On the fundamental class of an essentially smooth scheme-map

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ABSTRACT

Let $f: X \rightarrow Z$ be a separated, essentially finite-type, flat map of noetherian schemes and $\delta: X \rightarrow X \times_Z X$ the diagonal map. The *fundamental class* C_f (globalizing residues) is a map from the relative Hochschild functor $L\delta^*\delta_*f^*$ to the relative dualizing functor $f^!$. We show a compatibility between C_f and the derived tensor product. The main result is that, in a suitable sense, C_f generalizes Verdier's classical isomorphism for smooth f with fibers of dimension d , an isomorphism that binds $f^!$ to relative d -forms.

1. Introduction

1.1 Underlying duality theory

For a scheme X , we denote by $\mathbf{D}(X)$ the derived category of the abelian category of $\mathcal{O}_X$ -modules and by $\mathbf{D}_{qc}(X) \subset \mathbf{D}(X)$ (respectively, $\mathbf{D}_{qc}^+(X) \subset \mathbf{D}(X)$) the full subcategory spanned by the complexes C such that the cohomology modules $H^i(C)$ are all quasi-coherent (respectively, are all quasi-coherent *and* vanish for all but finitely many $i < 0$).

Grothendieck duality is concerned with a pseudofunctor $(-)^!$ over the category $\mathcal{E}$ of separated, essentially finite-type maps of noetherian schemes, taking values in $\mathbf{D}_{qc}^+(-)$. This pseudofunctor is uniquely determined up to isomorphism by the following three properties:

- (i) For formally étale $\mathcal{E}$ -maps f , the functor $f^!$ is the usual restriction functor f^* .
- (ii) (Duality) If f is a proper map of noetherian schemes, then $f^!$ is right-adjoint to Rf_* .
- (iii) Suppose given a fiber square in $\mathcal{E}$

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ g \downarrow & \clubsuit & \downarrow f \\ Z' & \xrightarrow[u]{} & Z, \end{array} \quad (1.1.1)$$

with f (hence g) proper and u (hence v) formally étale. Then the functorial *base-change*

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map

$$\beta_{\clubsuit}(F): v^* f^! F \longrightarrow g^! u^* F, \quad F \in \mathbf{D}_{\text{qc}}^+(Z), \quad (1.1.2)$$

defined to be adjoint to the natural composition

$$Rg_* v^* f^! F \xrightarrow{\sim} u^* Rf_* f^! F \longrightarrow u^* F$$

(see (2.6.7) below), is identical to the natural composite isomorphism

$$v^* f^! F = v^! f^! F \xrightarrow{\sim} (fv)^! F = (ug)^! F \xrightarrow{\sim} g^! u^! F = g^! u^* F. \quad (1.1.3)$$

(Note that the composite isomorphism (1.1.3) exists for *any* $\mathcal{E}$ -maps f, g, u, v with u and v étale and $fv = ug$.)

The case of finite-type maps is treated in [Lip09, Theorem 4.8.1] from the category-theoretic viewpoint of Verdier and Deligne¹, and extended to essentially finite-type maps in [Nay09, § 5.2]. The pseudofunctor $(-)^!$ expands so as to take values in $\mathbf{D}_{\text{qc}}(-)$ if one restricts to proper maps or to $\mathcal{E}$ -maps of finite flat dimension [AJL11, §§ 5.7–5.9]—and even without such restrictions if one substitutes “oplax functor” for “pseudofunctor,” that is, if for an $\mathcal{E}$ -diagram $W \xrightarrow{g} X \xrightarrow{f} Z$, the associated map $(fg)^! \rightarrow g^! f^!$ need not be an isomorphism; see [Nee17]. For flat $\mathcal{E}$ -maps, the maps with which we shall be mainly concerned, the oplax $(-)^!$ agrees with the preceding pseudofunctor, as results from [Nee17, Proposition 13.11].

1.2 Verdier’s isomorphism, fundamental class and the “Ideal Theorem”

In the original elaboration [Har66] (amended in [Con00, Con01]) of the duality theory conceived by Grothendieck and Verdier, the main result, Corollary 3.4 on p. 383, is roughly that in the presence of residual complexes, when confined to finite-type scheme-maps and complexes with coherent homology, the pseudofunctor $(-)^!$ is obtained by pasting together concrete realizations of its restrictions to smooth maps and to finite maps. This is a special case of the “Ideal Theorem” in [Har66, Introduction, p. 6].

For *finite* $f: X \rightarrow Z$, the canonical concrete realization is induced by the usual sheafified duality isomorphism (see (2.6.5))

$$Rf_* f^! F = Rf_* R\mathcal{H}om(\mathcal{O}_X, f^! F) \xrightarrow{\sim} R\mathcal{H}om(f_* \mathcal{O}_X, F), \quad F \in \mathbf{D}_{\text{qc}}^+(Z). \quad (1.2.1)$$

For *formally smooth* $f: X \rightarrow Z$ in $\mathcal{E}$ of relative dimension d (Section 2.3), a canonical concrete realization is given by an isomorphism

$$f^! F \xrightarrow{\sim} \Omega_f^d[d] \otimes_X f^* F, \quad F \in \mathbf{D}_{\text{qc}}^+(Z), \quad (1.2.2)$$

where Ω_f^d is the sheaf of relative d -forms and $[d]$ denotes d -fold translation (shift) in $\mathbf{D}(X)$.

The initial avatar of such an isomorphism uses a trace map for residual complexes (that are assumed to exist). In particular, when f is proper, there results an explicit (but somewhat complicated) description of Serre–Grothendieck duality. (See, for example, [Con00, § 3.4].)

A definition of an isomorphism with the same source and target as (1.2.2), but not requiring residual complexes, was given by Verdier in [Ver69, proof of Theorem 3]. We review and expand upon this classical isomorphism in Section 3. In particular, Proposition 3.4.5 explicates the compatibility of the isomorphism with derived tensor product.

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¹The frequent references in this paper to [Lip09] are due much more to the approach and convenience of that source than to its originality.

For any *flat* $\mathcal{E}$ -map $f: X \rightarrow Z$, let $\delta = \delta_f: X \rightarrow X \times_Z X$ be the diagonal of f . There is a functorial $\mathbf{D}_{\text{qc}}^+(X)$ -map, the *fundamental class of f* ,

$$C_f: \delta^* \delta_* f^* \longrightarrow f^!,$$

from the relative Hochschild functor to the relative dualizing functor; see Definition 4.2.

The fundamental class is compatible with derived tensor product; see Proposition 4.3.

When the flat map f is *equidimensional of relative dimension d* , applying the homology functor H^{-d} to C_f leads to a map c_f from the target of the map (1.2.2) to the source (Section 4.4). Proposition 2.4.2 in [AJL14] asserts that if, moreover, f is *formally smooth*, then c_f is an isomorphism. The proof uses Verdier’s isomorphism, and of course begs the question of whether that isomorphism is inverse to c_f .

That this does hold is our main result, Theorem 4.5.

Thus, the map c_f extends the inverse of Verdier’s isomorphism from the class of formally smooth $\mathcal{E}$ -maps to the class of arbitrary flat equidimensional $\mathcal{E}$ -maps.

Remarks 1.2.3. (a) The above discussion has been limited, for simplicity, to $\mathbf{D}_{\text{qc}}^+$, but the results will be established for $\mathbf{D}_{\text{qc}}$.

(b) The isomorphisms in play are not quite canonical: there are choices involved that affect the resulting homology maps up to sign. For example, for a scheme Y and an $\mathcal{O}_Y$ -ideal $\mathcal{I}$, there are two natural identifications of $\mathcal{I}/\mathcal{I}^2$ with $\text{Tor}_1^{\mathcal{O}_Y}(\mathcal{O}_Y/\mathcal{I}, \mathcal{O}_Y/\mathcal{I})$, one the negative of the other; we will have to choose one of them (see the lines following (3.2.1)). We will also have to assign different roles to the two projections from $X \times_Z X$ to X , necessitating another arbitrary choice. See also [Sas04, § 7.1].

Our choices minimize sign considerations, under the constraint of respect for the usual triangulated structure on the derived sheaf-hom functor (see the remarks following equations (3.1.5) and (3.2.5)). Other choices might have worked as well.

(c) In the present vein, we propose as a nontrivial exercise (that, as far as we know, no one has carried out) to specify the relation between the above “initial avatar” of the fundamental class and Verdier’s isomorphism. (Both isomorphisms have been known for fifty years or so.)

We believe that the fundamental class is important enough, historically and technically, to merit the kind of scrutiny it gets in this paper. For instance, familiarity with some of its aspects could well prove useful in establishing that the pseudofunctor $(-)^!$ together with the isomorphisms in (1.2.1) and (1.2.2) satisfy VAR 1–6 (*mutatis mutandis*) in [Har66, Chapter III, Theorem 8.7]—a version of the “Ideal Theorem” over all of $\mathcal{E}$. It is hoped that a full treatment of this application will materialize in the not too distant future.

1.3 Additional background: fundamental class and residues

More history and motivation behind the fundamental class can be found in [AJL14, § 0.6].

A preliminary version of the fundamental class, with roots in the dualizing properties of differentials on normal varieties, appears in [Gro60, p. 114], followed by some brief hints about connections with residues. In [Lip84], there is a concrete treatment of the case when $S = \text{Spec}(k)$ with k a perfect field and $f: X \rightarrow \text{Spec } k$ an integral algebraic k -scheme. The principal result (the “Residue Theorem”) reifies c_f as a globalization of the local residue maps at the points of X , leading to explicit versions of local and global duality and the relation between them. This is generalized to certain maps of noetherian schemes in [HS93].

