space $X \times_{\mathbb{A}^1} \mathbb{A}^2$ is the toric variety associated with the cone Σ spanned by $\{e_1, e_2, e_3, e_1 + e_3 - e_2\}$. The space $\widetilde{X(1)}$ is obtained by adding the vector $e_1 + e_3$ and refining Σ ; the space $X(1)$ is obtained by simply adding the 2-dimensional face spanned by e_1 and e_3 to Σ . One can easily see that $X(1)$ is a resolution of $X \times_{\mathbb{A}^1} \mathbb{A}^2$ but not the unique choice.

2.3 Standard families of expanded degenerations and pairs: gluing

For general locally simple degenerations $\pi: X \rightarrow \mathbb{A}^1$, we glue the local models together.

Étale locally around a point $p \in D$, the family $\pi: X \rightarrow \mathbb{A}^1$ is of the standard form; that is, there exist an étale neighborhood V_p of $0 \in \mathbb{A}^1$ and a common étale neighborhood $p \in U_p$ of $X \times_{\mathbb{A}^1} V_p$ and $\text{Spec}(\mathbb{C}[x, y, t, \vec{z}]/(xy - t)) \times_{\text{Spec } \mathbb{C}[t]} V_p$, where $\vec{z} = (z_1, \dots, z_n)$ with $n = \dim D$. Denote this local family by $\pi_p: U_p \rightarrow V_p$. Let $U_p(1)$ be the restriction to U_p of the standard local model defined in the previous subsection.

Étale locally around a point $p \notin D$, we can find an étale neighborhood U_p of p with $U_p \cap D = \emptyset$. In this case, we just define $U_p(1) := U_p \times_{\mathbb{A}^1} \mathbb{A}^2$, which is already smooth.

Now, $\{U_p \mid p \in X\}$ forms an étale covering $U = \coprod_p U_p \rightarrow X$ consisting of standard local models. We have the underlying relation

$$R := U \times_X U \rightrightarrows U \longrightarrow X,$$

where we denote the two projection maps by q_1 and q_2 and the inverse map switching the two factors by $i: R \rightarrow R$.

Now, we define $p: U(1) \rightarrow U$ and $\pi: U(1) \rightarrow \mathbb{A}^2$ just as the disjoint union of all $U_p(1)$. Let $R(1) := R \times_{q_1, U} U(1)$; then we have the relation

$$R(1) \rightrightarrows U(1),$$

where the upper arrow is $\text{pr}_2: R(1) = R \times_{q_1, U} U(1) \rightarrow U(1)$ and the lower arrow is $\text{pr}_2 \circ i$.

It is easy to check that $R(1) \rightrightarrows U(1)$ satisfies the axioms of a groupoid scheme in the sense of [LM00] and the map $R(1) \rightarrow U(1) \times U(1)$ is separated and quasi-compact. Thus, by Proposition 4.3.1 of [LM00], the stack-theoretic quotient $X(1) := [R(1) \rightrightarrows U(1)]$ defines a Deligne–Mumford stack of finite type. The maps $p: U(1) \rightarrow U$ and $\pi: U(1) \rightarrow \mathbb{A}^2$ also glue globally on the stack.

DEFINITION 2.8. Define $X(1) := [R(1) \rightrightarrows U(1)]$, the associated stack of the groupoid scheme. We have the projection $p: X(1) \rightarrow X$ and the family map $\pi: X(1) \rightarrow \mathbb{A}^2$, giving the commutative

diagram

$$\begin{array}{ccc} X(1) & \xrightarrow{p} & X \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{A}^2 & \xrightarrow{m} & \mathbb{A}^1. \end{array} \quad (2.1)$$

We call $\pi: X(1) \rightarrow \mathbb{A}^2$ the *standard family of length 1-expanded degenerations* with respect to $X \rightarrow \mathbb{A}^1$.

Note that if we compose $\pi: X(1) \rightarrow \mathbb{A}^2$ with the second projection $\text{pr}_1: \mathbb{A}^2 \rightarrow \mathbb{A}^1$, defined by $(t_0, t_1) \mapsto t_1$, the resulting $\pi_1: X(1) \rightarrow \mathbb{A}^1$ is still a locally simple degeneration. This can be checked on the local model in explicit coordinates.

Now, from $X(k)$ we proceed by induction to construct $X(k+1)$. Suppose that we already have $\pi: X(k) \rightarrow \mathbb{A}^{k+1}$ and that if projected to the last factor, the composite $\pi_k: X(k) \rightarrow \mathbb{A}^{k+1} \xrightarrow{\text{pr}_k} \mathbb{A}^1$ is a locally simple degeneration. Applying the $(k=1)$ -procedure, we obtain the 2-dimensional family

$$\pi_k(1): X(k)(1) \rightarrow \mathbb{A}^2,$$

and we take $X(k+1) = X(k)(1)$ to be the $(k+1)$ th space.

It remains to define the family map. Consider the projection to the first k factors $\pi_k^c: X(k) \xrightarrow{\pi} \mathbb{A}^{k+1} \xrightarrow{\text{pr}_{1,\dots,k}} \mathbb{A}^k$, and take the composite

$$p(k) \circ \pi_k^c: X(k+1) = X(k)(1) \rightarrow X(k) \rightarrow \mathbb{A}^k.$$

Combine the two maps:

$$\pi = (\pi_k(1), p(k) \circ \pi_k^c): X(k+1) \rightarrow \mathbb{A}^k \times \mathbb{A}^2 \cong \mathbb{A}^{k+2}.$$

This is the standard length $k+1$ -expanded degeneration family.

DEFINITION 2.9. We have constructed $X(k)$ with projection $p: X(k) \rightarrow X$ and the family map $\pi: X(k) \rightarrow \mathbb{A}^{k+1}$, giving the commutative diagram

$$\begin{array}{ccc} X(k) & \xrightarrow{p} & X \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{A}^{k+1} & \xrightarrow{m} & \mathbb{A}^1, \end{array} \quad (2.2)$$

where the map $m: \mathbb{A}^{k+1} \rightarrow \mathbb{A}^1$ is the multiplication $(t_0, \dots, t_k) \mapsto t_0 t_1 \cdots t_k$.

We call $\pi: X(k) \rightarrow \mathbb{A}^{k+1}$ the *standard family of length k -expanded degenerations* with respect to $X \rightarrow \mathbb{A}^1$.

Now, let us construct the standard families of expanded pairs. There are two equivalent ways of doing this. Let (Y, D) be a locally smooth pair.

Approach 1. Consider $Y(1) := \text{Bl}_{D \times 0}(Y \times \mathbb{A}^1)$, the blow-up of $Y \times \mathbb{A}^1$ at the closed substack $D \times 0$. We have the projection $\pi: Y(1) \rightarrow \mathbb{A}^1$, which is a locally simple degeneration, with central fiber $Y \cup_D \Delta$. Let $D(1)$ be the proper transform of $D \times \mathbb{A}^1$, which is still isomorphic to $D \times \mathbb{A}^1$.

Apply the $X(k)$ -construction as above with respect to this locally simple degeneration, and let $Y(k) := Y(1)(k-1)_{\text{deg}}$, where we put the subscript “deg” to indicate that it is the same construction as above, rather than the “ $Y(k)$ ”-construction we are defining for locally smooth

pairs. Here, we choose Y to be the “−” piece and Δ to be the “+” piece. We have the projection $\pi: Y(k) \rightarrow \mathbb{A}^k$ and contraction map $p: Y(k) \rightarrow Y(1) \rightarrow Y$, which is the composition of the contraction in the degeneration case and the blow-up. Let $D(k) \subset Y(k)$ be the base change of $D(1)$ in the diagram (2.2), which is isomorphic to $D \times \mathbb{A}^k$ since $D(1)$ does not intersect the singular divisor in the central fiber. Then $(Y(k), D(k))$ forms a locally smooth pair.

DEFINITION 2.10. We have constructed the family $\pi: Y(k) \rightarrow \mathbb{A}^k$, with contraction map $p: Y(k) \rightarrow Y$, and the locally smooth divisor $D(k) \subset Y(k)$. We call $\pi: Y(k) \rightarrow \mathbb{A}^k$ the *standard family of length k -expanded pairs* with respect to (Y, D) . We call $D(k) \subset Y(k)$ the *distinguished divisor*.

Approach 2 (Successive blow-up construction). Again, we let $Y(1) := \text{Bl}_{D \times 0}(Y \times \mathbb{A}^1)$. Assume that we already have $(Y(k), D(k))$, which is a locally smooth pair. Define $Y(k+1) := \text{Bl}_{D(k) \times 0}(Y(k) \times \mathbb{A}^1)$. The family map $\pi: Y(k+1) \rightarrow \mathbb{A}^{k+1}$ is defined in the obvious way. The following proposition says that these two approaches are actually equivalent.

PROPOSITION 2.11. *Let $(Y(k), D(k))$ and $\pi: Y(k) \rightarrow \mathbb{A}^k$ be defined as in Approach 1. There is an isomorphism $Y(k) \cong \text{Bl}_{D(k-1) \times 0}(Y(k-1) \times \mathbb{A}^1)$ making the following diagram commutative:*

$$\begin{array}{ccc} Y(k) & \xrightarrow{\sim} & \text{Bl}_{D(k-1) \times 0}(Y(k-1) \times \mathbb{A}^1) \\ \pi \downarrow & & \downarrow (\pi, \text{Id}_{\mathbb{A}^1}) \\ \mathbb{A}^k & \xrightarrow{\text{Id}} & \mathbb{A}^{k-1} \times \mathbb{A}^1. \end{array}$$

The difference of the two definitions lies in the contraction $p: Y(k) \rightarrow Y(k-1)$. If we look at this contraction on the base, the first one is given by $(t_1, \dots, t_k) \mapsto (t_1, \dots, t_{k-2}, t_{k-1}t_k)$ and the other is $(t_1, \dots, t_k) \mapsto (t_1, \dots, t_{k-1})$. In other words, the contraction $Y(k) \cong \text{Bl}_{D(k-1) \times 0}(Y(k-1) \times \mathbb{A}^1) \rightarrow Y(k-1)$ is different from the contraction p in Approach 1.

The successive blow-up definition, although easier to describe and appearing more natural, appears less compatible with the $X(k)$ -construction. One can see this later via the different $(\mathbb{C}^*)^k$ -actions.

Both approaches appear in J. Li’s work on degeneration theories, and both will be used during the proof of the properness of Quot-stacks.

Remark 2.12. Before stating the properties of the standard families, we introduce the following notation. Denote by $(Y(k)^\circ, \pi^\circ)$ the “reverse” standard family of expanded pairs. Let $Y(k)^\circ$ be the same as $Y(k)$, but $\pi^\circ := r \circ \pi: Y(k)^\circ \rightarrow \mathbb{A}^k$, where $r: \mathbb{A}^k \rightarrow \mathbb{A}^k$ is the “order-reversing” map $(t_1, \dots, t_k) \mapsto (t_k, \dots, t_1)$. For example, $Y(1)^\circ = \text{Bl}_{0 \times D}(\mathbb{A}^1 \times Y)$, and for larger k , the construction is conducted via base change from the left.

2.4 Properties of standard families

Let us fix some notation on $\mathbb{A}^{k+1}$ for later use.

1) There is a $(\mathbb{C}^*)^k$ -action on the base $\mathbb{A}^{k+1}$. For $\lambda = (\lambda_1, \dots, \lambda_k) \in (\mathbb{C}^*)^k$ and $t = (t_0, \dots, t_k) \in \mathbb{A}^{k+1}$, let

$$\lambda \cdot t := \left(\lambda_1 t_0, \frac{\lambda_2}{\lambda_1} t_1, \dots, \frac{\lambda_k}{\lambda_{k-1}} t_{k-1}, \lambda_k^{-1} t_k \right). \quad (2.3)$$

Under this action (and the trivial action on $\mathbb{A}^1$) and the multiplicative group homomorphism

$(\mathbb{C}^*)^k \rightarrow \mathbb{C}^*$ given by $(\lambda_1, \dots, \lambda_k) \mapsto \lambda_1 \cdots \lambda_k$, the multiplication map $m: \mathbb{A}^{k+1} \rightarrow \mathbb{A}^1$ is equivariant.

2) For a subset $I = \{i_0, \dots, i_l\} \subset \{0, \dots, k\}$ with $|I| = l + 1 \leq k + 1$ and $i_0 < \dots < i_l$, let $\tau_I: \mathbb{A}^{l+1} \rightarrow \mathbb{A}^{k+1}$ be the embedding given by

$$t_i = \begin{cases} t_{i_p} & \text{if } i = i_p \text{ for some } p, \\ 1 & \text{if } i \neq i_p \text{ for any } p. \end{cases}$$

3) Again for a subset $I \subset \{0, \dots, k\}$, let

$$U_I := \{(t_0, \dots, t_k) \mid t_i \neq 0 \forall i \notin I\},$$

which is a product of factors $\mathbb{A}^1$ and factors $\mathbb{G}_m$. We have the natural open immersion $\tilde{\tau}_I: U_I \rightarrow \mathbb{A}^{k+1}$.

For two subsets $I, I' \subset \{0, \dots, k\}$ as above with $|I| = |I'|$, we have the natural isomorphism given by reordering the coordinates:

$$\tilde{\tau}_{I,I'}: U_I \xrightarrow{\sim} U_{I'}.$$

4) For $0 \leq i \leq k$, denote by $\text{pr}_i: \mathbb{A}^{k+1} \rightarrow \mathbb{A}^1$ the i th projection $(t_0, \dots, t_k) \mapsto t_i$.

Now, we state the properties of standard families of expanded degenerations $\pi: X(k) \rightarrow \mathbb{A}^{k+1}$. They can be proved by induction.

PROPOSITION 2.13. 1) *The stack $X(k)$ is a separated Deligne–Mumford stack of finite type over $\mathbb{C}$, of dimension $n + k + 1$, where $n = \dim X_0$. Moreover, if $\pi: X \rightarrow \mathbb{A}^1$ is proper, the family map $\pi: X(k) \rightarrow \mathbb{A}^{k+1}$ is proper; if X is smooth, $X(k)$ is smooth.*

2) *For $t = (t_0, \dots, t_k) \in \mathbb{A}^{k+1}$, if $t_0 \cdots t_k \neq 0$, the fiber of the family over t is isomorphic to the generic fibers X_c , for $c \neq 0$, in the original family $\pi: X \rightarrow \mathbb{A}^1$.*

Let $I \subset \{0, \dots, k\}$ be a subset with $1 \leq |I| = l + 1 \leq k + 1$. If $t_i = 0$ for all $i \in I$ and $t_j \neq 0$ for all $j \notin I$, then the fiber over t is isomorphic to $X_0[l]$. In particular, the fiber over $0 \in \mathbb{A}^{k+1}$ is isomorphic to the length k -expanded degeneration $X_0[k]$.

3) *There is a $(\mathbb{C}^*)^k$ -action on $X(k)$, making the diagram (2.2) $(\mathbb{C}^*)^k$ -equivariant. This action gives isomorphisms of fibers in the same strata as described above. The induced action of the stabilizer on a fiber isomorphic to $X_0[l]$ is the same as described in Remark 2.5.*

4) *There are also discrete symmetries in $X(k)$, away from the central fiber. For two subsets $I, I' \subset \{0, \dots, k\}$ with $|I| = |I'|$, the natural isomorphisms $\tilde{\tau}_{I,I'}: U_I \xrightarrow{\sim} U_{I'}$ induce isomorphisms on the families*

$$\tilde{\tau}_{I,I',X}: X(k)|_{U_I} \xrightarrow{\sim} X(k)|_{U_{I'}}$$

extending those given by $(\mathbb{C}^*)^k$ -actions on smooth fibers.

Moreover, after restricting to the embedding $\tau_I: \mathbb{A}^{l+1} \rightarrow \mathbb{A}^{k+1}$, we have $X(k) \times_{\mathbb{A}^{k+1}, \tau_I} \mathbb{A}^{l+1} \cong X(l)$, and under the $(\mathbb{C}^*)^{k-l}$ -“translations” $U_I \cong \mathbb{A}^{l+1} \times (\mathbb{C}^*)^{k-l}$, we have the identification

$$X(k)|_{U_I} \cong X(l) \times (\mathbb{C}^*)^{k-l}.$$

Remark 2.14. Here, $(t_0, \dots, t_k) \in \mathbb{A}^{k+1}$ means a closed point $\mathbb{A}^{k+1}$. But all definitions and descriptions can be easily generalized to an arbitrary point $\text{Spec } K \rightarrow \mathbb{A}^{k+1}$. In this case, the statement $t_i \neq 0$ or $t_i = 1$ means that the image of $t_i \in \mathbb{C}[t_0, \dots, t_k]$ in K is nonzero or 1, respectively.

Remark 2.15. Let us say something about the $(\mathbb{C}^*)^k$ -action. It is easy to define the action on local models. To glue the actions on local models together, say for $X(1)$, the key observation is that

the action is actually trivial on the X -factor and only nontrivial on the extra $\mathbb{A}^1$ -factor. Hence, starting from an étale covering of X , one can get a natural $\mathbb{C}^*$ -equivariant covering of $X(1)$, which easily descends to a $\mathbb{C}^*$ -action on $X(1)$.

For $\pi: Y(k) \rightarrow \mathbb{A}^k$, again we need some notation on $\mathbb{A}^k$. Here, we use $(t_1, \dots, t_k)$ for coordinates of the base $\mathbb{A}^k$. We adopt almost the same notation τ_I , $\tilde{\tau}_I$ and $\tilde{\tau}_{I,I'}$, except that the indices range from 1 to k instead of 0 to k .

The only difference is the group action. The group $(\mathbb{C}^*)^k$ (instead of $(\mathbb{C}^*)^{k-1}$) acts on the base $\mathbb{A}^k$, but differently for the two different constructions. Consider $\lambda = (\lambda_1, \dots, \lambda_k) \in (\mathbb{C}^*)^k$ and $t = (t_1, \dots, t_k) \in \mathbb{A}^k$. For Approach 1, the action is

$$\lambda \cdot t := \left(\lambda_1 t_1, \frac{\lambda_2}{\lambda_1} t_2, \dots, \frac{\lambda_k}{\lambda_{k-1}} t_k \right), \quad (2.4)$$

whereas for the successive blow-up construction (Approach 2), the action is

$$\lambda \cdot t := (\lambda_1 t_1, \lambda_2 t_2, \dots, \lambda_k t_k).$$

The standard families of expanded pairs have the following properties, some of which follow from the properties of $X(k)$; the others can be easily proved by induction.

PROPOSITION 2.16. 1) *The stack $Y(k)$ is a separated Deligne–Mumford stack of finite type over $\mathbb{C}$, of dimension $n+k$, where $n = \dim Y$. If Y is proper, then the map π is proper. Moreover, if (Y, D) is a smooth pair, so is $(Y(k), D(k))$.*

We have $D(k) \cong D \times \mathbb{A}^k$, and the restrictions of p and π to $D(k)$ can be identified with the two projections to D and $\mathbb{A}^k$, respectively.

2) *For $t = (t_1, \dots, t_k) \in \mathbb{A}^k$, if $t_1 \cdots t_k \neq 0$, then the fiber of the pair over t is isomorphic to the original pair (Y, D) .*

Let $I \subset \{1, \dots, k\}$ be a subset with $1 \leq |I| = l \leq k$. If $t_i = 0$ for all $i \in I$ and $t_j \neq 0$ for all $j \notin I$, then the fiber over t is isomorphic to $(Y[l], D[l])$.

In particular, the fiber over $0 \in \mathbb{A}^k$ is isomorphic to the length k -expanded pair $(Y[k], D[k])$.

3) *There is a $(\mathbb{C}^*)^k$ -action on $Y(k)$, in both constructive approaches, compatible with the corresponding actions on $\mathbb{A}^k$. This action gives isomorphisms of fibers in the same strata as described above. In particular, the induced action of the stabilizer on a fiber isomorphic to $Y[l]$ is the same as described in Remark 2.5.*

4) *There are also discrete symmetries in $Y(k)$, away from the central fiber. For two subsets $I, I' \subset \{1, \dots, k\}$ with $|I| = |I'| = l$, the natural isomorphisms $\tilde{\tau}_{I,I'}: U_I \xrightarrow{\sim} U_{I'}$ induce isomorphisms on the families*

$$\tilde{\tau}_{I,I',Y}: Y(k)|_{U_I} \xrightarrow{\sim} Y(k)|_{U_{I'}}$$

extending those given by $(\mathbb{C}^)^k$ -actions on smooth fibers.*

The discrete actions restricted to $D(k) \cong D \times \mathbb{A}^k$ are just a reordering of the coordinates of the $\mathbb{A}^k$ -factors. Moreover, after restricting to the embedding $\tau_I: \mathbb{A}^l \rightarrow \mathbb{A}^k$, we have $Y(k) \times_{\mathbb{A}^k, \tau_I} \mathbb{A}^l \cong Y(l)$, and under the $(\mathbb{C}^)^{k-l}$ -“translations” $U_I \cong \mathbb{A}^l \times (\mathbb{C}^*)^{k-l}$, we have the identification*

$$Y(k)|_{U_I} \cong Y(l) \times (\mathbb{C}^*)^{k-l}.$$

Remark 2.17. For the $(\mathbb{C}^*)^k$ -action, we have one more factor than in the $X(k)$ -case. By construction, the action on $X(k)$ gives a $(\mathbb{C}^*)^{k-1}$ -action on $Y(k)$. The last $\mathbb{C}^*$ comes from the original action in $Y(1) = \text{Bl}_{D \times 0}(Y \times \mathbb{A}^1)$. One can see again that an arbitrary étale covering of Y would

give us a $\mathbb{C}^*$ -equivariant covering of $Y(1)$, and hence a $(\mathbb{C}^*)^k$ -equivariant covering of $Y(k)$, which makes the action available. In the successive blow-up construction, the action is more obvious—the blow-up process brings one $\mathbb{C}^*$ -factor each time.

Standard families of expanded degenerations and pairs are closely related to each other. Consider a locally simple degeneration $\pi: X \rightarrow \mathbb{A}^1$, with central fiber $X_0 \cong Y_- \cup_D Y_+$.

1) (Restriction to hyperplanes). Let $H_i \subset \mathbb{A}^{k+1}$ be the coordinate hyperplane defined by $t_i = 0$ for $0 \leq i \leq k$. The restriction of the family $\pi: X(k) \rightarrow \mathbb{A}^{k+1}$ to H_i is a “smoothing” of all divisors except D_i , in the following sense.

PROPOSITION 2.18. *Composing π with the projection $\text{pr}_i: \mathbb{A}^{k+1} \rightarrow \mathbb{A}^1$ to the factor t_i , we obtain $\pi_i: X(k) \rightarrow \mathbb{A}^1$. Then π_i is a locally simple degeneration.*

Denote the singular divisor by $D_i(k)$ and the central fiber decomposition by $X(k) \times_{\pi_i, \mathbb{A}^1} 0 =: X(k)_-^i \cup_{D_i(k)} X(k)_+^i$. Then one has

$$D_i(k) \cong \mathbb{A}^i \times D \times \mathbb{A}^{k-i}, \quad X(k)_-^i \cong Y_-(i) \times \mathbb{A}^{k-i}, \quad X(k)_+^i \cong \mathbb{A}^i \times Y_+(k-i)^\circ.$$

Moreover, the restrictions of the map $\pi: X(k) \rightarrow \mathbb{A}^{k+1}$ to these two components are given in the following diagrams:

$$\begin{array}{ccccc} & X(k)_-^i & & X(k)_+^i & \\ \pi_{0,\dots,i-1} \swarrow & \downarrow \pi_i & \searrow \text{Id}_{\mathbb{A}^{k-i}} & \swarrow \text{Id}_{\mathbb{A}^i} & \searrow \pi_{i+1,\dots,k}^\circ \\ \mathbb{A}^i & 0 & \mathbb{A}^{k-i} & \mathbb{A}^i & 0 & \mathbb{A}^{k-i} \end{array}$$

where the maps from $Y_-(i)$ and $Y_+(k-i)^\circ$ to the corresponding bases are just the maps of standard families of expanded pairs.

The gluing is along $D_i(k) \cong D_-(i) \times \mathbb{A}^{k-i} \cong \mathbb{A}^i \times D \times \mathbb{A}^{k-i} \cong \mathbb{A}^i \times D_+(k-i)^\circ$, where the order of the copies of $\mathbb{A}^1$ here reflects the map $\pi|_D$; we have $D_i(k) \cap D_j(k) = \emptyset$ for $i \neq j$.

All statements above are compatible with the corresponding $(\mathbb{C}^)^k$ -actions.*

2) (Restriction to lines). Let $L_i \subset \mathbb{A}^{k+1}$ be the coordinate line corresponding to t_i , for $0 \leq i \leq k$. The restriction of the family $\pi: X(k) \rightarrow \mathbb{A}^{k+1}$ to L_i is a “smoothing” of the divisor D_i , in the following sense.

PROPOSITION 2.19. *The restriction $\pi|_{L_i}: X(k)|_{L_i} \rightarrow L_i \cong \mathbb{A}^1$ is a locally simple degeneration, with central fiber decomposition $X_0[k] = Y_-[i] \cup_{D_i} Y_+[k-i]$ and generic fiber isomorphic to $X_0[k-1]$. Moreover, we have a description of the total space:*

$$X(k)|_{L_0} = Y_-(1) \cup_{D_-(1)} (\mathbb{A}^1 \times Y_+[k-1]);$$

$$X(k)|_{L_i} = Y_-[i](1) \cup_{D_-[i](1)} (\mathbb{A}^1 \times Y_+[k-i-1]) \cong (Y_-[i-1] \times \mathbb{A}^1) \cup_{D_-[i-1] \times \mathbb{A}^1} Y_+[k-i](1)^\circ$$

for $1 \leq i \leq k-1$;

$$X(k)|_{L_k} = (Y_-[k-1] \times \mathbb{A}^1) \cup_{D_-[k-1] \times \mathbb{A}^1} Y_+(1)^\circ.$$

2.5 Stacks of expanded degenerations and pairs

Now, we can define the stacks parameterizing expanded degenerations and pairs. We start with the relative case.

Let (Y, D) be a locally smooth pair. We have constructed the standard family $\pi: Y(k) \rightarrow \mathbb{A}^k$, with $(\mathbb{C}^*)^k$ -action on both $Y(k)$ and the base, making the family maps equivariant. We also have

isomorphisms on open substacks $\tilde{\tau}_{I,I',Y}: Y(k)|_{U_I} \xrightarrow{\sim} Y(k)|_{U_{I'}}$, compatible with the isomorphisms $\tilde{\tau}_{I,I'}: U_I \xrightarrow{\sim} U_{I'}$, where $I, I' \subset \{1, \dots, k\}$ have the same number of elements: $|I| = |I'| = l$.

These discrete symmetries do not form a group action on $Y(k)$ or $\mathbb{A}^k$, but they give étale equivalence relations. We rephrase the discrete symmetries as the relations

$$\begin{aligned} R_{I,I',\mathbb{A}^k} &:= \mathbb{A}^l \times (\mathbb{C}^*)^{k-l} \rightrightarrows \mathbb{A}^k, \\ R_{I,I',Y(k)} &:= Y(l) \times (\mathbb{C}^*)^{k-l} \rightrightarrows Y(k), \end{aligned} \quad (2.5)$$

where the maps are given by open immersions $\mathbb{A}^l \times (\mathbb{C}^*)^{k-l} \xrightarrow{\sim} U_I \hookrightarrow \mathbb{A}^k$ and $\mathbb{A}^l \times (\mathbb{C}^*)^{k-l} \xrightarrow{\sim} U_{I'} \hookrightarrow \mathbb{A}^k$, respectively, and similarly for $Y(k)$.

The étale equivalence relation on the whole space is just the union

$$R_{d,\mathbb{A}^k} := \coprod_{1 \leq |I|=|I'| \leq k} R_{I,I',\mathbb{A}^k}, \quad R_{d,Y(k)} := \coprod_{1 \leq |I|=|I'| \leq k} R_{I,I',Y(k)},$$

where “ d ” refers to “discrete.”

In other words, we have $R_{d,Y(k)} = \pi^* R_{d,\mathbb{A}^k}$ for $\pi: Y(k) \rightarrow \mathbb{A}^k$, or the following *Cartesian* diagram:

$$\begin{array}{ccc} R_{d,Y(k)} & \longrightarrow & Y(k) \times Y(k) \\ \downarrow & & \downarrow \\ R_{d,\mathbb{A}^k} & \longrightarrow & \mathbb{A}^k \times \mathbb{A}^k. \end{array}$$

Let S_k be the symmetric group acting on $\mathbb{A}^k$ by permuting the coordinates. The discrete relation on $\mathbb{A}^k$ is a subrelation of the S_k -action

$$R_{d,\mathbb{A}^k} \hookrightarrow S_k \times \mathbb{A}^k \rightrightarrows \mathbb{A}^k.$$

Hence, the two maps $R_{d,\mathbb{A}^k} \rightrightarrows \mathbb{A}^k$ are open immersions followed by a disjoint trivial covering, which is quasi-affine and étale.

Now, let us describe the combined equivalence relation. The group $S_k \subset \mathrm{GL}(k)$ acts on $(\mathbb{C}^*)^k$ by conjugation, and we can form the semidirect product $(\mathbb{C}^*)^k \rtimes S_k$. We have the following smooth equivalence relation $\sim$ generated by the $(\mathbb{C}^*)^k$ -action and discrete symmetries, which is quasi-compact and separated:

$$R_{\sim,\mathbb{A}^k} := (\mathbb{C}^*)^k \times R_{d,\mathbb{A}^k} \hookrightarrow (\mathbb{C}^*)^k \rtimes S_k \times \mathbb{A}^k \rightrightarrows \mathbb{A}^k.$$

We also have the similar relation $R_{\sim,Y(k)}$ on $Y(k)$, and these equivalence relations form a *non-Cartesian* diagram

$$\begin{array}{ccc} R_{\sim,Y(k)} & = & (\mathbb{C}^*)^k \times R_{d,Y(k)} \rightrightarrows Y(k) \\ \downarrow & & \downarrow \\ R_{\sim,\mathbb{A}^k} & = & (\mathbb{C}^*)^k \times R_{d,\mathbb{A}^k} \rightrightarrows \mathbb{A}^k. \end{array}$$

The equivalence relations $R_{\sim,\mathbb{A}^k}$ and $R_{\sim,Y(k)}$ are the ones generated by the $(\mathbb{C}^*)^k$ -action and discrete symmetries.

Consider the quotients, Artin stacks $[\mathbb{A}^k/R_{\sim,\mathbb{A}^k}]$ of dimension 0 and $[Y(k)/R_{\sim,Y(k)}]$ of dimension $(\dim Y - k)$, with induced 1-morphism $\pi: [Y(k)/R_{\sim,Y(k)}] \rightarrow [\mathbb{A}^k/R_{\sim,\mathbb{A}^k}]$.

Now, consider the embeddings $\tau_I: \mathbb{A}^l \hookrightarrow \mathbb{A}^k$ and $\tau_{I,Y}: Y(l) \hookrightarrow Y(k)$, which are compatible with the equivalence relation. Thus, we have embeddings of Artin stacks $[\mathbb{A}^l/R_{\sim, \mathbb{A}^l}] \rightarrow [\mathbb{A}^k/R_{\sim, \mathbb{A}^k}]$ and $[Y(l)/R_{\sim, Y(l)}] \rightarrow [Y(k)/R_{\sim, Y(k)}]$. In fact, they are open immersions. To see this, one can use the property $Y(k)|_{U_I} \cong Y(l) \times (\mathbb{C}^*)^{k-l}$ and take the smooth covers $\mathbb{A}^l \times (\mathbb{C}^*)^{k-l} \rightarrow \mathbb{A}^l$ and $Y(l) \times (\mathbb{C}^*)^{k-l} \rightarrow Y(l)$. Now, it is clear that these immersions form inductive systems, leading to the following definition.

DEFINITION 2.20. Define $\mathfrak{A} := \varinjlim [\mathbb{A}^k/R_{\sim, \mathbb{A}^k}]$ to be the *stack of expanded pairs*, with respect to (Y, D) , and let $\mathfrak{Y} := \varinjlim [Y(k)/R_{\sim, Y(k)}]$ be the *universal family of expanded pairs*. There is a family map $\pi: \mathfrak{Y} \rightarrow \mathfrak{A}$, which is of Deligne–Mumford type and is proper.

We interpret the definition in the categorical sense. For a fixed map $\xi_0: S \rightarrow \mathbb{A}^k$, pulling back the standard family $Y(k)$, one obtains $\pi: \mathcal{Y}_S := \xi_0^* Y(k) \rightarrow S$, a *family of expanded pairs* over (S, ξ_0) . The equivalence relation acts on ξ_0 by acting on the target, and for maps related by this relation, we get isomorphic families of expanded pairs. In this way, one can think of a map ξ_0 as a family of expanded pairs over S , up to the action of the equivalence relation.

Given a morphism of schemes $f: T \rightarrow S$ and a map $\xi: S \rightarrow \mathbb{A}^k$, one can take the composite $\xi \circ f: T \rightarrow \mathbb{A}^k$. The corresponding family of expanded pairs over T is just given by $\mathcal{Y}_T = f^* \mathcal{Y}_S$. By the following lemma, we see that it is unique up to unique isomorphism.