The close relation between the fundamental class and residues becomes clearer, and more general, over formal schemes, where local and global duality merge into a single theory with fundamental classes and residues conjoined. (See [Lip02, § 5.5]; a complete exposition has yet to appear.)

2. Preliminaries

In this section, we describe those parts of the duality machinery that we will subsequently use, along with a few of their basic properties. The reader is advised to skip to the next section, referring back to this one as needed.

2.1. Unless otherwise specified, we will be working exclusively with functors between full subcategories of categories of the form $\mathbf{D}(X)$ (see the beginning of Section 1.1); so to reduce clutter, we will write

- f_* instead of Rf_* (f a scheme-map),
- f^* instead of Lf^* (f a scheme-map),
- $\otimes$ instead of $\otimes^L$,
- $\mathcal{H}om$ instead of $R\mathcal{H}om$ (this is the derived *sheaf-hom* functor).

2.2. For any scheme-map $f: X \rightarrow Z$, the functor $f_*: \mathbf{D}(X) \rightarrow \mathbf{D}(Z)$ is *symmetric monoidal*; this entails, in particular, the existence of a functorial map

$$f_*E_1 \otimes_Z f_*E_2 \longrightarrow f_*(E_1 \otimes_X E_2), \quad E_i \in \mathbf{D}(X); \quad (2.2.1)$$

see, for example, [Lip09, 3.4.4(b)].

The functor $f^*: \mathbf{D}(Z) \rightarrow \mathbf{D}(X)$ is *pseudofunctorially* left-adjoint to f_* [Lip09, 3.6.7(d)]. This means that for scheme-maps $X \xrightarrow{f} Z \xrightarrow{g} W$, the canonical isomorphisms $(gf)_* \xrightarrow{\sim} g_*f_*$ and $f^*g^* \xrightarrow{\sim} (gf)^*$ are *conjugate*; see [Lip09, 3.3.7(a)].

For fixed $B \in \mathbf{D}(X)$, the functor $- \otimes_X B$ is left-adjoint to $\mathcal{H}om_X(B, -)$ (see, for example, [Lip09, (2.6.1)']). The resulting counit map

$$\mathrm{ev}(X, B, C): \mathcal{H}om_X(B, C) \otimes_X B \longrightarrow C, \quad C \in \mathbf{D}(X) \quad (2.2.2)$$

is referred to as “evaluation.”

2.3. A scheme-map $f: X \rightarrow Z$ is *essentially of finite type* if every $z \in Z$ has an affine open neighborhood $\mathrm{Spec} A$ whose inverse image is a union of finitely many affine open subschemes $\mathrm{Spec} B_i$ such that each B_i is a localization of a finitely generated A -algebra.

Throughout, $\mathcal{E}$ will be the category of separated, essentially finite-type maps of noetherian schemes. The category $\mathcal{E}$ is closed under scheme-theoretic fiber product.

An $\mathcal{E}$ -map $f: X \rightarrow Z$ is *essentially smooth* if it is flat and it has geometrically regular fibers or, equivalently, if f is *formally smooth*, that is, for each $x \in X$ and $z := f(x)$, the local ring $\mathcal{O}_{X,x}$ is formally smooth over $\mathcal{O}_{Z,z}$ for the discrete topologies; see [EGAIV, 17.1.2, 17.1.6] and [EGAIV, Chapter 0, 19.3.3, 19.3.5(iv) and 22.6.4 a) $\Leftrightarrow$ c)]. (For this equivalence, which involves only local properties, it can be assumed that $X = \mathrm{Spec} A$ and $Z = \mathrm{Spec} B_S$, where B is a finite-type A -algebra and $S \subset B$ is a multiplicatively closed subset; then the relevant local rings

are the same for f or for the finite-type map $g: \operatorname{Spec} B \rightarrow \operatorname{Spec} A$, so that one only needs the equivalence for g , as given by [EGAIV, 6.8.6].)

For essentially smooth $f: X \rightarrow Z$, the diagonal map $X \rightarrow X \times_Z X$ is a regular immersion; see [EGAIV, 16.9.2], [EGAIV, 16.10.2] and [EGAIV, 16.10.5]—whose proof is valid for $\mathcal{E}$ -maps.

Arguing as in [EGAIV, 17.10.2], one gets that for essentially smooth f , the relative differential sheaf Ω_f is locally free over $\mathcal{O}_X$; moreover, when f is of finite type, the rank of Ω_f at $x \in X$ is the dimension at x of the fiber $f^{-1}f(x)$. For any essentially smooth f , the (locally constant) rank of Ω_f will be referred to as the *relative dimension of f* .

An $\mathcal{E}$ -map is *essentially étale* if it is essentially smooth and of relative dimension 0 (cf. [EGAIV, 17.6.1]).

2.4. For a scheme X , an $\mathcal{O}_X$ -complex E is *perfect* if each $x \in X$ has an open neighborhood $U = U_x$ such that the restriction $E|_U$ is $\mathbf{D}(U)$ -isomorphic to a bounded complex of finite-rank free $\mathcal{O}_U$ -modules.

When applied to $\mathcal{E}$ -maps, the term “perfect” means “having finite tor-dimension” (or “finite flat dimension”).

Perfection of maps is preserved under composition and under flat base change; see, for example, [III71, 3.4, p. 243, and 3.5.1, p. 245].

2.5. Let $\mathbf{D}_{\text{qc}}^+(X) \subset \mathbf{D}_{\text{qc}}(X) \subset \mathbf{D}(X)$ be as at the beginning of Section 1.1.

For any $\mathcal{E}$ -map $f: X \rightarrow Z$, there exists a functor $f^\times: \mathbf{D}(Z) \rightarrow \mathbf{D}_{\text{qc}}(X)$ that is bounded below and right-adjoint to $\mathbf{R}f_*$. (See, for example, [Lip09, §4.1].) In particular, for id_X the identity map of X , one has a functor $\operatorname{id}_X^\times$, the *derived quasi-coherator*, right-adjoint to the inclusion $\mathbf{D}_{\text{qc}}(X) \hookrightarrow \mathbf{D}(X)$.

For any complexes A and B in $\mathbf{D}_{\text{qc}}(X)$, and with the notation as in Section 2.1, set

$$\operatorname{Hom}_X^{\text{qc}}(A, B) := \operatorname{id}_X^\times \operatorname{Hom}_X(A, B) \in \mathbf{D}_{\text{qc}}(X). \quad (2.5.1)$$

Recall that $\mathbf{D}_{\text{qc}}(X)$ is a (symmetric, monoidal) closed category, with multiplication $\otimes_X$ and internal hom $\operatorname{Hom}_X^{\text{qc}}$ (cf., for example, [Lip09, 3.5.2(d)]).

For $C \in \mathbf{D}_{\text{qc}}(X)$, the counit map is a $\mathbf{D}(X)$ -isomorphism $\operatorname{id}_X^\times C \xrightarrow{\sim} C$. So for $A, B \in \mathbf{D}(X)$, the counit map $\operatorname{Hom}_X^{\text{qc}}(A, B) \rightarrow \operatorname{Hom}_X(A, B)$ is an *isomorphism* when $\operatorname{Hom}_X(A, B) \in \mathbf{D}_{\text{qc}}(X)$, for example, when $B \in \mathbf{D}_{\text{qc}}^+(X)$ and the cohomology sheaves $H^i A$ are coherent for all i and vanishing for $i \gg 0$; see [Har66, 3.3, p. 92].

2.6. We will use some standard functorial maps, gathered here, that are associated to an $\mathcal{E}$ -map $f: X \rightarrow Z$ and objects $E_\bullet \in \mathbf{D}_{\text{qc}}(X)$, $F_\bullet \in \mathbf{D}_{\text{qc}}(Z)$, $G_\bullet \in \mathbf{D}_{\text{qc}}(Z)$. For the most part, the definitions of these maps emerge category-theoretically from Section 2.2; cf. [Lip09, §3.5.4].

(a) The map

$$f^*(F \otimes_Z G) \longrightarrow f^*F \otimes_X f^*G \quad (2.6.1)$$

is adjoint to the natural composite

$$F \otimes_Z G \longrightarrow f_* f^* F \otimes_Z f_* f^* G \xrightarrow{(2.2.1)} f_*(f^* F \otimes_X f^* G).$$

The map (2.6.1) is an *isomorphism* [Lip09, 3.2.4].

(b) The *sheafified adjunction isomorphism*

$$f_* \mathcal{H}om_X(f^* G, E) \xrightarrow{\sim} \mathcal{H}om_Z(G, f_* E) \quad (2.6.2)$$

is right-conjugate [ILN15, § 1.6], for each fixed G , to the isomorphism (2.6.1).

(c) The *projection isomorphisms*

$$f_* E \otimes_Z G \xrightarrow{\sim} f_*(E \otimes_X f^* G) \quad \text{and} \quad G \otimes_Z^{\mathbb{L}} f_* E \xrightarrow{\sim} f_*(f^* G \otimes_X^{\mathbb{L}} E) \quad (2.6.3)$$

(see, for example, [Lip09, 3.9.4]) are the natural composites

$$\begin{aligned} f_* E \otimes_Z G &\longrightarrow f_* E \otimes_Z f_* f^* G \xrightarrow{(2.2.1)} f_*(E \otimes_X f^* G), \\ G \otimes_Z f_* E &\longrightarrow f_* f^* G \otimes_Z f_* E \xrightarrow{(2.2.1)} f_*(f^* G \otimes_X E). \end{aligned}$$

(This makes sense for arbitrary scheme-maps, though in that generality, the composites are not always isomorphisms.)

(d) The map

$$f^* \mathcal{H}om_Z(G, F) \longrightarrow \mathcal{H}om_X(f^* G, f^* F) \quad (2.6.4)$$

is (f^*-f_*) -adjoint to the natural composite map

$$\mathcal{H}om_Z(G, F) \longrightarrow \mathcal{H}om_Z(G, f_* f^* F) \xrightarrow[(2.6.2)]{\sim} f_* \mathcal{H}om_X(f^* G, f^* F).$$

This is an isomorphism if the map f is perfect, G is homologically bounded above, with coherent homology sheaves, and $F \in \mathbf{D}_{\text{qc}}^+(Z)$; see [Lip09, Proposition 4.6.6].

The map (2.6.4) is $(\mathcal{H}om \otimes)$ -adjoint to the natural composite map

$$f^* \mathcal{H}om_Z(G, F) \otimes_X f^* G \xrightarrow[(2.6.1)]{\sim} f^*(\mathcal{H}om_Z(G, F) \otimes_Z G) \longrightarrow f^* F;$$

see [Lip09, Exercise 3.5.6(a)].

(e) The *duality isomorphism*

$$\zeta(E, F): f_* \mathcal{H}om_X^{\text{qc}}(E, f^\times F) \xrightarrow{\sim} \mathcal{H}om_Z^{\text{qc}}(f_* E, F) \quad (2.6.5)$$

is right-conjugate, for each fixed E , to the projection isomorphism

$$f_*(f^* G \otimes_X^{\mathbb{L}} E) \xleftarrow{\sim} G \otimes_Z^{\mathbb{L}} f_* E.$$

(f) The bifunctorial map

$$\chi_f(F, G): f^! F \otimes_X f^* G \longrightarrow f^!(F \otimes_Z G) \quad (2.6.6)$$

is defined in [Nee17, 13.1–13.3] (denoted by σ instead of χ), and shown in [Nee17, 13.11] to be an *isomorphism* whenever the map f is perfect

When f is proper, $f^!$ is right-adjoint to f_* and the map χ_f is adjoint to the natural composite

$$f_*(f^! F \otimes_X f^* G) \xrightarrow[(2.6.3)]{\sim} f_* f^! F \otimes_Z G \longrightarrow F \otimes_Z G.$$

In particular, $\chi_f(\mathcal{O}_Z, G)$ identifies with a map of triangulated functors

$$f^\# G := f^! \mathcal{O}_Z \otimes_X f^* G \rightarrow f^! G.$$

Note, however, that whereas the isomorphism $f^!(G[1]) \xrightarrow{\sim} (f^! G)[1]$ associated to the triangulated structure on $f^!$ is the identity map, the same is not true of $f^\#$. (See [Lip09, just before 1.5.5].)

(g) The *base-change map*

$$\beta_{\clubsuit}(F): v^* f^! F \longrightarrow g^! u^* F \quad (2.6.7)$$

is associated to a fiber square in $\mathcal{E}$

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ g \downarrow & \clubsuit & \downarrow f \\ Z' & \xrightarrow[u]{} & Z, \end{array}$$

with f any $\mathcal{E}$ -map and u (hence v) flat. This map is defined in [Nee17, 11.6 and 11.7] and denoted by $\theta: \mathfrak{R} \rightarrow \mathfrak{S}$, where $\mathfrak{R}$ and $\mathfrak{S}$ are the maps taking the cartesian square to $v^* f^!$ and $g^! u^*$, respectively.

In general, $\beta_{\clubsuit}(F)$ is not an isomorphism. However, if $F \in \mathbf{D}_{\text{qc}}^+(Z)$, then $\beta_{\clubsuit}(F)$ is the isomorphism of [Lip09, Theorem 4.8.3] and [Nay09, Theorem 5.3], and if f is perfect, then for any $F \in \mathbf{D}_{\text{qc}}(Z)$, the map $\beta_{\clubsuit}(F)$ is the isomorphism of [AJL11, § 5.8.4]. (It actually suffices for g to be perfect; see [Nee17, 11.13(i)].)

(h) The bifunctorial map

$$\kappa_X(E_1, E_2, E_3): \mathcal{H}om_X(E_1, E_2) \otimes_X E_3 \longrightarrow \mathcal{H}om_X(E_1, E_2 \otimes_X E_3) \quad (2.6.8)$$

is adjoint to the natural composite (with ev the evaluation map (2.2.2))

$$\mathcal{H}om_X(E_1, E_2) \otimes_X E_3 \otimes_X E_1 \xrightarrow{\sim} \mathcal{H}om_X(E_1, E_2) \otimes_X E_1 \otimes_X E_3 \xrightarrow{\text{via ev}} E_2 \otimes_X E_3.$$

The map (2.6.8) is an *isomorphism* if the complex E_1 is *perfect*. Indeed, the question is local on X , so one can assume that E_1 is a bounded complex of finite-rank, free $\mathcal{O}_X$ -modules, and conclude via a simple induction on the number of degrees in which E_1 does not vanish (cf. the second to last paragraph of the proof of [Lip09, 4.6.7]).

Similarly, (2.6.8) is an isomorphism if E_3 is perfect.

The projection map (2.6.3) is compatible with derived tensor product, in the following sense.

LEMMA 2.7. *Let $f: X \rightarrow Z$ be a scheme-map, $E \in \mathbf{D}(X)$ and $F, G \in \mathbf{D}(Z)$. The following natural diagram, where $\otimes$ stands for $\otimes_X$ or $\otimes_Z$, commutes:*

$$\begin{array}{ccccc} (f_* E \otimes F) \otimes G & \xrightarrow{(2.6.3)} & f_*(E \otimes f^* F) \otimes G & \xrightarrow{(2.6.3)} & f_*((E \otimes f^* F) \otimes f^* G) \\ \simeq \downarrow & & & & \downarrow \simeq \\ f_* E \otimes (F \otimes G) & \xrightarrow{(2.6.3)} & f_*(E \otimes f^*(F \otimes G)) & \xrightarrow{(2.6.1)} & f_*(E \otimes (f^* F \otimes f^* G)). \end{array}$$

Proof. After expanding each instance of (2.6.3) according to its definition and recalling (2.2.1), one reduces to proving the commutativity of the border of the following diagram (whose maps

are the obvious ones):

$$\begin{array}{ccccc}
 (f_*E \otimes F) \otimes G & \longrightarrow & (f_*E \otimes f_*f^*F) \otimes G & \longrightarrow & f_*(E \otimes f^*F) \otimes G \\
 \downarrow & & \searrow & \searrow & \downarrow \\
 f_*E \otimes (F \otimes G) & \longrightarrow & f_*E \otimes (f_*f^*F \otimes G) & & f_*(E \otimes f^*F) \otimes f_*f^*G \\
 \downarrow & & \searrow & \searrow & \downarrow \\
 f_*E \otimes f_*f^*(F \otimes G) & \longrightarrow & f_*E \otimes f_*(f^*F \otimes f^*G) & \text{⑦} & f_*((E \otimes f^*F) \otimes f^*G) \\
 \downarrow & & \searrow & & \downarrow \\
 f_*(E \otimes f^*(F \otimes G)) & \longrightarrow & f_*(E \otimes (f^*F \otimes f^*G)) & & f_*(E \otimes (f^*F \otimes f^*G))
 \end{array}$$

⑥ $f_*E \otimes (f_*f^*F \otimes f_*f^*G) \longrightarrow f_*(E \otimes f^*F) \otimes f_*f^*G$

The commutativity of subdiagram ⑥ follows directly from the definition of (2.6.1). The commutativity of ⑦ is given by the symmetric monoidality of f_* ; see Section 2.2. The commutativity of the other subdiagrams is pretty obvious, whence the conclusion. $\square$

For the map (2.6.8) we will need a “transitivity” property, an instance of the Kelly–Mac Lane coherence theorem [KM71, Theorem 2.4, p. 107].