LEMMA 2.21. *Suppose that one has the following 2-commutative diagram:*

$$\begin{array}{ccc} \mathcal{Y}_T & \xrightarrow{F} & \mathcal{Y}_S \\ \downarrow & & \downarrow \\ T & \xrightarrow{f} & S. \end{array}$$

Then, the 1-morphism of stacks F is representable. As a consequence, F has no nontrivial 2-isomorphisms.

Proof. By [AV02, Lemma 4.4.3], it suffices to prove that the homomorphisms between isotropy groups of geometric points are monomorphisms. Since maps between fibers are just maps between expanded pairs $Y[k] \rightarrow Y[l]$, it suffices to prove the statement for those maps. But any map $Y[k] \rightarrow Y[l]$ is a successive composition of contractions of bubbles and embeddings, which is obviously representable. $\square$

Let S be a scheme. An object $\bar{\xi} \in \mathfrak{A}(S)$ is a compatible system of objects $\xi_k \in [\mathbb{A}^k/R_{\sim, \mathbb{A}^k}](S)$. By construction, an object $\xi \in [\mathbb{A}^k/R_{\sim, \mathbb{A}^k}](S)$ is given by a “descent datum” over S , that is, a map $\xi: S_\xi \rightarrow \mathbb{A}^k$, where $S_\xi = \coprod S_i \rightarrow S$ is a surjective étale covering of S which satisfies the descent compatibility on overlaps; in other words, we have a map $r_\xi: R_\xi := S_\xi \times_S S_\xi \rightarrow R_{\sim, \mathbb{A}^k}$ compatible with the groupoid structure. By the interpretation above, one can view this as a family of expanded pairs over S_ξ .

The 2-isomorphisms are as follows. Given another $\xi': S_{\xi'} \rightarrow \mathbb{A}^{k'}$ representing the same object, we can embed $\mathbb{A}^k$ and $\mathbb{A}^{k'}$ into a larger base $\mathbb{A}^{k''}$ for $k'' \geq k, k'$ and pass to the refinement $S_{\xi\xi'} := S_\xi \times_S S_{\xi'}$. Then for sufficiently large k'' , the two resulting maps $S_{\xi\xi'} \rightarrow \mathbb{A}^{k''}$ give the same object on the k'' -level. In other words, they factor through the relation $R_{\sim, \mathbb{A}^{k''}} \rightarrow \mathbb{A}^{k''} \times \mathbb{A}^{k''}$.

For a map of schemes $f: T \rightarrow S$, passing to the étale cover, we have $f_\xi: T_\xi := T \times_S S_\xi \rightarrow S_\xi$. Then the 1-arrow is defined by the composition $\xi \circ f$.

In particular, for a fixed object $\bar{\xi} \in \mathfrak{A}(S)$, the stabilizer group of this object is as follows. Take a representative $\xi: S_\xi \rightarrow \mathbb{A}^k$. The stabilizer group of this representative is the group scheme $\text{Aut}_\sim(\xi, k)$ over S_ξ in the following Cartesian diagram:

$$\begin{array}{ccccc} \text{Aut}_\sim(\xi, k) & \longrightarrow & S_\xi & \longrightarrow & S \\ \downarrow & & \downarrow \Delta \circ \xi & & \\ R_{\sim, \mathbb{A}^k} & \longrightarrow & \mathbb{A}^k \times \mathbb{A}^k & & \end{array}$$

Note that this group scheme actually does not depend on k for large k , and by descent theory of affine morphisms, it glues to a group scheme over S . We denote it by $\text{Aut}_\sim(\bar{\xi})$, which is independent of S_ξ and k for large k . As a result, one can see that $\mathfrak{A}$ is actually a 0-dimensional Artin stack.

Now, let us apply the same procedure to a locally simple degeneration $\pi: X \rightarrow \mathbb{A}^1$. We have constructed the standard family $\pi: X(k) \rightarrow \mathbb{A}^{k+1}$, again with compatible $(\mathbb{C}^*)^k$ -actions on $X(k)$ and the base. We introduce smooth equivalence relations on $X(k)$ and $\mathbb{A}^{k+1}$ in the same manner. For $k \geq 0$, the Artin stacks $[\mathbb{A}^{k+1}/R_{\sim, \mathbb{A}^{k+1}}]$ and $[X(k)/R_{\sim, X(k)}]$ also form inductive systems.

DEFINITION 2.22. Define $\mathfrak{C} := \varinjlim [\mathbb{A}^{k+1}/R_{\sim, \mathbb{A}^{k+1}}]$ to be the *stack of expanded degenerations*, with respect to the locally simple degeneration $\pi: X \rightarrow \mathbb{A}^1$, and let $\mathfrak{X} := \varinjlim [X(k)/R_{\sim, X(k)}]$ be the *universal family of expanded degenerations*. There is a family map $\pi: \mathfrak{X} \rightarrow \mathfrak{C}$, of Deligne–Mumford type and proper, making the following diagram commutative (but not Cartesian):

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{p} & X \\ \pi \downarrow & & \downarrow \pi \\ \mathfrak{C} & \xrightarrow{p} & \mathbb{A}^1. \end{array}$$

Both $\mathfrak{C}$ and $\mathfrak{X}$ can be viewed as Artin stacks over $\mathbb{A}^1$. The stack $\mathfrak{C}$ is 1-dimensional, and $\mathfrak{X}$ is of dimension $\dim X$.

Given an $\mathbb{A}^1$ -map $\xi: S \rightarrow \mathbb{A}^{k+1}$, a *family of expanded degenerations* $\pi: \mathcal{X}_S \rightarrow S$ over (S, ξ) is obtained by pull back of the standard family. Given a morphism of $\mathbb{A}^1$ -schemes $f: (T, \eta) \rightarrow (S, \xi)$, the pull-back 1-morphism is just given by $f^*\mathcal{X}_S = \mathcal{X}_T$ and is unique up to 2-isomorphisms.

For an $\mathbb{A}^1$ -scheme S , an object $\bar{\xi} \in \mathfrak{C}(S)$ is represented by an $\mathbb{A}^1$ -map $\xi: S_\xi \rightarrow \mathbb{A}^{k+1}$, where $S_\xi = \coprod S_i$ is a surjective étale covering of S . Different representatives of the same object are related by embeddings into a common larger base $\mathbb{A}^{k'+1}$ and the equivalence relation over that base.

Remark 2.23. Given a family of expanded pairs (respectively, degenerations) $\pi: \mathcal{Y} \rightarrow S$ (respectively, $\mathcal{X} \rightarrow S$), each fiber of the family is an expanded pair (respectively, degeneration) with respect to (Y, D) (respectively, $\pi: X \rightarrow \mathbb{A}^1$). Given a base change $f: (T, \eta) \rightarrow (S, \xi)$, the induced maps between fibers are given by maps of the corresponding expanded pairs (respectively, degenerations). In this way, we see that the definition makes sense for parameterizing all expanded pairs (respectively, degenerations).

3. Admissible sheaves and stable quotients

In this section, we introduce the notion of admissible sheaves, the correct objects we need to consider for the relative Donaldson–Thomas theory. Again, our definitions are direct generalizations of J. Li and B. Wu [LW15].

Let W be a separated Deligne–Mumford stack of finite type, and let $Z \subset W$ be a closed substack. Let $\mathcal{F}$ be a coherent sheaf on W .

DEFINITION 3.1. The sheaf $\mathcal{F}$ is said to be *normal* to Z if $\mathrm{Tor}_1^{\mathcal{O}_W}(\mathcal{F}, \mathcal{O}_Z) = 0$. Moreover, we say that $\mathcal{F}$ is *normal* to Z at a point $p \in Z$ if there is an étale neighborhood $i: U_p \rightarrow W$ of p such that $i^*\mathcal{F}$ is normal to $Z \times_W U_p$.

According to the following lemma, most properties of the normality of coherent sheaves can be directly generalized to Deligne–Mumford stacks.

LEMMA 3.2. *The sheaf $\mathcal{F}$ is normal to Z if and only if it is normal to Z at every point $p \in Z$. In other words, normality is a local property in the étale topology.*

Proof. Let $\mathcal{I}$ be the ideal sheaf of Z . The sheaf $\mathcal{F}$ is normal to Z if and only if the map $\mathcal{I} \otimes \mathcal{F} \rightarrow \mathcal{F}$ is injective, which is a local property in the étale topology. $\square$

REMARK 3.3. From the definition, we note that normality can be checked on stalks or completion over local rings.

In the following, we will consider two cases we mostly care about, the relative case and the degeneration case.

3.1 Relative case

This is the case where Y is a separated Deligne–Mumford stack of finite type and $D \subset Y$ is an effective Cartier divisor. Take an affine étale neighborhood $U = \mathrm{Spec} A$ of $p \in D$, where $D|_U$ is defined by some non-zero-divisor $f \in A$. Then there is a map $U \rightarrow \mathbb{A}^1 = \mathrm{Spec} \mathbb{C}[y]$, given by $y \mapsto f$. A coherent sheaf $\mathcal{F}$ on U is represented by an A -module M . The following lemma gives a local description of normality in this case, which is a restatement of the definition by the flatness criteria in commutative algebra.

LEMMA 3.4. *The following are equivalent:*

- 1) *The sheaf $\mathcal{F}|_U$ is normal to $D|_U$.*
- 2) *The map $M \xrightarrow{\times f} M$ is injective.*
- 3) *The A -module M is flat over $\mathbb{A}^1$ at the point $0 \in \mathbb{A}^1$; that is, the stalk of M at $0 \in \mathbb{A}^1$ is flat over the local ring $\mathcal{O}_{\mathbb{A}^1, 0} = \mathbb{C}[y]_{(y)}$.*

Let us look closer into the coherent sheaf $\mathcal{F}$ and analyze its normality. Let $\mathcal{I} \subset \mathcal{O}_Y$ be the ideal sheaf of $i: D \hookrightarrow Y$. Define $\mathcal{F}_{\mathcal{I}} := i^!\mathcal{F}$ to be the maximal subsheaf of $\mathcal{F}$ supported on D . More precisely, locally in an affine open U , for $\mathcal{I}(U) = I$,

$$\mathcal{F}_{\mathcal{I}}(U) := \{m \in M \mid \mathrm{ann}(m) \supset I^k, \text{ for some } k \in \mathbb{Z}_+\}.$$

One can easily check by descent theory that this definition defines a global coherent sheaf on Y . We call $\mathcal{F}_{\mathcal{I}}$ the torsion subsheaf of $\mathcal{F}$ along D .

Another way to define the torsion subsheaf is to take $\mathcal{F}_{\mathcal{I}}(U) = M_I = \ker(M \xrightarrow{\times f} M)$, which is the torsion part of M as a $\mathbb{C}[y]$ -module. In this viewpoint, the torsion-free quotient is

$$\mathcal{F}^{\mathrm{tf}} := \mathcal{F}/\mathcal{F}_{\mathcal{I}} = \mathrm{coker}(M \xrightarrow{\times y} M),$$

and we have the short exact sequence

$$0 \longrightarrow \mathcal{F}_{\mathcal{I}} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^{\mathrm{tf}} \longrightarrow 0.$$

The following proposition is a direct result of Lemma 3.4.

PROPOSITION 3.5. *The sheaf $\mathcal{F}$ is normal to D at $p \in D$ if and only if $(\mathcal{F}_{\mathcal{I}})_p = 0$, where the subscript p stands for the stalk in the étale topology. In particular, $\mathcal{F}$ is normal to D if and only if $\mathcal{F}_{\mathcal{I}} = 0$.*

The following proposition tells us that normality is an open condition.

PROPOSITION 3.6. *Let S be a scheme, and let $\mathcal{F}$ be a coherent sheaf on $Y \times S$ which is flat over S . Then the set $\{s \in S \mid \mathcal{F}|_s := \mathcal{F} \otimes_{\mathcal{O}_S} k(s) \text{ is normal to } D \times s\}$ is open in S .*

In order to prove this proposition, we use the fiberwise criterion for flatness, which is a consequence of [GD66, Theorem 11.3.10] (also see [Sta17, Theorem 36.16.2, Tag 039C]).

LEMMA 3.7 (Fiberwise criterion for flatness of coherent sheaves). *Let S, X, Y be locally noetherian schemes, and let $g: X \rightarrow S$ and $h: Y \rightarrow S$ be two morphisms of schemes. Let $f: X \rightarrow Y$ be an S -morphism, and let $\mathcal{F}$ be a coherent sheaf on X . Let $x \in X$, and set $y = f(x)$ and $s = h(y) = g(x)$. Assume that the stalk $\mathcal{F}_x$ is nonzero. Then the following are equivalent:*

- 1) *The sheaf $\mathcal{F}$ is flat over S at x , and $\mathcal{F}|_s$ is flat over Y_s at $x \in X_s$.*
- 2) *The scheme Y is flat over S at y , and $\mathcal{F}$ is flat over Y at x .*

Proof for Proposition 3.6. Suppose that for $s \in S$, the fiber $\mathcal{F}|_s$ is normal to D . Then for all $p \in D$, the fiber $\mathcal{F}|_s$ is normal to D in some affine étale neighborhood U_p of p . Consider the local diagram

$$\begin{array}{ccc} U_p \times S & \xrightarrow{\quad} & \mathbb{A}^1 \times S \\ & \searrow \quad \swarrow & \\ & S, & \end{array}$$

where $U_p \rightarrow \mathbb{A}^1$ is as given above.

We know that $\mathcal{F}|_s$, as a sheaf on $U_p \times \{s\}$, is flat over $\mathbb{A}^1 \times \{s\}$ at (p, s) . By the fiberwise criterion, $\mathcal{F}$ is flat over $\mathbb{A}^1 \times S$ at this point; in other words, there exist affine étale neighborhoods of $p \in Y$, which we still call U_p , and V_p of $s \in S$, such that on $U_p \times V_p \subset Y \times S$, the sheaf $\mathcal{F}$ is flat over $\text{Spec } \mathcal{O}_{\mathbb{A}^1, 0} \times V_p$. Since D is quasi-compact, we can take a finite subcover $U_i \times V_i$, for $1 \leq i \leq N$, such that $\{U_i\}$ covers D , and take $V := V_1 \times_S \cdots \times_S V_N$. Then we have the inclusion $V \subset \{s \in S \mid \mathcal{F}|_s \text{ is normal to } D\}$, which proves the proposition. $\square$

3.2 Degeneration case

In this case, we consider $X_0 = Y_- \cup_D Y_+$, where the two separated finite-type Deligne–Mumford stacks $Y_{\pm}$ intersect transversally along an effective Cartier divisor and X_0 is the gluing along D . By the transversality, we mean that étale locally around $p \in D$, we have a common affine neighborhood $U = \text{Spec } A$, as follows:

$$T \times \text{Spec } \mathbb{C}[x, y]/(xy) \xleftarrow{\text{ét}} U \xrightarrow{\text{ét}} X_0,$$

where T is a scheme.

Hence, we have a map $U \rightarrow \text{Spec } \mathbb{C}[x, y]/(xy)$. The ring can be rewritten as $\mathbb{C}[x, y]/(xy) \cong \mathbb{C}[x] \times_{\mathbb{C}} \mathbb{C}[y] = \{(f, g) \in \mathbb{C}[x] \times \mathbb{C}[y] \mid f(0) = g(0)\}$.

A coherent sheaf $\mathcal{F}$ on U is represented by an A -module M . We have the following lemma.

LEMMA 3.8. *The following are equivalent:*

- 1) *The scheme M is flat over the local ring $(\mathbb{C}[x, y]/(xy))_{(x, y)}$.*
- 2) *The quotient M/yM is flat over $\mathbb{C}[x]_{(x)}$, and M/xM is flat over $\mathbb{C}[y]_{(y)}$.*
- 3) *The restrictions $\mathcal{F}|_{U \cap Y_{\pm}}$ are normal to $D|_U \subset U \cap Y_{\pm}$.*
- 4) *The sheaf $\mathcal{F}|_U$ is normal to $D|_U$.*

In particular, 1)–4) imply

- 5) $M \cong (M/yM) \times_{M/(x, y)M} (M/xM)$.

Proof. The equivalence $2) \Leftrightarrow 3)$ is a consequence of the relative case. The equivalence $1) \Leftrightarrow 4)$ is true since statement 4) is equivalent to the injectivity of the map $(x, y) \otimes M \rightarrow M$, which is equivalent to statement 1) since (x, y) is the maximal ideal, and the map is obviously injective when localized at other maximal ideals. The implication $1) \Rightarrow 2)$ is the base-change property of flatness.

Let us prove $2) \Rightarrow 5)$. Suppose that statement 2) holds. We can easily see that the obvious map $\varphi: M \rightarrow (M/yM) \times_{M/(x, y)M} (M/xM)$ is surjective. In fact, given $(\bar{m}_1, \bar{m}_2)$ in the target, one can choose $m_1, m_2 \in M$ in the preimages of $\bar{m}_1, \bar{m}_2$, respectively. By construction, there exist $a, b \in M$ such that $m_1 - m_2 = ax + by$. Then $m_1 - by = m_2 + ax \in M$ is in the preimage of $(\bar{m}_1, \bar{m}_2)$.

For the injectivity, φ is identity at points $(x, y) \neq (0, 0)$; thus, it remains to check its injectivity at $(0, 0)$ or, equivalently, over the local ring at $(0, 0)$. If for some $m \in M_{(x, y)}$, we have $\varphi(m) = 0$, then there are some $m_1, m_2 \in M_{(x, y)}$ such that $m = xm_1 = ym_2$. Then by statement 2), the equality $xm_1 = ym_2$ implies $m_1 \in yM_{(x, y)}$ and $m_2 \in xM_{(x, y)}$, which implies $m = 0$. Thus, φ is injective and statement 5) is proved.

Finally, let us prove $2) \Rightarrow 1)$. We just need to check the injectivity of the map $(x, y) \otimes M_{(x, y)} \rightarrow M_{(x, y)}$. We can view (x, y) as $(x) \times_{\mathbb{C}} (y)$, the maximal ideal of $(\mathbb{C}[x, y]/(xy))_{(x, y)} \cong \mathbb{C}[x]_{(x)} \times_{\mathbb{C}} \mathbb{C}[y]_{(y)}$. By the implication $2) \Rightarrow 5)$, we have $M \cong (M/yM) \times_{M/(x, y)M} (M/xM)$; thus, it suffices to check the injectivity on the two components, which reduces to the flatness stated in item 2). $\square$

COROLLARY 3.9. *The sheaf $\mathcal{F}$ is normal to D if and only if $\mathcal{F}|_{Y_{\pm}}$ are normal to $D \subset Y_{\pm}$, respectively. Moreover, in this case, $\mathcal{F} \cong \ker(\mathcal{F}|_{Y_-} \oplus \mathcal{F}|_{Y_+} \rightarrow \mathcal{F}|_D)$.*

Again, one can consider the maximal torsion subsheaf supported on D , denoted by $\mathcal{F}_{\mathcal{J}}$, where $\mathcal{J} \subset \mathcal{O}_{X_0}$ is the ideal sheaf of $D \subset X_0$. Let $\mathcal{I}_{\pm}$ be the ideal sheaves of $D \subset Y_{\pm}$, and let $\mathcal{F}_{\mathcal{I}_{\pm}}$ be the corresponding torsion subsheaves defined in the relative case. We have the short exact sequence

$$0 \longrightarrow \mathcal{F}_{\mathcal{J}} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^{\text{tf}} \longrightarrow 0.$$

Now, the case is a little more complicated. In general, $\mathcal{F}^{\text{tf}}$ is not normal to D , but it satisfies the splitting property as in Lemma 3.8(5).

LEMMA 3.10. *If $\mathcal{F}_{\mathcal{J}} = 0$, then $\mathcal{F} \cong \ker(\mathcal{F}|_{Y_-} \oplus \mathcal{F}|_{Y_+} \rightarrow \mathcal{F}|_D)$.*

Proof. Locally, the map $\varphi: M \rightarrow (M/yM) \times_{M/(x, y)M} (M/xM)$ is surjective, where M, x, y are the same as in Lemma 3.8. Take $v \in M$ in the kernel. By definition, there are $m, n \in M$ such that $v = xm = yn$. Thus, we have $(x, y) \cdot v = 0$, which implies $v \in M_{\mathcal{J}} = 0$. $\square$

To determine whether $\mathcal{F}$ is normal to D , we look at the quotient $\mathcal{F}^{\text{tf}}$. It is normal to D if and only if $\mathcal{F}^{\text{tf}}|_{Y_{\pm}}$ is normal to D . Thus we have the following.

COROLLARY 3.11. *The sheaf $\mathcal{F}$ is normal to D if and only if $\mathcal{F}_{\mathcal{J}} = (\mathcal{F}^{\text{tf}}|_{Y_-})_{\mathcal{I}_-} = (\mathcal{F}^{\text{tf}}|_{Y_+})_{\mathcal{I}_+} = 0$.*

Combining Corollary 3.9 with Proposition 3.6, we also have the following.

COROLLARY 3.12. *Let S be a scheme, and let $\mathcal{F}$ be a coherent sheaf on $X_0 \times S$ which is flat over S . Then the set $\{s \in S \mid \mathcal{F}|_s := \mathcal{F} \otimes_{\mathcal{O}_S} k(s) \text{ is normal to } D \times s\}$ is open in S .*

3.3 Admissible sheaves

Consider a locally smooth pair (Y, D) for the relative case and a locally simple degeneration $\pi: X \rightarrow \mathbb{A}^1$, $X_0 = Y_- \cup_D Y_+$ for the degeneration case. We make the following definition.

DEFINITION 3.13. A coherent sheaf $\mathcal{F}$ on $Y[k]$ (respectively, $X_0[k]$) is called *admissible* if it is normal to each $D_i \subset Y[k]$ (respectively, $D_i \subset X_0[k]$) for $0 \leq i \leq k$.

Let $\mathcal{Y}_S \rightarrow S$ (respectively, $\mathcal{X}_S \rightarrow S$) be a family of expanded pairs (respectively, degenerations), and let $\mathcal{F}$ be a coherent sheaf on $\mathcal{Y}_S$ (respectively, $\mathcal{X}_S$), flat over S . We call $\mathcal{F}$ *admissible* if for every point $s \in S$, the fiber $\mathcal{F}|_s$ is admissible on the fiber $\mathcal{Y}_{S,s}$ (respectively, $\mathcal{X}_{S,s}$).

Remark 3.14. The difference of the definitions of admissibility on $X_0[k]$ and $Y[k]$ is that we include the distinguished divisor D_k in the relative case.

We would like to prove that admissibility is an open condition. Before that, we need a criterion for normality on a locally simple degeneration.

LEMMA 3.15. *Consider the standard simple degeneration $\pi: X = \text{Spec } \mathbb{C}[x, y, t]/(xy - t) \rightarrow \mathbb{A}^1 = \text{Spec } \mathbb{C}[t]$, with singular divisor of the central fiber $D = \text{Spec } \mathbb{C}$. Let M be a coherent sheaf on X , flat over $\mathbb{A}^1$. Then M is normal to $D \subset X$ if and only if $M|_0$ is normal to $D \subset X_0$.*

Proof. This is an immediate consequence of the slicing criterion for flatness, which can be found in [Eis95, Corollary 6.9]. $\square$

PROPOSITION 3.16. *Let $\mathcal{Y}_S$ (respectively, $\mathcal{X}_S$) be a family of expanded pairs (respectively, degenerations). Let $\mathcal{F}$ be a coherent sheaf on $\mathcal{Y}_S$, flat over S . Then the set $\{s \in S \mid \mathcal{F}|_s \text{ is admissible}\}$ is open in S .*

Proof. For standard families $Y(k) \rightarrow \mathbb{A}^k$, it suffices to prove that the points $s \in \mathbb{A}^k$ over which $\mathcal{F}|_s$ is normal to all $D_i \in \mathcal{Y}_s$ form an open subset. By Lemma 3.15, those are exactly the points $s \in \mathbb{A}^k$ where $\mathcal{F}$ is normal to all $D_i(k)$ at every $p \in \mathcal{Y}_s$. From the properties of $Y(k)$, we know that after projection to the i th coordinate, for $1 \leq i \leq k$, the 1-dimensional family $\pi_i := \text{pr}_i \circ \pi: Y(k) \rightarrow \mathbb{A}^1$ is also a locally simple degeneration. In other words, locally around $D_i(k)$, the scheme $Y(k)$ has a common étale neighborhood with the local model $D_i(k) \times \text{Spec } \mathbb{C}[x, y, t_i]/(xy - t_i) \cong D_i \times \mathbb{A}^{k-1} \times \text{Spec } \mathbb{C}[x, y, t_i]/(xy - t_i)$. At points where $t_i \neq 0$, the normality of $\mathcal{F}$ is trivial. Moreover, by Corollary 3.12, the points $s \in H_i = (t_i = 0)$ where $\mathcal{F}$ is not normal to D_i form a closed subset. Thus, it is still closed in $\mathbb{A}^k$, whose complement is open. The proof for the distinguished divisor $D(k)$ is similar. For general families $\mathcal{Y}_S$ over S , the proof works by pulling everything back to S . $\square$

3.4 Stable quotients

Now, we come to the stability condition and the notion of stable quotients, following the convention of J. Li and B. Wu [LW15].

In this subsection, let (Y, D) and $\pi: X \rightarrow \mathbb{A}^1$ be as above. We have the contractions $p: Y[k] \rightarrow Y$ and $p: X_0[k] \rightarrow X_0$ contracting the extra bubbled components. Let $\mathcal{V}$ be a vector bundle of finite rank on Y (respectively, X).

Consider quotient sheaves of the form $\phi: p^*\mathcal{V} \rightarrow \mathcal{F}$ on $Y[k]$ (respectively, $X_0[k]$). For two quotients $\phi_1: p^*\mathcal{V} \rightarrow \mathcal{F}_1$ and $\phi_2: p^*\mathcal{V} \rightarrow \mathcal{F}_2$, an equivalence between them is defined as a pair (σ, ψ) , where $\sigma: Y[k] \rightarrow Y[k]$ (respectively, $X_0[k] \rightarrow X_0[k]$) is an isomorphism *induced from the* $(\mathbb{C}^*)^k$ -*action*, and $\psi: \mathcal{F}_1 \xrightarrow{\sim} \sigma^*\mathcal{F}_2$ is an isomorphism of coherent sheaves making the following diagram commute:

$$\begin{array}{ccc} p^*\mathcal{V} & \xrightarrow{\phi_1} & \mathcal{F}_1 \\ \text{Id} \downarrow & & \downarrow \psi \\ \sigma^*p^*\mathcal{V} = p^*\mathcal{V} & \xrightarrow{\sigma^*\phi_2} & \sigma^*\mathcal{F}_2. \end{array}$$

Let $\text{Aut}(\phi)$ be the group of autoequivalences of a fixed quotient $\phi: p^*\mathcal{V} \rightarrow \mathcal{F}$. There is a map $\text{Aut}(\phi) \rightarrow (\mathbb{C}^*)^k$ defined by $(\sigma, \psi) \mapsto \sigma$, forgetting the second component. This is an injection since for a fixed quotient, $\sigma = 1$ will just identify $p^*\mathcal{V}$ with itself and induce Id on $\mathcal{F}$. As a result, $\text{Aut}(\phi) \subset (\mathbb{C}^*)^k$ is a subgroup.

Remark 3.17. An equivalent definition of the equivalence of quotients is to consider the kernels of the quotients, say $0 \rightarrow \mathcal{K}_i \rightarrow p^*\mathcal{V} \rightarrow \mathcal{F}_i \rightarrow 0$, and define an equivalence as an isomorphism of the two kernels as subsheaves of $p^*\mathcal{V}$. It is clear from this definition that $\text{Aut}(\phi) \subset (\mathbb{C}^*)^k$ is a subgroup.

DEFINITION 3.18. Let $\phi: p^*\mathcal{V} \rightarrow \mathcal{F}$ be a quotient sheaf on $Y[k]$ or $X_0[k]$. The quotient ϕ is called *stable* if $\mathcal{F}$ is admissible and $\text{Aut}(\phi)$ is finite.

For standard families $Y(k)$ and $X(k)$, we have contraction maps $p: Y(k) \rightarrow Y$ and $p: X(k) \rightarrow X$, which pass to general families.

DEFINITION 3.19. Let $\mathcal{Y}_S \rightarrow S$ (respectively, $\mathcal{X}_S \rightarrow S$) be a family of expanded pairs (respectively, degenerations), with contraction map $p: \mathcal{Y}_S \rightarrow Y$ (respectively, $p: \mathcal{X}_S \rightarrow X$), and let $\phi: p^*\mathcal{V} \rightarrow \mathcal{F}$ be a quotient sheaf on $\mathcal{Y}_S$ (respectively, $\mathcal{X}_S$), with $\mathcal{F}$ flat over S . Then ϕ is called *stable* if for every point $s \in S$, the fiber $\mathcal{F}|_s$ is stable on the fiber $\mathcal{Y}_{S,s}$ (respectively, $\mathcal{X}_{S,s}$).

The $(\mathbb{C}^*)^k$ -actions on $Y(k)$ and $\mathbb{A}^k$ induce a $(\mathbb{C}^*)^k$ -action on the Quot-space $\text{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}}$, which is a separated algebraic space, locally of finite type by [OS03]. For a fixed object (ξ, ϕ) represented by $\xi: S \rightarrow \mathbb{A}^k$ and an S -flat quotient ϕ on $\mathcal{Y}_S$, the stabilizer is given by

$$\begin{array}{ccc} \text{Aut}_{*,S}(\xi, \phi, k) & \xrightarrow{\quad} & S \\ \downarrow & & \downarrow \Delta \circ (\xi, \phi) \\ (\mathbb{C}^*)^k \times \text{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}} & \longrightarrow & \text{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}} \times \text{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}} \end{array} \quad (3.1)$$

We see that $\text{Aut}_{*,S}(\xi, \phi, k) \hookrightarrow (\mathbb{C}^*)^k \times S$ is a subgroup scheme over S . Thus, it is quasi-compact and separated over S , which means that the action is quasi-compact and separated.

We also have the open condition property.

PROPOSITION 3.20. Let $\mathcal{Y}_S \rightarrow S$ (respectively, $\mathcal{X}_S \rightarrow S$) be a family of expanded pairs (respectively, degenerations), with contraction map $p: \mathcal{Y}_S \rightarrow Y$ (respectively, $p: \mathcal{X}_S \rightarrow X$), and

let $\phi: p^*\mathcal{V} \rightarrow \mathcal{F}$ be a quotient sheaf on $\mathcal{Y}_S$ (respectively, $\mathcal{X}_S$), with $\mathcal{F}$ flat over S . The set $\{s \in S \mid \phi_s := p^*\mathcal{V} \rightarrow \mathcal{F}|_s \text{ is stable}\}$ is open in S .

Proof. By Proposition 3.16, it suffices to prove that the finiteness of the autoequivalence group is an open condition. We take the relative case; the degeneration case is similar.

Now, $(\mathbb{C}^*)^k$ acts on $\text{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}}$ by (3.1). Given an object $(\xi, \phi) \in \text{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}}(S)$, the stabilizer group scheme of the action is just $\text{Aut}_{*,S}(\xi, \phi, k)$. Thus, the set $\{s \in S \mid \phi_s := p^*\mathcal{V} \rightarrow \mathcal{F}|_s \text{ is stable}\}$ is the set where $\text{Aut}_{*,S}(\xi, \phi, k) \rightarrow S$ is quasi-finite, which is open. $\square$

We still need some concepts for later discussions of sheaves. The following definitions can be found in [Lie07] and [Nir08a].

DEFINITION 3.21. Let $\mathcal{F}$ be a coherent sheaf on a Deligne–Mumford stack W . The *support* of $\mathcal{F}$ is the closed substack defined by the ideal sheaf $0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_W \rightarrow \mathcal{E}nd_{\mathcal{O}_W}(\mathcal{F})$. The *dimension* of $\mathcal{F}$ is the dimension of its support. The sheaf $\mathcal{F}$ is called *pure* if any proper subsheaf of it has the same dimension as $\mathcal{F}$.

DEFINITION 3.22. Let $T_i(\mathcal{F})$ be the maximal subsheaf of $\mathcal{F}$ of dimension at most i . Then we have the *torsion filtration*

$$0 \subset T_0(\mathcal{F}) \subset \cdots \subset T_d(\mathcal{F}) = \mathcal{F},$$

where $d = \dim \mathcal{F}$ and each factor $T_i(\mathcal{F})/T_{i-1}(\mathcal{F})$ is 0 or pure of dimension i .

PROPOSITION 3.23. Let $\mathcal{F}$ be a coherent sheaf on a locally smooth pair (Y, D) . Then $\mathcal{F}$ is normal to D if and only if no irreducible components of any nonempty $\text{Supp } T_i(\mathcal{F})$ lie in D .

Proof. If $\mathcal{F}$ is normal to D and some irreducible component of $T_i(\mathcal{F})$ lies in D , then for some point in that component, $\mathcal{F}$ is not flat along the normal direction to D since the multiplication in the normal direction vanishes on $T_i(\mathcal{F})$. This gives a contradiction.