LEMMA 2.8. *With $\kappa := \kappa_X$ as in (2.6.8), the map $\kappa(E_1, E_2, E_3 \otimes E_4)$ factors as*

$$\begin{aligned}
 \mathcal{H}om_X(E_1, E_2) \otimes E_3 \otimes E_4 & \xrightarrow{\kappa(E_1, E_2, E_3) \otimes \text{id}} \mathcal{H}om_X(E_1, E_2 \otimes E_3) \otimes E_4 \\
 & \xrightarrow{\kappa(E_1, E_2 \otimes E_3, E_4)} \mathcal{H}om_X(E_1, E_2 \otimes E_3 \otimes E_4).
 \end{aligned}$$

Proof. The assertion results from the commutativity of the following natural diagram, where $[-, -]$ denotes $\mathcal{H}om_X(-, -)$ and $\otimes$ denotes $\otimes_X$, and various associativity isomorphisms are omitted:

$$\begin{array}{ccccc}
 [E_1, E_2] \otimes E_3 \otimes E_4 \otimes E_1 & \xrightarrow{\sim} & [E_1, E_2] \otimes E_3 \otimes E_1 \otimes E_4 & \xrightarrow{\sim} & [E_1, E_2] \otimes E_1 \otimes E_3 \otimes E_4 \\
 \downarrow \text{via } \kappa & \text{①} & \downarrow \text{via } \kappa & \text{②} & \downarrow \text{via ev} \\
 [E_1, E_2 \otimes E_3] \otimes E_4 \otimes E_1 & \xrightarrow{\sim} & [E_1, E_2 \otimes E_3] \otimes E_1 \otimes E_4 & \xrightarrow{\sim} & E_2 \otimes E_3 \otimes E_4 \\
 & \text{via } \kappa & \text{③} & \text{via ev} & \\
 & & [E_1, E_2 \otimes E_3 \otimes E_4] \otimes E_1 & \xrightarrow{\sim} & E_2 \otimes E_3 \otimes E_4
 \end{array}$$

The commutativity of subdiagram ① is clear, and that of ② and ③ follows from the definitions of $\kappa(E_1, E_2, E_3)$ and $\kappa(E_1, E_2 \otimes E_3, E_4)$, respectively. $\square$

2.9. Let $\delta: X \rightarrow Y$ be an $\mathcal{E}$ -map and $p: Y \rightarrow X$ a scheme-map such that $p\delta = \text{id}_X$. We will be using the bifunctorial isomorphism

$$\psi = \psi(\delta, p, E, F): \delta^* \delta_* E \otimes_X F \xrightarrow{\sim} \delta^* \delta_*(E \otimes_X F), \quad E, F \in \mathbf{D}_{\text{qc}}(X)$$

that is defined as the natural composite

$$\begin{array}{ccc}
 \delta^* \delta_* E \otimes_X F & & \delta^* \delta_*(E \otimes_X F) \\
 \simeq \downarrow & & \uparrow \simeq \\
 \delta^* \delta_* E \otimes_X \delta^* p^* F & \xrightarrow[\text{(2.6.1)}]{\sim} \delta^*(\delta_* E \otimes_Y p^* F) & \xrightarrow[\text{(2.6.3)}]{\sim} \delta^* \delta_*(E \otimes_X \delta^* p^* F)
 \end{array} \tag{2.9.1}$$

(cf. [AJL14, (2.2.6)]).

The rest of this subsection brings out properties of ψ needed later on.

LEMMA 2.9.2. *With the preceding notation, the isomorphism $\psi(\delta, p, E, \delta^* p^* F)$ factors as*

$$\delta^* \delta_* E \otimes_X \delta^* p^* F \xrightarrow[\text{(2.6.1)}]{\sim} \delta^*(\delta_* E \otimes_Y p^* F) \xrightarrow[\text{(2.6.3)}]{\sim} \delta^* \delta_*(E \otimes_X \delta^* p^* F).$$

Proof. This results from the fact that the natural composites

$$\delta^* \delta_* E \otimes F \xrightarrow{\sim} \delta^* \delta_* E \otimes \delta^* p^* F \xrightarrow{\psi(\delta, p, E, \delta^* p^* F)} \delta^* \delta_*(E \otimes \delta^* p^* F) \xrightarrow{\sim} \delta^* \delta_*(E \otimes F)$$

and

$$\delta^* \delta_* E \otimes F \xrightarrow{\sim} \delta^* \delta_* E \otimes \delta^* p^* F \xrightarrow{(\text{2.6.3}) \circ (\text{2.6.1})} \delta^* \delta_*(E \otimes \delta^* p^* F) \xrightarrow{\sim} \delta^* \delta_*(E \otimes F)$$

are both equal to $\psi(\delta, p, E, F)$, the first by the functoriality of ψ and the second by definition. $\square$

COROLLARY 2.9.3. *The following natural diagram commutes:*

$$\begin{array}{ccc}
 \delta^* \delta_* E \otimes F & \xrightarrow{\psi} & \delta^* \delta_*(E \otimes F) \\
 & \searrow \quad \swarrow & \\
 & E \otimes F &
 \end{array}$$

Proof. One can replace F with the isomorphic complex $\delta^* p^* F$, whereupon the assertion follows from Lemma 2.9.2 via [Lip09, 3.4.6.2]. $\square$

LEMMA 2.9.4. *For $E, F_1, F_2 \in \mathbf{D}_{\text{qc}}(X)$, with $\psi(-, -) := \psi(\delta, p, -, -)$ and $\otimes := \otimes_X$, the isomorphism $\psi(E, F_1 \otimes F_2)$ factors (modulo associativity isomorphisms) as*

$$\begin{aligned}
 \delta^* \delta_* E \otimes F_1 \otimes F_2 & \xrightarrow[\psi(E, F_1) \otimes \text{id}]{\sim} \delta^* \delta_*(E \otimes F_1) \otimes F_2 \\
 & \xrightarrow[\psi(E \otimes F_1, F_2)]{\sim} \delta^* \delta_*(E \otimes F_1 \otimes F_2).
 \end{aligned}$$

Proof. The lemma asserts the commutativity of the border of the following diagram, in which

$\otimes$ stands for $\otimes_X$ or $\otimes_Y$ and the maps are the obvious ones:

$$\begin{array}{ccccc}
 \delta^* \delta_* E \otimes F_1 \otimes F_2 & \xrightarrow{\quad} & \delta^*(\delta_* E \otimes p^* F_1) \otimes F_2 & & \\
 \downarrow & & \swarrow & \searrow & \downarrow \\
 & \textcircled{1} & \delta^*(\delta_* E \otimes p^* F_1) \otimes \delta^* p^* F_2 & & \delta^* \delta_*(E \otimes p^* F_1) \otimes F_2 \\
 \delta^*(\delta_* E \otimes p^*(F_1 \otimes F_2)) & \swarrow & \downarrow & \swarrow & \downarrow \\
 & \delta^*(\delta_* E \otimes p^* F_1 \otimes p^* F_2) & & \delta^* \delta_*(E \otimes \delta^* p^* F_1) \otimes \delta^* p^* F_2 & \delta^* \delta_*(E \otimes F_1) \otimes F_2 \\
 \downarrow & \downarrow & \searrow & \downarrow & \downarrow \\
 \delta^* \delta_*(E \otimes \delta^* p^*(F_1 \otimes F_2)) & \swarrow & \delta^* \delta_*(E \otimes \delta^* p^* F_1 \otimes p^* F_2) & \swarrow & \delta^* \delta_*(E \otimes F_1) \otimes \delta^* p^* F_2 \\
 & \textcircled{2} & & & \\
 \delta^* \delta_*(E \otimes \delta^* p^*(F_1 \otimes F_2)) & \downarrow & \delta^*(\delta_* (E \otimes \delta^* p^* F_1) \otimes p^* F_2) & \searrow & \downarrow \\
 & \delta^* \delta_*(E \otimes \delta^* p^* F_1 \otimes \delta^* p^* F_2) & & & \delta^* \delta_*(E \otimes F_1) \otimes \delta^* p^* F_2 \\
 \downarrow & \downarrow & \swarrow & \searrow & \downarrow \\
 \delta^* \delta_*(E \otimes F_1 \otimes F_2) & \xleftarrow{\quad} & \delta^* \delta_*(E \otimes F_1 \otimes \delta^* p^* F_2) & \xleftarrow{\quad} & \delta^*(\delta_* (E \otimes F_1) \otimes p^* F_2) . \\
 & \textcircled{3} & & &
 \end{array}$$

The commutativity of the unlabeled subdiagrams is easy to verify. Subdiagram ① expands, naturally, to

$$\begin{array}{ccccc}
 \delta^* \delta_* E \otimes F_1 \otimes F_2 & \xrightarrow{\quad} & \delta^* \delta_* E \otimes \delta^* p^* F_1 \otimes F_2 & \xrightarrow{\quad} & \delta^*(\delta_* E \otimes p^* F_1) \otimes F_2 \\
 \downarrow & & \downarrow & & \downarrow \\
 \delta^* \delta_* E \otimes \delta^* p^*(F_1 \otimes F_2) & \xrightarrow{\quad} & \delta^* \delta_* E \otimes \delta^* p^* F_1 \otimes \delta^* p^* F_2 & & \\
 \downarrow & \searrow & \swarrow & \searrow & \downarrow \\
 \delta^*(\delta_* E \otimes p^*(F_1 \otimes F_2)) & \xrightarrow{\quad} & \delta^*(\delta_* E \otimes p^* F_1 \otimes p^* F_2) & \xleftarrow{\quad} & \delta^*(\delta_* E \otimes p^* F_1) \otimes \delta^* p^* F_2 . \\
 & \textcircled{4} & \textcircled{5} & &
 \end{array}$$

For the commutativity of ④, see the proof of [Lip09, 3.6.10]. For that of ⑤, see [Lip09, 3.4.4(b)]. The commutativity of the other two subdiagrams is clear, and so ① commutes. The commutativity of ② is given by Lemma 2.7. Finally, for the commutativity of ③, see again the proof of [Lip09, 3.6.10]. $\square$

COROLLARY 2.9.5. *Upon identifying, as one may, $G \otimes \mathcal{O}_X$ with G for all $G \in \mathbf{D}(X)$, one has that $\psi := \psi(E, \mathcal{O}_X)$ is the identity map of $\delta^* \delta_* E$.*

Proof. The case $F_1 = F_2 = \mathcal{O}_X$ of Lemma 2.9.4 implies $\psi^2 = \psi$, so $\psi = \psi^2 \psi^{-1} = \psi \psi^{-1} = \text{id}$. $\square$

3. Verdier's isomorphism

Recall from Section 2.1 that, unless otherwise specified, all the functors that appear are functors between derived categories.

3.1. Let $U \xrightarrow{i} X \xrightarrow{f} Z$ be $\mathcal{E}$ -maps, with i essentially étale, and let

$$\begin{array}{ccc} Y := U \times_Z X & \xrightarrow{p_2} & X \\ p_1 \downarrow & \spadesuit & \downarrow f \\ U & \xrightarrow{fi} & Z \end{array}$$

be the resulting fiber square, where p_1 and p_2 are the canonical projections. With δ_U the diagonal map and id_U the identity map of U , the composite

$$g: U \xrightarrow{\delta_U} U \times_Z U \xrightarrow{\text{id}_U \times i} U \times_Z X \quad (3.1.1)$$

is the graph of i ; that is, $p_1 g = \text{id}_U$ and $p_2 g = i$.

In [Ver69, Theorem 3], the map fi is essentially smooth, of relative dimension, say, d (see Section 2.3 above). That theorem says that *there is a $\mathbf{D}(U)$ -isomorphism*

$$(fi)^! \mathcal{O}_Z \xrightarrow{\sim} \Omega_{fi}^d[d] := (\wedge^d \Omega_{fi})[d].$$

A slight elaboration of Verdier's approach produces, as follows, a functorial isomorphism

$$v_{f,i}(F): (fi)^! F \xrightarrow{\sim} \Omega_{fi}^d[d] \otimes_U (fi)^* F, \quad F \in \mathbf{D}_{\text{qc}}^+(Z). \quad (3.1.2)$$

In particular, making the allowable identifications

$$\Omega_{fi}^d[d] \otimes_X (fi)^* \mathcal{O}_Z = \Omega_{fi}^d[d] \otimes_X \mathcal{O}_X = \Omega_{fi}^d[d],$$

we consider $v_{f,i}(\mathcal{O}_Z)$ to be a map from $(fi)^! \mathcal{O}_Z$ to $\Omega_{fi}^d[d]$.

(In Section 3.4, the definition of $v_{f,i}(F)$ will be extended to all $F \in \mathbf{D}_{\text{qc}}(Z)$.)

The construction of $v_{f,i}(F)$ uses two maps. The first is the following natural composite map ϑ , which exists whenever fi is flat, with χ_g as in (2.6.6) and $\beta_{\spadesuit}$ as in (2.6.7) (note that $\beta_{\spadesuit}$ is an isomorphism since $F \in \mathbf{D}_{\text{qc}}^+(Y)$):

$$\begin{aligned} \vartheta: g^! \mathcal{O}_Y \otimes_U (fi)^! F &\xrightarrow{\sim} g^! \mathcal{O}_Y \otimes_U i^* f^! F \xrightarrow{\sim} g^! \mathcal{O}_Y \otimes_U g^* p_2^* f^! F \\ &\xrightarrow{\chi_g} g^! p_2^* f^! F \\ &\xrightarrow{\beta_{\spadesuit}} g^! p_1^! (fi)^* F \\ &\xrightarrow{\sim} (fi)^* F. \end{aligned} \quad (3.1.3)$$

If fi is essentially smooth, then in (3.1.1), both $\text{id}_U \times i$ and the regular immersion δ_U are perfect, whence so is g (see Section 2.4), so that χ_g is an isomorphism². Thus, in this case, ϑ is an isomorphism.

²As δ_U is finite and $\text{id}_U \times i$ is essentially étale, one can see this more concretely by showing that $\delta_{U*}(\chi)$ is isomorphic to the natural map $\mathcal{H}\text{om}(\delta_{U*} \mathcal{O}_U, \mathcal{O}_{U \times_Z U}) \otimes G \rightarrow \mathcal{H}\text{om}(\delta_{U*} \mathcal{O}_U, G)$ for $G := (\text{id}_U \times i)^* p_2^* f^! F$.

The second map is an isomorphism that holds when fi is essentially smooth of relative dimension d ,

$$g^! \mathcal{O}_Y \xrightarrow{\sim} \mathcal{H}om_U(\Omega_{fi}^d, \mathcal{O}_U)[-d]; \quad (3.1.4)$$

it is described in [Har66, Corollary 7.3, p. 180] or [Con00, § 2.5] via the “fundamental local isomorphism” and Cartan–Eilenberg resolutions. An avatar (3.2.8) of (3.1.4) is reviewed in Section 3.2 below.

The $\mathcal{O}_U$ -module Ω_{fi}^d is invertible, so the complexes $\mathcal{H}om_U(\Omega_{fi}^d, \mathcal{O}_U)[-d]$ and $\mathcal{H}om_U(\Omega_{fi}^d[d], \mathcal{O}_U)$ are naturally isomorphic in $\mathbf{D}(U)$ to the complex G which is $(\Omega_{fi}^d)^{-1}$ in degree d and 0 elsewhere. Modulo these isomorphisms, the isomorphism

$$\mathcal{H}om_U(\Omega_{fi}^d, \mathcal{O}_U)[-d] \xrightarrow{\sim} \mathcal{H}om_U(\Omega_{fi}^d[d], \mathcal{O}_U), \quad (3.1.5)$$

resulting from the usual triangulated structure on the functor $\mathcal{H}om(-, \mathcal{O}_U)$, is given by scalar multiplication in G by $(-1)^{d^2+d(d-1)/2} = (-1)^{d(d+1)/2}$ (cf. [Con00, (1.3.8), p. 11]).

Let

$$\gamma: g^! \mathcal{O}_Y \xrightarrow{\sim} \mathcal{H}om_U(\Omega_{fi}^d[d], \mathcal{O}_U) \quad (3.1.6)$$

be the isomorphism obtained by composition from (3.1.4) and (3.1.5).

When fi is essentially smooth, so that Ω_{fi}^d is an *invertible* $\mathcal{O}_U$ -module, one thus has a chain of natural functorial isomorphisms, where the first is inverse to the evaluation map (2.2.2):

$$\begin{aligned} (fi)^! F &\xrightarrow{\sim} \mathcal{H}om_U(\Omega_{fi}^d[d], (fi)^! F) \otimes_U \Omega_{fi}^d[d] \\ &\xrightarrow[\text{(2.6.8)}]{\sim} \mathcal{H}om_U(\Omega_{fi}^d[d], \mathcal{O}_U) \otimes_U (fi)^! F \otimes_U \Omega_{fi}^d[d] \\ &\xrightarrow[\text{(3.1.6)}]{\sim} g^! \mathcal{O}_Y \otimes_U (fi)^! F \otimes_U \Omega_{fi}^d[d] \\ &\xrightarrow[\text{(3.1.3)}]{\sim} (fi)^* F \otimes_U \Omega_{fi}^d[d] \xrightarrow{\sim} \Omega_{fi}^d[d] \otimes_U (fi)^* F. \end{aligned} \quad (3.1.7)$$

The map $v_{f,i}(F)$ is defined to be the composition of this chain.

3.2. Expanding a bit on [LS81, § 2, II], we review the relation (3.2.7) between the normal bundle of a regular immersion $\delta: U \rightarrow W$ and the relative dualizing complex $\delta^! \mathcal{O}_W$ and from that, deduce the isomorphism (3.1.4).

Let $\delta: U \rightarrow W$ be any closed immersion of schemes, and let I be the kernel of the associated surjective map $s: \mathcal{O}_W \rightarrow \delta_* \mathcal{O}_U$. Then

$$\mathcal{O}_U \cong H^0 \delta^* \mathcal{O}_W \xrightarrow{H^0 \delta^* s} H^0 \delta^* \delta_* \mathcal{O}_U \cong \mathcal{O}_U$$

is an *isomorphism*.

The natural triangle

$$\delta^* I \longrightarrow \delta^* \mathcal{O}_W \xrightarrow{\delta^* s} \delta^* \delta_* \mathcal{O}_U \xrightarrow{+}$$

gives rise to an exact sequence of $\mathcal{O}_U$ -modules

$$\begin{array}{ccccc} H^{-1} \delta^* \mathcal{O}_W & \longrightarrow & H^{-1} \delta^* \delta_* \mathcal{O}_U & \xrightarrow{t} & H^0 \delta^* I & \longrightarrow & \ker(H^0 \delta^* s) \\ \parallel & & & & \parallel & & \parallel \\ 0 & & & & I/I^2 & & 0. \end{array}$$

Clearly, t is an isomorphism, and so one has the natural $\mathcal{O}_U$ -isomorphism

$$t^{-1}: I/I^2 \xrightarrow{\sim} H^{-1}\delta^*\delta_*\mathcal{O}_U. \quad (3.2.1)$$

(Note that the isomorphism t is induced by the projection $C[-1] \rightarrow P$, where $K \rightarrow I$ is a flat resolution of I and C is the mapping cone of the composite map $K \rightarrow I \hookrightarrow \mathcal{O}_W$. It is the *negative* of the usual connecting homomorphism

$$\mathrm{Tor}_1^{\mathcal{O}_W}(\delta_*\mathcal{O}_U, \delta_*\mathcal{O}_U) \longrightarrow \mathrm{Tor}_0^{\mathcal{O}_W}(\delta_*\mathcal{O}_U, I)$$

attached to the natural exact sequence $0 \rightarrow I \rightarrow \mathcal{O}_W \rightarrow \delta_*\mathcal{O}_U \rightarrow 0$; see [Lip09, end of § 1.4].)