Conversely, suppose that no irreducible components of any $T_i(\mathcal{F})$ lie in D and $\mathcal{F}$ is not normal to D . Then $\mathcal{F}_{\mathcal{I}} \neq 0$, where $\mathcal{I}$ is the ideal sheaf of $D \subset Y$. Suppose $\dim \mathcal{F}_{\mathcal{I}} = d'$; then $\text{Supp } \mathcal{F}_{\mathcal{I}} \subset D$ contains some irreducible components of $\text{Supp } T_{d'}(\mathcal{F})$. This gives a contradiction. $\square$

The following are some examples for stable quotients.

EXAMPLE 3.24. Let $\mathcal{V} = \mathcal{O}_Y$ (and likewise with $\mathcal{O}_{X_0}$) be the structure sheaf. A quotient sheaf is just a closed substack Z of $Y[k]$.

If $\dim Z = 0$, in other words, Z is just a collection of (possibly multiple or gerby) points, then $\mathcal{O}_{Y[k]} \rightarrow \mathcal{O}_Z$ is admissible if and only if none of the points of Z support on any divisor D_i for $0 \leq i \leq k$. It is stable if, moreover, $Z \cap \Delta_i \neq \emptyset$ for $1 \leq i \leq k$.

If $\dim Z = 1$ and Z is pure, then Z is admissible if and only if none of its irreducible components lie entirely in any divisor D_i for $0 \leq i \leq k$. The substack Z is stable if, furthermore, it is not “entirely fiberwise”; in other words, restricted to any extra bubbled component of $Y[k]$, it is not a fiber (or a union of fibers) of the corresponding (possibly orbi) $\mathbb{P}^1$ -bundle.

3.5 $\mathbb{C}^*$ -equivariant flat limits

To conclude this section, we describe the completion of flat families of quotients. Let $\mathcal{W}$ be a separated Deligne–Mumford stack of finite type, with a vector bundle $\mathcal{V}$, and let $\pi: \mathcal{W} \rightarrow \mathbb{A}^1$ be proper and flat. Let $\phi^\circ: \mathcal{V}|_{\mathbb{C}^*} \rightarrow \mathcal{F}^\circ$ be a quotient of coherent sheaves on $\mathcal{W}|_{\mathbb{C}^*}$, flat over $\mathbb{C}^*$.

By the properness of the Quot-space $\mathrm{Quot}_{\mathcal{W}/\mathbb{A}^1}^{\mathcal{V}}$ [OS03, Theorem 1.5], there is a unique quotient $\phi: \mathcal{V} \rightarrow \mathcal{F}$ on $\mathcal{W}$, flat over $\mathbb{A}^1$, whose restriction to $\mathcal{W}|_{\mathbb{C}^*}$ is the given family. We call ϕ the *flat completion* and $\phi|_0$ the *flat limit* of the given family at $0 \in \mathbb{A}^1$.

Consider the following special case. Let W be a proper Deligne–Mumford stack, with a vector bundle V . Consider the family $\pi: W \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$, with the other projection $p: W \times \mathbb{A}^1 \rightarrow W$. Then $\mathbb{C}^*$ acts in the obvious way on the $\mathbb{A}^1$ -factor of the family $W \times \mathbb{A}^1$. Let $\phi^\circ: p^*V|_{W \times \mathbb{C}^*} \rightarrow \mathcal{F}^\circ$ be a $\mathbb{C}^*$ -equivariant and $\mathbb{C}^*$ -flat quotient of coherent sheaves on $W \times \mathbb{C}^*$; we have the flat completion ϕ , which is still $\mathbb{C}^*$ -equivariant.

LEMMA 3.25. *We have $\phi \cong p^*\phi|_0$.*

Proof. We rephrase the situation in terms of the language of Quot-spaces. The map ϕ° defines a $\mathbb{C}^*$ -equivariant map $F^\circ: \mathbb{C}^* \rightarrow \mathrm{Quot}_W$. Here, the $\mathbb{C}^*$ -action on the Quot-space is the trivial action. Hence F° is a constant map, defined by restricting ϕ° to any fiber. By the properness of Quot_W , the unique extension of F° to $\mathbb{A}^1$ is the constant map, which gives the trivial family $p^*\phi|_0$. $\square$

Let (Y, D) be a locally smooth pair as in previous sections, and let $\mathcal{V}$ be a vector bundle on Y . Let $\Delta = \mathbb{P}_D(\mathcal{O} \oplus N) \xrightarrow{p} D$ be a bubble component.

PROPOSITION 3.26. *Let $\phi: p^*\mathcal{V} \rightarrow \mathcal{F}$ be an admissible $\mathbb{C}^*$ -equivariant quotient on Δ , with $\mathbb{C}^*$ acting along the fiber. Assume furthermore that $\mathbb{C}^*$ acts trivially on $V|_D$; that is, for all $p \in D$, the fiber $V|_p$ is trivial as a $\mathbb{C}^*$ -representation. Then $\mathcal{F} \cong p^*\mathcal{F}|_D$.*

Proof. Locally, $\Delta - D_-$ has the form $W \times \mathbb{A}^1$ with W an affine chart. The admissibility implies the flatness of $\mathcal{F}$ over $\mathbb{A}^1$. By Lemma 3.25, there is an isomorphism $\mathcal{F} \cong p^*\mathcal{F}|_D$ on this affine chart. The separatedness of Quot_W in Lemma 3.25 ensures that the isomorphisms on local affine charts are unique and hence glue to a global isomorphism. $\square$

We still need a technical lemma for later use. Consider $\mathrm{Spec} R := \mathrm{Spec} \mathbb{C}[x, y, t]/(xy)$. Let $R_0 = R/tR$, $R_- = R/yR$ and $R_{0,-} = R_0/yR_0$. Let $J = (x, y) \subset R$, $J_0 = (x, y) \subset R_0$, $I_- = (x) \subset R_-$ and $I_{0,-} = (x) \subset R_0/yR_0$ be ideals. Let $\mathbb{C}^*$ acts on t by weight $a < 0$, on x by weight $b > 0$, and trivially on y . Let $\phi: R^{\oplus r} \rightarrow M \rightarrow 0$ be a $\mathbb{C}^*$ -equivariant quotient, flat over $\mathbb{C}[t]$. Let $\phi_-: R_-^{\oplus r} \rightarrow M_- \rightarrow 0$ be its restriction on R_- .

LEMMA 3.27 ([LW15, Lemmas 3.13 and 3.14]). 1) $M_J \otimes_R R_0 \cong (M \otimes_R R_0)_{J_0}$.

2) Let M_-^* be the flat limit of the restriction of M_- on $\mathrm{Spec} \mathbb{C}[x, x^{-1}, t]$ along the x -direction to the locus $(x = 0)$, and let $M_{0,-}^*$ be the flat limit of the restriction of $M_{0,-} := M_- \otimes_R R_{0,-}$ on $\mathrm{Spec} \mathbb{C}[x, x^{-1}]$ along the x -direction to the locus $(x = 0)$. Then M_-^* is the pull-back of $M_{0,-}^*$ along the projection $\mathrm{Spec} \mathbb{C}[x, x^{-1}, t] \rightarrow \mathrm{Spec} \mathbb{C}[x, x^{-1}]$.

Proof. The proof is exactly the same as in [LW15]. $\square$

4. Moduli of stable quotients

4.1 The Quot-functor of stable quotients

Now, let us consider stable quotients on the stacks of expanded pairs and degenerations. Let (Y, D) be a locally smooth pair, and let $\pi: X \rightarrow \mathbb{A}^1$ be a locally simple degeneration. Let $\mathcal{V}$ be a vector bundle of finite rank on Y or X . We have the stacks $\mathfrak{A}$ and $\mathfrak{C}$, with universal families $\mathfrak{Y}$ and $\mathfrak{X}$.

First, let us describe the equivalence relation on the Quot-spaces. Recall that we have the discrete symmetries given in the following *Cartesian* diagram:

$$\begin{array}{ccc} R_{d,Y(k)} & \rightrightarrows & Y(k) \\ \downarrow & & \downarrow \\ R_{d,\mathbb{A}^k} & \rightrightarrows & \mathbb{A}^k. \end{array}$$

One has an étale equivalence relation

$$R_{d,\text{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}}} := \text{Quot}_{R_{d,Y(k)}/R_{d,\mathbb{A}^k}}^{p^*\mathcal{V}} \rightrightarrows \text{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}},$$

induced by the two projections in the Cartesian diagram above. Here, the maps are well defined because $R_{d,\mathbb{A}^k} \rightrightarrows \mathbb{A}^k$ are actually disjoint unions of open immersions.

These are the induced discrete symmetries on the Quot-space. Two representatives (ξ, ϕ) and (ξ', ϕ') , with $\xi, \xi': S \rightarrow \mathbb{A}^k$, represent the same object if the associated map factors through $R_{d,\text{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}}}$.

Recall that we also have the $(\mathbb{C}^*)^k$ -action

$$(\mathbb{C}^*)^k \times \text{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}} \rightrightarrows \text{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}}.$$

The equivalence relations combine to give a smooth equivalence relation on the Quot-space, still as a subrelation of the group action by the semidirect product $(\mathbb{C}^*)^k \rtimes S_k$,

$$R_{\sim, \text{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}}} := (\mathbb{C}^*)^k \times R_{d, \text{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}}} \hookrightarrow (\mathbb{C}^*)^k \rtimes S_k \times \text{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}} \rightrightarrows \text{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}}.$$

In particular, for a fixed object represented by a map $\xi: S \rightarrow \mathbb{A}^k$ and a quotient ϕ on $\mathcal{Y}_S$, its isotropy group is the pull-back in the following Cartesian diagram:

$$\begin{array}{ccc} \text{Aut}_{\sim, S}(\xi, \phi, k) & \xrightarrow{\quad} & S \\ \downarrow & & \downarrow \Delta \circ (\xi, \phi) \\ R_{\sim, \text{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}}} & \xrightarrow{\quad} & \text{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}} \times \text{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}} \end{array} \quad (4.1)$$

Here, $\text{Aut}_{\sim, S}(\xi, \phi, k)$ is a subgroup scheme over S of $(\mathbb{C}^*)^k \rtimes S_k \times S$, which is quasi-compact and separated. We know that the smooth relation $R_{\sim, \text{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}}}$ is quasi-compact and separated.

Now, we have the following Cartesian diagram of closed or open embeddings, for $|I| = l \leq k$:

$$\begin{array}{ccccc} Y(l) & \xhookrightarrow{\tau_{I,Y}} & Y(k)|_{U_I} & \xhookrightarrow{\tilde{\tau}_{I,Y}} & Y(k) \\ p \downarrow & & \downarrow & & \downarrow p \\ \mathbb{A}^l & \xhookrightarrow{\tau_I} & U_I & \xhookrightarrow{\tilde{\tau}_I} & \mathbb{A}^k. \end{array}$$

This diagram induces the closed embedding $\text{Quot}_{Y(l)/\mathbb{A}^l}^{p^*\mathcal{V}} \hookrightarrow \text{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}}$ and the open embedding

$(\mathbb{C}^*)^{k-l} \times \mathrm{Quot}_{Y(l)/\mathbb{A}^l}^{p^*\mathcal{V}} \cong \mathrm{Quot}_{Y(k)|_{U_I}/U_I}^{p^*\mathcal{V}} \hookrightarrow \mathrm{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}}$. Hence, we have the embeddings

$$\begin{array}{ccccc} R_{\sim, \mathrm{Quot}_{Y(l)/\mathbb{A}^l}^{p^*\mathcal{V}}} & \hookrightarrow & (\mathbb{C}^*)^{k-l} \times R_{\sim, \mathrm{Quot}_{Y(l)/\mathbb{A}^l}^{p^*\mathcal{V}}} & \hookrightarrow & R_{\sim, \mathrm{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}}} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \mathrm{Quot}_{Y(l)/\mathbb{A}^l}^{p^*\mathcal{V}} & \hookrightarrow & (\mathbb{C}^*)^{k-l} \times \mathrm{Quot}_{Y(l)/\mathbb{A}^l}^{p^*\mathcal{V}} & \hookrightarrow & \mathrm{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}} \end{array}$$

Let $\mathrm{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}, st} \subset \mathrm{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}}$ denote the subfunctor of the ordinary Quot-functor, consisting of stable quotients. By Proposition 3.20, this is an open subfunctor, thus represented by an algebraic subspace of the Quot-space. It is invariant under the equivalence relation.

Define the stable Quot-stack as

$$\mathfrak{Quot}_{\mathfrak{Y}/\mathfrak{A}}^{\mathcal{V}} := \varinjlim \left[\mathrm{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}, st} / R_{\sim, \mathrm{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}}} \right].$$

Similarly, in the degeneration case, one has the corresponding stable Quot-stack

$$\mathfrak{Quot}_{\mathfrak{X}/\mathfrak{C}}^{\mathcal{V}} := \varinjlim \left[\mathrm{Quot}_{X(k)/\mathbb{A}^{k+1}}^{p^*\mathcal{V}, st} / R_{\sim, \mathrm{Quot}_{X(k)/\mathbb{A}^{k+1}}^{p^*\mathcal{V}}} \right],$$

where the transition maps in the directed system are again open immersions and the limits make sense.

We interpret the objects and morphisms of $\mathfrak{Quot}_{\mathfrak{X}/\mathfrak{C}}^{\mathcal{V}}$ and $\mathfrak{Quot}_{\mathfrak{Y}/\mathfrak{A}}^{\mathcal{V}}$ in the categorical sense. For any scheme S , an object $(\bar{\xi}, \bar{\phi})$ of $\mathfrak{Quot}_{\mathfrak{Y}/\mathfrak{A}}^{\mathcal{V}}(S)$ is represented by some object $(\xi, \phi) \in [\mathrm{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}, st} / R_{\sim, \mathrm{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}}}] (S)$, for some k , possibly after some further étale base change; more precisely, passing to a surjective étale covering, we have the map $\xi: S_{\xi} = \coprod S_i \rightarrow \mathbb{A}^k$ and a stable quotient $\phi: p^*\mathcal{V} \rightarrow \mathcal{F}$ on the associated family of expanded pairs $\mathcal{Y}_{S_{\xi}}$.

Different representations are compatible in the following way. Given another (ξ', ϕ') with $\xi': S_{\xi'} \rightarrow \mathbb{A}^{k'}$, passing to the refinement $S_{\xi\xi'} := S_{\xi} \times_S S_{\xi'}$, denote the two induced families by $\mathcal{Y}_{S_{\xi\xi'}}$ and $\mathcal{Y}'_{S_{\xi\xi'}}$. By construction, they are isomorphic to each other via some $\sigma: \mathcal{Y} \xrightarrow{\sim} \mathcal{Y}'$, by embedding into a larger ambient space $Y(k'')$ for some $k'' \geq k, k'$ and pulling back the $\sim$ -equivalence relation. Then, the compatibility means that, passing to $S_{\xi\xi'}$, there is an isomorphism $\phi \xrightarrow{\sim} \sigma^*\phi'$.

Given $f: T \rightarrow S$, the map $\mathfrak{Quot}_{\mathfrak{Y}/\mathfrak{A}}^{\mathcal{V}}(S) \rightarrow \mathfrak{Quot}_{\mathfrak{Y}/\mathfrak{A}}^{\mathcal{V}}(T)$ is defined by pull back. Suppose $(\bar{\xi}, \bar{\phi}) \in \mathfrak{Quot}_{\mathfrak{Y}/\mathfrak{A}}^{\mathcal{V}}(S)$. For a representative $(\xi, \phi_{S_{\xi}})$, we have the corresponding map $f_{\xi}: T_{\xi} := T \times_S S_{\xi} \rightarrow S_{\xi}$. The 1-arrow between the two objects is defined by the composite $\eta = \xi \circ f_{\xi}$ and the pull-back $\mathcal{Y}_{T, \xi} = f_{\xi}^* \mathcal{Y}_{S, \xi}$, $\phi_{T, \xi} := f_{\xi}^* \phi_{S, \xi}$.

The functor in the degeneration case can be defined in the same manner. For any $\mathbb{A}^1$ -scheme S , $\mathfrak{Quot}_{\mathfrak{X}/\mathfrak{C}}^{\mathcal{V}}(S)$ is defined as the set of all pairs $(\bar{\xi}, \bar{\phi})$, where $\bar{\xi} \in \mathfrak{C}(S)$ and $\bar{\phi}$ is a stable quotient of sheaves on the family over $\bar{\xi}$.

THEOREM 4.1. 1) *The Quot-stack $\mathfrak{Quot}_{\mathfrak{Y}/\mathfrak{A}}^{\mathcal{V}}$ is a Deligne–Mumford stack, locally of finite type;*

2) *The Quot-stack $\mathfrak{Quot}_{\mathfrak{X}/\mathfrak{C}}^{\mathcal{V}}$ is a Deligne–Mumford stack, locally of finite type over $\mathbb{A}^1$.*

Proof. We only prove the relative case; the degeneration case is similar. The limit stack is defined by the stack associated with the limit groupoid, with objects and morphisms described as above. It remains to prove that it is Deligne–Mumford.

Look at the stabilizer of a fixed object $(\bar{\xi}, \bar{\phi}) \in \mathbf{Quot}_{\mathfrak{Y}/\mathfrak{X}}^{\mathcal{V}}(S)$, which is represented by $\xi: S_{\xi} \rightarrow \mathbb{A}^k$ and a stable quotient ϕ on $\mathcal{V}_{S_{\xi}}$. The stabilizer of this representative is given in the following Cartesian diagram:

$$\begin{array}{ccc} \mathrm{Aut}_{\sim}(\xi, \phi, k) & \xrightarrow{\quad} & S_{\xi} \xrightarrow{\quad} S \\ \downarrow & & \downarrow \Delta \circ (\xi, \phi) \\ R_{\sim, \mathrm{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}, st}} & \longrightarrow & \mathrm{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}, st} \times \mathrm{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}, st} \end{array}.$$

Again, one can see that for k sufficiently large, this stabilizer does not depend on k , since the limit is taken over open immersions of stacks, which are representable. It also does not depend on the choice of S_{ξ} since the lower horizontal map is affine, and by descent theory of affine morphisms, we glue to obtain a group scheme over S . Denote this group scheme by $\mathrm{Aut}_{\sim}(\bar{\xi}, \bar{\phi})$. It is the stabilizer of the object, independent of k and the choice of representatives for k large. Moreover, $\mathrm{Aut}_{\sim}(\bar{\xi}, \bar{\phi})$ is a quasi-compact and separated (actually affine) group scheme over S . Thus, $\mathbf{Quot}_{\mathfrak{Y}/\mathfrak{X}}^{\mathcal{V}}$ is an Artin stack.

It remains to check that each $[\mathrm{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}, st} / R_{\sim, \mathrm{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}}}]$ is a Deligne–Mumford stack, locally of finite type. Recall that $R_{\sim, \mathrm{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}}} \cong (\mathbb{C}^*)^k \times \mathrm{Quot}_{R_{d, Y(k)}/R_{d, \mathbb{A}^k}}^{p^*\mathcal{V}}$. The $(\mathbb{C}^*)^k$ -action has finite stabilizer due to the stability condition. On the other hand,

$$R_{d, \mathrm{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}}} \hookrightarrow S_k \times \mathrm{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}} \rightrightarrows \mathrm{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}}$$

is obviously étale. Therefore the stabilizer is finite and the stack is Deligne–Mumford. Since each Quot-space $\mathrm{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}}$ is of finite type, the stack we are considering is locally of finite type. $\square$

4.2 Projective Deligne–Mumford stacks

From now on, we need the notion of projective Deligne–Mumford stacks. The references for this are [OS03, Kre09, Nir08a]. First, we list some properties of coarse moduli spaces, which can be found in [AOV08, AV02, Nir08a]. We collect these results here just for later use. They are not stated in their full generality.

PROPOSITION 4.2. *Let W be a Deligne–Mumford stack, let $\underline{W}$ be a noetherian scheme, and let $c: W \rightarrow \underline{W}$ be a proper quasi-finite map.*

- 1) *If $\underline{W}$ is the coarse moduli space of W and $\underline{W}' \rightarrow \underline{W}$ is any morphism of schemes, then $\underline{W}'$ is the coarse moduli space of $\underline{W}' \times_{\underline{W}} W$.*
- 2) *If $\underline{W}' \rightarrow \underline{W}$ is a flat surjective morphism of schemes and $\underline{W}'$ is the coarse moduli space of $\underline{W}' \times_{\underline{W}} W$, then $\underline{W}$ is the coarse moduli space of W .*

PROPOSITION 4.3. *Let W be a separated Deligne–Mumford stack over $\mathbb{C}$, and let $c: W \rightarrow \underline{W}$ be its coarse moduli space.*

- 1) *There exists an étale covering $\coprod W_{\alpha} \rightarrow W$ such that $W \times_{\underline{W}} W_{\alpha} \cong [U_{\alpha}/\Gamma_{\alpha}]$, where each U_{α} is a scheme with finite group Γ_{α} acting on it.*
- 2) *The map c is proper. The functor c_* carries quasi-coherent sheaves to quasi-coherent sheaves, coherent sheaves to coherent sheaves, and is exact.*
- 3) *$c_*\mathcal{O}_W \cong \mathcal{O}_{\underline{W}}$.*

Now, we introduce the notion of generating sheaves, which can be understood as “relatively ample” sheaves of a stack over its coarse moduli space.

DEFINITION 4.4. Let W be a Deligne–Mumford stack over $\mathbb{C}$, with coarse moduli space $c: W \rightarrow \underline{W}$. A vector bundle $\mathcal{E}$ on W is called a *generating sheaf* on W if for every quasi-coherent sheaf $\mathcal{F}$, the morphism $\theta_{\mathcal{E}}(\mathcal{F}): c^*c_*\mathcal{H}om(\mathcal{E}, \mathcal{F}) \otimes_{\mathcal{O}_W} \mathcal{E} \rightarrow \mathcal{F}$ defined as the left adjoint of the map $\text{Id}: c_*(\mathcal{F} \otimes_{\mathcal{O}_W} \mathcal{E}^\vee) \rightarrow c_*(\mathcal{F} \otimes_{\mathcal{O}_W} \mathcal{E}^\vee)$ is surjective.

PROPOSITION 4.5. 1) A vector bundle $\mathcal{E}$ is a generating sheaf if and only if for every geometric point of W , the representation of the stabilizer group of that point on the fiber of $\mathcal{E}$ contains every irreducible representation.

2) Let $f: W' \rightarrow W$ be any morphism of algebraic spaces; then $\underline{W}'$ is the coarse moduli space of $W' := W \times_W \underline{W}'$, with map $f': W' \rightarrow W$. Suppose that $\mathcal{E}$ is a generating sheaf on W ; then $f^*\mathcal{E}$ is a generating sheaf on W' .

EXAMPLE 4.6 ([OS03, Proposition 5.2]). Let G be a finite group. A sheaf $\mathcal{F}$ on BG is equivalent to a complex G -representation. Then $\mathcal{F}$ is a generating sheaf if and only if it contains every irreducible representations of G as a G -submodule. In particular, the regular representation R of G is a generating sheaf.

More generally, let U be a scheme with a G -action. Let $f: [U/G] \rightarrow BG$ be the canonical 1-morphism. Then f^*R is a generating sheaf on $[U/G]$.

COROLLARY 4.7. Let W be a Deligne–Mumford stack over $\mathbb{C}$, with generating sheaf $\mathcal{E}$.

- 1) The dual $\mathcal{E}^\vee$ is still a generating sheaf.
- 2) If $f: W' \rightarrow W$ is a representable 1-morphism between Deligne–Mumford stacks, then $f^*\mathcal{E}$ is a generating sheaf on W' .

Proof. Statement 1) follows directly from Proposition 4.5(1). Statement 2) follows from Frobenius reciprocity and the property of representable maps that homomorphisms between isotropy groups are monomorphic. $\square$

DEFINITION 4.8. A Deligne–Mumford stack W over $\mathbb{C}$ is called *projective* if it has a coarse moduli space $c: W \rightarrow \underline{W}$, where $\underline{W}$ is a projective scheme, and it possesses a generating sheaf.

Let $\pi: W \rightarrow S$ be a Deligne–Mumford stack over S , with coarse moduli space $c: W \rightarrow \underline{W}$. Then $\pi: W \rightarrow S$ is called a *family of projective stacks* if the underlying map of schemes $\underline{\pi}: \underline{W} \rightarrow S$ is projective and W possesses a generating sheaf.

There are some equivalent definitions of a projective Deligne–Mumford stack.

PROPOSITION 4.9 ([Kre09, Proposition 5.1 and Corollary 5.4]). Let W be a proper Deligne–Mumford stack over $\mathbb{C}$; then the following are equivalent:

- 1) The stack W is a projective Deligne–Mumford stack.
- 2) The stack W has a projective coarse moduli space and is isomorphic to the quotient of a projective scheme by a reductive algebraic group acting linearly.
- 3) The stack W has a projective coarse moduli space, and every coherent sheaf admits a surjective morphism from a vector bundle.
- 4) The stack W admits a closed embedding into a smooth proper Deligne–Mumford stack with projective coarse moduli space.

In particular, a smooth Deligne–Mumford stack over $\mathbb{C}$ with projective coarse moduli space is always projective.

The following concept first appeared in [OS03], with the name “generalized Hilbert polynomial,” as an analog to the usual Hilbert polynomials on schemes to refine the ordinary Quot-spaces. We will also use this later, with a little modification.

DEFINITION 4.10. The *Hilbert homomorphism* of a coherent sheaf $\mathcal{F}$ on W with respect to $\mathcal{E}$ is the group homomorphism $P_{\mathcal{F}}^{\mathcal{E}}: K^0(W) \rightarrow \mathbb{Z}$ defined as

$$[V] \mapsto \chi(W, V \otimes_{\mathcal{O}_W} \mathcal{F} \otimes_{\mathcal{O}_W} \mathcal{E}^\vee),$$

where V is a vector bundle on W extended additively to $K^0(W)$.

The following concept is introduced in [Nir08a] to define the stability condition. We will also use it later for the boundedness.

DEFINITION 4.11. Let $\underline{H}$ be an ample line bundle on the projective coarse moduli space $\underline{W}$, and let $H = c^*\underline{H}$. The *modified Hilbert polynomial* of $\mathcal{F}$ with respect to the “polarization” $(\mathcal{E}, H)$ is the polynomial $P_{\mathcal{F}}^{\mathcal{E}, H}$ defined as $P_{\mathcal{F}}^{\mathcal{E}, H}(v) := P_{\mathcal{F}}^{\mathcal{E}}(H^{\otimes v})$.

One of the reasons that we are interested in generating sheaves is the following property, the consequence of a series of propositions in [Nir08a].

PROPOSITION 4.12. Let W be a projective Deligne–Mumford stack over $\mathbb{C}$, and let $c: W \rightarrow \underline{W}$ be its coarse moduli space, together with a generating sheaf $\mathcal{E}$. Let $\mathcal{F}$ be a coherent sheaf on W .

- 1) We have $c_*(\mathcal{F} \otimes_{\mathcal{O}_W} \mathcal{E}^\vee) = 0$ if and only if $\mathcal{F} = 0$.
- 2) The sheaf $c_*(\mathcal{F} \otimes_{\mathcal{O}_W} \mathcal{E}^\vee)$ on $\underline{W}$ has the same dimension as $\mathcal{F}$. Moreover, if $\mathcal{F}$ is pure, $\text{Supp } c_*(\mathcal{F} \otimes_{\mathcal{O}_W} \mathcal{E}^\vee) = c(\text{Supp } \mathcal{F})$.
- 3) We have $\mathcal{F} = 0$ if and only if $P_{\mathcal{F}}^{\mathcal{E}} = 0$, and also if and only if the polynomial $P_{\mathcal{F}}^{\mathcal{E}, H}$ is 0.

To end this section, we give a description of the coarse moduli space of the bubble.

PROPOSITION 4.13. Let (Y, D) be a locally smooth pair, and let $\Delta = \mathbb{P}(\mathcal{O}_D \oplus N_{D/Y})$ be the bubble. Let $\underline{D}$ and $\underline{\Delta}$ be the corresponding coarse moduli spaces. Then there is a $\mathbb{P}^1$ -bundle B over $\underline{D}$, fitting into the diagram

$$\begin{array}{ccc} & \underline{\Delta} & \\ \swarrow & & \searrow \\ \Delta & \xrightarrow{\quad} & B \\ \downarrow & & \downarrow \\ D & \xrightarrow{\quad} & \underline{D}, \end{array}$$

where the map $\Delta \rightarrow B$ is surjective, proper and quasi-finite, and $\underline{\Delta} \rightarrow B$ is finite.

In particular, $\underline{\Delta}$ is a projective scheme and Δ is projective as a Deligne–Mumford stack.

Proof. By definition, $\Delta = \text{Proj}_{\mathcal{O}_D}(\bigoplus_{d=0}^{\infty} N_{D/Y}^d)$. By [AGV08, Lemma 2.1.2], there is a smallest integer e such that $N_{D/Y}^e \cong c^*M$ for some line bundle M on the coarse moduli space. Take

$$B := \text{Proj}_{\mathcal{O}_{\underline{D}}}(\bigoplus_{d=0}^{\infty} N_{D/Y}^{de}) \cong \mathbb{P}_{\underline{D}}(\mathcal{O}_{\underline{D}} \oplus M).$$

There is a proper, quasi-finite and surjective map $\Delta \rightarrow B$ induced by the embedding of the graded algebras which, by the universal property of coarse moduli spaces, factors through $c: \Delta \rightarrow \underline{\Delta}$. The induced map $\underline{\Delta} \rightarrow B$ is also quasi-finite. By [Vis89, Proposition 2.6], there is a finite surjective map $A \rightarrow \underline{\Delta}$, where A is a scheme. Applying [Har77, II, Exercise 4.4] to the composition $A \rightarrow \underline{\Delta} \rightarrow B$ over B , we see that $\underline{\Delta}$ is proper, and hence finite over B . $\square$

4.3 Moduli of stable quotients with fixed topological data

To refine the Quot-stacks and get some finite-type spaces, we need to fix some topological data. From now on, let (Y, D) be a *smooth* pair, with Y *projective*, and let $\pi: X \rightarrow \mathbb{A}^1$ be a *simple degeneration*, which is a *family of projective Deligne–Mumford stacks*. Let $\mathcal{E}_X$ and $\mathcal{E}_Y$ be generating sheaves on X and Y , which we will just call $\mathcal{E}$ if there is no risk of confusion.

Consider the Grothendieck K-group of Y . Note that by smoothness and Proposition 4.9(3), we have $K_0(Y) \cong K^0(Y)$, which we just denote by $K(Y)$ for simplicity. Let $p: Y[k] \rightarrow Y$ be the contraction map, and let $\mathcal{F}$ be a coherent sheaf on $Y[k]$. We can define a homomorphism $P_{\mathcal{F}}^{\mathcal{E}}: K(Y) \rightarrow \mathbb{Z}$, still called the *Hilbert homomorphism* with respect to $\mathcal{E}$, as follows:

$$[V] \mapsto \chi(Y[k], p^*V \otimes_{\mathcal{O}_W} \mathcal{F} \otimes_{\mathcal{O}_W} p^*\mathcal{E}^\vee),$$

where V is a vector bundle and the homomorphism is naturally extended additively. Again, if one chooses an ample line bundle $\underline{H}$ on the coarse moduli space $c: Y \rightarrow \underline{Y}$, one can define the *modified Hilbert polynomial* as $P_{\mathcal{F}}^{\mathcal{E}, \underline{H}}(v) := P_{\mathcal{F}}^{\mathcal{E}}(p^*H^{\otimes v})$, where $H = c^*\underline{H}$.

Note that properties in Proposition 4.12 do not necessarily hold in this case. Although $p^*\mathcal{E}$ would still be a generating sheaf on $Y[k]$, the pull-back of H may not be ample. But we can still use them to refine the Quot-stacks.

Consider a family of expanded pairs $\pi: \mathcal{Y}_S \rightarrow S$, with a coherent sheaf $\mathcal{F}$ on $\mathcal{Y}_S$, flat over S . For each point $s \in S$, we have a Hilbert homomorphism on the fiber $P_{\mathcal{F}|_s}^{\mathcal{E}}: K(Y) \rightarrow \mathbb{Z}$. By the same argument as in [OS03, Lemma 4.3], if S is connected, there exists a group homomorphism $P: K(Y) \rightarrow \mathbb{Z}$ such that $P_{\mathcal{F}|_s}^{\mathcal{E}} = P$ for all $s \in S$.

Now, fix $P: K(Y) \rightarrow \mathbb{Z}$, and let $\mathbf{Quot}_{\mathcal{Y}/\mathcal{A}}^{\mathcal{V}, P}$ be the subfunctor of $\mathbf{Quot}_{\mathcal{Y}/\mathcal{A}}^{\mathcal{V}}$ parameterizing stable quotients with Hilbert homomorphism P . This is an open and closed subfunctor and is thus represented by a Deligne–Mumford stack, locally of finite type.

For the expanded degenerations, we make the similar definition. The only subtle difference is that we need to compose with the restriction $K(X) \rightarrow K(X_c)$ for $c \neq 0$ and $K(X) \rightarrow K(X_0)$ for the Hilbert homomorphisms on fibers. Also, for a fixed $P: K(X) \rightarrow \mathbb{Z}$, one can define $\mathbf{Quot}_{\mathcal{X}/\mathcal{C}}^{\mathcal{V}, P}$, which is an open and closed subfunctor of $\mathbf{Quot}_{\mathcal{X}/\mathcal{C}}^{\mathcal{V}}$. As for the modified Hilbert polynomial, one needs to fix a relatively ample line bundle H for $\pi: X \rightarrow \mathbb{A}^1$.