There is an alternating graded $\mathcal{O}_U$ -algebra structure on $\oplus_{n \geq 0} H^{-n}\delta^*\delta_*\mathcal{O}_U$, induced by the natural product map

$$\delta^*\delta_*\mathcal{O}_U \otimes_U \delta^*\delta_*\mathcal{O}_U \xrightarrow{\sim} \delta^*(\delta_*\mathcal{O}_U \otimes_W \delta_*\mathcal{O}_U) \longrightarrow \delta^*\delta_*(\mathcal{O}_U \otimes_U \mathcal{O}_U) \xrightarrow{\sim} \delta^*\delta_*\mathcal{O}_U$$

(cf., for example, [Bou07, Exercise 9(c), p. 201]). Hence (3.2.1) extends uniquely to a homomorphism of graded $\mathcal{O}_U$ -algebras

$$\oplus_{n \geq 0} \wedge^n(I/I^2) \longrightarrow \oplus_{n \geq 0} H^{-n}\delta^*\delta_*\mathcal{O}_U. \quad (3.2.2)$$

For example, over an affine open subset of W , if I is generated by a regular sequence of global sections $\mathbf{t} := (t_1, t_2, \dots, t_d)$, then a finite, free resolution of $\delta_*\mathcal{O}_U$ is provided by the Koszul complex $K(\mathbf{t}) := \otimes_{i=1}^d K_i$, where K_i is the complex which is $\mathcal{O}_W \xrightarrow{t_i} \mathcal{O}_W$ in degrees -1 and 0 and vanishes elsewhere; and there results a $\mathbf{D}(U)$ -map

$$\oplus_{n=0}^d \wedge^n(I/I^2)[n] \cong \delta^*K(\mathbf{t}) \xrightarrow{\sim} \delta^*\delta_*\mathcal{O}_U. \quad (3.2.3)$$

(Note that $\delta^*K(\mathbf{t})$ is just the exterior algebra, with vanishing differentials, on $\mathcal{O}_U^n$, which is isomorphic to I/I^2 via the natural map

$$\mathcal{O}_W^n = K^{-1}(\mathbf{t}) \hookrightarrow I \subset \mathcal{O}_W.)$$

One verifies that applying the functor H^{-n} to (3.2.3) produces the degree n component of (3.2.2), which is a map that *does not depend on the choice of the generating family $\mathbf{t}$* .

Next, for any $\mathcal{O}_U$ -complex E with $H^e E = 0$ for all $e < 0$, the natural map $H^0 E \rightarrow E$ induces a map

$$H^0 \mathcal{H}\mathrm{om}_U(E, \mathcal{O}_U) \longrightarrow H^0 \mathcal{H}\mathrm{om}(H^0 E, \mathcal{O}_U) =: \mathcal{H}\mathrm{om}^0(H^0 E, \mathcal{O}_U). \quad (3.2.4)$$

Hence for any integer n and any $\mathcal{O}_U$ -complex F such that $H^e F = 0$ for all $e < -n$, one has the natural composite

$$\begin{aligned} H^n \mathcal{H}\mathrm{om}_U(F, \mathcal{O}_U) &\xrightarrow{\sim} H^0 \mathcal{H}\mathrm{om}_U(F, \mathcal{O}_U)[n] \xrightarrow{\sim} H^0 \mathcal{H}\mathrm{om}_U(F[-n], \mathcal{O}_U) \\ &\xrightarrow[\text{(3.2.4)}]{\sim} \mathcal{H}\mathrm{om}^0(H^0(F[-n]), \mathcal{O}_U) \xrightarrow{\sim} \mathcal{H}\mathrm{om}^0(H^{-n}F, \mathcal{O}_U). \end{aligned} \quad (3.2.5)$$

(The second map, whose source and target are equal, is multiplication by $(-1)^{n(n+1)/2}$: replace $\mathcal{O}_U$ with a quasi-isomorphic injective complex and take $p = 0$ and $m = -n$ in the expression $(-1)^{pm+m(m-1)/2}$ appearing after equation (1.3.9) in [Con00, p. 11].)

Now, if the closed immersion δ is *regular*, of *codimension d* , that is, I is generated locally by regular sequences $\mathbf{t}$ of length d , so that $\delta_*\mathcal{O}_U$ is locally resolved by complexes of the form $K(\mathbf{t})$,

then we obtain the sequence of *isomorphisms*

$$\begin{aligned}
 H^d \delta^* \delta_* \delta^! \mathcal{O}_W &\xrightarrow[\text{(1.2.1)}]{\sim} H^d \delta^* \mathcal{H}om_W(\delta_* \mathcal{O}_U, \mathcal{O}_W) \\
 &\xrightarrow[\text{(2.6.4)}]{\sim} H^d \mathcal{H}om_U(\delta^* \delta_* \mathcal{O}_U, \mathcal{O}_U) \\
 &\xrightarrow[\text{(3.2.5)}]{\sim} \mathcal{H}om_U^0(H^{-d} \delta^* \delta_* \mathcal{O}_U, \mathcal{O}_U) \\
 &\xrightarrow[\text{(3.2.2)}]{\sim} \mathcal{H}om_U^0(\wedge^d(I/I^2), \mathcal{O}_U).
 \end{aligned} \tag{3.2.6}$$

In particular, there is a canonical $\mathcal{O}_U$ -*isomorphism*

$$H^d \delta^* \delta_* \delta^! \mathcal{O}_W \xrightarrow{\sim} \mathcal{H}om_U^0(\wedge^d(I/I^2), \mathcal{O}_U) =: \nu_\delta.$$

Moreover, since $\delta_* \delta^! \mathcal{O}_W \cong \mathcal{H}om_W(\delta_* \mathcal{O}_U, \mathcal{O}_W)$ has nonvanishing homology only in degree d , the same holds for $\delta^! \mathcal{O}_W$, whence the natural maps are isomorphisms:

$$\delta^! \mathcal{O}_W[d] \xrightarrow{\sim} H^d \delta^! \mathcal{O}_W \xleftarrow{\sim} H^d \delta^* \delta_* \delta^! \mathcal{O}_W.$$

Thus, when δ is a regular immersion, there is a canonical isomorphism

$$\boxed{\delta^! \mathcal{O}_W \xrightarrow{\sim} \nu_\delta[-d]}. \tag{3.2.7}$$

It is left to the interested reader to work out the precise relationship of (3.2.7) to the similar isomorphisms in [Har66, Corollary 7.3, p. 180] and [Con00, § 2.5].

Now, in (3.1.1), if fi is essentially smooth of relative dimension d , then $\delta_U: U \rightarrow W := U \times_Z U$ is a regular immersion and Ω_{fi}^d is locally free of rank one (see Section 2.3), so the natural map $\mathcal{H}om_U^0(\Omega_{fi}^d, \mathcal{O}_U) \xrightarrow{\sim} \mathcal{H}om_U(\Omega_{fi}^d, \mathcal{O}_U)$ is an isomorphism. Thus, there are natural isomorphisms, with $Y := U \times_Z X$,

$$\begin{aligned}
 g^! \mathcal{O}_Y &\xrightarrow{\sim} \delta_U^!(\text{id}_U \times i)^! \mathcal{O}_Y = \delta_U^!(\text{id}_U \times i)^* \mathcal{O}_Y = \delta_U^! \mathcal{O}_W \\
 &\xrightarrow[\text{(3.2.7)}]{\sim} \mathcal{H}om_U^0(\wedge^d(I/I^2), \mathcal{O}_U)[-d] = \mathcal{H}om_U(\Omega_{fi}^d, \mathcal{O}_U)[-d].
 \end{aligned} \tag{3.2.8}$$

The isomorphism (3.1.4) is defined to be the resulting composition.

3.3. Though it is not *a priori* clear, $v_{f,i}$ depends only on the map fi and not on its factorization into f and i . (In [Ver69], the map f is assumed proper.)

PROPOSITION 3.3.1. *For $U \xrightarrow{i} X \xrightarrow{f} Z$ as in Section 3.1, if fi is essentially smooth, one has $v_{f,i} = v_{fi, \text{id}_U}$.*

Proof. There is a commutative $\mathcal{E}$ -diagram, with i, f, p_1, p_2 and g as in Section 3.1, δ the diagonal map, and q_1 and q_2 the canonical projections. The labels $\spadesuit, \clubsuit$ and $\heartsuit$ indicate the front, top and rear faces of the cube, respectively. (These three faces are fiber squares; since fi is flat, so are p_2 and q_2 .)

Note that i is essentially étale, and therefore $i^! = i^*$ and $j^! = j^*$.

There is, as in (3.2.8), a natural composite isomorphism

$$\xi: g^! \mathcal{O}_Y \xrightarrow{\sim} \delta^! j^! \mathcal{O}_Y = \delta^! j^* \mathcal{O}_Y \xrightarrow{\sim} \delta^! \mathcal{O}_W.$$

A detailed examination of the definition of $v_{f,i}$ in (3.1.7), taking into account the definition (3.2.8) of the map (3.1.4), shows that it will suffice to prove the commutativity of the border of the natural functorial diagram (3.3.2) below.

The commutativity of subdiagrams ① and ⑤ results from the pseudofunctoriality of $(-)^*$ and $(-)^!$, respectively. That of ② is given by the transitivity of χ with respect to composition of maps; see [Nee17, 13.4] (or [Lip09, Exercices 4.7.3.4(d) and 4.9.3(d)]). The commutativity of ③ follows from the fact that for any $F \in \mathbf{D}_{\text{qc}}^+(Y)$, the following natural diagram commutes:

$$\begin{array}{ccccc} j^! \mathcal{O}_Y \otimes j^* F & \xlongequal{\quad} & j^* \mathcal{O}_Y \otimes j^* F & \xrightarrow{\sim} & \mathcal{O}_W \otimes j^* F \\ \chi_j \downarrow & \textcircled{3}_1 & \downarrow \simeq & \textcircled{3}_2 & \parallel \\ j^! (\mathcal{O}_Y \otimes F) & \xlongequal{\quad} & j^* (\mathcal{O}_Y \otimes F) & \xlongequal{\quad} & j^* F. \end{array}$$

For a sketch of the proof that ③₁ commutes, see [Lip09, 4.9.2.3]. As for commutativity of ③₂, replacing F with a quasi-isomorphic, flat complex reduces the problem to the context of ordinary (nonderived) functors, at which point the justification is left to the reader. The commutativity of ④ is given by the transitivity of β with respect to the juxtaposition of fiber squares (see [Nee17, Theorem 11.9] or [Lip09, Theorem 4.8.3]), as applied to the following decomposition of

the fiber square $\heartsuit$:

$$\begin{array}{ccccc}
 \bullet & \xrightarrow{q_2} & \bullet & & \\
 j \downarrow & & \clubsuit & & \downarrow i \\
 \bullet & \xrightarrow{p_2} & \bullet & & \\
 p_1 \downarrow & & \spadesuit & & \downarrow f \\
 \bullet & \xrightarrow{fi} & \bullet & &
 \end{array}$$

Here one needs that $\beta_{\clubsuit}$ is the canonical isomorphism $q_2^* i^* \xrightarrow{\sim} j^* p_2^*$ (see [Nee17, 11.4 and 11.5] with $p = p' = \text{identity map}$ in the diagram of [Nee17, 11.4(i)], or [Lip09, 4.8.8(i)]).

The commutativity of the remaining subdiagrams is easy to check, whence the assertion. $\square$

Henceforth, accordingly, we restrict ourselves to the case $U = X$ and $i = \text{id}_X$. The map $g: X \rightarrow X \otimes_Z X$ is then the diagonal.

$$\begin{array}{c}
 \begin{array}{ccccccc}
 g^! \mathcal{O}_Y \otimes (fi)^! & \xrightarrow{\text{via } \xi} & & & \delta^! \mathcal{O}_W \otimes (fi)^! & & \\
 \simeq \downarrow & & & & \downarrow \simeq & & \\
 g^! \mathcal{O}_Y \otimes i^* f^! & \longrightarrow & \delta^! j^! \mathcal{O}_Y \otimes i^* f^! & \xlongequal{\quad} & \delta^! j^* \mathcal{O}_Y \otimes i^* f^! & & \\
 \simeq \downarrow & & \textcircled{1} & & \downarrow \simeq & & \\
 g^! \mathcal{O}_Y \otimes g^* p_2^* f^! & \longrightarrow & \delta^! j^! \mathcal{O}_Y \otimes \delta^* j^* p_2^* f^! & \longrightarrow & \delta^! \mathcal{O}_W \otimes \delta^* q_2^* i^* f^! & & \\
 \chi_g \downarrow & \textcircled{2} & \chi_\delta \downarrow & & \downarrow \chi_\delta & & \\
 g^! (\mathcal{O}_Y \otimes g^* p_2^* f^!) & & \delta^! (j^! \mathcal{O}_Y \otimes j^* p_2^* f^!) & \longrightarrow & \delta^! (\mathcal{O}_W \otimes q_2^* i^* f^!) & & \\
 \parallel & & \delta^! \chi_j \downarrow & \nearrow & \parallel & & \\
 g^! p_2^* f^! & \longrightarrow & \delta^! j^! p_2^* f^! = \delta^! j^* p_2^* f^! & \longrightarrow & \delta^! q_2^* i^* f^! & \longrightarrow & \delta^! q_2^* (fi)^! \\
 g^! \beta_{\spadesuit} \downarrow & & \text{via } \beta_{\clubsuit} \downarrow & & \textcircled{4} & & \downarrow \delta^! \beta_{\heartsuit} \\
 g^! p_1^! (fi)^* & \longrightarrow & \delta^! j^! p_1^! (fi)^* & \longrightarrow & \delta^! (p_1 j)^! (fi)^* = \delta^! q_1^! (fi)^* & & \\
 & \searrow \simeq & \textcircled{5} & \swarrow \simeq & & & \\
 & & (fi)^* & & & &
 \end{array}
 \end{array}
 \tag{3.3.2}$$

3.4. Again, let $f: X \rightarrow Z$ be an essentially smooth $\mathcal{E}$ -map of relative dimension d . With χ_f as in (2.6.6), define $v_f(F)$ for $F \in \mathbf{D}_{\text{qc}}(Z)$ to be the composite isomorphism

$$v_f(F): f^! F \xrightarrow[\chi_f^{-1}]{\sim} f^! \mathcal{O}_Z \otimes_X f^* F \xrightarrow[v_{f, \text{id}_X}(\mathcal{O}_Z) \otimes \text{id}]{\sim} \Omega_f^d[d] \otimes_X f^* F. \tag{3.4.1}$$

As before (just after (3.1.2)), the identifications

$$\Omega_f^d[d] \otimes_X f^* \mathcal{O}_Z = \Omega_f^d[d] \otimes_X \mathcal{O}_X = \Omega_f^d[d]$$

allow us to consider $v_f(\mathcal{O}_Z)$ as a map from $f^! \mathcal{O}_Z$ to $\Omega_f^d[d]$.

PROPOSITION 3.4.2. *If $F \in \mathbf{D}_{qc}^+(Z)$, then $v_f(F) = v_{f, \text{id}_X}(F)$.*

Proof. Let $g: X \rightarrow Y := X \times_Z X$ be the diagonal map. Set $\omega := \Omega_f^d[d]$ (a perfect complex), and set $[A, B] := \mathcal{H}om_X(A, B)$ for $A, B \in \mathbf{D}(X)$. Let $\kappa = \kappa_X(\omega, -, -)$ be as in (2.6.8), ϑ as in (3.1.3), and γ as in (3.1.6) (the last two with $i := \text{id}_X$). Since f is essentially smooth, all of these maps are isomorphisms. By definition (see (3.1.7)), Proposition 3.4 says that the border of the following diagram (in which $\otimes := \otimes_X$) commutes:

$$\begin{array}{c}
 \begin{array}{ccc}
 f^! \mathcal{O}_Z \otimes f^* F & \xlongequal{\quad} & f^! \mathcal{O}_Z \otimes f^* F \\
 \swarrow \chi_f \quad \searrow \simeq \quad \nearrow \textcircled{1} & & \\
 f^! F & & g^! \mathcal{O}_Y \otimes f^! \mathcal{O}_Z \otimes \omega \otimes f^* F \\
 \downarrow \simeq & \searrow [\omega, f^! \mathcal{O}_Z] \otimes \omega \otimes f^* F & \downarrow v_{f, \text{id}_X}(\mathcal{O}_Z) \otimes \text{id} \\
 [\omega, f^! \mathcal{O}_Z \otimes f^* F] \otimes \omega & \xrightarrow{\simeq} & \omega \otimes f^* F \\
 \swarrow \chi_f & \searrow \kappa & \uparrow \gamma^{-1} \\
 [\omega, f^! F] \otimes \omega & \xrightarrow{\kappa^{-1}} & f^* \mathcal{O}_Z \otimes \omega \otimes f^* F \\
 \downarrow \kappa^{-1} & \nearrow \kappa^{-1} & \uparrow \vartheta \\
 [\omega, \mathcal{O}_X] \otimes f^! \mathcal{O}_Z \otimes f^* F \otimes \omega & \xrightarrow{\gamma^{-1}} & g^! \mathcal{O}_Y \otimes f^! \mathcal{O}_Z \otimes f^* F \otimes \omega \\
 \swarrow \chi_f & \searrow \chi_f & \downarrow \vartheta \\
 [\omega, \mathcal{O}_X] \otimes f^! F \otimes \omega & \xrightarrow{\gamma^{-1}} & g^! \mathcal{O}_Y \otimes f^! F \otimes \omega
 \end{array}
 \end{array}$$

$\textcircled{2}$ $\textcircled{3}$ $\textcircled{4}$

The commutativity of the unlabeled subdiagrams is straightforward to verify. The commutativity of $\textcircled{1}$ follows from the definition (3.1.7) of $v_{f, \text{id}_X}(\mathcal{O}_Z)$. The commutativity of $\textcircled{2}$ is essentially the definition of the map κ . The commutativity of $\textcircled{3}$ results from Lemma 2.8. For $\textcircled{4}$, it is enough to have the commutativity of the functorial diagram

$$\begin{array}{ccc}
 g^! \mathcal{O}_Y \otimes_X f^! \mathcal{O}_Z \otimes_X f^* & \xrightarrow{\text{via } \chi_f} & g^! \mathcal{O}_Y \otimes_X f^! \\
 \downarrow \text{via } \vartheta & & \downarrow \vartheta \\
 f^* \mathcal{O}_Z \otimes_X f^* & \xlongequal{\quad} & f^* ,
 \end{array}$$

that expands to the border of the natural diagram (3.4.3) below, in which the omitted subscripts

on the symbols $\otimes$ are the obvious ones, and, with reference to the standard fiber square

$$\begin{array}{ccc} X \times_Z X & \xrightarrow{p_2} & X \\ p_1 \downarrow & \diamond & \downarrow f \\ X & \xrightarrow{f} & Z, \end{array}$$

the map $\beta_\diamond$ is as in (2.6.7):