5. Properness of the moduli of 1-dimensional stable quotients

In this section, we prove the properness of the Quot-stacks $\mathbf{Quot}_{\mathcal{Y}/\mathcal{A}}^{\mathcal{V}, P}$ and $\mathbf{Quot}_{\mathcal{X}/\mathcal{C}}^{\mathcal{V}, P}$, for a fixed generalized Hilbert polynomial P . Since our main interest is the application in Donaldson–Thomas theory and Pandharipande–Thomas theory, we concentrate on the 1-dimensional case. We will use the valuative criteria for separatedness and properness of Deligne–Mumford stacks. Let (Y, D) and $\pi: X \rightarrow \mathbb{A}^1$, $\mathcal{E}$, H be fixed as in Section 4.3.

The following lemmas justify what we mean by the 1-dimensional case; that is, the dimension of the supports of the stable quotients is preserved in a family.

LEMMA 5.1. *Let $\mathcal{F}$ be an admissible sheaf on $Y[k]$ (respectively, $X_0[k]$), with contraction map $p: Y[k] \rightarrow Y$ (respectively, $p: X_0[k] \rightarrow X_0$). Then $\dim \mathcal{F} = \dim p_*\mathcal{F}$.*

Proof. It suffices to prove the result for $k = 1$, and the general case simply follows by induction. Let $\dim \mathcal{F} = d$. Then $\dim \mathcal{F}|_\Delta = d$ or $\dim \mathcal{F}|_Y = d$. If the latter is true, the lemma is proved.

Now, we assume $\dim \mathcal{F}|_\Delta = d$ and $\dim p_*\mathcal{F} \leq d-1$. For this to be true, $\mathcal{F}|_\Delta$ must have support along the fibers of Δ . Hence, $\dim \mathcal{F}|_D = \dim \mathcal{F}|_\Delta - 1 = d-1$. On the other hand, by admissibility, $(\mathcal{F}|_Y)|_D$ must have the same dimension as $\mathcal{F}|_D$, and therefore $\dim \mathcal{F}|_Y = \dim \mathcal{F}|_D + 1 = d$. This gives a contradiction. Thus $\dim p_*\mathcal{F} = d$, which proves the lemma. $\square$

LEMMA 5.2 ([LW15, Lemma 3.18]). *Let Δ be a bubble component with divisor $D = D_-$ and contraction $p: \Delta \rightarrow D$. Suppose that $\mathcal{F}$ on Δ is normal to D ; then $R^1p_*\mathcal{F} = 0$. Moreover, if $P_{\mathcal{F}}^{\mathcal{E},H} = P_{\mathcal{F}|_D}^{\mathcal{E},H}$, then $\mathcal{F} \cong p^*\mathcal{F}|_D$.*

Proof. Choose a surjection $p^*\mathcal{V} \rightarrow \mathcal{F} \rightarrow 0$, where some $\mathcal{V}$ is a vector bundle on D , and let $\mathcal{K}$ be the kernel. Look at the long exact sequence

$$\cdots \rightarrow R^1p_*p^*\mathcal{V} \rightarrow R^1p_*\mathcal{F} \rightarrow R^2p_*\mathcal{K} \rightarrow \cdots$$

It is clear that $R^1p_*p^*\mathcal{V} = 0$ and $R^2p_*\mathcal{K} = 0$ for dimensional reasons. Thus $R^1p_*\mathcal{F} = 0$.

Now, suppose that we have $P_{\mathcal{F}}^{\mathcal{E},H} = P_{\mathcal{F}|_D}^{\mathcal{E},H}$. By admissibility, we have the short exact sequence $0 \rightarrow \mathcal{F}(-D) \rightarrow \mathcal{F} \rightarrow \mathcal{F}|_D \rightarrow 0$. By $R^1p_*\mathcal{F}(-D) = 0$, we have the surjection $p_*\mathcal{F} \rightarrow \mathcal{F}|_D \rightarrow 0$. Since H is ample on D , the equality $P_{\mathcal{F}}^{\mathcal{E},H} = P_{\mathcal{F}|_D}^{\mathcal{E},H}$ implies that this is actually an isomorphism $p_*\mathcal{F} \cong \mathcal{F}|_D$. Taking the pull-back of the inverse map, we have $p^*\mathcal{F}|_D \cong p^*p_*\mathcal{F} \rightarrow \mathcal{F}$, which we can assume to be surjective if one replaces $\mathcal{F}$ by its twist with a sufficiently relatively ample line bundle at the beginning. Again by comparison of the modified Hilbert polynomials, one concludes that this map is also injective and $p^*\mathcal{F}|_D \cong \mathcal{F}$. $\square$

COROLLARY 5.3. *Let $\mathcal{F}$ be an admissible sheaf on $Y[k]$ or $X_0[k]$. Then the Hilbert homomorphism and modified Hilbert polynomial can be computed using $p_*\mathcal{F}$ on Y or X_0 , respectively, that is, $P_{\mathcal{F}}^{\mathcal{E}} = P_{p_*\mathcal{F}}^{\mathcal{E}}$ and $P_{\mathcal{F}}^{\mathcal{E},H} = P_{p_*\mathcal{F}}^{\mathcal{E},H}$.*

Proof. This follows from $R^1p_*\mathcal{F} = 0$. $\square$

5.1 Boundedness

In this subsection, for a fixed group homomorphism $P: K(Y) \rightarrow \mathbb{Z}$ (respectively, $P: K(X) \rightarrow \mathbb{Z}$), we require that the associate modified Hilbert polynomial $v \mapsto P(H^{\otimes v})$ has degree 1 or less. We denote it by $f(v) = av + b$, with $a \geq 0$. Consider Quot-stacks $\mathbf{Quot}_{\mathfrak{Y}/\mathfrak{A}}^{\mathcal{V},P}$ and $\mathbf{Quot}_{\mathfrak{X}/\mathfrak{C}}^{\mathcal{V},P}$. By Lemmas 5.1 and 5.2, one can see that the objects parameterized by them have $\dim \mathcal{F} = \deg f$. By boundedness, we mean the quasi-compactness of the Quot-stacks.

PROPOSITION 5.4. *The stack $\mathbf{Quot}_{\mathfrak{Y}/\mathfrak{A}}^{\mathcal{V},P}$ (respectively, $\mathbf{Quot}_{\mathfrak{X}/\mathfrak{C}}^{\mathcal{V},P}$) is of finite type over $\mathbb{C}$ (respectively, over $\mathbb{A}^1$).*

Recall that $\coprod_{k \geq 0} [\mathbf{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V},st,P} / R_{\sim, \mathbf{Quot}_{Y(k)/\mathbb{A}^k}^{p^*\mathcal{V}}}]$ forms an étale covering of $\mathbf{Quot}_{\mathfrak{Y}/\mathfrak{A}}^{\mathcal{V},P}$. It suffices to prove the following.

PROPOSITION 5.5. *Fix a polynomial $f(v) = av + b$ with $a, b \in \mathbb{Z}$ and $a \geq 0$ and a polarization $(\mathcal{E}, H)$ on Y (respectively, X). There exists a constant $N = N(f, \mathcal{E}, H)$ such that for any $k \geq 0$, and any stable quotient $\phi: p^*\mathcal{V} \rightarrow \mathcal{F}$ on $Y[k]$ (respectively, $X[k]$), with modified Hilbert polynomial $P_{\mathcal{F}}^{\mathcal{E},H} = f$, one has $k \leq N$.*

The rest of this subsection is used to prove this proposition. We mainly concentrate on the relative case; for the degeneration case, the proof will be similar. Let us first state some results that will be useful.

LEMMA 5.6 ([CG97, Proposition 5.9.3], [Kol96, Corollary VI.2.3]). *Let W be an n -dimensional scheme over $\mathbb{C}$, with the topological filtration $0 = \Gamma_{-1} \subset \Gamma_0 \subset \cdots \subset \Gamma_n = K_0(W)$, where Γ_i consists of the classes generated by coherent sheaves of dimension at most i . Let $\mathcal{F}$ be a coherent sheaf on W with $\dim \text{Supp } \mathcal{F} = d$. Then, we have $[\mathcal{F}] \in \Gamma_d$ and*

$$[\mathcal{F}] = \sum_{Z \subset \text{Supp } \mathcal{F}} \text{mult}(\mathcal{F}; Z)[\mathcal{O}_Z] \mod \Gamma_{d-1}$$

in $K_0(W)$, where the sum is over all d -dimensional irreducible components of $\text{Supp } \mathcal{F}$ and $\text{mult}(\mathcal{F}; Z)$ is the length of $\mathcal{F}$ as a module over the generic point of Z .

We need a result of Grothendieck on the boundedness of families of sheaves.

LEMMA 5.7 ([Gro95, Lemma 2.6]). *Let W be a projective scheme over a noetherian scheme S , and let $\mathcal{O}_W(1)$ be ample on W relative to S . Let E be a family of isomorphism classes of coherent sheaves on the fibers of W/S , contained in the family of isomorphism classes of the quotients of a certain fixed coherent sheaf on W , such that for all $\mathcal{F} \in E$, the Hilbert polynomial of $\mathcal{F}$ is*

$$P_{\mathcal{F}}(v) = a_{\mathcal{F}} \frac{v^r}{r!} + b_{\mathcal{F}} \frac{v^{r-1}}{(r-1)!} + \cdots$$

and $a_{\mathcal{F}}$ is bounded. Then $b_{\mathcal{F}}$ is bounded from below.

Moreover, if $b_{\mathcal{F}}$ is bounded, then the family consisting of all $\mathcal{F}_{(r)}$ is bounded, where $\mathcal{F}_{(r)} := \mathcal{F}/T_r(\mathcal{F})$ is $\mathcal{F}$ quotient by its torsion subsheaf.

Proof of Proposition 5.5. By Corollary 3.9, given a stable quotient $\phi: p^*\mathcal{V} \rightarrow \mathcal{F}$ on $Y[k]$, we have the short exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{F}|_Y \oplus \bigoplus_{i=1}^k \mathcal{F}|_{\Delta_i} \rightarrow \bigoplus_{i=1}^k \mathcal{F}|_{D_{i-1}} \rightarrow 0.$$

Thus,

$$\begin{aligned} av + b &= \chi(Y[k], \mathcal{F} \otimes p^*(H^v \otimes \mathcal{E}^\vee)) \\ &= \chi(Y, \mathcal{F}|_Y \otimes H^v \otimes \mathcal{E}^\vee) + \sum_{i=1}^k \chi(\Delta_i, \mathcal{F}|_{\Delta_i} \otimes p^*(H^v \otimes \mathcal{E}^\vee)) \\ &\quad - \sum_{i=1}^k \chi(D_{i-1}, \mathcal{F}|_{D_{i-1}} \otimes p^*(H^v \otimes \mathcal{E}^\vee)) \\ &= \chi(Y, \mathcal{F}|_Y \otimes H^v \otimes \mathcal{E}^\vee) + \sum_{i=1}^k \chi(\Delta_i, \mathcal{F}|_{\Delta_i}(-D_{i-1}) \otimes p^*(H^v \otimes \mathcal{E}^\vee)) \\ &=: (a_0(k, \mathcal{F})v + b_0(k, \mathcal{F})) + \sum_{i=1}^k (a_i(k, \mathcal{F})v + b_i(k, \mathcal{F})), \end{aligned} \tag{5.1}$$

where in (5.1), we use the exact sequence $0 \rightarrow \mathcal{F}|_{\Delta_i}(-D_{i-1}) \rightarrow \mathcal{F}|_{\Delta_i} \rightarrow \mathcal{F}|_{D_{i-1}} \rightarrow 0$, by admissibility. Here, the coefficients a and b also depend on the polarization $\mathcal{E}, H$, but we only

emphasize that on k and $\mathcal{F}$. Let

$$\Lambda := \{i \mid 1 \leq i \leq k, a_i(k, \mathcal{F}) > 0\},$$

and write

$$av + b = (a_0(k, \mathcal{F})v + b_0(k, \mathcal{F})) + \sum_{i \in \Lambda} (a_i(k, \mathcal{F})v + b_i(k, \mathcal{F})) + \sum_{i \notin \Lambda} b_i(k, \mathcal{F}). \quad (5.2)$$

Since $a_i \geq 1$ for $i \in \Lambda$ and $a = \sum_{i=0}^k a_i$, we can bound the size of Λ by $|\Lambda| \leq a$.

For $i \notin \Lambda$, we know $b_i(k, \mathcal{F}) \neq 0$, because otherwise by Lemma 5.2, the restriction $\mathcal{F}|_{\Delta_i}$ would be a pull-back from D_{i-1} and thus unstable. Thus $b_i(k, \mathcal{F}) \geq 1$.

Now, by the following Lemma 5.8, for $i \in \Lambda \cup \{0\}$, we have the lower bound $b_i(k, \mathcal{F}) \geq -M$. By (5.2), we have $b \geq (|\Lambda| + 1)(-M) + k - |\Lambda|$. Thus, $k \leq b + M + (M + 1)|\Lambda| \leq b + M + (M + 1)a$, which proves the proposition. $\square$

LEMMA 5.8. *In the proof of Proposition 5.5, there exists $M = M(f, \mathcal{E}, H) > 0$, which does not depend on k and $\mathcal{F}$, such that $b_i(k, \mathcal{F}) \geq -M$ for all $i \in \Lambda \cup \{0\}$.*

Proof. Consider $i \in \Lambda$. Let $\Delta \cong \Delta_i$, with divisor $D_- \cong D_{i-1}$. By Proposition 4.13, there is a diagram

$$\begin{array}{ccc} & \underline{\Delta} & \\ c \nearrow & & \searrow \\ \Delta & \xrightarrow{c_B} & B \\ p \downarrow & & \downarrow p_B \\ D & \xrightarrow{c} & \underline{D}, \end{array}$$

where the underlined objects are corresponding coarse moduli spaces, B is a $\mathbb{P}^1$ -bundle over $\underline{D}$ and $\underline{\Delta}$ is finite over B .

Let us compute the terms in (5.1). For simplicity, let $\mathcal{G}_i := (c_B)_*(\mathcal{F}|_{\Delta_i}(-D_{i-1}) \otimes p^*\mathcal{E}^\vee)$ for $1 \leq i \leq k$ and $\mathcal{G}_0 := c_*(\mathcal{F}|_Y \otimes \mathcal{E}^\vee)$. Then

$$\begin{aligned} a_i(k, \mathcal{F})v + b_i(k, \mathcal{F}) &= \chi(\Delta, \mathcal{F}|_{\Delta_i}(-D_{i-1}) \otimes p^*\mathcal{E}^\vee \otimes p^*H^v) \\ &= \chi(\underline{\Delta}, c_*(\mathcal{F}|_{\Delta_i}(-D_{i-1}) \otimes p^*\mathcal{E}^\vee) \otimes \underline{p}^*\underline{H}^v) \\ &= \chi(B, \mathcal{G}_i \otimes p_B^*\underline{H}^v), \end{aligned}$$

where we used $p^*c^* = c^*p^*$ and the projection formula.

By Corollary 4.7, we see that $(p^*\mathcal{E}^\vee)^\vee$ is still a generating sheaf on Δ , and by Proposition 4.12(2), we have $\dim \text{Supp } \mathcal{G}_i = \dim \mathcal{F}|_{\Delta_i}$. Thus, by Lemma 5.6,

$$a_i(k, \mathcal{F})v = \sum_{Z \subset \text{Supp } \mathcal{G}_i} \text{mult}(\mathcal{G}_i; Z) \cdot \chi(Z, \mathcal{O}_Z \otimes p_B^*\underline{H}^v) \mod \Gamma_0,$$

where Z ranges over the 1-dimensional irreducible components of $\text{Supp } \mathcal{G}_i$. We see (by choosing a divisor for $\underline{H}$ and computing the leading coefficients by exact sequence) that $\chi(Z, \mathcal{O}_Z \otimes p_B^*\underline{H}^v)$ has leading term $vc_1(p_B^*\underline{H}) \cdot [Z]$, where one can view $c_1(p_B^*\underline{H})$ as an element of $N^1(B)$ and $[Z]$ as an element of $N_1(B)$. Thus

$$a_i(k, \mathcal{F}) = c_1(p_B^*\underline{H}) \cdot \sum_Z \text{mult}(\mathcal{G}_i; Z) \cdot [Z] =: c_1(p_B^*\underline{H}) \cdot [\mathcal{G}_i], \quad (5.3)$$

where $[\mathcal{G}_i]$ is the numerical 1-cycle determined by the support of $\mathcal{G}_i$.

However, $p_B^* \underline{H}$ is not ample on B , and the above is not a Hilbert polynomial, as required by Lemma 5.7. We need to introduce some ample line bundles on B . Replace $\underline{H}$ by its sufficiently large power such that $p_B^* \underline{H} \otimes \mathcal{O}_B(\underline{D})$ is ample on B . Consider $h := c_B^*(p_B^* \underline{H} \otimes \mathcal{O}_B(\underline{D})) = H \otimes c_B^* \mathcal{O}_B(\underline{D})$ on Δ . Note that this procedure does not depend on k .

Now, let us compute the modified Hilbert polynomial with respect to h . Let $a'_i(k, \mathcal{F})v + b'_i(k, \mathcal{F}) := \chi(\Delta, \mathcal{F}|_{\Delta_i}(-D_{i-1}) \otimes p^* \mathcal{E}^\vee \otimes h^v)$ and $a'_0v + b'_0 := a_0v + b_0$. By a similar computation, we arrive at

$$a'_i(k, \mathcal{F}) = (c_1(p_B^* \underline{H}) + [\underline{D}]) \cdot [\mathcal{G}_i].$$

Combining this with (5.3), we get

$$a'_i(k, \mathcal{F}) = a_i(k, \mathcal{F}) + [\underline{D}] \cdot [\mathcal{G}_i] \leq a + [\underline{D}] \cdot [\mathcal{G}_i]. \quad (5.4)$$

Now, the lemma follows from the following claim and Lemma 5.7. A uniform upper bound for all a'_i would imply the existence of a uniform lower bound for all b'_i . Since $b'_i = b_i$ by a Riemann–Roch computation, we get a uniform lower bound for all b_i .

Claim: There is a uniform upper bound for a'_i for all i .

We prove this by induction. For $i = 0$, we have $a'_0(k, \mathcal{F}) = a_0(k, \mathcal{F}) \leq a$. Then, analogously to (5.3), we have

$$a_0(k, \mathcal{F}) = c_1(\underline{H}) \cdot [\mathcal{G}_0]. \quad (5.5)$$

Since $\underline{H}$ is ample on $\underline{Y}$, the boundedness of $a_0(k, \mathcal{F})$ implies that $\{[\mathcal{G}_0] \mid k, \mathcal{F}\}$ is bounded as a subset in $N_1(\underline{Y})$. Thus, the pairing $[\underline{D}] \cdot [\mathcal{G}_0]$ is bounded, say by some constant $C_0 > 0$. On the other hand, we have

$$[\underline{D}] \cdot [\mathcal{G}_0] = [\underline{D}] \cdot [\mathcal{G}_1],$$

which follows from the definition and restrictions of $\mathcal{F}$ to $D \cong D_0$ from Y and Δ_1 , respectively. Thus by (5.4), we have

$$a'_1(k, \mathcal{F}) - a_1(k, \mathcal{F}) \leq C_0.$$

Now, one can proceed by induction. Assume $a'_i(k, \mathcal{F}) - a_i(k, \mathcal{F}) \leq C_i$ for some uniform bound C_i . Then $\{[\mathcal{G}_i] \mid k, \mathcal{F}\}$ is bounded as a subset in $N_1(\underline{\Delta})$. By admissibility,

$$[\underline{D}] \cdot [\mathcal{G}_{i+1}] = [\underline{D}_+] \cdot [\mathcal{G}_i]$$

is bounded by some uniform bound C_{i+1} .

Now, we look at $i + 1$. If $i + 1 \in \Lambda$, then

$$a'_{i+1}(k, \mathcal{F}) - a_{i+1}(k, \mathcal{F}) = [\underline{D}] \cdot [\mathcal{G}_{i+1}] \leq C_{i+1}$$

is bounded.

If $i + 1 \notin \Lambda$, then $\mathcal{G}_{i+1}$ is of dimension 0 in $\underline{\Delta}_{i+1}$, which by admissibility does not intersect with the divisors. Thus in this case, one has

$$a'_{i+1}(k, \mathcal{F}) - a_{i+1}(k, \mathcal{F}) = 0.$$

We keep proceeding by induction, and finally we have that the set of numbers $\{a'_i(k, \mathcal{F}) - a_i(k, \mathcal{F}) \mid 0 \leq i \leq k\}$ is bounded by $C := \max\{C_i \mid 0 \leq i \leq k\}$, where there are only $|\Lambda|$ many nonzero C_i , and each nonzero C_i is determined by the previous nonzero one in some way independent of k and $\mathcal{F}$ (the process only depends on the intersections and bounds on $N_1(B)$). Thus, $a + C$ does not depend on k and $\mathcal{F}$ (but depends on the choice of $\underline{H}$, $\mathcal{E}$) and bounds all $a'_i(k, \mathcal{F})$. $\square$

5.2 Separatedness

In this section, we prove the following.

PROPOSITION 5.9. *The stack $\mathbf{Quot}_{\mathfrak{y}/\mathfrak{A}}^{\mathcal{V},P}$ (respectively, $\mathbf{Quot}_{\mathfrak{x}/\mathfrak{C}}^{\mathcal{V},P}$) is separated over $\mathbb{C}$ (respectively, over $\mathbb{A}^1$).*

We use the valuative criterion of separatedness for Deligne–Mumford stacks. Consider $S = \operatorname{Spec} R$, where R is a valuation ring, with fractional field K . Let $\eta = \operatorname{Spec} K$ be the generic point, and let $\eta_0 = \operatorname{Spec} k$ be the closed point. Since we have proved that the Quot-stack is of finite type, by [LM00, Proposition 7.8], it suffices to treat the case where R is a complete discrete valuation ring with an algebraically closed residue field k . Let $u \in R$ be the uniformizer. Again, we concentrate on the relative case; the proof for the degeneration case is similar.

Remark 5.10. By our description of the objects, we need to consider representatives $\xi: S_\xi \rightarrow \mathbb{A}^k$, where $S_\xi := \coprod S_i \rightarrow S$ is an étale covering of S . Since R is a discrete valuation ring, for each $S_i \rightarrow S$ to be étale, it must be an isomorphism. Therefore, we can always assume $S_\xi = S$.

Suppose that we are given two arrows $(\bar{\xi}, \bar{\phi}), (\bar{\xi}', \bar{\phi}'): S \rightarrow \mathbf{Quot}_{\mathfrak{y}/\mathfrak{A}}^{\mathcal{V},P}$, represented by (ξ, ϕ) and (ξ', ϕ') , whose restrictions to η are isomorphic. The maps $\xi: S \rightarrow \mathbb{A}^k$ and $\xi': S \rightarrow \mathbb{A}^{k'}$ are of the form

$$\begin{aligned} \mathbb{C}[t_1, \dots, t_k] &\rightarrow R, & t_i &\mapsto c_i(u)u^{e_i}, \\ \mathbb{C}[t_1, \dots, t_{k'}] &\rightarrow R, & t_i &\mapsto c'_i(u)u^{e'_i}, \end{aligned}$$

where $c_i(u)$ and $c'_i(u)$ are either 0 or invertible in R and $e_i, e'_i \geq 0$. Embedding them into a larger target and applying the $\sim$ -equivalence relation, we can assume $k = k'$ and that the maps are of the form

$$\begin{aligned} (t_1, \dots, t_k) &\mapsto (u^{e_1}, \dots, u^{e_l}, 0, \dots, 0), \\ (t_1, \dots, t_{k'}) &\mapsto (u^{e'_1}, \dots, u^{e'_l}, 0, \dots, 0), \end{aligned}$$

where the two expressions have the same number of zeros, due to the isomorphism over η . The proof of the separatedness then reduces to the following lemma.

LEMMA 5.11. *Consider the map $\xi: S \rightarrow \mathbb{A}^1$ given by $\mathbb{C}[t_1] \rightarrow R, t_1 \mapsto u^e, e \geq 1$. Let $\tilde{\xi}: S \rightarrow \mathbb{A}^e$ be $\mathbb{C}[t_1, \dots, t_e] \rightarrow R, (t_1, \dots, t_e) \mapsto (u, \dots, u)$. Let $\mathcal{Y}_S, \tilde{\mathcal{Y}}_S$ be the corresponding associated families of expanded pairs. Then, there exist a map $h: \tilde{\mathcal{Y}}_S \rightarrow \mathcal{Y}_S$ which is an isomorphism on η and a contraction $Y[e] \rightarrow Y[1]$ on η_0 , where the only uncontracted component is Δ_e .*

Proof. This is due to the construction of $X(k)$. By the following diagram and the universal property of fiber product, we get the map h :

$$\begin{array}{ccccc} \tilde{\mathcal{Y}}_S & \longrightarrow & Y(e) & \xrightarrow{p} & Y(1) \\ \downarrow & & \downarrow & & \downarrow \\ S & \xrightarrow{\tilde{\xi}} & \mathbb{A}^e & \longrightarrow & \mathbb{A}^1. \end{array}$$

The map on the base is multiplication, and the map on the central fiber is the contraction of the inserted components. $\square$

Proof of Proposition 5.9. We need to prove that the isomorphism on η extends to S . Let $E := \sum_{i=1}^l e_i$ and $E' := \sum_{i=1}^l e'_i$. First, we consider the special case $E = E'$.

Consider the map $\tilde{\xi}: S \rightarrow \mathbb{A}^{E+k-l}$ given by $(t_1, \dots, t_E, \dots, t_{E+k-l}) \mapsto (u, \dots, u, 0, \dots, 0)$. Repeatedly applying Lemma 5.11, we see that there is a map between associated families $h: \tilde{\mathcal{Y}}_S \rightarrow \mathcal{Y}_S$ such that $h|_\eta$ is an isomorphism and $h|_{\eta_0}: Y[E+k-l] \rightarrow Y[k]$ is a contraction of components, where the uncontracted components are $\Delta_{e_1}, \Delta_{e_1+e_2}, \dots, \Delta_E$ and the last $(k-l)$ ones correspond to the zeros. Applying the same with ξ' , we also have a map $h': \tilde{\mathcal{Y}}_S \rightarrow \mathcal{Y}'_S$ contracting the corresponding components for ξ' .

Consider quotient sheaves $h^*\phi$ and $h'^*\phi'$ on $\tilde{\mathcal{Y}}_S$. Now, $(\tilde{\xi}, h^*\phi)$ and $(\tilde{\xi}, h'^*\phi')$ are isomorphic over η (since $h|_\eta$ and $h'|_\eta$ are isomorphisms) and flat over S by admissibility and Lemma 3.15. By the separatedness of the ordinary Quot-space $\text{Quot}_{Y(E+k-l)/\mathbb{A}^{E+k-l}}^{p^*\mathcal{V}}$, they are also isomorphic over S . On the other hand, $h^*\phi|_{\eta_0}$ and $h'^*\phi'|_{\eta_0}$ are stable on uncontracted components of $Y[E+k-l]$ but are unstable on contracted ones. For them to be isomorphic, the contracted components for h and h' must coincide, which implies $e_i = e'_i$ for all $1 \leq i \leq l$, in which case h and h' must be isomorphisms and the conclusion follows.

For the general case, assume $E \leq E'$. We take the embeddings

$$\begin{aligned} \tilde{\xi}: S &\rightarrow \mathbb{A}^{E+k-l}, & (t_1, \dots, t_E, \dots, t_{E+k-l}) &\mapsto (u, \dots, u, 0, \dots, 0), \\ \tilde{\xi}': S &\rightarrow \mathbb{A}^{E'+k-l}, & (t_1, \dots, t_E, \dots, t_{E'}, \dots, t_{E'+k-l}) &\mapsto (u, \dots, u, u, \dots, u, 0, \dots, 0), \end{aligned}$$

with corresponding associated families $\tilde{\mathcal{Y}}_S$ and $\tilde{\mathcal{Y}}'_S$. By the successive blow-up construction (Proposition 2.11), there is a map $c: \tilde{\mathcal{Y}}'_S \rightarrow \tilde{\mathcal{Y}}_S$ which is a contraction of components if restricted to η_0 . This map c is given by the diagram

$$\begin{array}{ccccc} \tilde{\mathcal{Y}}'_S & \longrightarrow & Y(E'+k-l) & \xrightarrow{p_{\text{Bl}}} & Y(E+k-l) \\ \downarrow & & \downarrow & & \downarrow \\ S & \xrightarrow{\tilde{\xi}'} & \mathbb{A}^{E'+k-l} & \xrightarrow{\text{pr}} & \mathbb{A}^{E+k-l}, \end{array}$$

where p_{Bl} is the contraction map for the successive blow-up construction and the projection $\text{pr}: \mathbb{A}^{E'+k-l} \rightarrow \mathbb{A}^{E+k-l}$ forgets the components $t_{E+1}, \dots, t_{E'}$. Again by the separatedness of the ordinary Quot-space, we have an isomorphism $c^*h^*\phi \cong h'^*\phi'$. Comparing the stable components of the two quotients, one obtains the conclusion $e_i = e'_i$ for all $1 \leq i \leq l$. $\square$

5.3 Numerical criterion for admissibility

Before getting into the proof of properness, we need a numerical criterion for a sheaf to be admissible. Let $\mathcal{F}$ be a coherent sheaf on $Y[k]$ (respectively, $X_0[k]$), with generating sheaf $\mathcal{E}$ on Y and a line bundle H which is a pull-back of an ample line bundle from the coarse moduli space. Let $\mathcal{I}_i^-$ and $\mathcal{I}_i^+$ be the ideal sheaf of the divisors $D_{i-1}, D_i \subset \Delta_i$, and let $\mathcal{I}_0^+$ be the ideal sheaf of $D_0 \subset Y$. Let $\mathcal{J}_i$ be the ideal sheaf of $D_i \subset Y[k]$. Let $\mathcal{F}^{\text{tf}}$ be the quotient in the exact sequence

$$0 \longrightarrow \bigoplus_{i=0}^{k-1} \mathcal{F}_{\mathcal{J}_i} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^{\text{tf}} \longrightarrow 0.$$

DEFINITION 5.12. Define the i th error of $\mathcal{F}$ to be the following polynomial (here, we write the subscripts in parentheses, making them easy to read).

In the relative case:

$$\begin{aligned} \text{Err}_i^{\mathcal{E}, H}(\mathcal{F})(v) &:= P^{\mathcal{E}, H}(\mathcal{F}_{\mathcal{J}_i})(v) + P^{\mathcal{E}, H}(\mathcal{F}^{\text{tf}}|_{\Delta_i, \mathcal{I}_i^+})(v) + P^{\mathcal{E}, H}(\mathcal{F}^{\text{tf}}|_{\Delta_{i+1}, \mathcal{I}_{i+1}^-})(v) \\ &\quad - \frac{1}{2} P^{\mathcal{E}, H}((\mathcal{F}^{\text{tf}}|_{\Delta_i, \mathcal{I}_i^+})|_{D_i}) - \frac{1}{2} P^{\mathcal{E}, H}((\mathcal{F}^{\text{tf}}|_{\Delta_{i+1}, \mathcal{I}_{i+1}^-})|_{D_i}) \end{aligned}$$

for $0 \leq i \leq k-1$, and

$$\text{Err}_k^{\mathcal{E},H}(\mathcal{F})(v) := P^{\mathcal{E},H}(\mathcal{F}^{\text{tf}}|_{\Delta_k, \mathcal{I}_k^+})(v),$$

where we set $\Delta_0 := Y$ and $\mathcal{F}^{\text{tf}}|_{\Delta_i, \mathcal{I}_i^+} := (\mathcal{F}^{\text{tf}}|_{\Delta_i})_{\mathcal{I}_i^+}$.

In the degeneration case:

$$\begin{aligned} \text{Err}_i^{\mathcal{E},H}(\mathcal{F})(v) &:= P^{\mathcal{E},H}(\mathcal{F}_{\mathcal{I}_i})(v) + P^{\mathcal{E},H}(\mathcal{F}^{\text{tf}}|_{\Delta_i, \mathcal{I}_i^+})(v) + P^{\mathcal{E},H}(\mathcal{F}^{\text{tf}}|_{\Delta_{i+1}, \mathcal{I}_{i+1}^-})(v) \\ &\quad - \frac{1}{2}P^{\mathcal{E},H}((\mathcal{F}^{\text{tf}}|_{\Delta_i, \mathcal{I}_i^+})|_{D_i}) - \frac{1}{2}P^{\mathcal{E},H}((\mathcal{F}^{\text{tf}}|_{\Delta_{i+1}, \mathcal{I}_{i+1}^-})|_{D_i}) \end{aligned}$$

for $0 \leq i \leq k$, where we denote $\Delta_0 := Y_-$ and $\Delta_{k+1} := Y_+$.

Moreover, we define the total error as

$$\text{Err}^{\mathcal{E},H}(\mathcal{F})(v) := \sum_{i=0}^k \text{Err}_i^{\mathcal{E},H}(\mathcal{F})(v).$$

For simplicity, we will just omit the $\mathcal{E}, H$ and write $\text{Err}_i(\mathcal{F})$ and $\text{Err}(\mathcal{F})$. Here, the polynomials are actual Hilbert polynomials and there is no problem with the ampleness of the pull-back of H , since the definition only involves the restriction of H to D_i . Hence, $\mathcal{F}$ is admissible if and only if $\text{Err}(\mathcal{F}) = 0$. In other words, the error polynomial measures the failure for a sheaf to be admissible.

For Δ_i with $0 \leq i \leq k$, let $(\mathcal{F}^{\text{tf}}|_{\Delta_i})^{\text{tf}}$ be the quotient in the exact sequences

$$\begin{aligned} 0 \longrightarrow \mathcal{F}^{\text{tf}}|_{\Delta_i, \mathcal{I}_i^-} \oplus \mathcal{F}^{\text{tf}}|_{\Delta_i, \mathcal{I}_i^+} \longrightarrow \mathcal{F}^{\text{tf}}|_{\Delta_i} \longrightarrow (\mathcal{F}^{\text{tf}}|_{\Delta_i})^{\text{tf}} \longrightarrow 0, \quad 1 \leq i \leq k; \\ 0 \longrightarrow \mathcal{F}^{\text{tf}}|_{Y, \mathcal{I}_0^+} \longrightarrow \mathcal{F}^{\text{tf}}|_Y \longrightarrow (\mathcal{F}^{\text{tf}}|_Y)^{\text{tf}} \longrightarrow 0, \quad i = 0. \end{aligned}$$

It is clear that the $(\mathcal{F}^{\text{tf}}|_{\Delta_i})^{\text{tf}}$ are admissible. Restricting to D_i , we get the sequences

$$\begin{aligned} 0 \longrightarrow (\mathcal{F}^{\text{tf}}|_{\Delta_i, \mathcal{I}_i^-})|_{D_{i-1}} \longrightarrow \mathcal{F}^{\text{tf}}|_{D_{i-1}} \longrightarrow (\mathcal{F}^{\text{tf}}|_{\Delta_i})^{\text{tf}}|_{D_{i-1}} \longrightarrow 0, \\ 0 \longrightarrow (\mathcal{F}^{\text{tf}}|_{\Delta_i, \mathcal{I}_i^+})|_{D_i} \longrightarrow \mathcal{F}^{\text{tf}}|_{D_i} \longrightarrow (\mathcal{F}^{\text{tf}}|_{\Delta_i})^{\text{tf}}|_{D_i} \longrightarrow 0. \end{aligned}$$

Let

$$\begin{aligned} \delta_i^-(\mathcal{F}) &:= P((\mathcal{F}^{\text{tf}}|_{\Delta_i})^{\text{tf}}) - P((\mathcal{F}^{\text{tf}}|_{\Delta_i})^{\text{tf}}|_{D_{i-1}}), \quad 1 \leq i \leq k, \\ \delta_i^+(\mathcal{F}) &:= P((\mathcal{F}^{\text{tf}}|_{\Delta_i})^{\text{tf}}) - P((\mathcal{F}^{\text{tf}}|_{\Delta_i})^{\text{tf}}|_{D_i}), \quad 0 \leq i \leq k-1, \end{aligned}$$

and set

$$\delta_0^-(\mathcal{F}) := P((\mathcal{F}^{\text{tf}}|_Y)^{\text{tf}}), \quad \delta_k^+(\mathcal{F}) := P((\mathcal{F}^{\text{tf}}|_{\Delta_k})^{\text{tf}}).$$

We have the following lemma.

LEMMA 5.13. 1) $P_{\mathcal{F}} = \text{Err}(\mathcal{F}) + \frac{1}{2} \sum_{i=0}^k (\delta_i^-(\mathcal{F}) + \delta_i^+(\mathcal{F}))$.

2) For $1 \leq i \leq k-1$, the leading coefficients of $\delta_i^{\pm}$ are nonnegative. They vanish if and only if $(\phi|_{\Delta_i})^{\text{tf}}$ is $\mathbb{C}^*$ -equivariant, and also if and only if $(\phi|_{\Delta_i})^{\text{tf}}$ is a pull-back from D_i or D_{i-1} .

Proof. Part 2) simply follows from Lemma 5.2. For part 1), let us compute the modified Hilbert polynomial. Recall that Lemma 3.10 implies $\mathcal{F}^{\text{tf}} = \ker(\bigoplus_{i=0}^k \mathcal{F}^{\text{tf}}|_{\Delta_i} \rightarrow \bigoplus_{i=0}^{k-1} \mathcal{F}^{\text{tf}}|_{D_i})$. Thus,

$$P(\mathcal{F}) = \sum_{i=0}^{k-1} P(\mathcal{F}_{\mathcal{I}_i}) + \sum_{i=0}^k P(\mathcal{F}^{\text{tf}}|_{\Delta_i}) - \frac{1}{2} \sum_{i=1}^k P(\mathcal{F}^{\text{tf}}|_{D_{i-1}}) - \frac{1}{2} \sum_{i=0}^{k-1} P(\mathcal{F}^{\text{tf}}|_{D_i})$$

$$\begin{aligned}
 &= \sum_{i=0}^{k-1} P(\mathcal{F}_{\mathcal{I}_i}) + \sum_{i=1}^k P(\mathcal{F}^{\text{tf}}|_{\Delta_i, \mathcal{I}_i^-}) + \sum_{i=0}^k P(\mathcal{F}^{\text{tf}}|_{\Delta_i, \mathcal{I}_i^+}) + \sum_{i=0}^k P((\mathcal{F}^{\text{tf}}|_{\Delta_i})^{\text{tf}}) \\
 &\quad - \frac{1}{2} \sum_{i=1}^k P((\mathcal{F}^{\text{tf}}|_{\Delta_i, \mathcal{I}_i^-})|_{D_{i-1}}) - \frac{1}{2} \sum_{i=0}^{k-1} P((\mathcal{F}^{\text{tf}}|_{\Delta_i, \mathcal{I}_i^+})|_{D_i}) - \frac{1}{2} \sum_{i=1}^k P((\mathcal{F}^{\text{tf}}|_{\Delta_i})^{\text{tf}}|_{D_{i-1}}) \\
 &\quad - \frac{1}{2} \sum_{i=0}^{k-1} P((\mathcal{F}^{\text{tf}}|_{\Delta_i})^{\text{tf}}|_{D_i}).
 \end{aligned}$$

By definition, we have

$$\begin{aligned}
 \text{Err}(\mathcal{F}) &= \sum_{i=0}^{k-1} P(\mathcal{F}_{\mathcal{I}_i}) + \sum_{i=0}^k P(\mathcal{F}^{\text{tf}}|_{\Delta_i, \mathcal{I}_i^+}) + \sum_{i=1}^k P(\mathcal{F}^{\text{tf}}|_{\Delta_i, \mathcal{I}_i^-}) - \frac{1}{2} \sum_{i=0}^{k-1} P((\mathcal{F}^{\text{tf}}|_{\Delta_i, \mathcal{I}_i^+})|_{D_i}) \\
 &\quad - \frac{1}{2} \sum_{i=1}^k P((\mathcal{F}^{\text{tf}}|_{\Delta_i, \mathcal{I}_i^-})|_{D_{i-1}}).
 \end{aligned}$$

Then

$$\begin{aligned}
 P(\mathcal{F}) - \text{Err}(\mathcal{F}) &= \sum_{i=0}^k P((\mathcal{F}^{\text{tf}}|_{\Delta_i})^{\text{tf}}) - \frac{1}{2} \sum_{i=1}^k P((\mathcal{F}^{\text{tf}}|_{\Delta_i})^{\text{tf}}|_{D_{i-1}}) - \frac{1}{2} \sum_{i=0}^{k-1} P((\mathcal{F}^{\text{tf}}|_{\Delta_i})^{\text{tf}}|_{D_i}) \\
 &= \frac{1}{2} \sum_{i=0}^k (\delta_i^-(\mathcal{F}) + \delta_i^+(\mathcal{F})). \quad \square
 \end{aligned}$$

To compare the polynomials, we introduce the following order in the set of Hilbert polynomials. For two polynomials $f(v) = a_r v^r / r! + \dots$ and $g(v) = b_s v^s / s! + \dots$, where $a_r, b_s \in \mathbb{Z}_+$, we say $f \prec g$ if $r < s$ or $r = s$ and $a_r < b_s$; we say $f \preceq g$ if the same holds with $<$ replaced by $\leq$. Note that modified Hilbert polynomials are actually of this kind, since they can be viewed as usual Hilbert polynomials on coarse moduli spaces. Another observation is that under this order, the space of Hilbert polynomials satisfies the strict descending condition: any strictly descending chain $f_1 \succ f_2 \succ \dots$ must attain 0 at some finite step.

5.4 Properness

In this subsection we prove the properness.

THEOREM 5.14. *The Quot-stack $\mathbf{Quot}_{\mathfrak{y}/\mathfrak{A}}^{\mathcal{V}, P}$ (respectively, $\mathbf{Quot}_{\mathfrak{x}/\mathfrak{C}}^{\mathcal{V}, P}$) is a complete Deligne–Mumford stack over $\mathbb{C}$ (respectively, over $\mathbb{A}^1$). In particular, if the associated modified Hilbert polynomial $P(H^{\otimes v})$ has degree 0 or 1, the stack is proper.*

Remark 5.15. Here, by complete, we mean that it satisfies the valuative criterion for properness but is not necessarily quasi-compact. In other words, it is separated and universally closed. We expect the “dimension ≤ 1 ” condition to be superfluous, and the properness is true for moduli of stable quotients in any dimensions, which is treated in [LW15], but for simplicity we just pose this assumption and restrict to the 1-dimensional case.

Again, let $S = \text{Spec } R$, where R is a complete discrete valuation ring with uniformizer $u \in R$, generic point $\eta = \text{Spec } K$ and closed point $\eta_0 = \text{Spec } k$. As remarked in the separatedness part, we also work on S directly instead of passing to an étale covering. Let $(\bar{\xi}_\eta, \bar{\phi}_\eta): \eta \rightarrow \mathbf{Quot}_{\mathfrak{y}/\mathfrak{A}}^{\mathcal{V}, P}$ be an object of $\mathbf{Quot}_{\mathfrak{y}/\mathfrak{A}}^{\mathcal{V}, P}(\eta)$, represented by some map $\xi_\eta: \eta \rightarrow \mathbb{A}^k$ and a stable quotient $\phi_\eta: p^* \mathcal{V} \rightarrow \mathcal{F}_\eta$

on $\mathcal{Y}_\eta$. By the valuative criterion, it suffices to extend this object over η_0 after some finite base change of S .

Applying the equivalence relation $\sim$, one can always make ξ_η in the form

$$\xi_\eta: \eta \rightarrow \mathbb{A}^k, \quad \mathbb{C}[t_1, \dots, t_k] \rightarrow K, \quad (t_1, \dots, t_k) \mapsto (1, \dots, 1, 0, \dots, 0),$$

where $t_i \mapsto 1$ for the first l (possibly 0) coordinates. Note that if $l > 0$, then ξ_η actually factors through the standard embedding $\{(1, \dots, 1)\} \times \mathbb{A}^{k-l} \hookrightarrow \mathbb{A}^k$. In this case, we can pull everything back to $\mathbb{A}^{k-l}$ and work on $\mathbb{A}^{k-l}$ instead of $\mathbb{A}^k$. Thus, we can assume $l = 0$.

Now, for this ξ_η , there is a naive extension $\xi: S \rightarrow \mathbb{A}^k$ which maps S constantly to $0 \in \mathbb{A}^k$. Then $\mathcal{Y}_S \cong S \times Y[k]$, and by the completeness of the ordinary Quot-space $\text{Quot}_{Y[k]}^{p^*\mathcal{V}}$ [OS03, Theorem 1.1]), ϕ_η extends to some ϕ over S . If $\phi|_{\eta_0}$ is stable, then we are done; otherwise, we need to modify our extension ξ . We will see that a good modification is always available.

Step 1. Normal to the distinguished divisor.

LEMMA 5.16. *Let $\xi: S \rightarrow \mathbb{A}^k$ be a map, and let $\phi: p^*\mathcal{V} \rightarrow \mathcal{F}$ be a quotient on $\mathcal{Y}_S$, flat over S , such that $\phi|_\eta$ is admissible. Then there exists (ξ', ϕ') , with $\xi': S \rightarrow \mathbb{A}^{k'}$ and S -flat quotient ϕ' on $\mathcal{Y}'_S$, such that $(\xi', \phi')|_\eta \cong (\xi, \phi)|_\eta$ and ϕ' is normal to the distinguished divisor $\mathcal{D}'_S \subset \mathcal{Y}'_S$.*

Proof. Consider $D[k] \subset \mathcal{Y}_{\eta_0} \cong Y[k]$. If $\phi|_{\eta_0}$ is normal to $D[k] \times \eta_0$, then we are done. Suppose that this is not the case. Let us look at what this means in a local chart. Étale locally near a point in $D[k] \times \eta_0$, the local model of $\mathcal{Y}_S$ can be taken as $U = \text{Spec } A := \text{Spec } R[y, \bar{z}]$, where $(D[k] \times \eta_0)|_U$ is defined by the ideal (u, y) . Geometrically, $\bar{z}$ stands for coordinates in D , and $y = 0$ is the local defining equation for D .

The quotient ϕ is represented by a sequence $0 \rightarrow K \rightarrow A^{\oplus r} \rightarrow M \rightarrow 0$. Assume $K = (f_1(u, y), \dots, f_m(u, y))$, with generators

$$f_i(u, y) = c_i + u^{\alpha_i} g_i(u) + y h_i(u, y) \in A^{\oplus r}, \quad (5.6)$$

where $c_i \in k[\bar{z}]^{\oplus r}$, $g_i \in k[\bar{z}]^{\oplus r}[u]$, $h_i \in k[\bar{z}]^{\oplus r}[u, y]$, $g_i(0) \neq 0$ and $\alpha_i \geq 1$. By generic normality, $c_i + u^{\alpha_i} g_i(u) \neq 0$. Assume that these generators are minimal in the sense that m is as small as possible. One can easily observe that M is flat over R if and only if for all i , we have $u \nmid f_i(u, y)$, that is, $c_i + y h_i(0, y) \neq 0$.

Restricting to η_0 and using flatness, we see that $\phi|_{\eta_0}$ is given by the sequence $0 \rightarrow K_0 \rightarrow A_0^{\oplus r} \rightarrow M_0 \rightarrow 0$, with $K_0 = (f_1(0, y), \dots, f_m(0, y))$. The subscript 0 here means restriction to η_0 . By our assumption, M_0 is not normal to $D[k]$, which is equivalent to $y \mid f_i(0, y)$, that is, $c_i = 0$, for some i .

To modify this family, we apply successive blow-ups at the divisor $D[k] \times \eta_0 \subset \mathcal{Y}_S$; in other words, we take the modified family as

$$\mathcal{Y}'_S \cong \text{Bl}_{D[k] \times \eta_0} \mathcal{Y}_S.$$

This is still a family of expanded pairs, as it fits into the diagram

$$\begin{array}{ccccc} \mathcal{Y}'_S & \longrightarrow & Y(k+1) & \xrightarrow{p_{\text{Bl}}} & Y(k) \\ \downarrow & & \downarrow & & \downarrow \\ S & \xrightarrow{\xi'} & \mathbb{A}^{k+1} & \xrightarrow{\text{pr}_{1, \dots, k}} & \mathbb{A}^k, \end{array}$$

where ξ' is given by $(t_1, \dots, t_k, t_{k+1}) \mapsto (u^{e_1}, \dots, u^{e_k}, u)$.

The local model of $\mathcal{Y}'_S$ is the blow-up of $\text{Spec } R[y, \bar{z}]$ at the ideal (u, y) . Thus, étale locally around a point in the new distinguished divisor $D[k+1] \times \eta_0 \subset \mathcal{Y}'_S$, the local model is $\text{Spec } B := \text{Spec } R[w, \bar{z}]$, with $y = uw$.

By the completeness of the ordinary Quot-space, $\phi|_\eta$ extends to an S -flat quotient ϕ' on the new family. Locally, ϕ' is given by a sequence $0 \rightarrow K' \rightarrow B^{\oplus r} \rightarrow M' \rightarrow 0$. By construction of the flat limit, one has

$$K' = K_u \cap B^{\oplus r},$$

where we view $A \subset B \subset B_u = A_u$ as submodules. Then

$$\begin{aligned} K' &= (f_1(u, uw), \dots, f_m(u, uw))A_u^{\oplus r} \cap B^{\oplus r}, \\ f_i(u, uw) &= u^{\alpha_i} g_i(u) + uw h_i(u, uw), \end{aligned}$$

from which we see that $u \mid f_i(u, uw)$.

Take $\gamma_i := \min\{\alpha_i, \text{ord}(h) + 1\} \geq 1$, where $\text{ord}(h)$ is the minimal degree of the monomials of h in u and y . We have

$$f_i(u, uw) = u^{\gamma_i} \left(u^{\alpha_i - \gamma_i} g_i(u) + w \frac{h_i(u, uw)}{u^{\gamma_i - 1}} \right) = u^{\gamma_i} (u^{\alpha_i - \gamma_i} g_i(u) + w h'_i(u, w)),$$

where h'_i is some polynomial with $u \nmid (u^{\alpha_i - \gamma_i} g_i(u) + w h'_i(u, w))$.

After localizing at u and intersection with $B^{\oplus r}$, we may take

$$f'_i(u, w) := u^{\alpha'_i} g_i(u) + w h'_i(u, w)$$

as the generators of K' , where $\alpha'_i := \alpha_i - \gamma_i < \alpha_i$.

If $\alpha'_i > 0$ for some i , then c'_i , defined similarly to c_i in (5.6) but for f'_i , is still 0. We can repeat the procedure above to further decrease α'_i . After finitely many steps, we get all α'_i to be 0; that is, all c'_i are nonzero. In other words, after successive blow-ups, $\phi'|_{\eta_0}$ would be normal to $D[k] \times \eta_0$. $\square$

Step 2. Admissibility. The crucial lemma in this step is the following.

LEMMA 5.17. *Let $\xi: S \rightarrow \mathbb{A}^k$ be a map, and let $\phi: p^*\mathcal{V} \rightarrow \mathcal{F}$ be a quotient on $\mathcal{Y}_S$, flat over S , such that $\mathcal{Y}_S|_\eta$ is smooth over η and $\phi|_\eta$ is admissible. Then, there exist a finite base change $S' \rightarrow S$ and (ξ', ϕ') , with $\xi': S' \rightarrow \mathbb{A}^{k'}$ and S' -flat quotient ϕ' on $\mathcal{Y}_{S'}$, such that $(\xi', \phi') \cong (\xi, \phi) \times_\eta \eta'$, where η' is the generic point of S' , the quotient ϕ' is admissible and $[\text{Aut}(\phi'|_{\eta'_0}) : \text{Aut}(\phi|_{\eta_0})]$ is finite.*

One can assume that $\xi: S \rightarrow \mathbb{A}^k$ is of the form $\mathbb{C}[t_1, \dots, t_k] \rightarrow R$ given by

$$(t_1, \dots, t_k) \mapsto (u^{e_1}, \dots, u^{e_k}),$$

where $e_i \geq 1$. Moreover, by Step 1, we may assume that $\phi|_{\eta_0}$ is normal to $D[k]$.

Pick $1 \leq l \leq k-1$ such that $\deg \text{Err}_l(\phi|_{\eta_0}) = \deg \text{Err}(\phi|_{\eta_0})$. We would like to modify the family and reduce this l th error. Our strategy is to embed η into a larger $\mathbb{A}^{k+1}$, take the flat limit in the larger target and somehow resolve the error to the newly introduced divisor. However, the generic point can approach $0 \in \mathbb{A}^{k+1}$ in many different ways. To parameterize those different directions, we apply the following procedure, analogous to the construction of $X(1)$.

Consider the map $m: S \times S \rightarrow S$ given by $R \rightarrow R \otimes R$, $u \mapsto v \otimes w$, which fits in the diagram

$$\begin{array}{ccc} S & \xrightarrow{\alpha} & S \times S \xrightarrow{\tilde{\xi}} \mathbb{A}^{k+1} \\ m \downarrow & & \downarrow (t_l, t_{l+1}) \mapsto t_l t_{l+1} \\ S & \xrightarrow{\xi} & \mathbb{A}^k. \end{array}$$

Then $\tilde{\xi}: S \times S \rightarrow \mathbb{A}^{k+1}$ is given by

$$\begin{aligned} \mathbb{C}[t_1, \dots, t_{k+1}] &\rightarrow R \otimes R, \\ (t_1, \dots, t_{k+1}) &\mapsto (u^{e_1}, \dots, u^{e_{l-1}}, v^{e_l}, w^{e_l}, u^{e_{l+1}}, \dots, u^{e_k}), \end{aligned}$$

where $u = vw$ and we view (v, w) as uniformizers of $S \times S$. Let $\mathbb{C}^*$ act on $S \times S$ by $\lambda \cdot (v, w) := (\lambda v, \lambda^{-1} w)$.

Any map $\alpha: S \rightarrow S \times S$ will give an arrow $\tilde{\xi} \circ \alpha: S \rightarrow \mathbb{A}^{k+1}$, which is isomorphic to ξ_η over the generic point. Note that there are two obvious such α from the standard embedding, $\alpha: S \hookrightarrow \{1\} \times S$ and $\alpha: S \hookrightarrow S \times \{1\}$, and if $e_l = 1$, they are the only ones. (Here, by $1 \in S$ we mean the $\mathbb{C}$ -point $\text{Spec } \mathbb{C} \rightarrow \eta$ given by $K \rightarrow \mathbb{C}$, $u \mapsto 1$.) In general $m \circ \alpha: S \rightarrow S$ is a finite base change. The map $\tilde{\xi}: S \times S \rightarrow \mathbb{A}^{k+1}$ contains the information of all possible further embeddings of the original family.

By the $\mathbb{C}^*$ -action, the stable quotient ϕ_η on $\mathcal{Y}_\eta \cong \mathcal{Y}_{\{1\} \times \eta}$ extends to a $\mathbb{C}^*$ -equivariant stable quotient on $\mathcal{Y}_{\eta \times \eta}$, which furthermore extends to an equivariant family of quotients on $\mathcal{Y}_{S \times S - \eta_0 \times \eta_0}$, by the original extension given in the assumption of Lemma 5.17. This extension on $\mathcal{Y}_{S \times S - \eta_0 \times \eta_0}$ is stable over the points $\eta \times \eta_0$ and $\eta_0 \times \eta$.

Now, for the origin $\eta_0 \times \eta_0$, the problem comes from the fact that different choices of α give different flat limits. In other words, the equivariant family of quotients over $\mathcal{Y}_{S \times S - \eta_0 \times \eta_0}$ is equivalent to an equivariant map $f: S \times S - \eta_0 \times \eta_0 \rightarrow \text{Quot}_{Y(k+1)/\mathbb{A}^{k+1}}^{p^* \mathcal{V}, P}$, but it does not necessarily extend to the codimension 2 point $\eta_0 \times \eta_0$.

We need the resolution of indeterminacy. Here, $S \times S - \eta_0 \times \eta_0$ is a smooth surface, and $\text{Quot}_{Y(k+1)/\mathbb{A}^{k+1}}^{p^* \mathcal{V}, P}$ is a projective scheme over $\mathbb{A}^{k+1}$ by [OS03, Theorem 1.5]. In this case, f extends to some $\tilde{f}: V \rightarrow \text{Quot}_{Y(k+1)/\mathbb{A}^{k+1}}^{p^* \mathcal{V}, P}$, where $V \rightarrow S \times S$ is a composite of successive blow-ups at points and the exceptional divisor E is a chain of rational curves. Moreover, $\tilde{f}$ is $\mathbb{C}^*$ -equivariant with respect to the canonical $\mathbb{C}^*$ -actions on both sides.

Let us describe the family $(\mathcal{Y}_V, \tilde{\phi})$ induced by $\tilde{f}$. Let $E =: \Sigma_1 \cup \dots \cup \Sigma_m$ be the exceptional divisor of $V \rightarrow S \times S$; let Σ_0 and Σ_{m+1} be the proper transforms of $S \times \eta_0$ and $\eta_0 \times S$, respectively. The only intersections are $q_i := \Sigma_i \cap \Sigma_{i+1}$, for $0 \leq i \leq m$. Then by properties of $X(k)$ (Proposition 2.19), we have

$$\begin{aligned} \mathcal{Y}_V|_{\Sigma_0} &\cong \text{Bl}_{D_l \times \eta_0}(\cup_l \Delta \times S) \cup_{D \times S} (Y[k-l] \times S), \\ \mathcal{Y}_V|_{\Sigma_{m+1}} &\cong (Y[l] \times S) \cup_{D \times S} \text{Bl}_{D \times \eta_0}(\cup_{k-l} \Delta \times S) \end{aligned}$$

and $\mathcal{Y}_V|_{\Sigma_i} \cong \mathbb{P}^1 \times Y[k+1]$ for $1 \leq i \leq m$. Then $\tilde{\phi}: p^* \mathcal{V} \rightarrow \tilde{\mathcal{F}}$ is a $\mathbb{C}^*$ -equivariant quotient $\mathcal{Y}_V$, flat over V , obtained as the pull-back via $\tilde{f}$ of the universal family on the Quot-scheme.

We have $\tilde{\phi}|_a \cong \phi|_{\eta_0}$ for every $\mathbb{C}$ -point a in $\eta \times \eta_0 = \Sigma_0 - q_0$ or $\eta_0 \times \eta = \Sigma_{m+1} - q_m$, and the restriction of $\tilde{\phi}$ to E parameterizes various flat limits of $\phi|_{S \times S - \eta_0 \times \eta_0}$ from different directions. For convenience, we denote the singular divisors in $Y[k+1]$ by

$$D_0, \dots, D_{l-1}, D_l^-, D_l^+, D_{l+1}, \dots, D_k$$

and the bubble components by

$$\Delta_1, \dots, \Delta_l, \tilde{\Delta}, \Delta_{l+1}, \dots, \Delta_k.$$

We see that $\mathcal{Y}_V|_{\Sigma_0}$ is a smoothing of D_l^- and $\mathcal{Y}_V|_{\Sigma_{m+1}}$ is a smoothing of D_l^+ . The group $\mathbb{C}^*$ acts by weights $\pm e_l$ on Σ_0 and Σ_{m+1} , and by weights $\pm 2e_l$ on other Σ_i and the newly inserted components $\tilde{\Delta}$ along the fiber. It acts trivially on other components.

LEMMA 5.18. *For any $0 \leq i \leq m-1$, pick $a \in \Sigma_{i+1} - \{q_i, q_{i+1}\}$.*

- 1) *We have $\text{Err}_{l-}(\tilde{\mathcal{F}}|_a) = \text{Err}_{l-}(\tilde{\mathcal{F}}|_{q_i})$, $\text{Err}_{l+}(\tilde{\mathcal{F}}|_a) = \text{Err}_{l+}(\tilde{\mathcal{F}}|_{q_{i+1}})$, where $\text{Err}_{l\pm}$ denote the error at $D_l^\pm$; moreover, $\text{Err}_{l+}(\tilde{\mathcal{F}}|_{q_0}) = \text{Err}_{l-}(\tilde{\mathcal{F}}|_{q_m}) = \text{Err}_l(\mathcal{F}|_{\eta_0})$.*
- 2) *We have $\text{Err}_j(\tilde{\mathcal{F}}|_{q_i}) = \text{Err}_j(\tilde{\mathcal{F}}|_a) = \text{Err}_j(\tilde{\mathcal{F}}|_{q_{i+1}})$ for any $j \neq l, l+1$.*
- 3) *$\text{Err}_{l-}(\tilde{\mathcal{F}}|_{q_0}) = \text{Err}_{l+}(\tilde{\mathcal{F}}|_{q_m}) = 0$.*

Proof. For statement 1), we observe that near $D_l^- \times \{q_i\}$, there is an étale local chart $\text{Spec } A := \text{Spec } \mathbb{C}[\tilde{z}, x, y, t]/(xy)$ of $\mathcal{Y}_{\Sigma_{i+1}-q_{i+1}}$, where t is the coordinate in Σ_{i+1} and x and y are coordinates along $\tilde{\Delta}$ and Δ_l , respectively. The group $\mathbb{C}^*$ acts on x and t by weights of opposite signs. We are in the situation of Lemma 3.27. Globally, this implies that for $a \in \Sigma_{i+1} - \{q_i, q_{i+1}\}$,

$$P((\tilde{\mathcal{F}}|_a)_{\mathcal{J}_{l-}}) = P(\tilde{\mathcal{F}}_{\tilde{\mathcal{J}}_{l-}}|_a) = P(\tilde{\mathcal{F}}_{\tilde{\mathcal{J}}_{l-}}|_{q_i}) = P((\tilde{\mathcal{F}}|_{q_i})_{\mathcal{J}_{l-}}),$$

where $\tilde{\mathcal{J}}_{l-}$ denotes the ideal sheaf of $D_l^- \times \Sigma_{i+1} \subset Y[k+1] \times \Sigma_{i+1}$. The first equality follows from the isomorphism given by the $\mathbb{C}^*$ -action over $\Sigma_{i+1} - \{q_i, q_{i+1}\}$, the second by flatness and the third from Lemma 3.27. One has the similar equality for $\tilde{\mathcal{F}}^{\text{tf}}|_{\Delta_l, \mathcal{I}_{l+}}$, $\tilde{\mathcal{F}}^{\text{tf}}|_{\tilde{\Delta}, \mathcal{I}_{D_l^-} \subset \tilde{\Delta}}$ and their restriction to D_l^- . Thus, statement 1) holds.

Statement 2) follows from the $\mathbb{C}^*$ -action on Σ_{i+1} .

For statement 3), one needs to prove that $\tilde{\mathcal{F}}|_{q_0}$ is normal to D_l^- . Let $\Theta := \text{Bl}_{D_l \times \eta_0}(\Delta_l \times S)$ be the irreducible component of $\mathcal{Y}_{\Sigma_0}$, and set $\Theta^* := \Theta - (\text{proper transform of } D_{l-1} \times S \cup D_l \times S)$. Consider the map

$$g: \Theta^* \hookrightarrow \Theta = \text{Bl}_{D_l \times \eta_0}(\Delta_l \times S) \xrightarrow{p_{\text{Bl}}} \Delta_l \times S \rightarrow \Delta_l,$$

which is the contraction map of $\mathcal{Y}_{\Sigma_0}$ to the original family $\mathcal{Y}_{\eta_0} \times S$, restricted to the involved component Θ .

Consider $\tilde{\phi}|_{\Theta^*}$. From the $\mathbb{C}^*$ -equivariance, there is an isomorphism between $g^*((\phi|_{\eta_0})^{\text{tf}}|_{\Delta_l})^{\text{tf}}$ and $\tilde{\phi}|_{\Theta^*}$ over $\Theta^* - \mathcal{Y}_{q_0}$. Since they are both flat over Σ_0 (because any torsion over the closed point would contradict the flatness of $g^*((\phi|_{\eta_0})^{\text{tf}}|_{\Delta_l})^{\text{tf}}$ on $\Delta \times S$), they must have the same flat limit over q_0 . Thus, it suffices to prove that $g^*((\phi|_{\eta_0})^{\text{tf}}|_{\Delta_l})^{\text{tf}}$, restricted to $\mathcal{Y}_{q_0}$, is normal to D_l^- , which is obvious from the construction. $\square$

Proof of Lemma 5.17. If $\phi|_{\eta_0}$ is stable, then we have nothing to do. Suppose otherwise; that is, $\phi|_{\eta_0}$ is normal to $D[k]$, but not admissible. Take l as above, with $1 \leq l \leq k-1$. We apply the procedure as above.

Since $\tilde{\mathcal{F}}$ is flat over V , the fiber $\tilde{\mathcal{F}}|_a$ has the same modified Hilbert polynomial, for every $\mathbb{C}$ -point $a \in V$. By Lemma 5.13(1) and Lemma 5.18(2), the following is constant for each $a \in \Sigma_1 \cup \dots \cup \Sigma_m$:

$$\text{Err}_{l-}(\tilde{\mathcal{F}}|_a) + \text{Err}_{l+}(\tilde{\mathcal{F}}|_a) + \frac{1}{2}(\delta_{\sim}^-(\tilde{\mathcal{F}}|_a) + \delta_{\sim}^+(\tilde{\mathcal{F}}|_a)),$$

where $\delta_{\sim}^\pm$ denote the δ -polynomials defined for the component $\tilde{\Delta}$.

Since $\tilde{\phi}$ is $\mathbb{C}^*$ -equivariant and each $\tilde{\mathcal{Y}}_{q_i}$ is $\mathbb{C}^*$ -invariant under the action, $\tilde{\phi}|_{q_i}$ is $\mathbb{C}^*$ -equivariant. By Lemma 5.13(2), we have $\delta_{\sim}^{\pm}(\tilde{\mathcal{F}}|_{q_i}) = 0$ for $0 \leq i \leq m$ and $\delta_{\sim}^{\pm}(\tilde{\mathcal{F}}|_a)$ have nonnegative leading coefficients. Hence,

$$\text{Err}_{l-}(\tilde{\mathcal{F}}|_{q_j}) + \text{Err}_{l+}(\tilde{\mathcal{F}}|_{q_j}) = \text{Err}_l(\tilde{\mathcal{F}}|_{\eta_0}) \quad \forall 0 \leq j \leq m.$$

On the other hand, we have $\text{Err}_{l-}(\tilde{\mathcal{F}}|_{q_0}) = 0$ and $\text{Err}_{l-}(\tilde{\mathcal{F}}|_{q_m}) = \text{Err}_l(\tilde{\mathcal{F}}|_{\eta_0}) \neq 0$. Thus, there exists some i such that $\text{Err}_{l-}(\tilde{\mathcal{F}}|_{q_i}) \prec \text{Err}_{l-}(\tilde{\mathcal{F}}|_{q_{i+1}})$. Then for $a \in \Sigma_{i+1} - \{q_i, q_{i+1}\}$, by Lemma 5.18, we have

$$\begin{aligned} \text{Err}_{l-}(\tilde{\mathcal{F}}|_a) + \text{Err}_{l+}(\tilde{\mathcal{F}}|_a) &= \text{Err}_{l-}(\tilde{\mathcal{F}}|_{q_i}) + \text{Err}_{l+}(\tilde{\mathcal{F}}|_{q_{i+1}}) \\ &\prec \text{Err}_{l-}(\tilde{\mathcal{F}}|_{q_{i+1}}) + \text{Err}_{l+}(\tilde{\mathcal{F}}|_{q_{i+1}}) \\ &= \text{Err}_l(\tilde{\mathcal{F}}|_{\eta_0}). \end{aligned}$$

Thus $\text{Err}(\tilde{\mathcal{F}}|_a) \prec \text{Err}(\tilde{\mathcal{F}}|_{\eta_0})$. Moreover, since the total Hilbert polynomial is constant, we must have $\delta_{\sim}^{\pm}(\tilde{\mathcal{F}}|_a) \succ 0$; in particular, $\text{Aut}(\tilde{\mathcal{F}}|_a)$ is finite on the new bubble $\tilde{\Delta}$.

Since V is smooth, one can pick a curve $S' \subset V$ that contains a , such that the map $S' \rightarrow V \rightarrow S \times \mathbb{A}^1 \rightarrow S$ is finite. Take the map $\xi': S' \rightarrow V \rightarrow \mathbb{A}^{k+1}$ and the quotient pulled back from the universal family. We obtain a quotient ϕ' on $\mathcal{Y}_{S'}$, with $\phi'|_{\eta'} \cong \phi|_{\eta} \times_{\eta} \eta'$ and $\text{Err}(\mathcal{F}'|_{\eta'_0}) = \text{Err}(\tilde{\mathcal{F}}|_a) \prec \text{Err}(\mathcal{F}|_{\eta_0})$, and only finitely many new autoequivalences arise. We repeat this process, and finally we will get $\text{Err}(\mathcal{F}'|_{\eta'_0}) = 0$. $\square$

Step 3. Finite autoequivalences.

LEMMA 5.19. *Consider the family $\Delta \times S \rightarrow S$, viewed as a component of the family $\mathcal{Y}_S$ associated with the map $\xi: S \rightarrow \mathbb{A}^1$, where ξ maps S constantly to $0 \in \mathbb{A}^1$. Let ϕ be an S -flat quotient on $\Delta \times S$ such that $\phi|_{\eta}$ is stable but $\phi|_{\eta_0}$ is $\mathbb{C}^*$ -equivariant. Then there is another S -flat quotient ψ such that $\psi|_{\eta}$ is related to $\phi|_{\eta}$ via the $\mathbb{C}^*$ -action on $Y(1) \rightarrow \mathbb{A}^1$ and $\mathcal{F}|_{\eta_0}$ is normal to D_- .*

Proof. First, note that $\delta_{\Delta}^{\pm} = 0$ by Lemma 5.13. Thus $\text{Err}(\mathcal{F}|_{\eta_0}) \neq 0$; that is, the sheaf must be inadmissible. Suppose that $\mathcal{F}|_{\eta_0}$ is not normal to $D_- \times \eta_0$.

Étale locally around a point in $D_- \times \eta_0$, the product $\Delta \times S$ has an affine local chart $\text{Spec } A := \text{Spec } R[y, \bar{z}]$, where $D_- \times \eta_0$ is defined by the ideal (u, y) . The quotient ϕ is represented by a sequence $0 \rightarrow K \rightarrow A^{\oplus r} \rightarrow M \rightarrow 0$. Assume $K = (f_1(u, y), \dots, f_m(u, y))$, with generators

$$f_i(u, y) = c_i + u^{\alpha_i} g_i(u) + y h_i(u, y) \in A^{\oplus r},$$

where $c_i \in k[\bar{z}]^{\oplus r}$, $g_i \in k[\bar{z}]^{\oplus r}[u]$, $h_i \in k[\bar{z}]^{\oplus r}[u, y]$, $g_i(0) \neq 0$ and $\alpha_i \geq 1$. As in Step 1, one concludes that

$$f_i(u, y) = u^{\alpha_i} g_i(u) + y h_i(u, y) \in A^{\oplus r}.$$

Now, we take some $N \geq \max\{\alpha_i\}$ and apply the $\mathbb{C}^*$ -action induced by

$$\eta \rightarrow \mathbb{C}^*, \quad \mathbb{C}[t_1, t_1^{-1}] \rightarrow K, \quad t_1 \mapsto u^N,$$

which acts as $y \mapsto u^N y$ on Δ .

The family $\phi|_{\eta}$ becomes a new family ψ_{η} represented by $0 \rightarrow K'_u \rightarrow A_u^{\oplus r} \rightarrow M'_u \rightarrow 0$, where the generators become

$$f_i(u, u^N y) = u^{\alpha_i} g_i(u) + u^N y h_i(u, u^N y),$$

and we may also take

$$f'_i(u, y) = g_i(u) + u^{N-\alpha_i} y h_i(u, u^N y)$$

as the generators of K'_u .

Let us look at its flat limit ψ , or a new sequence $0 \rightarrow K' \rightarrow A^{\oplus r} \rightarrow M' \rightarrow 0$. By construction, we have

$$K' = K'_u \cap A^{\oplus r},$$

with generators f'_i . Now, for every i , the generator f'_i is not divisible by y ; that is, the flat limit $\psi|_{\eta_0}$ is normal to $D_- \times \eta_0$. $\square$

LEMMA 5.20. *Let $\Delta \times S$ be given as in Lemma 5.19. Let ϕ be an S -flat quotient on $\Delta \times S$ such that $\phi|_{\eta}$ is stable. Then, there is another S -flat quotient ψ such that $\psi|_{\eta}$ is related to $\phi|_{\eta}$ via the $\mathbb{C}^*$ -action on $Y(1) \rightarrow \mathbb{A}^1$, with $\text{Aut}(\phi|_{\eta_0})$ finite.*

Proof. Suppose that $\text{Aut}(\phi|_{\eta_0})$ is not finite. Then $\phi|_{\eta_0}$ is $\mathbb{C}^*$ -equivariant, and it must be inadmissible. By Lemma 5.19, up to a $\mathbb{C}^*$ -action, we can assume that it is normal to D_+ . By that same lemma, there is another quotient $\psi|_{\eta}$, related to $\phi|_{\eta}$ via a $\mathbb{C}^*$ -action $\lambda: \eta \rightarrow \mathbb{C}^*$, such that $\psi|_{\eta_0}$ is normal to D_- .

We apply an argument similar to Step 2. Consider the 2-dimensional family $\Delta \times S \times S$ and two different embeddings of $\Delta \times S$ into it. The first is the standard one induced by $\{1\} \times S \rightarrow S \times S$, that is, $u \mapsto (1, u)$, and the second is given by $u \mapsto (\lambda \cdot u, 1)$. Let $\mathbb{C}^*$ act on $S \times S$ via multiplication, that is, $\mu \cdot (u, v) := (\lambda \cdot \mu u, \mu^{-1} v)$. Then as in Step 2, there is a $\mathbb{C}^*$ -equivariant quotient on $S \times S - \eta_0 \times \eta_0$ whose restrictions to the two embeddings are $\phi|_{\eta}$ and $\psi|_{\eta}$, respectively.

Again, we pass to a successive blow-up $V \rightarrow S \times S$ and obtain a $\mathbb{C}^*$ -equivariant quotient $\tilde{\phi}$ on $\Delta \times V$. The exceptional fiber $E = \Sigma_1 \cup \dots \cup \Sigma_m$ is a chain of projective lines $\mathbb{P}^1$. Let $\Sigma_0 := \{1\} \times S$ and $\Sigma_{m+1} := S \times \{1\}$. Again, Σ_{m+1} is viewed as twisted by λ . Consider the quotients on these $\Delta \times \Sigma_i$.

For a in Σ_0 or Σ_{m+1} , the fiber $\tilde{\phi}|_a$ is isomorphic to $\phi|_{\eta_0}$ or $\psi|_{\eta_0}$, respectively, whose Err-polynomials concentrate on D_- and D_+ , respectively. By same arguments as in Step 2, one concludes that there are some $1 \leq i \leq m$ and $a \in \Sigma_i^\circ$ such that $\text{Err}(\tilde{\mathcal{F}}|_a) \prec \text{Err}(\mathcal{F}|_{\eta_0})$. Equivalently, this means $\delta_\Delta^\pm(\tilde{\mathcal{F}}|_a) \neq 0$ and that $\text{Aut}(\tilde{\phi}|_a)$ is finite. Again, one picks a curve $S' \subset V$ passing through a which is finite over S and projects it to S . The lemma follows. $\square$

Proof of Theorem 5.14. The proof is a combination of the three steps. As mentioned at the beginning of the subsection, we have a naive extension (ξ, ϕ) with $\xi: S \rightarrow \mathbb{A}^k$, mapping constantly to $0 \in \mathbb{A}^k$, and $\phi: p^*\mathcal{V} \rightarrow \mathcal{F}$ on $\mathcal{Y}_S \cong S \times Y[k]$ such that $\phi|_{\eta}$ is stable but $\phi|_{\eta_0}$ is not necessarily stable. By Lemma 5.20, we can assume that $\text{Aut}(\phi|_{\eta_0})$ is finite. Thus, we just need admissibility.

Let us modify ξ . We have

$$\mathcal{Y}_S \cong S \times Y[k] = (S \times Y) \cup_{S \times D} \dots \cup_{S \times D} (S \times \Delta_k).$$

Apply Lemmas 5.16 and 5.17 to each smooth pair $(\Delta_j, D_{j-1} \cup D_j)$. After a finite base change $S' \rightarrow S$, for each $0 \leq j \leq k$, one can find $\xi': S' \rightarrow \mathbb{A}^{k'}$ and extend $\phi|_{\eta} \times_{\eta} \eta'$ to an admissible quotient ϕ'_i on $\mathcal{Y}_{j,S'}$, where $\mathcal{Y}_{j,S'}$ is the associated family of expanded pairs with respect to the pair $(\Delta_j, D_{j-1} \cup D_j)$. Note that ϕ'_i for adjacent Δ must coincide on the intersecting divisors because of admissibility and the uniqueness of the flat limit. Hence, these quotients glue together to an admissible quotient (ξ', ϕ') on $\mathcal{Y}_{S'}$, and $\phi'|_{\eta'} \cong \phi|_{\eta} \times_{\eta} \eta'$.

During each step, the autoequivalence group remains finite, and for the base changes we can take the fiber products and pass to a common finite base change. The resulting $\phi'|_{\eta'}$ is stable. $\square$

6. Orbifold Donaldson–Thomas theory

In this section, we consider the (absolute) Donaldson–Thomas theory on 3-dimensional smooth Deligne–Mumford stacks. Let S be a scheme, let $\pi: W \rightarrow S$ be a *smooth* family of projective Deligne–Mumford stacks over $\mathbb{C}$, of relative dimension 3, and let $c: W \rightarrow \underline{W}$ be the coarse moduli space. Let $\mathcal{E}$ be a generating sheaf on W , and let $H = c^*\underline{H}$ be a pull-back of a relatively ample line bundle $\underline{H}$ on $\underline{W} \rightarrow S$.

Let $P: K(W) \rightarrow \mathbb{Z}$ be a group homomorphism. We require that the associated Hilbert polynomial $P(H^{\otimes v})$ has degree not greater than 1. One can form the Hilbert scheme $\mathcal{M} := \text{Hilb}_{W/S}^P$ parameterizing certain closed substacks of W over S . Then $\mathcal{M}$ is a projective scheme over S by [OS03, Theorem 4.4]. The following lemma gives another description of $\mathcal{M}$.

LEMMA 6.1. *The Hilbert scheme $\mathcal{M}$ is the fine moduli space of torsion-free coherent sheaves $\mathcal{I}$ on W with relative Hilbert homomorphism $P_{\mathcal{I}} = P_{\mathcal{O}_W} - P$ and $\det \mathcal{I}$ trivial.*

Proof. We sketch the idea of the proof. Let $\mathcal{M}'$ denote the functor parameterizing such sheaves on W ; that is, given any $T \rightarrow S$, the category $\mathcal{M}'(T)$ consists of T -flat families $\mathcal{I}_T$ of coherent sheaves on $T \times_S W$ whose fibers are torsion free, have Hilbert homomorphism $P_{\mathcal{O}_W} - P$ and trivial determinant. There is a natural transformation $\mathcal{M} \rightarrow \mathcal{M}'$ sending a flat family of closed substacks to its ideal sheaf. This is obviously fully faithful. One can check that this is a bijection over closed points by embedding a torsion-free sheaf into its double dual and applying [OSS80, Lemma 1.1.15]. For the proof in families, one can [Kol90, Lemma 6.13]. $\square$

In the following, we define a perfect relative obstruction theory on $\mathcal{M}$ over S , in the sense of [BF97]. The construction follows from the idea of D. Huybrechts and R. P. Thomas [HT10] and is already adopted by A. Gholampour and H.-H. Tseng in [GT13].

Let $p: \mathcal{M} \times_S W \rightarrow \mathcal{M}$ and $q: \mathcal{M} \times_S W \rightarrow W$ be the projections, and let $\mathcal{Z} \subset \mathcal{M} \times_S W$ be the universal family of the Hilbert scheme. Let $\mathbb{I} \subset \mathcal{O}_{\mathcal{M} \times_S W}$ be the universal ideal sheaf. We have maps (in the derived category) $\text{Id}: \mathcal{O}_{\mathcal{M} \times_S W} \rightarrow R\mathcal{H}om(\mathbb{I}, \mathbb{I})$ and $\text{tr}: R\mathcal{H}om(\mathbb{I}, \mathbb{I}) \rightarrow \mathcal{O}_{\mathcal{M} \times_S W}$ which give a splitting of $R\mathcal{H}om(\mathbb{I}, \mathbb{I})$.

Let $L_{\mathcal{M} \times_S W/S}^\bullet$ be Illusie’s cotangent complex of $\mathcal{M} \times_S W \rightarrow S$. We have the Atiyah class

$$\text{At}_{\mathcal{M} \times_S W/S}(\mathbb{I}) \in \text{Ext}_{\mathcal{M} \times_S W}^1(\mathbb{I}, \mathbb{I} \otimes L_{\mathcal{M} \times_S W/S}^\bullet),$$

which can be seen as a map $\mathbb{I} \rightarrow \mathbb{I} \otimes L_{\mathcal{M} \times_S W/S}^\bullet[1]$. By functoriality, if we compose it with the natural projection $L_{\mathcal{M} \times_S W/S}^\bullet \rightarrow p^*L_{\mathcal{M}/S}^\bullet$, we get the Atiyah class of $\mathbb{I}$ as a module over $\mathcal{M}$,

$$\text{At}_{\mathcal{M} \times_S W/W}(\mathbb{I}) \in \text{Ext}_{\mathcal{M} \times_S W}^1(\mathbb{I}, \mathbb{I} \otimes p^*L_{\mathcal{M}/S}^\bullet).$$

After restriction to the traceless part, we get $R\mathcal{H}om(\mathbb{I}, \mathbb{I})_0 \rightarrow p^*L_{\mathcal{M}/S}^\bullet[1]$. We then tensor with the dualizing sheaf $R\mathcal{H}om(\mathbb{I}, \mathbb{I})_0 \otimes q^*\omega_{W/S} \rightarrow p^*L_{\mathcal{M}/S}^\bullet \otimes q^*\omega_{W/S}[1] \cong p^!L_{\mathcal{M}/S}^\bullet[-2]$, where the last isomorphism is the Serre duality for Deligne–Mumford stacks [Nir08b]. We have obtained the map

$$\Phi: E^\bullet := Rp_*(R\mathcal{H}om(\mathbb{I}, \mathbb{I})_0 \otimes q^*\omega_{W/S})[2] \rightarrow L_{\mathcal{M}/S}^\bullet.$$

THEOREM 6.2. *The map $\Phi: E^\bullet \rightarrow L_{\mathcal{M}/S}^\bullet$ is a perfect relative obstruction theory on $\mathcal{M}$ over S , in the sense of [BF97].*

Proof. First, let us prove that Φ is an obstruction theory. Consider a square-zero extension $T \hookrightarrow \bar{T}$ of schemes with ideal sheaf J and a map $g: T \rightarrow \mathcal{M}$. The canonical map

$$g^*L_{\mathcal{M}/S}^\bullet \rightarrow L_{T/S}^\bullet \rightarrow L_{T/\bar{T}}^\bullet \rightarrow \tau^{\geq -1}L_{T/\bar{T}}^\bullet \cong J[1]$$

gives an element $\omega(g) \in \text{Ext}^1(g^*L_{\mathcal{M}/S}^\bullet, J)$ which is the obstruction to extending the map g to $\bar{T}$. Composing with Φ , we get $\Phi^*\omega(g) \in \text{Ext}^1(g^*E^\bullet, J)$.

By the relative version of [BF97, Theorem 4.5], it suffices to prove that $\Phi^*\omega(g)$ is also an obstruction to the extension of g to $\bar{T}$, that is, $\Phi^*\omega(g) = 0$ if and only if an extension $\bar{g}$ of g to $\bar{T}$ exists, and in that case all extensions form a torsor under $\text{Ext}^0(g^*E^\bullet, J)$.

Denote by p_T and q_T the corresponding projections from $T \times_S W$ to T and W , respectively. By the construction above, $\Phi^*\omega(g)$ is the composite of

$$g^*\Phi: g^*E^\bullet = R p_{T*}(R\mathcal{H}om(g^*\mathbb{I}, g^*\mathbb{I})_0 \otimes q_T^*\omega_{W/S})[2] \rightarrow g^*L_{\mathcal{M}/S}^\bullet,$$

with $g^*L_{\mathcal{M}/S}^\bullet \rightarrow L_{T/S}^\bullet$, and the Kodaira–Spencer map

$$\kappa(T/\bar{T}/S): L_{T/S}^\bullet \rightarrow L_{T/\bar{T}}^\bullet \rightarrow \tau^{\geq -1}L_{T/\bar{T}}^\bullet \cong J[1].$$

By functoriality and Serre duality, the first composition

$$g^*E^\bullet = R p_{T*}(R\mathcal{H}om(g^*\mathbb{I}, g^*\mathbb{I})_0 \otimes q_T^*\omega_{W/S})[2] \rightarrow g^*L_{\mathcal{M}/S}^\bullet \rightarrow L_{T/S}^\bullet$$

is just the traceless part of the Atiyah class

$$\text{At}_{T \times_S W/W}(g^*\mathbb{I}) \in \text{Ext}_{T \times_S W}^1(g^*\mathbb{I}, g^*\mathbb{I} \otimes p_T^*L_{T/S}^\bullet).$$

Hence, viewed as an element in the Ext^2 group below, the element

$$\Phi^*\omega(g) \in \text{Ext}_T^{-1}(R p_{T*}(R\mathcal{H}om(g^*\mathbb{I}, g^*\mathbb{I})_0 \otimes q_T^*\omega_{W/S}), J) \cong \text{Ext}_{T \times_S W}^2(g^*\mathbb{I}, g^*\mathbb{I} \otimes p_T^*J)_0$$

is just the product of the traceless Atiyah class and Kodaira–Spencer class

$$p_T^*\kappa(T/\bar{T}/S) \circ \text{At}_{T \times_S W/W}(g^*\mathbb{I}).$$

We claim that this is an obstruction as desired. Note that (as in the proof of [BF97, Theorem 4.5]) it suffices to work under the further assumption that T is affine. Now, we can apply Proposition 3.1.8 of [Ill71], which says that the obstruction class to extending $\mathbb{I}$ from T to $\bar{T}$, as a coherent sheaf, is exactly the product of Atiyah class and Kodaira–Spencer class. Now, similar arguments as in [Tho00] show that the traceless part is exactly the obstruction with $\det \mathbb{I}$ fixed.

By Lemma 6.1, a map $g: T \rightarrow \mathcal{M}$ corresponds exactly to an ideal sheaf given by the pull-back of the universal ideal sheaf $g^*\mathbb{I}$, and g extends to $\bar{T}$ if and only if $g^*\mathbb{I}$ extends in $\mathcal{M}$. As an open condition, the torsion-free condition poses no restriction on possible deformations. Thus, $\Phi^*\omega(g)$ gives precisely the obstruction to the extension of g , and Φ is an obstruction theory.

It remains to prove that Φ is perfect. The argument is completely the same as in [HT10] and [PT09]. It suffices to prove that $R p_* R\mathcal{H}om(\mathbb{I}, \mathbb{I})_0$ is quasi-isomorphic to a perfect complex with amplitude in $[1, 2]$. Pick a finite complex of locally free sheaves $A^\bullet$ resolving $R\mathcal{H}om(\mathbb{I}, \mathbb{I})_0$ such that $R^i p_* A^j = 0$ for all $i \neq 0$ and all j . Then each $F^j := p_* A^j$ is locally free and the complex $F^\bullet$ is a resolution of $R p_* R\mathcal{H}om(\mathbb{I}, \mathbb{I})_0$.

By the following Lemma 6.3 and the cohomology and base-change theorem, $F^\bullet$ has nontrivial cohomology only in degrees 1 and 2. Suppose that F^j with $j > 2$ is the last nonzero term of $F^\bullet$. We can replace $F^{j-1} \rightarrow F^j \rightarrow 0$ by $\ker(F^{j-1} \rightarrow F^j) \rightarrow 0 \rightarrow 0$. We can do likewise with the nonzero terms before 1. Finally, one obtains a two-term perfect complex. $\square$

We still need the following lemma to finish the proof.

LEMMA 6.3. *Let W be a 3-dimensional smooth projective Deligne–Mumford stack over $\mathbb{C}$, and let $Z \subset W$ be a 0- or 1-dimensional closed substack, with ideal sheaf $\mathcal{I}$. Then $\mathrm{Ext}_W^i(\mathcal{I}, \mathcal{I})_0 = 0$ for all $i \neq 1, 2$.*

Proof. It suffices to show $\mathrm{Ext}_W^i(\mathcal{I}, \mathcal{I})_0 = 0$ for $i = 0, 3$ or, in other words, $\mathrm{Ext}_W^i(\mathcal{I}, \mathcal{I}) = H^i(W, \mathcal{O}_W)$ for $i = 0, 3$. For $0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_W \rightarrow \mathcal{O}_Z \rightarrow 0$, we have the long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathcal{H}\mathrm{om}_W(\mathcal{O}_Z, \mathcal{I}) \longrightarrow \mathcal{H}\mathrm{om}_W(\mathcal{O}_Z, \mathcal{O}_W) \longrightarrow \mathcal{H}\mathrm{om}_W(\mathcal{O}_Z, \mathcal{O}_Z) \\ \longrightarrow \mathcal{E}\mathrm{xt}_W^1(\mathcal{O}_Z, \mathcal{I}) \longrightarrow \mathcal{E}\mathrm{xt}_W^1(\mathcal{O}_Z, \mathcal{O}_W), \end{aligned}$$

where $\mathcal{H}\mathrm{om}_W(\mathcal{O}_Z, \mathcal{O}_W) = \mathcal{E}\mathrm{xt}_W^1(\mathcal{O}_Z, \mathcal{O}_W) = 0$ since $\mathrm{codim} Z \geq 2$ (see [HL10, Proposition 1.1.6]). Hence, $\mathcal{H}\mathrm{om}_W(\mathcal{O}_Z, \mathcal{I}) = 0$ and $\mathcal{E}\mathrm{xt}_W^1(\mathcal{O}_Z, \mathcal{I}) = \mathcal{O}_Z$. Then, we look at

$$\begin{aligned} 0 = \mathcal{H}\mathrm{om}_W(\mathcal{O}_Z, \mathcal{I}) \longrightarrow \mathcal{H}\mathrm{om}_W(\mathcal{O}_W, \mathcal{I}) \longrightarrow \mathcal{H}\mathrm{om}_W(\mathcal{I}, \mathcal{I}) \\ \longrightarrow \mathcal{E}\mathrm{xt}_W^1(\mathcal{O}_Z, \mathcal{I}) \longrightarrow \mathcal{E}\mathrm{xt}_W^1(\mathcal{O}_W, \mathcal{I}) = 0, \end{aligned}$$

Hence, $\mathcal{H}\mathrm{om}_W(\mathcal{I}, \mathcal{I})$ differs with $\mathcal{I}$ by a sheaf $\mathcal{O}_Z$ of codimension 2, and we have $\mathrm{Hom}(\mathcal{I}, \mathcal{I}) = H^0(W, \mathcal{I}) = H^0(W, \mathcal{O}_W)$. The last equality is because $\mathcal{I}$ coincides with $\mathcal{O}_W$ up to codimension 2.

Tensoring the sequence above with ω_W , we see that $\mathcal{H}\mathrm{om}_W(\mathcal{I}, \mathcal{I} \otimes \omega_W)$ coincides with $\mathcal{I} \otimes \omega_W$, and hence ω_W , up to codimension 2. Thus, by Serre duality, $\mathrm{Ext}_W^3(\mathcal{I}, \mathcal{I}) = \mathrm{Hom}_W(\mathcal{I}, \mathcal{I} \otimes \omega_W)^\vee = H^0(W, \mathcal{I} \otimes \omega_W)^\vee = H^0(W, \omega_W)^\vee = H^3(W, \mathcal{O}_W)$. $\square$

COROLLARY 6.4. *There is a virtual fundamental class $[\mathcal{M}]^{\mathrm{vir}} \in A_{\mathrm{vdim} + \dim S}(\mathcal{M})$ of virtual dimension $\mathrm{vdim} = \mathrm{rk} E^\bullet$.*

The following gives the deformation invariance of the perfect obstruction theory.

PROPOSITION 6.5. *Assume that the base S is smooth of constant dimension. For each closed point $s \in S$, the pull-back $i_s^* E^\bullet$ is a perfect (absolute) obstruction theory on the fiber $\mathcal{M}_s$ and $[\mathcal{M}_s]^{\mathrm{vir}} = i_s^! [\mathcal{M}]^{\mathrm{vir}}$.*

Proof. Apply [BF97, Proposition 7.2]. $\square$

Following [Edi13] and [Tse10, Appendix A], we introduce the following notion of Chern character. Let $\mathcal{X}$ be a smooth proper Deligne–Mumford stack, and let $I\mathcal{X}$ be the inertia stack. Connected components of $I\mathcal{X}$ are gerbes over their coarse moduli spaces. A vector bundle V on $I\mathcal{X}$ splits into a direct sum of eigenbundles $\bigoplus_\zeta V^{(\zeta)}$ of the gerbe actions, where $V^{(\zeta)}$ has eigenvalue ζ .

DEFINITION 6.6. Define $\rho: K(I\mathcal{X}) \rightarrow K(I\mathcal{X})_{\mathbb{C}}$ as

$$\rho(V) := \sum_{\zeta} \zeta V^{(\zeta)} \in K(I\mathcal{X})_{\mathbb{C}}.$$

Define $\tilde{\mathrm{ch}}: K(\mathcal{X})_{\mathbb{Q}} \rightarrow A^*(I\mathcal{X})_{\mathbb{C}}$ as

$$\tilde{\mathrm{ch}}(V) := \mathrm{ch}(\rho(\pi^* V)),$$

where $\pi: I\mathcal{X} \rightarrow \mathcal{X}$ is the usual projection and ch is the usual Chern character.

It is easy to see that $\widetilde{\text{ch}}$ is a ring homomorphism. If $\mathcal{X}$ is furthermore projective and thus satisfies the resolution property, then by the splitting principle, one can check that

$$\widetilde{\text{ch}}(V^\vee) = \widetilde{\text{ch}}^\dagger(V) := \overline{\text{ch}^\vee(\rho(\pi^*V))},$$

where $\text{ch}^\vee$ means the usual dual of the Chern character, and the bar over it means the conjugate with respect to the natural real structure $K(I\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{R} \subset K(I\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{C}$.

Now, for a 3-dimensional smooth projective Deligne–Mumford stack W over $\mathbb{C}$, we can define the Donaldson–Thomas invariants. Again, let $\mathcal{M} = \text{Hilb}_W^P$, with P fixed as above. Consider $p: \mathcal{M} \times W \rightarrow \mathcal{M}$, $q: \mathcal{M} \times W \rightarrow W$, the universal family $\mathcal{Z} \subset \mathcal{M} \times W$ and the universal ideal sheaf $\mathbb{I} \subset \mathcal{O}_{\mathcal{M} \times W}$.

The product $\mathcal{M} \times W$ is projective, thus $\mathbb{I}$ admits locally free resolutions of finite length and the Chern character $\text{ch}(\mathbb{I}) \in A^*(\mathcal{M} \times W)$ in the operational cohomology. Similarly, on inertia stacks, we have the modified Chern character $\widetilde{\text{ch}}(\mathbb{I}) \in A^*(\mathcal{M} \times IW)$.

Consider the orbifold or Chen–Ruan cohomology A_{orb}^* as in [AGV08, Tse10], defined as

$$A_{\text{orb}}^*(W) := \bigoplus_i A^{*- \text{age}_i}(W_i),$$

where age_i is the degree shift number and W_i is a connected component of IW . Then, we can define an orbifold version of the Chern character operator

$$\widetilde{\text{ch}}^{\text{orb}}(\mathbb{I}) \in A_{\text{orb}}^*(\mathcal{M} \times W)$$

just by shifting the degrees.

Given $\gamma \in A_{\text{orb}}^l(W)$, define the operator

$$\widetilde{\text{ch}}_{k+2}^{\text{orb}}(\gamma): A_*(\mathcal{M}) \rightarrow A_{*-k+1-l}(\mathcal{M})$$

as

$$\widetilde{\text{ch}}_{k+2}^{\text{orb}}(\gamma)(\xi) := p_*(\widetilde{\text{ch}}_{k+2}^{\text{orb}}(\mathbb{I}) \cdot \iota^* q^* \gamma \cap p^* \xi),$$

where p and q here are the projections on $\mathcal{M} \times IW$ and $\iota: IW \rightarrow IW$ is the canonical involution map. Note that the orbifold degrees match well thanks to the identity

$$\text{age}_i + \text{age}_{\iota(i)} = \text{codim}(W_i, W).$$

DEFINITION 6.7. For $\gamma_i \in A_{\text{orb}}^*(W)$ with $1 \leq i \leq r$, define the Donaldson–Thomas invariants with descendents as

$$\left\langle \prod_{i=1}^r \tau_{k_i}(\gamma_i) \right\rangle_W^P := \deg \left[\prod_{i=1}^r \widetilde{\text{ch}}_{k_i+2}^{\text{orb}}(\gamma_i) \cdot [\mathcal{M}]^{\text{vir}} \right]_0 \in \mathbb{C}.$$

If the dimensions do not match, we simply define the invariant to be 0.

Note that when $r = 0$, that is, there are no insertions, the invariants are integers.

7. Degeneration formula: cycle version

In this section, we consider the Donaldson–Thomas theory on Hilbert stacks of the moduli of simple degenerations and relative pairs. This will lead to a degeneration formula.

7.1 Modified versions of the stacks and decomposition of central fibers

Let $\pi: X \rightarrow C$ be a locally simple degeneration, where X is a finite-type separated Deligne–Mumford stack over $\mathbb{C}$, with central fiber $X_0 = Y_- \cup_D Y_+$. Recall that we have the stacks $\mathfrak{C}$ parameterizing expanded degenerations, $\mathfrak{X}$ the universal family, and $\mathfrak{A}$ and $\mathfrak{Y}$ in the relative case. There is $\pi: \mathfrak{X} \rightarrow \mathfrak{C}$, where the stacks are defined as the limits of $[X(k)/R_{\sim, X(k)}]$ and $[\mathbb{A}^{k+1}/R_{\sim, \mathbb{A}^{k+1}}]$.

We define a stack $\mathfrak{X}_0^\dagger$ as follows. Let $H_i \subset \mathbb{A}^{k+1}$ be the hyperplane defined by $t_i = 0$. Recall that we have $p: X(k) \rightarrow \mathbb{A}^{k+1}$ and by Proposition 2.18,

$$X(k)|_{H_i} \cong (Y_-(i) \times \mathbb{A}^{k-i}) \cup_{\mathbb{A}^i \times D \times \mathbb{A}^{k-i}} (\mathbb{A}^i \times Y_+(k-i)^\circ). \quad (7.1)$$

The equivalence relation restricts naturally to H_i and $X(k)|_{H_i}$. Consider the stacks

$$\mathfrak{C}_0^\dagger := \varinjlim \left[\prod_{i=0}^k H_i / R_{\sim, \coprod H_i} \right], \quad \mathfrak{X}_0^\dagger := \varinjlim \left[\prod_{i=0}^k X(k)|_{H_i} / R_{\sim, \coprod X(k)|_{H_i}} \right].$$

Remark 7.1. The stack $\mathfrak{X}_0^\dagger$ parameterizes all families of expanded degenerations with singular fibers, with one distinguished nodal divisor D_i . More precisely, for any $\mathbb{A}^1$ -map $S \rightarrow H_i \subset \mathbb{A}^{k+1}$, the associated family $\mathcal{X}_S$ has a decomposition $\mathcal{Y}_{S,-} \cup_{\mathcal{D}_S} \mathcal{Y}_{S,+}$.

To keep track of the Hilbert homomorphisms of the fibers, we need to introduce the weighted version of the classifying stacks.

DEFINITION 7.2 ([LW15, Definition 2.14]). Let $\Lambda := \text{Hom}(K(X), \mathbb{Z})$. Consider $X_0[k] = \cup_{i=0}^{k+1} \Delta_i$, where $\Delta_0 = Y_-$ and $\Delta_{k+1} = Y_+$.

- 1) A *weight assignment* on $X_0[k]$ is a function

$$w: \{\Delta_0, \dots, \Delta_{k+1}, D_1, \dots, D_k\} \rightarrow \Lambda$$

such that $w(\Delta_i) \neq 0$ for all $i \in \Lambda$.

- 2) For any $0 \leq a \leq b \leq k+1$, the *total weight* of the segment $\cup_{i=a}^b \Delta_i$ is defined as

$$w(\cup_{i=a}^b \Delta_i) := \sum_{i=a}^b w(\Delta_i) - \sum_{i=a+1}^b w(D_i).$$

- 3) For a family of expanded degeneration $\pi: \mathcal{X}_S \rightarrow S$, a *continuous weight assignment* on $X(k)$ is an assignment of a weight function on each fiber that is continuous over S .

More precisely, if for some curve $C \subset S$ and $s_0, s \in C$, the general fiber $\mathcal{X}_s \cong X_0[m]$ specializes to the special fiber $\mathcal{X}_{s_0} \cong X_0[n]$, with $m \leq n$, in which $\Delta_i \subset \mathcal{X}_s$ specializes to $\cup_{j=a}^b \Delta_j \subset \mathcal{X}_{s_0}$, then one must have $w_s(\Delta_i) = w_{s_0}(\cup_{j=a}^b \Delta_j)$.

Remark 7.3. An alternative definition is as follows. Firstly, for a standard family $X(k) \rightarrow \mathbb{A}^{k+1}$, one specifies the weight assignment on the central fiber $X_0[k]$. Then for any $s \in L_i$, where $L_i \subset \mathbb{A}^{k+1}$ is the coordinate line corresponding to t_i , define

$$w_s(\Delta_j) = \begin{cases} w_0(\Delta_i) + w_0(\Delta_{i+1}) - w_0(D_i) & \text{if } j = i, \\ w_0(\Delta_j) & \text{if } j \neq i. \end{cases}$$

For s in lower strata, one can choose a slice in the base and define inductively. For a general family $\mathcal{X}_S \rightarrow S$, just take the map $S \rightarrow \mathbb{A}^{k+1}$ for some k and pull back the weight assignment on the standard family $X(k)$.

Let $P \in \Lambda$ be fixed. We define $\mathfrak{C}^P$ to be the stack parameterizing weighted families of expanded degenerations, with total weight P , and $\mathfrak{X}^P$ to be the universal family. More precisely, let $(\mathbb{A}^{k+1})^P$ be the disjoint union of all $\mathbb{A}^{k+1}$ indexed by all possible continuous weight assignments, and let $X(k)^P$ be the universal family. Let $R_{\sim, \mathbb{A}^{k+1}}^P$ be the equivalence relation generated by the original relations on each copy of $\mathbb{A}^{k+1}$ and identifications between different copies respecting weight assignments. Then we can define

$$\mathfrak{C}^P := \varinjlim [(\mathbb{A}^{k+1})^P / R_{\sim, \mathbb{A}^{k+1}}^P], \quad \mathfrak{X}^P := \varinjlim [X(k)^P / R_{\sim, X(k)}^P].$$

Similarly, we can define $\mathfrak{X}_0^{\dagger, P} \rightarrow \mathfrak{C}_0^{\dagger, P}$, the weighted version of $\mathfrak{X}_0^\dagger \rightarrow \mathfrak{C}_0^\dagger$.

Let Λ_P^{spl} be the set of splitting data

$$\Lambda_P^{\text{spl}} := \{\theta = (\theta_-, \theta_+, \theta_0) \mid \theta_\pm, \theta_0 \in \Lambda, \theta_- + \theta_+ - \theta_0 = P\}.$$

Given $\theta \in \Lambda_P^{\text{spl}}$, let $\mathfrak{C}_0^{\dagger, \theta} \subset \mathfrak{C}_0^{\dagger, P}$ be the open and closed substacks parameterizing those families whose Hilbert homomorphism splits according to the datum θ on the fiber decompositions $\mathcal{Y}_{S,-} \cup_{\mathcal{D}_S} \mathcal{Y}_{S,+}$.

Similarly, in the relative case, given a smooth pair (Y, D) and $P \in \text{Hom}(K(Y), \mathbb{Z})$, we also have the stacks $\mathfrak{A}^P$ and $\mathfrak{Y}^P$. Given (P, P') with $P' \prec P$, one can also define the substacks $\mathfrak{A}^{P, P'}$ and $\mathfrak{Y}^{P, P'}$ parameterizing those families whose Hilbert homomorphism is P on each fiber and P' on the distinguished divisor of each fiber. The following splitting result follows from (7.1).

PROPOSITION 7.4 ([LW15, Proposition 2.20]). *Given $\theta \in \Lambda_P^{\text{spl}}$, we have the following isomorphisms:*

$$\begin{array}{ccc} (\mathfrak{Y}_-^{\theta_-, \theta_0} \times \mathfrak{A}^{\theta_+, \theta_0}) \cup_{\mathfrak{A}^{\theta_-, \theta_0} \times D \times \mathfrak{A}^{\theta_+, \theta_0}} (\mathfrak{A}^{\theta_-, \theta_0} \times (\mathfrak{Y}_+^\circ)^{\theta_+, \theta_0}) & \xrightarrow{\cong} & \mathfrak{X}_0^{\dagger, \theta} \\ \downarrow & & \downarrow \\ \mathfrak{A}^{\theta_-, \theta_0} \times \mathfrak{A}^{\theta_+, \theta_0} & \xrightarrow{\cong} & \mathfrak{C}_0^{\dagger, \theta}. \end{array}$$

The next proposition describes the relationship between these stacks.

PROPOSITION 7.5 ([LW15, Proposition 2.19]). *Given $\theta \in \Lambda_P^{\text{spl}}$, there is a pair (L_θ, s_θ) , with L_θ a line bundle on $\mathfrak{C}^P$ and s_θ a section of L_θ , such that*

1) *we have*

$$\bigotimes_{\theta \in \Lambda_P^{\text{spl}}} L_\theta \cong \mathcal{O}_{\mathfrak{C}^P}, \quad \prod_{\theta \in \Lambda_P^{\text{spl}}} s_\theta = \pi^* t,$$

where $\pi: \mathfrak{C}^P \rightarrow \mathbb{A}^1 = \text{Spec } \mathbb{C}[t]$ is the canonical map;

2) *the stack $\mathfrak{C}_0^{\dagger, \theta}$ is the closed substack in $\mathfrak{C}^P$ defined by $(s_\theta = 0)$.*

Proof. The stack $\mathfrak{C}^P$ has an étale covering consisting of $U_k := [(\mathbb{A}^{k+1})^P / R_{\sim, \mathbb{A}^{k+1}}^P]$. It suffices to specify L_θ on each chart.

Connected components of $(\mathbb{A}^{k+1})^P$ are indexed by various copies of $\mathbb{A}^{k+1}$ equipped with different continuous weight assignments w whose total weight is P . We denote such a copy by $(\mathbb{A}^{k+1}, w)$. Restricted to $H_i \subset \mathbb{A}^{k+1}$, the family $X(k)$ is of the form

$$\mathcal{Y}_{-,i} \cup_{\mathcal{D}_i} \mathcal{Y}_{+,i} := (Y_-(i) \times \mathbb{A}^{k-i}) \cup_{\mathbb{A}^i \times D \times \mathbb{A}^{k-i}} (\mathbb{A}^i \times Y_+(k-i)^\circ).$$

On each $\mathcal{Y}_{\pm,i}$, for any $s \in H_i$, the weight assignment w gives a splitting datum

$$\theta_{i,\pm} := w_s(\mathcal{Y}_{\pm,i}), \quad \theta_{i,0} := w_s(\mathcal{D}_i),$$

which is locally constant in s . They satisfy $\theta_{i,-} + \theta_{i,+} - \theta_{i,0} = w_s(X(k)) = P$, that is, $(\theta_{i,\pm}, \theta_{i,0}) \in \Lambda_P^{\text{spl}}$. Moreover, one can easily check that $(\theta_{i,\pm}, \theta_{i,0}) \neq (\theta_{j,\pm}, \theta_{j,0})$ for $i \neq j$.

We define the line bundles as follows. Given a fixed $\theta \in \Lambda_P^{\text{spl}}$ on the connected component $(\mathbb{A}^{k+1}, w)$, if $(\theta_{i,\pm}, \theta_{i,0}) \neq \theta$ for any i , then let $L_\theta|_{(\mathbb{A}^{k+1}, w)} := \mathcal{O}_{\mathbb{A}^{k+1}}$ and $s_\theta|_{(\mathbb{A}^{k+1}, w)} := 1$. Otherwise, if $(\theta_{i,\pm}, \theta_{i,0}) = \theta$ for some i , then let $L_\theta|_{(\mathbb{A}^{k+1}, w)} := \mathcal{O}_{\mathbb{A}^{k+1}}(H_i)$, and let $s_\theta|_{(\mathbb{A}^{k+1}, w)}$ be the image of 1 under the canonical map $\mathcal{O} \rightarrow \mathcal{O}(H_i)$.

It is clear that this defines an $R_{\sim, \mathbb{A}^{k+1}}^P$ -invariant line bundle on $(\mathbb{A}^{k+1})^P$ and thus a line bundle on U_k . One can check that these data actually define line bundles with sections (L_θ, s_θ) on $\mathfrak{C}^P$ satisfying the properties stated in the proposition. Note that $\mathcal{O}(H_i)$ is trivial as a line bundle on $\mathbb{A}^{k+1}$. But written in this way, it is clear how one can glue those data together to form a global line bundle. $\square$

Now, we assume furthermore that $\pi: X \rightarrow \mathbb{A}^1$ is a family of *projective* Deligne–Mumford stacks, with $c: X \rightarrow \underline{X}$ the coarse moduli space and $\mathcal{E}, H$ a fixed polarization. Let $P: K(X) \rightarrow \mathbb{Z}$ be a group homomorphism such that the associated Hilbert polynomial $P(H^{\otimes v})$ has degree at most 1. Let $\mathcal{V}$ be a vector bundle of finite rank on X .

In previous sections, we have defined the Quot-stack $\mathbf{Quot}_{\mathfrak{X}/\mathfrak{C}}^{\mathcal{V}, P}$ parameterizing stable quotients on the universal family $\mathfrak{X} \rightarrow \mathfrak{C}$, with Hilbert homomorphism P . This is a proper Deligne–Mumford stack over $\mathbb{A}^1$ by results of Section 5. One has the map to the base $\mathbf{Quot}_{\mathfrak{X}/\mathfrak{C}}^{\mathcal{V}, P} \rightarrow \mathfrak{C}^P$, defined by the usual map to the base and the weight assignments induced by the stable quotients.

Similarly, we have the proper Deligne–Mumford stack $\mathbf{Quot}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}^{\mathcal{V}, P}$. Given $\theta \in \Lambda_P^{\text{spl}}$, let

$$\mathbf{Quot}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}^{\mathcal{V}, \theta} \subset \mathbf{Quot}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}^{\mathcal{V}, P}$$

be the open and closed substacks parameterizing those stable quotients whose Hilbert homomorphism splits in the type θ on the fiber decomposition.

In the relative case, given a smooth pair (Y, D) , we also have the proper Deligne–Mumford stack $\mathbf{Quot}_{\mathfrak{Y}/\mathfrak{A}}^{\mathcal{V}, P}$. Given (P, P') with $P' \preceq P$, one can also define the substack $\mathbf{Quot}_{\mathfrak{Y}/\mathfrak{A}}^{\mathcal{V}, P, P'}$ parameterizing those stable quotients whose Hilbert homomorphism is P on each fiber and is P' on the distinguished divisor of each fiber.

Now, the splitting results naturally lead to a morphism

$$\begin{array}{ccc} \mathbf{Quot}_{\mathfrak{Y}_-/\mathfrak{A}}^{\mathcal{V}, \theta_-, \theta_0} \times_{\mathbf{Quot}_D^{\mathcal{V}, \theta_0}} \mathbf{Quot}_{\mathfrak{Y}_+/\mathfrak{A}}^{\mathcal{V}, \theta_+, \theta_0} & \xrightarrow{\Phi_\theta} & \mathbf{Quot}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}^{\mathcal{V}, \theta} \\ \downarrow & & \downarrow \\ \mathfrak{A}^{\theta_-, \theta_0} \times \mathfrak{A}^{\theta_+, \theta_0} & \xrightarrow{\cong} & \mathfrak{C}_0^{\dagger, \theta}, \end{array}$$

where we just glue two families of expanded pairs by (7.1) to obtain an object in $\mathfrak{X}_0^\dagger$ and the stable quotients also glue together since the sheaves are admissible. One can easily see that the following holds.

PROPOSITION 7.6. *The morphism Φ_θ is an isomorphism.*

We have the following results, whose proof is essentially the same as before.

PROPOSITION 7.7 ([LW15, Theorem 5.27]). *For $\theta \in \Lambda_P^{\text{spl}}$, let (L_θ, s_θ) be the line bundle and section on $\mathfrak{C}$ defined earlier. Consider the map $F: \mathbf{Quot}_{\mathfrak{X}/\mathfrak{C}}^{\mathcal{V}, P} \rightarrow \mathfrak{C}^P$.*

1) We have

$$\bigotimes_{\theta \in \Lambda_P^{\text{spl}}} F^* L_\theta \cong \mathcal{O}_{\text{Quot}_{\mathfrak{X}/\mathfrak{C}}^{\mathcal{V}, P}}, \quad \prod_{\theta \in \Lambda_P^{\text{spl}}} F^* s_\theta = F^* \pi^* t,$$

where $\pi: \mathfrak{C}^P \rightarrow \mathbb{A}^1 = \text{Spec } \mathbb{C}[t]$ is the canonical map.

2) The stack $\text{Quot}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}^{\mathcal{V}, \theta}$ is the closed substack in $\text{Quot}_{\mathfrak{X}/\mathfrak{C}}^{\mathcal{V}, P}$ defined by $(F^* s_\theta = 0)$.

7.2 Perfect obstruction theory on Hilbert stacks

From now on, we make the further assumption that $\pi: X \rightarrow \mathbb{A}^1$ is of relative dimension 3 and take $\mathcal{V} = \mathcal{O}_X$. Then, we have the Hilbert stack

$$\mathcal{M}^P := \text{Quot}_{\mathfrak{X}/\mathfrak{C}}^{\mathcal{O}_X, P}.$$

Similarly, we denote

$$\mathcal{M}^\theta := \text{Quot}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}^{\mathcal{O}_X, \theta}, \quad \mathcal{N}_\pm^{\theta_\pm, \theta_0} := \text{Quot}_{\mathfrak{Y}_\pm/\mathfrak{A}}^{\mathcal{O}_X, \theta_\pm, \theta_0}.$$

We look for a virtual fundamental class on $\mathcal{M}^P$. Let $p: \mathcal{M}^P \times_{\mathfrak{C}^P} \mathfrak{X}^P \rightarrow \mathcal{M}^P$ and $q: \mathcal{M}^P \times_{\mathfrak{C}^P} \mathfrak{X}^P \rightarrow \mathfrak{X}^P$ be the projections. Let $\mathcal{Z} \subset \mathcal{M}^P \times_{\mathfrak{C}^P} \mathfrak{X}^P$ be the universal family of the Hilbert stack, and let $\mathbb{I} \subset \mathcal{O}_{\mathfrak{X}^P \times_{\mathfrak{C}^P} \mathcal{M}^P}$ be the universal ideal sheaf. In the derived category, we have maps $\text{Id}: \mathcal{O}_{\mathcal{M}^P \times_{\mathfrak{C}^P} \mathfrak{X}^P} \rightarrow R\mathcal{H}om(\mathbb{I}, \mathbb{I})$ and $\text{tr}: R\mathcal{H}om(\mathbb{I}, \mathbb{I}) \rightarrow \mathcal{O}_{\mathcal{M}^P \times_{\mathfrak{C}^P} \mathfrak{X}^P}$ which give a splitting of $R\mathcal{H}om(\mathbb{I}, \mathbb{I})$.

Let $L_{\mathcal{M}^P \times_{\mathfrak{C}^P} \mathfrak{X}^P/\mathfrak{C}^P}^\bullet$ be the cotangent complex of $\mathcal{M}^P \times_{\mathfrak{C}^P} \mathfrak{X}^P \rightarrow \mathfrak{C}^P$, which is of Deligne–Mumford type. Consider the Atiyah class

$$\text{At}_{\mathcal{M}^P \times_{\mathfrak{C}^P} \mathfrak{X}^P/\mathfrak{C}^P}(\mathbb{I}): \mathbb{I} \rightarrow \mathbb{I} \otimes L_{\mathcal{M}^P \times_{\mathfrak{C}^P} \mathfrak{X}^P/\mathfrak{C}^P}^\bullet[1].$$

Composing with the projection and restricting to the traceless part, we get

$$\text{At}_{\mathcal{M}^P \times_{\mathfrak{C}^P} \mathfrak{X}^P/\mathfrak{X}^P}(\mathbb{I}): R\mathcal{H}om(\mathbb{I}, \mathbb{I})_0 \rightarrow p^* L_{\mathcal{M}^P/\mathfrak{C}^P}^\bullet[1].$$

By the Serre duality of simple normal crossing families, we have the map

$$\Phi: E^\bullet := Rp_*(R\mathcal{H}om(\mathbb{I}, \mathbb{I})_0 \otimes q^* \omega_{\mathfrak{X}^P/\mathfrak{C}^P})[2] \rightarrow L_{\mathcal{M}^P/\mathfrak{C}^P}^\bullet,$$

where $\omega_{\mathfrak{X}^P/\mathfrak{C}^P}$ is the relative dualizing line bundle of $\mathfrak{X}^P \rightarrow \mathfrak{C}^P$.

THEOREM 7.8. *The map $\Phi: E^\bullet \rightarrow L_{\mathcal{M}^P/\mathfrak{C}^P}^\bullet$ is a perfect relative obstruction theory on $\mathcal{M}^P$ over $\mathfrak{C}^P$, in the sense of [BF97].*

Proof. The proof is almost the same as that of Theorem 6.2. Given a square-zero extension $T \hookrightarrow \overline{T}$ of schemes with ideal sheaf J and a map $g: T \rightarrow \mathcal{M}^P$, one has an element $\Phi^* \omega(g) \in \text{Ext}^1(g^* E^\bullet, J)$. To say that Φ is an obstruction theory is the same as saying that $\Phi^* \omega(g)$ is an obstruction to extending g from T to $\overline{T}$.

The induced element $\Phi^* \omega(g)$ is the product of the Atiyah class and Kodaira–Spencer map, which are

$$\text{At}_{T \times_{\mathfrak{C}^P} \mathfrak{X}^P/\mathfrak{X}^P}(g^* \mathbb{I}): g^* E^\bullet = Rp_{T*}(R\mathcal{H}om(g^* \mathbb{I}, g^* \mathbb{I})_0 \otimes q^* \omega_{\mathfrak{X}^P/\mathfrak{C}^P})[2] \rightarrow g^* L_{\mathcal{M}^P/\mathfrak{C}^P}^\bullet \rightarrow L_{T/\mathfrak{C}^P}^\bullet$$

and

$$\kappa(T/\overline{T}/\mathfrak{C}^P): L_{T/\mathfrak{C}^P}^\bullet \rightarrow L_{T/\overline{T}}^\bullet \rightarrow \tau^{\geq -1} L_{T/\overline{T}}^\bullet \cong J[1],$$

where p_T is the projection $T \times_{\mathfrak{C}^P} \mathfrak{X}^P \rightarrow T$. In other words,

$$\Phi^* \omega(g) = p_T^* \kappa(T/\overline{T}/\mathfrak{C}^P) \circ \text{At}_{T \times_{\mathfrak{C}^P} \mathfrak{X}^P/\mathfrak{X}^P}(g^* \mathbb{I})$$

in the group

$$\text{Ext}_T^{-1}(Rp_{T*}(R\mathcal{H}\text{om}(g^* \mathbb{I}, g^* \mathbb{I})_0 \otimes q^* \omega_{\mathfrak{X}^P/\mathfrak{C}^P}), J) \cong \text{Ext}_{T \times_{\mathfrak{C}^P} \mathfrak{X}^P}^2(g^* \mathbb{I}, g^* \mathbb{I} \otimes p_T^* J)_0.$$

Again, we may assume that T is affine, and [III71, Proposition 3.1.8] says that $\Phi^* \omega(g)$ is the obstruction to extending $g^* \mathbb{I}$ from T to $\overline{T}$ with $\det \mathbb{I}$ fixed.

By construction, a map $g: T \rightarrow \mathcal{M}^P$ can be represented by a family of expanded degenerations $\mathcal{X}_T \rightarrow T$, with a closed substack $\mathcal{Z}_T \subset \mathcal{X}_T$, where $\mathcal{Z}_T$ is the pull-back of the universal family $\mathcal{Z} \subset \mathcal{M}^P \times_{\mathfrak{C}^P} \mathfrak{X}^P$. But since $\mathcal{Z}$ and $\mathcal{Z}_T$ are admissible, one can see that $g^* \mathbb{I}$ is just the ideal sheaf of $\mathcal{Z}_T \subset \mathcal{X}_T$. Hence, $\Phi^* \omega(g)$ is also the obstruction to extending $\mathcal{Z}_T$ or, equivalently, g , from T to $\overline{T}$, which proves that Φ is an obstruction theory.

The proof that Φ is perfect is also the same as in Theorem 6.2, with the following lemma in place of Lemma 6.3. $\square$

LEMMA 7.9. *Let $X \rightarrow \mathbb{A}^1$ be a simple degeneration of relative dimension 3. Consider $X_0[k]$ for some $k \geq 0$, and let $Z \subset X_0[k]$ be a 1-dimensional admissible closed substack, with ideal sheaf $\mathcal{I}$. Then $\text{Ext}_{X_0[k]}^i(\mathcal{I}, \mathcal{I})_0 = 0$ for all $i \neq 1, 2$.*

Proof. We just prove the result for $k = 0$; the proof for general k is exactly the same. Let $X_0 = Y_- \cup_D Y_+$. Since Z is admissible, the ideal sheaf $\mathcal{I}$ is also admissible; it fits in the exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{I}|_{Y_-} \oplus \mathcal{I}|_{Y_+} \longrightarrow \mathcal{I}|_D \longrightarrow 0.$$

Applying $\text{Hom}(\mathcal{I}, -)$, we have

$$0 \longrightarrow \text{Hom}(\mathcal{I}, \mathcal{I}) \longrightarrow \text{Hom}(\mathcal{I}|_{Y_-}, \mathcal{I}|_{Y_-}) \oplus \text{Hom}(\mathcal{I}|_{Y_+}, \mathcal{I}|_{Y_+}) \longrightarrow \text{Hom}(\mathcal{I}|_D, \mathcal{I}|_D).$$

Then $\text{Hom}(\mathcal{I}, \mathcal{I})_0 = 0$ follows from $\text{Hom}(\mathcal{I}|_{Y_{\pm}}, \mathcal{I}|_{Y_{\pm}})_0 = 0$, as the sequence respects the trace map. The same arguments applied to the sequence tensored with the dualizing sheaf lead to the vanishing of $\text{Ext}^3(\mathcal{I}, \mathcal{I})_0$. $\square$

COROLLARY 7.10. *There is a virtual fundamental class $[\mathcal{M}^P]^{\text{vir}} \in A_*(\mathcal{M}^P)$.*

In the same way, one can prove that there are perfect obstruction theories on $\mathcal{N}_{\pm}^{\theta_{\pm}, \theta_0} \rightarrow \mathfrak{A}^{\theta_{\pm}, \theta_0}$ for a given $\theta \in \Lambda_P^{\text{spl}}$. Again, let p, q be the projections of $\mathcal{N}_{\pm}^{\theta_{\pm}, \theta_0} \times_{\mathfrak{A}^{\theta_{\pm}, \theta_0}} \mathfrak{Y}_{\pm}^{\theta_{\pm}, \theta_0}$ to its two factors. The obstruction theory is given by

$$\Phi_{\pm}: E_{\pm}^{\bullet} := Rp_*(R\mathcal{H}\text{om}(\mathbb{I}_{\pm}, \mathbb{I}_{\pm})_0 \otimes q^* \omega_{\mathfrak{Y}_{\pm}^{\theta_{\pm}, \theta_0}/\mathfrak{A}^{\theta_{\pm}, \theta_0}})[2] \rightarrow L_{\mathcal{N}_{\pm}^{\theta_{\pm}, \theta_0}/\mathfrak{A}^{\theta_{\pm}, \theta_0}}^{\bullet},$$

where $\mathbb{I}_{\pm}$ is the corresponding universal ideal sheaf. We have the virtual fundamental classes $[\mathcal{N}_{\pm}^{\theta_{\pm}, \theta_0}]^{\text{vir}} \in A_*(\mathcal{N}_{\pm}^{\theta_{\pm}, \theta_0})$.

Also on $\mathcal{M}^{\theta} \rightarrow \mathfrak{C}_0^{\dagger, \theta}$, the Cartesian diagram

$$\begin{array}{ccc} \mathcal{M}^{\theta} & \longrightarrow & \mathcal{M}^P \\ \downarrow & & \downarrow \\ \mathfrak{C}_0^{\dagger, \theta} & \xrightarrow{\iota_{\theta}} & \mathfrak{C}^P \end{array}$$

and [BF97, Proposition 7.2] imply that the restriction of everything from $\mathcal{M}^P \rightarrow \mathfrak{C}^P$ to the θ -piece is a perfect obstruction theory and $[\mathcal{M}^\theta]^{\text{vir}} = \iota_\theta^! [\mathcal{M}^P]^{\text{vir}}$.

7.3 Degeneration formula: cycle version

Now, everything is ready for a cycle-version degeneration formula.

By Proposition 7.6, we have the Cartesian diagram

$$\begin{array}{ccc}
 \mathcal{M}^\theta & \xleftarrow[\Phi_\theta]{\cong} \mathcal{N}_{-}^{\theta-, \theta_0} \times_{\text{Hilb}_D^{\theta_0}} \mathcal{N}_{+}^{\theta+, \theta_0} & \longrightarrow \mathcal{N}_{-}^{\theta-, \theta_0} \times \mathcal{N}_{+}^{\theta+, \theta_0} \\
 & \downarrow g & \downarrow \\
 & \text{Hilb}_D^{\theta_0} \times \mathfrak{C}_0^{\dagger, \theta} & \longrightarrow \text{Hilb}_D^{\theta_0} \times \text{Hilb}_D^{\theta_0} \times \mathfrak{C}_0^{\dagger, \theta} \\
 & \downarrow & \downarrow \\
 & \text{Hilb}_D^{\theta_0} & \xrightarrow{\Delta} \text{Hilb}_D^{\theta_0} \times \text{Hilb}_D^{\theta_0},
 \end{array} \tag{7.2}$$

with vertical arrows in the upper row given by the natural forgetful maps $\mathcal{N}_{\pm}^{\theta\pm, \theta_0} \rightarrow \text{Hilb}_D^{\theta_0}$ and $\mathcal{N}_{\pm}^{\theta\pm, \theta_0} \rightarrow \mathfrak{A}^{\theta\pm, \theta_0}$.

For a family $\pi: X \rightarrow \mathbb{A}^1$ and a point $c \in \mathbb{A}^1$, let $i_c: \{c\} \hookrightarrow \mathbb{A}^1$ be the inclusion. For $c \neq 0$, the fiber X_c is smooth. The restriction to X_c gives a perfect obstruction theory, with virtual fundamental class $[\text{Hilb}_{X_c}^P]^{\text{vir}}$. One has the following two Cartesian diagrams:

$$\begin{array}{ccc}
 \text{Hilb}_{X_c}^P & \longrightarrow & \mathcal{M}^P \\
 \downarrow & & \downarrow \\
 \{c\} & \hookrightarrow & \mathbb{A}^1,
 \end{array}
 \quad
 \begin{array}{ccc}
 \mathcal{M}^\theta & \xrightarrow{\iota_\theta} & \mathfrak{Quot}_{\mathfrak{X}_0/\mathfrak{C}_0}^{\mathcal{O}_{X,P}} \hookrightarrow \mathcal{M}^P \\
 \downarrow & & \downarrow \\
 \{0\} & \hookrightarrow & \mathbb{A}^1,
 \end{array}$$

where the map $\mathcal{M}^P \rightarrow \mathbb{A}^1$ is the composition $\mathcal{M}^P \rightarrow \mathfrak{C}^P \rightarrow \mathbb{A}^1$.

THEOREM 7.11 (Degeneration formula: cycle version). *We have*

$$i_c^! [\mathcal{M}^P]^{\text{vir}} = [\text{Hilb}_{X_c}^P]^{\text{vir}}, \tag{7.3}$$

$$i_0^! [\mathcal{M}^P]^{\text{vir}} = \sum_{\theta \in \Lambda_P^{\text{sp1}}} \iota_{\theta*} \Delta^! ([\mathcal{N}_{-}^{\theta-, \theta_0}]^{\text{vir}} \times [\mathcal{N}_{+}^{\theta+, \theta_0}]^{\text{vir}}), \tag{7.4}$$

where the classes in the second row are viewed in $0 \times_{\mathbb{A}^1} \mathcal{M}^P$.

Proof. Let $\mathcal{Z}^\theta \subset \mathcal{M}^\theta \times_{\mathfrak{C}_0^{\dagger, \theta}} \mathfrak{X}_0^{\dagger, \theta}$, $\mathcal{Z}_{\pm} \subset \mathcal{N}_{\pm}^{\theta\pm, \theta_0} \times_{\mathfrak{A}^{\theta\pm, \theta_0}} \mathfrak{Y}_{\pm}^{\theta\pm, \theta_0}$ and $\mathcal{Z}_D \subset \text{Hilb}_D^{\theta_0} \times D$ be the universal families of the classifying stacks. Let $\mathbb{I}^\theta$, $\mathbb{I}_{\pm}$ and $\mathbb{I}_D$ be the corresponding ideal sheaves. By admissibility, we have the gluing

$$\mathcal{Z}^\theta \cong (\mathcal{Z}_{-} \times \mathfrak{A}^{\theta+, \theta_0}) \cup_{\mathfrak{A}^{\theta-, \theta_0} \times D \times \mathfrak{A}^{\theta+, \theta_0}} (\mathfrak{A}^{\theta-, \theta_0} \times \mathcal{Z}_{+})$$

and the exact sequence

$$0 \longrightarrow \mathbb{I}^\theta \longrightarrow \mathbb{I}_{-} \boxplus \mathbb{I}_{+} \longrightarrow j_{D*} \text{ev}_\theta^* \mathbb{I}_D \longrightarrow 0$$

in the total spaces

$$\mathcal{M}^\theta \times_{\mathfrak{C}_0^{\dagger, \theta}} \mathfrak{X}_0^{\dagger, \theta} \cong \mathcal{M}^\theta \times_{\mathfrak{C}_0^{\dagger, \theta}} ((\mathfrak{Y}_{-}^{\theta-, \theta_0} \times \mathfrak{A}^{\theta+, \theta_0}) \cup_{\mathfrak{A}^{\theta-, \theta_0} \times D \times \mathfrak{A}^{\theta+, \theta_0}} (\mathfrak{A}^{\theta-, \theta_0} \times (\mathfrak{Y}_{+}^{\theta+, \theta_0})^\circ)).$$

Here, $\mathbb{I}_- \boxplus \mathbb{I}_+$ means $\Delta^*(p_-^* \mathbb{I}_- \oplus p_+^* \mathbb{I}_+)$, where $p_\pm$ stands for (base change to the total space of) the projection $\mathcal{N}_-^{\theta_-, \theta_0} \times \mathcal{N}_+^{\theta_+, \theta_0} \rightarrow \mathcal{N}_\pm^{\theta_\pm, \theta_0}$. The map $\text{ev}_\theta: \mathcal{M}^\theta \rightarrow \text{Hilb}_D^{\theta_0}$ is the evaluation map and also denotes its base change to the total space. The map j_D is the inclusion of the universal distinguished divisor $\mathcal{M}^\theta \times D \subset \mathcal{M}^\theta \times_{\mathfrak{C}_0^{\dagger, \theta}} \mathfrak{X}_0^{\dagger, \theta}$.

Now, $\Delta^* p_-^* \mathbb{I}_-$ is the universal ideal sheaf on $\mathcal{M}^\theta \times_{\mathfrak{C}_0^{\dagger, \theta}} (\mathfrak{Y}_-^{\theta_-, \theta_0} \times \mathfrak{A}_+^{\theta_+, \theta_0})$ and similarly for $\Delta^* p_+^* \mathbb{I}_+$; furthermore, $\Delta^*(\text{ev}_- \times \text{ev}_+)^* \mathbb{I}_D = \text{ev}_\theta^* \mathbb{I}_D$ is the ideal sheaf of $\mathcal{M}^\theta \times D$.

Let p denote the projections from the universal families to classifying stacks $\mathcal{M}^\theta \times_{\mathfrak{C}_0^{\dagger, \theta}} \mathfrak{X}_0^{\dagger, \theta} \rightarrow \mathcal{M}^\theta$, $\mathcal{N}_\pm^{\theta_\pm, \theta_0} \times_{\mathfrak{A}_\pm^{\theta_\pm, \theta_0}} \mathfrak{Y}_\pm^{\theta_\pm, \theta_0} \rightarrow \mathcal{N}_\pm^{\theta_\pm, \theta_0}$ and $\text{Hilb}_D^{\theta_0} \times D \rightarrow \text{Hilb}_D^{\theta_0}$. Then, applying $Rp_* R\mathcal{H}om(-, \mathbb{I}^\theta)$, we get the diagram of distinguished triangles

$$\begin{array}{ccccc} Rp_* R\mathcal{H}om(\mathbb{I}_D, \mathbb{I}_D)_0^\vee & \longrightarrow & Rp_* R\mathcal{H}om(\mathbb{I}_-, \mathbb{I}_-)_0^\vee \boxplus Rp_* R\mathcal{H}om(\mathbb{I}_+, \mathbb{I}_+)_0^\vee & \longrightarrow & R\pi_* R\mathcal{H}om(\mathbb{I}^\theta, \mathbb{I}^\theta)_0^\vee \\ \downarrow & & \downarrow & & \downarrow \\ L_{\mathcal{M}^\theta / \mathcal{N}_-^{\theta_-, \theta_0} \times \mathcal{N}_+^{\theta_+, \theta_0}}^\bullet[-1] & \longrightarrow & L_{\mathcal{N}_-^{\theta_-, \theta_0} / \mathfrak{A}_-^{\theta_-, \theta_0}}^\bullet \boxplus L_{\mathcal{N}_+^{\theta_+, \theta_0} / \mathfrak{A}_+^{\theta_+, \theta_0}}^\bullet & \longrightarrow & L_{\mathcal{M}^\theta / \mathfrak{C}_0^{\dagger, \theta}}^\bullet, \end{array}$$

where the upper row is obtained from the exact sequence of ideal sheaves and the lower row is the distinguished triangle for the map $\mathcal{M}^\theta \rightarrow \mathcal{N}_-^{\theta_-, \theta_0} \times \mathcal{N}_+^{\theta_+, \theta_0}$, relative to $\mathfrak{C}_0^{\dagger, \theta}$.

It is easy to check that $\text{Ext}^2(I, I)_0 = 0$ for an ideal sheaf I on a 2-dimensional smooth Deligne–Mumford stack. Thus, we have

$$Rp_* R\mathcal{H}om(\mathbb{I}_D, \mathbb{I}_D)_0^\vee \cong \Omega_{\text{Hilb}_D^{\theta_0}} \cong L_{\text{Hilb}_D^{\theta_0} / \text{Hilb}_D^{\theta_0} \times \text{Hilb}_D^{\theta_0}}^\bullet[-1],$$

and the first column in the diagram is the same as the canonical map

$$g^* L_{\text{Hilb}_D^{\theta_0} / \text{Hilb}_D^{\theta_0} \times \text{Hilb}_D^{\theta_0}}^\bullet \rightarrow L_{\mathcal{M}^\theta / \mathcal{N}_-^{\theta_-, \theta_0} \times \mathcal{N}_+^{\theta_+, \theta_0}}^\bullet,$$

from (7.2).

In other words, the perfect obstruction theories on $\mathcal{M}^\theta$ and $\mathcal{N}_-^{\theta_-, \theta_0} \times \mathcal{N}_+^{\theta_+, \theta_0}$ form a compatibility datum in the sense of [BF97], relative to $\mathfrak{C}_0^{\dagger, \theta}$. As a result,

$$[\mathcal{M}^\theta]^{\text{vir}} = \Delta^!([\mathcal{N}_-^{\theta_-, \theta_0}]^{\text{vir}} \times [\mathcal{N}_+^{\theta_+, \theta_0}]^{\text{vir}}).$$

Now, the conclusion follows from the following splitting result by Proposition 7.7:

$$i_0^! [\mathcal{M}^P]^{\text{vir}} = \sum_{\theta \in \Lambda_P^{\text{spl}}} \iota_{\theta*} [\mathcal{M}^\theta]^{\text{vir}}. \quad \square$$

8. Degeneration formula: numerical version

8.1 Relative orbifold Donaldson–Thomas theory

We defined the orbifold Donaldson–Thomas invariant for a 3-dimensional smooth projective Deligne–Mumford stack in Section 6. Now, we can define the relative Donaldson–Thomas invariant for a smooth pair (Y, D) , where Y is a 3-dimensional smooth projective Deligne–Mumford stack and $D \subset Y$ is a smooth divisor. Fix a polarization $(\mathcal{E}, H)$ of Y and P as before.

Consider $K(Y) := K(Y)_\mathbb{Q}$. The pairing $\chi: K(Y) \times K(Y) \rightarrow \mathbb{Z}$ is nondegenerate because of the projectivity. We can identify $\text{Hom}(K(Y), \mathbb{Q})$ with $K(Y)$ and view P as an element of $K(Y)$. The dimension condition is that $P \in F_1 K(Y)$, where $F_\bullet$ is the natural topological filtration.

For those P represented by admissible sheaves, $P_0 = i^!P$ lies in $F_0K(D)$ by admissibility, where $i: D \hookrightarrow X$ is the inclusion.

Consider the Hilbert stack $\mathcal{N}^{P,P_0} := \mathfrak{Hilb}_{\mathfrak{Y}/\mathfrak{A}}^{P,P_0} \rightarrow \mathfrak{A}^{P,P_0}$ parameterizing the stable quotients on the classifying stacks of expanded pairs. As in Section 7, we have a perfect obstruction theory and thus a virtual fundamental class $[\mathcal{N}^{P,P_0}]^{\text{vir}} \in A_*(\mathcal{N}^{P,P_0})$.

Let p, q be the projections of $\mathcal{N}^{P,P_0} \times_{\mathfrak{A}^{P,P_0}} \mathfrak{Y}^{P,P_0}$ to its two factors, and let $\mathbb{I}$ be the universal ideal sheaf. Given $\gamma \in A_{\text{orb}}^l(Y)$, we still have the operator $\widetilde{\text{ch}}^{\text{orb}}(\gamma): A_*(\mathcal{N}^{P,P_0}) \rightarrow A_{*-k+1-l}(\mathcal{N}^{P,P_0})$ defined via the diagram

$$\begin{array}{ccccc} \mathcal{N} \times_{\mathfrak{A}} (I_{\mathfrak{A}}\mathfrak{Y}) & \longrightarrow & I_{\mathfrak{A}}\mathfrak{Y} & \longrightarrow & IY \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{N} \times_{\mathfrak{A}} \mathfrak{Y} & \xrightarrow{q} & \mathfrak{Y} & \longrightarrow & Y \\ \downarrow p & & \downarrow & & \\ \mathcal{N} & \longrightarrow & \mathfrak{A}, & & \end{array}$$

where for simplicity we have omitted the superscripts P and P_0 , and $I_{\mathfrak{A}}\mathfrak{Y} \rightarrow \mathfrak{Y}$ is the inertia stack of $\mathfrak{Y}$ over $\mathfrak{A}$. Let $\text{ev}: \mathcal{N}^{P,P_0} \rightarrow \text{Hilb}^{P_0}(D)$ be the evaluation map.

DEFINITION 8.1. Given $\gamma_i \in A_{\text{orb}}^*(Y)$, for $1 \leq i \leq r$, and $C \in A^*(\text{Hilb}^{P_0}(D))$, define the relative Donaldson–Thomas invariant as

$$\left\langle \prod_{i=1}^r \tau_{k_i}(\gamma_i) \middle| C \right\rangle_{X,D}^P := \deg \left[\text{ev}^*(C) \cdot \prod_{i=1}^r \widetilde{\text{ch}}_{k_i+2}^{\text{orb}}(\gamma_i) \cdot [\mathcal{N}^{P,P_0}]^{\text{vir}} \right]_0.$$

8.2 Degeneration formula

Now, let us consider the case of a simple degeneration. Let $\pi: X \rightarrow \mathbb{A}^1$ be a family of smooth projective Deligne–Mumford stacks, which is a simple degeneration of relative dimension 3, with central fiber $X_0 = Y_- \cup_D Y_+$. Take $0 \neq c \in \mathbb{A}^1$ and $X_c = \pi^{-1}(c)$. Fix $P \in F_1K(X_c)$. Let $P_0 = i^!P \in F_0K(D)$.

Let $\gamma \in A_{\text{orb}}^*(X_c)$ be in the image of the restriction from $A_{\text{orb}}^*(X)$, and let $\gamma_{\pm}$ and γ_0 be its restrictions to $Y_{\pm}$ and D , respectively. We abuse this notation to also denote their pushforwards to X ; therefore, $\gamma = \gamma_- + \gamma_+ - \gamma_0$. Let $\{C_k\}$ be a basis of $A^*(\text{Hilb}^{P_0}(D))$, with cup product

$$\int_{\text{Hilb}^{P_0}(D)} C_k \cup C_l = g_{kl}.$$

Let (g^{kl}) be the inverse matrix.

We have the numerical version of the degeneration formula in the following.

THEOREM 8.2 (Degeneration formula: numerical version). *Given $P \in F_1K(X_c)$, assume that $\gamma_{i,\pm}$ are disjoint with D . We have*

$$\left\langle \prod_{i=1}^r \tau_{k_i}(\gamma_i) \right\rangle_{X_c}^P = \sum_{\substack{\theta_- + \theta_+ - P_0 = P, \\ S \subset \{1, \dots, r\}, k, l}} \left\langle \prod_{i \in S} \tau_{k_i}(\gamma_{i,-}) \middle| C_k \right\rangle_{Y_-, D}^{\theta_-} g^{kl} \left\langle \prod_{i \notin S} \tau_{k_i}(\gamma_{i,+}) \middle| C_l \right\rangle_{Y_+, D}^{\theta_+},$$

where $\theta_{\pm} \in F_1K(Y_{\pm})$ range over all configurations that satisfy $\theta_- + \theta_+ - P_0 = P$.

Proof of Theorem 8.2. Use the cycle-version degeneration formula. Apply $\tilde{\text{ch}}_{k+2}^{\text{orb}}(\gamma_i)$ to (7.3) and take the degree 0 part; one gets the left-hand side of the formula. For the right-hand side, we apply $\tilde{\text{ch}}_{k+2}^{\text{orb}}(\gamma_i)$ to (7.4). Let p and q be the projections from $\mathcal{M}^\theta \times_{\mathfrak{C}_0^{\dagger,\theta}} \mathfrak{X}_0^{\dagger,\theta}$ to the two factors. Let $\mathcal{Z}^\theta$ be the universal family, and let $\mathbb{I}^\theta$ be the universal ideal sheaf. Consider the embedding $\mathcal{M}^\theta \hookrightarrow \mathcal{N}_-^{\theta-, \theta_0} \times \mathcal{N}_+^{\theta+, \theta_0}$.

Recall the sequence on $\mathcal{M}^\theta \times_{\mathfrak{C}_0^{\dagger,\theta}} \mathfrak{X}_0^{\dagger,\theta}$,

$$0 \longrightarrow \mathbb{I}^\theta \longrightarrow \mathbb{I}_- \oplus \mathbb{I}_+ \longrightarrow j_{D*} \text{ev}_\theta^* \mathbb{I}_D \longrightarrow 0,$$

where j_D is the embedding $\mathcal{M}^\theta \times D \subset \mathcal{M}^\theta \times_{\mathfrak{C}_0^{\dagger,\theta}} \mathfrak{X}_0^{\dagger,\theta}$ and $\text{ev}_\theta: \mathcal{M}^\theta \rightarrow \text{Hilb}^{\theta_0}(D)$. Hence

$$\begin{aligned} \tilde{\text{ch}}_{k+2}^{\text{orb}}(\gamma) \cdot i_0^! [\mathcal{M}^P]^{\text{vir}} &= \sum_{\theta \in \Lambda_P^{\text{spl}}} p_* (q^* \gamma \cap \tilde{\text{ch}}_{k+2}^{\text{orb}}(\mathbb{I}^\theta) \cdot p^* \iota_{\theta*} \Delta^!([\mathcal{N}_-^{\theta-, \theta_0}]^{\text{vir}} \times [\mathcal{N}_+^{\theta+, \theta_0}]^{\text{vir}})) \\ &= \sum_{\theta \in \Lambda_P^{\text{spl}}} p_* \iota_{\theta*} \Delta^!((q^* \gamma_- \cap \tilde{\text{ch}}_{k+2}^{\text{orb}}(\mathbb{I}_-) \cdot p^* [\mathcal{N}_-^{\theta-, \theta_0}]^{\text{vir}}) \times p^* [\mathcal{N}_+^{\theta+, \theta_0}]^{\text{vir}}) \\ &\quad + \sum_{\theta \in \Lambda_P^{\text{spl}}} p_* \iota_{\theta*} \Delta^!(p^* [\mathcal{N}_-^{\theta-, \theta_0}]^{\text{vir}} \times (q^* \gamma_+ \cap \tilde{\text{ch}}_{k+2}^{\text{orb}}(\mathbb{I}_+) \cdot p^* [\mathcal{N}_+^{\theta+, \theta_0}]^{\text{vir}})) \\ &\quad - \sum_{\theta \in \Lambda_P^{\text{spl}}} p_* (q^* \gamma \cap \tilde{\text{ch}}_{k+2}^{\text{orb}}(j_{D*} \text{ev}_\theta^* \mathbb{I}_D) \cdot p^* [\mathcal{M}^\theta]^{\text{vir}}). \end{aligned}$$

Note that since γ is disjoint from D , the last term actually vanishes.

Therefore, we have proved the identity

$$\begin{aligned} \tilde{\text{ch}}_{k+2}^{\text{orb}}(\gamma) \cdot i_0^! [\mathcal{M}^P]^{\text{vir}} &= \sum_{\theta \in \Lambda_P^{\text{spl}}} p_* (q^* \gamma \cap p^* \iota_{\theta*} \Delta^!((\tilde{\text{ch}}_{k+2}^{\text{orb}}(\mathbb{I}_-) \cdot [\mathcal{N}_-^{\theta-, \theta_0}]^{\text{vir}}) \times [\mathcal{N}_+^{\theta+, \theta_0}]^{\text{vir}})) \\ &\quad + \sum_{\theta \in \Lambda_P^{\text{spl}}} p_* (q^* \gamma \cap p^* \iota_{\theta*} \Delta^!([\mathcal{N}_-^{\theta-, \theta_0}]^{\text{vir}} \times (\tilde{\text{ch}}_{k+2}^{\text{orb}}(\mathbb{I}_+) \cdot [\mathcal{N}_+^{\theta+, \theta_0}]^{\text{vir}}))). \end{aligned}$$

Take $\theta_0 = P_0$; the data $\theta \in \Lambda_P^{\text{spl}}$ can be identified with $(\theta_-, \theta_+, P_0)$ satisfying the condition in the assumption.

For a basis $\{C_k\}$ of $A^*(\text{Hilb}^m(D))$, we have Kunnet decomposition of the diagonal

$$[\Delta] = \sum_{k,l} g^{kl} C_k \otimes C_l.$$

Apply this to the equality, and the degeneration formula follows. $\square$

We are particularly interested in a special type of curve classes. Let (Y, D) be a smooth pair.

DEFINITION 8.3. A class $P \in K(Y)$ is called *multi-regular* if it can be represented by some coherent sheaf such that the associated representation of the stabilizer group at the generic point is a multiple of the regular representation.

Denote by $F_1^{\text{mr}} K(Y) \subset F_1 K(Y)$ the subgroup generated by the multi-regular classes. Let $(\beta, \varepsilon) \in F_1^{\text{mr}} K(Y)/F_0 K(Y) \oplus F_0 K(Y)$ be the image of P in the associated graded K-group. One can check that P is multi-regular if and only if β is a pull-back from a curve class in the coarse moduli space.

Let $F_0^{\text{mr}}K(D)$ be the subgroup generated by the 0-dimensional substacks whose associated representations are multi-regular. Then, $F_0^{\text{mr}}K(D) \cong F_0K(\underline{D}) \cong \mathbb{Z}$, where $\underline{D}$ is the coarse moduli space. Let $P \in F_1^{\text{mr}}K(Y)$ be represented by some admissible curve. Then $P_0 = i^*P \in F_0^{\text{mr}}K(D)$ only depends on β . Let m be the number such that $\beta \cdot D = m[\mathcal{O}_x]$, where $x \in D$ is the preimage of a point in $\underline{D}$.

Now, for a simple degeneration $\pi: X \rightarrow \mathbb{A}^1$, classes $(\beta_1, \varepsilon_1) \in F_1^{\text{mr}}K(Y_-)/F_0K(Y_-) \oplus F_0K(Y_-)$ and $(\beta_2, \varepsilon_2) \in F_1^{\text{mr}}K(Y_+)/F_0K(Y_+) \oplus F_0K(Y_+)$ come from a splitting datum if $\beta_1 + \beta_2 = \beta$ and $\varepsilon_1 + \varepsilon_2 - m = \varepsilon$.

THEOREM 8.4 (Degeneration formula: numerical version for multi-regular case). *Given $\beta \in F_1^{\text{mr}}K(X_c)/F_0K(X_c)$, assume that $\gamma_{i,\pm}$ are disjoint with D . We have*

$$\left\langle \prod_{i=1}^r \tau_{k_i}(\gamma_i) \right\rangle_{X_c}^{\beta, \varepsilon} = \sum_{\substack{\beta_- + \beta_+ = \beta, \\ \varepsilon_- + \varepsilon_+ = \varepsilon + m, \\ S \subset \{1, \dots, r\}, k, l}} \left\langle \prod_{i \in S} \tau_{k_i}(\gamma_{i,-}) \middle| C_k \right\rangle_{Y_-, D}^{\beta_-, \varepsilon_-} g^{kl} \left\langle \prod_{i \notin S} \tau_{k_i}(\gamma_{i,+}) \middle| C_l \right\rangle_{Y_+, D}^{\beta_+, \varepsilon_+},$$

where $\beta_- \in F_1^{\text{mr}}K(Y_-)/F_0K(Y_-)$ and $\beta_+ \in F_1^{\text{mr}}K(Y_+)/F_0K(Y_+)$ range over all curve classes that coincide on D and satisfy $\beta_- + \beta_+ = \beta$.

Define the descendent Donaldson–Thomas partition function of X_c as

$$Z_\beta \left(X_c; q \middle| \prod_{i=1}^r \tau_{k_i}(\gamma_i) \right) := \sum_{\varepsilon \in F_0K(X_c)} \left\langle \prod_{i=1}^r \tau_{k_i}(\gamma_i) \right\rangle_{X_c}^{\beta, \varepsilon} q^\varepsilon.$$

Similarly, for a pair (Y, D) and $C \in A^*(\text{Hilb}^{(\beta \cdot D)[\mathcal{O}_x]}(D))$, define the relative Donaldson–Thomas partition function as

$$Z_{\beta, C} \left(Y, D; q \middle| \prod_{i=1}^r \tau_{k_i}(\gamma_i) \right) := \sum_{\varepsilon \in F_0K(Y)} \left\langle \prod_{i=1}^r \tau_{k_i}(\gamma_i) \middle| C \right\rangle_{Y, D}^{\beta, \varepsilon} q^\varepsilon.$$

COROLLARY 8.5. *Given $\beta \in F_1^{\text{mr}}K(X_c)/F_0K(X_c)$, assume that $\gamma_{i,\pm}$ are disjoint with D . Then,*

$$\begin{aligned} Z_\beta \left(X_c; q \middle| \prod_{i=1}^r \tau_{k_i}(\gamma_i) \right) &= \sum_{\substack{\beta_- + \beta_+ = \beta \\ S \subset \{1, \dots, r\}, k, l}} \frac{g^{kl}}{q^m} Z_{\beta_-, C_k} \left(Y_-, D; q \middle| \prod_{i \in S} \tau_{k_i}(\gamma_{i,-}) \right) \\ &\quad \times Z_{\beta_+, C_l} \left(Y_+, D; q \middle| \prod_{i \notin S} \tau_{k_i}(\gamma_{i,+}) \right). \end{aligned}$$

Remark 8.6. In practice, one has to find a good basis $\{C_k\}$ for the cohomology of the Hilbert scheme of an orbifold surface. In the orbifold case, depending on the specific problem, there are usually natural choices of such bases. For example, when a torus action is involved, to work in the equivariant setting and take the fixed-point basis is one such choice.

Another important case is that the orbifold surface D is of ADE type, that is, of the form $[\mathbb{C}^2/\Gamma]$, where Γ is a finite subgroup of $\text{SL}(2, \mathbb{C})$. Then $\text{Hilb}(D)$ has a structure of Nakajima quiver variety, which provides a natural basis of its cohomology.

ACKNOWLEDGEMENTS

I would like to express sincere gratitude to my advisor Chiu-Chu Melissa Liu, who led me into this exciting area and provided plenty of help and guidance. Her encouragement and patience are really important to the author. Special thanks go to Jun Li and Baosen Wu, for their preceding work and helpful discussions. The author would also like to thank Daveshe Maulik, Jason Starr and Zhengyu Zong for useful discussions.

REFERENCES

- ACFW13 D. Abramovich, C. Cadman, B. Fantechi and J. Wise, *Expanded degenerations and pairs*, Comm. Algebra **41** (2013), no. 6, 2346–2386; <https://doi.org/10.1080/00927872.2012.658589>.
- AF16 D. Abramovich and B. Fantechi, *Orbifold techniques in degeneration formulas*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **16** (2016), no. 2, 519–579; https://doi.org/10.2422/2036-2145.201408_006.
- AGV08 D. Abramovich, T. Graber and A. Vistoli, *Gromov–Witten theory of Deligne–Mumford stacks*, Amer. J. Math. **130** (2008), no. 5, 1337–1398; <https://doi.org/10.1353/ajm.0.0017>.
- AOV08 D. Abramovich, M. Olsson and A. Vistoli, *Tame stacks in positive characteristic*, Ann. Inst. Fourier (Grenoble) **58** (2008), no. 4, 1057–1091; <https://doi.org/10.5802/aif.2378>.
- AV02 D. Abramovich and A. Vistoli, *Compactifying the space of stable maps*, J. Amer. Math. Soc. **15** (2002), no. 1, 27–75; <https://doi.org/10.1090/S0894-0347-01-00380-0>.
- BCY12 J. Bryan, C. Cadman and B. Young, *The orbifold topological vertex*, Adv. Math. **229** (2012), no. 1, 531–595; <https://doi.org/10.1016/j.aim.2011.09.008>.
- BF97 K. Behrend and B. Fantechi, *The intrinsic normal cone*, Invent. Math. **128** (1997), no. 1, 45–88; <https://doi.org/10.1007/s002220050136>.
- BG09 J. Bryan and T. Graber, *The crepant resolution conjecture*, Algebraic Geometry (Seattle 2005, Part 1), Proc. Sympos. Pure Math., vol. 80 (Amer. Math. Soc., Providence, RI, 2009), 23–42; <https://doi.org/10.1090/pspum/080.1/2483931>.
- BP08 J. Bryan and R. Pandharipande, *The local Gromov–Witten theory of curves*, J. Amer. Math. Soc. **21** (2008), no. 1, 101–136; <https://doi.org/10.1090/S0894-0347-06-00545-5>.
- CG97 N. Chriss and V. Ginzburg, *Representation theory and complex geometry* (Birkhäuser Boston, Inc., Boston, MA, 1997); <https://doi.org/10.1007/978-0-8176-4938-8>.
- Edi13 D. Edidin, *Riemann–Roch for Deligne–Mumford stacks*, A Celebration of Algebraic Geometry, Clay Math. Proc., vol. 18 (Amer. Math. Soc., Providence, RI, 2013), 241–266.
- Eis95 D. Eisenbud, *Commutative algebra: with a view toward algebraic geometry*, Grad. Texts in Math., vol. 150 (Springer-Verlag, New York, 1995); <https://doi.org/10.1007/978-1-4612-5350-1>.
- GD66 A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. Troisième partie*, Publ. Math. Inst. Hautes Études Sci. (1966), no. 28, 5–255; http://www.numdam.org/item?id=PMIHES_1966__28__5_0.
- Gro95 A. Grothendieck, *Techniques de construction et théorèmes d’existence en géométrie algébrique. IV. Les schémas de Hilbert*, Séminaire Bourbaki, Vol. 6 (Soc. Math. France, Paris, 1995), Exp. No. 221, 249–276; http://www.numdam.org/item?id=SB_1961-1962__7__221_0.
- GT13 A. Gholampour and H.-H. Tseng, *On Donaldson–Thomas invariants of threefold stacks and gerbes*, Proc. Amer. Math. Soc. **141** (2013), no. 1, 191–203; <https://doi.org/10.1090/S0002-9939-2012-11346-2>.

- Har77 R. Hartshorne, *Algebraic geometry*, Grad. Texts in Math., vol. 52 (Springer-Verlag, New York – Heidelberg, 1977); <http://dx.doi.org/10.1007/978-1-4757-3849-0>.
- HL10 D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, 2nd ed., Cambridge Mathematical Library (Cambridge Univ. Press, Cambridge, 2010); <https://doi.org/10.1017/CB09780511711985>.
- HT10 D. Huybrechts and R. P. Thomas, *Deformation-obstruction theory for complexes via Atiyah and Kodaira–Spencer classes*, Math. Ann. **346** (2010), no. 3, 545–569; <https://doi.org/10.1007/s00208-009-0397-6>.
- Ill71 L. Illusie, *Complexes cotangent et déformations. I*, Lecture Notes in Math., vol. 239 (Springer-Verlag, Berlin – New York, 1971); <https://doi.org/10.1007/BFb0059052>.
- Kol90 J. Kollár, *Projectivity of complete moduli*, J. Differential Geom. **32** (1990), no. 1, 235–268; <https://doi.org/10.4310/jdg/1214445046>.
- Kol96 ———, *Rational curves on algebraic varieties*, Ergebn. Math. Grenzgeb. (3), vol. 32 (Springer-Verlag, Berlin, 1996); <https://doi.org/10.1007/978-3-662-03276-3>.
- Kre09 A. Kresch, *On the geometry of Deligne–Mumford stacks*, Algebraic Geometry (Seattle 2005, Part 1), Proc. Sympos. Pure Math., vol. 80 (Amer. Math. Soc., Providence, RI, 2009), 259–271; <https://doi.org/10.1090/pspum/080.1/2483938>.
- Li01 J. Li, *Stable morphisms to singular schemes and relative stable morphisms*, J. Differential Geom. **57** (2001), no. 3, 509–578; <https://doi.org/10.4310/jdg/1090348132>.
- Lie07 M. Lieblich, *Moduli of twisted sheaves*, Duke Math. J. **138** (2007), no. 1, 23–118; <https://doi.org/10.1215/S0012-7094-07-13812-2>.
- LM00 G. Laumon and L. Moret-Bailly, *Champs algébriques*, Ergebn. Math. Grenzgeb. (3), vol. 39 (Springer-Verlag, Berlin, 2000).
- LW15 J. Li and B. Wu, *Good degeneration of Quot-schemes and coherent systems*, Comm. Anal. Geom. **23** (2015), no. 4, 841–921; <https://doi.org/10.4310/CAG.2015.v23.n4.a5>.
- Mau09 D. Maulik, *Gromov–Witten theory of A_n -resolutions*, Geom. Topol. **13** (2009), no. 3, 1729–1773; <https://doi.org/10.2140/gt.2009.13.1729>.
- MNOP06a D. Maulik, N. Nekrasov, A. Okounkov and R. Pandharipande, *Gromov–Witten theory and Donaldson–Thomas theory. I*, Compos. Math. **142** (2006), no. 5, 1263–1285; <https://doi.org/10.1112/S0010437X06002302>.
- MNOP06b ———, *Gromov–Witten theory and Donaldson–Thomas theory. II*, Compos. Math. **142** (2006), no. 5, 1286–1304; <https://doi.org/10.1112/S0010437X06002314>.
- MO09a D. Maulik and A. Oblomkov, *Donaldson–Thomas theory of $A_n \times P^1$* , Compos. Math. **145** (2009), no. 5, 1249–1276; <https://doi.org/10.1112/S0010437X09003972>.
- MO09b ———, *Quantum cohomology of the Hilbert scheme of points on A_n -resolutions*, J. Amer. Math. Soc. **22** (2009), no. 4, 1055–1091; <https://doi.org/10.1090/S0894-0347-09-00632-8>.
- Nir08a F. Nironi, *Moduli spaces of semistable sheaves on projective Deligne–Mumford stacks*, 2008, [arXiv:0811.1949](https://arxiv.org/abs/0811.1949).
- Nir08b ———, *Grothendieck duality for Deligne–Mumford stacks*, 2008, [arXiv:0811.1955](https://arxiv.org/abs/0811.1955).
- OP10 A. Okounkov and R. Pandharipande, *The local Donaldson–Thomas theory of curves*, Geom. Topol. **14** (2010), no. 3, 1503–1567; <https://doi.org/10.2140/gt.2010.14.1503>.
- OS03 M. Olsson and J. Starr, *Quot functors for Deligne–Mumford stacks*, Comm. Algebra **31** (2003), no. 8, 4069–4096; <https://doi.org/10.1081/AGB-120022454>.
- OSS80 C. Okonek, M. Schneider and H. Spindler, *Vector bundles on complex projective spaces*, Progress in Math., vol. 3 (Birkhäuser, Boston, Mass., 1980); <https://doi.org/10.1007/978-3-0348-0151-5>.
- PT09 R. Pandharipande and R. P. Thomas, *Curve counting via stable pairs in the derived category*, Invent. Math. **178** (2009), no. 2, 407–447; <https://doi.org/10.1007/s00222-009-0203-9>.

- Ros15 D. Ross, *On the Gromov–Witten/Donaldson–Thomas correspondence and Ruan’s conjecture for Calabi–Yau 3-orbifolds*, Comm. Math. Phys. **340** (2015), no. 2, 851–864; <https://doi.org/10.1007/s00220-015-2438-1>.
- Ros17 ———, *Donaldson–Thomas theory and resolutions of toric A -singularities*, Selecta Math. (N.S.) **23** (2017), no. 1, 15–37; <https://doi.org/10.1007/s00029-016-0234-1>.
- Rua06 Y. Ruan, *The cohomology ring of crepant resolutions of orbifolds*, Gromov–Witten theory of spin curves and orbifolds, Contemp. Math., vol. 403 (Amer. Math. Soc., Providence, RI, 2006), 117–126; <https://doi.org/10.1090/conm/403/07597>.
- RZ13 D. Ross and Z. Zong, *The gerby Gopakumar–Mariño–Vafa formula*, Geom. Topol. **17** (2013), no. 5, 2935–2976.
- RZ15 ———, *Cyclic Hodge integrals and loop Schur functions*, Adv. Math. **285** (2015), 1448–1486; <https://doi.org/10.1016/j.aim.2015.08.023>.
- Sta17 The Stacks Project Authors, *Stacks Project*, version 2017, available at <http://stacks.math.columbia.edu>.
- Tho00 R. P. Thomas, *A holomorphic Casson invariant for Calabi–Yau 3-folds, and bundles on $K3$ fibrations*, J. Differential Geom. **54** (2000), no. 2, 367–438; <https://doi.org/10.4310/jdg/1214341649>.
- Tse10 H.-H. Tseng, *Orbifold quantum Riemann–Roch, Lefschetz and Serre*, Geom. Topol. **14** (2010), no. 1, 1–81; <https://doi.org/10.2140/gt.2010.14.1>.
- Vis89 A. Vistoli, *Intersection theory on algebraic stacks and on their moduli spaces*, Invent. Math. **97** (1989), no. 3, 613–670; <https://doi.org/10.1007/BF01388892>.
- Zon15 Z. Zong, *Generalized Mariño–Vafa formula and local Gromov–Witten theory of orbicurves*, J. Differential Geom. **100** (2015), no. 1, 161–190; <https://doi.org/10.4310/jdg/1427202767>.

Zijun Zhou zz2224@stanford.edu

Department of Mathematics, Columbia University, New York, NY 10027, USA

Current address: Department of Mathematics, Stanford University, 450 Serra Mall, Stanford, CA 94305, USA