$$\begin{array}{ccc} g^! \mathcal{O}_Y \otimes f^! \mathcal{O}_Z \otimes f^* & \xrightarrow{\text{via } \chi_f} & g^! \mathcal{O}_Y \otimes f^! \\ \downarrow \simeq & \searrow \text{via } \chi_f & \downarrow \simeq \\ & g^! \mathcal{O}_Y \otimes g^* p_2^* (f^! \mathcal{O}_Z \otimes f^*) \xrightarrow{\text{via } \chi_f} g^! \mathcal{O}_Y \otimes g^* p_2^* f^! & \\ \downarrow \simeq & \downarrow \simeq & \downarrow \text{via } \beta_\diamond \\ g^! \mathcal{O}_Y \otimes g^* p_2^* f^! \mathcal{O}_Z \otimes g^* p_2^* f^* & \xrightarrow{\simeq} g^! \mathcal{O}_Y \otimes g^* (p_2^* f^! \mathcal{O}_Z \otimes p_2^* f^*) & \\ \downarrow \text{via } \beta_\diamond & \downarrow \text{via } \beta_\diamond & \downarrow \text{via } \beta_\diamond \\ g^! \mathcal{O}_Y \otimes g^* p_1^! f^* \mathcal{O}_Z \otimes g^* p_2^* f^* & \xrightarrow{\simeq} g^! \mathcal{O}_Y \otimes g^* (p_1^! f^* \mathcal{O}_Z \otimes p_1^* f^*) & \\ \downarrow \text{via } \chi_g & \downarrow \text{via } \chi_g & \downarrow \text{via } \chi_{p_1} \\ g^! p_1^! f^* \mathcal{O}_Z \otimes g^* p_1^* f^* & \xrightarrow{\chi_g} g^! (p_1^! f^* \mathcal{O}_Z \otimes p_1^* f^*) & \\ \downarrow \simeq & \downarrow \text{via } \chi_{p_1} & \downarrow \text{via } \chi_{p_1} \\ (p_1 g)^! f^* \mathcal{O}_Z \otimes (p_1 g)^* f^* & \xrightarrow{\chi_{p_1 g}} (p_1 g)^! (f^* \mathcal{O}_Z \otimes f^*) & \\ \downarrow \simeq & \downarrow \simeq & \downarrow \simeq \\ f^* \mathcal{O}_Z \otimes f^* & \xrightarrow{\simeq} f^* & \end{array} \quad (3.4.3)$$

For the commutativity of subdiagram ⑤, see the last two paragraphs of [Lip09, § 3.6]. The commutativity of ⑥ results from [Nee17, 13.7] applied to the diagram $\diamond$ (cf. [Lip09, 4.9.3(c)]). That of ⑦, ⑧ and ⑨ is left to the reader (cf. [Lip09, Exercises 4.7.3.4(a), (d) and (b)], as modified in [Lip09, 4.9.3(d)], describing the behavior of χ vis-à-vis the associativity of the tensor product, the composition of maps, and identity maps, respectively). For more details on ⑧, see [Nee17, 13.4]. The commutativity of the remaining subdiagrams is easy to check, whence the assertion. $\square$

Remark 3.4.4 (Base change). Suppose given a fiber square in $\mathcal{E}$

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Z' & \xrightarrow{g} & Z, \end{array}$$

in which f , and hence f' , is essentially smooth of relative dimension d . The isomorphism $v_f(\mathcal{O}_Z)$ induces an $\mathcal{O}_X$ -isomorphism

$$\omega_f := H^{-d} f^! \mathcal{O}_Z \xrightarrow{\sim} H^{-d}(\Omega_f^d[d]) = \Omega_f^d,$$

and similarly for f' . The resulting composite isomorphism

$$g'^* \omega_f \xrightarrow{\sim} g'^* \Omega_f^d \xrightarrow{\sim} \Omega_{f'}^d \xrightarrow{\sim} \omega_{f'}$$

is discussed in [Sas04, Theorem 2.3.5, p. 740].

The following proposition expresses the compatibility between v_f and the (derived) tensor product.

PROPOSITION 3.4.5. *For any $F_1, F_2 \in \mathbf{D}_{\text{qc}}(Z)$, the following diagram, where $\otimes := \otimes_X$, commutes:*

$$\begin{array}{ccc} f^! F_1 \otimes f^* F_2 & \xrightarrow{v_f(F_1) \otimes \text{id}} & \Omega_f^d[d] \otimes f^* F_1 \otimes f^* F_2 \\ \chi_f(F_1, F_2) \downarrow & & \downarrow \text{natural} \\ f^!(F_1 \otimes F_2) & \xrightarrow{v_f(F_1 \otimes F_2)} & \Omega_f^d[d] \otimes f^*(F_1 \otimes F_2). \end{array}$$

Sketch of a proof. The definition of v_f makes it enough to prove the commutativity of the following diagram (which expresses the transitivity of χ_f for any $F_0 \in \mathbf{D}_{\text{qc}}(Z)$), and then to take $F_0 := \mathcal{O}_Z$:

$$\begin{array}{ccc} f^!(F_0 \otimes F_1) \otimes f^* F_2 & \xleftarrow{\chi_f(F_0, F_1) \otimes \text{id}} & f^! F_0 \otimes f^* F_1 \otimes f^* F_2 \\ \chi_f(F_0 \otimes F_1, F_2) \downarrow & & \downarrow \text{natural} \\ f^!(F_0 \otimes F_1 \otimes F_2) & \xleftarrow{\chi_f(F_0, F_1 \otimes F_2)} & f^! F_0 \otimes f^*(F_1 \otimes F_2). \end{array}$$

For this, via a compactification of f , one easily reduces to the case where f is proper, a case dealt with (in outline) in [Lip09, Exercise 4.7.3.4(a)]. $\square$

4. The fundamental class

4.1. Let $f: X \rightarrow Z$ be a flat $\mathcal{E}$ -map. Set $Y := X \times_Z X$, let $\delta: X \rightarrow Y$ be the diagonal map and $p_i: Y \rightarrow X$ for $i = 1, 2$ the projections onto the first and second factors, respectively. We then have the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\delta} & Y & \xrightarrow{p_2} & X \\ & & p_1 \downarrow & \clubsuit & \downarrow f \\ & & X & \xrightarrow{f} & Z, \end{array} \tag{4.1.1}$$

with fiber square $\clubsuit$. The maps p_i are flat.

There are maps of $\mathbf{D}_{\text{qc}}$ -functors

$$\mu_i: \delta_* \rightarrow p_i^!, \quad i = 1, 2 \tag{4.1.2}$$

adjoint, respectively, to the natural maps $\text{id} = (p_i \delta)^! \rightarrow \delta^! p_i^!$. Thus μ_i is the natural composite map

$$\delta_* = \delta_*(p_i \delta)^! \longrightarrow \delta_* \delta^! p_i^! \longrightarrow p_i^!.$$

Associated to $\clubsuit$ is the functorial *base change isomorphism* (see (2.6.7))

$$\beta = \beta_{\clubsuit}: p_2^* f^! F \xrightarrow{\sim} p_1^! f^* F, \quad F \in \mathbf{D}_{\text{qc}}(Z).$$

DEFINITION 4.2. With the preceding notation, the *fundamental class* of f ,

$$\mathbf{C}_f: \delta^* \delta_* f^* \longrightarrow f^!,$$

a map between functors from $\mathbf{D}_{\text{qc}}(Z)$ to $\mathbf{D}_{\text{qc}}(X)$, is given by the composite

$$\delta^* \delta_* f^* \xrightarrow[\text{via } \mu_1]{} \delta^* p_1^! f^* \xrightarrow[\delta^* \beta^{-1}]{\sim} \delta^* p_2^* f^! \xrightarrow[\text{natural}]{\sim} f^!.$$

Remarks (Not used elsewhere). It results from [AJL14, 2.5 and 3.1] that *the fundamental class commutes with essentially étale localization on X* . That is, if $g: X' \rightarrow X$ is essentially étale, then $\mathbf{C}_{fg}$ is obtained from $\mathbf{C}_f$ by applying the functor g^* and then making canonical identifications.

See also [AJL14, Theorem 5.1] for the behavior of $\mathbf{C}_f$ under flat base change.

These results imply that if $u: U \hookrightarrow X$ and $v: V \hookrightarrow Z$ are open immersions such that $f(U) \subset V$, and $f_0: U \rightarrow V$ is the restriction of f , then $u^*(\mathbf{C}_f)$ can be identified with $\mathbf{C}_{f_0}$. Locally, $\mathbf{C}_f$ then reduces to the fundamental class of a flat $\mathcal{E}$ -map of affine schemes, in which case a simple explicit description is given in [ILN15, Theorem 4.2.4].

The following proposition expresses the compatibility between $\mathbf{C}_f$ and the derived tensor product.

PROPOSITION 4.3. *Let $\otimes := \otimes_X$, and let χ_f be as in (2.6.6). For any $F_1, F_2 \in \mathbf{D}_{\text{qc}}(Z)$, the following diagram, with $\psi := \psi(\delta, p_1, f^* F_1, f^* F_2)$, commutes:*

$$\begin{array}{ccc} \delta^* \delta_* f^* F_1 \otimes f^* F_2 & \xrightarrow{\mathbf{C}_f(F_1) \otimes \text{id}} & f^! F_1 \otimes f^* F_2 \\ \psi \downarrow & & \downarrow \chi_f(F_1, F_2) \\ \delta^* \delta_*(f^* F_1 \otimes f^* F_2) & & \\ \textcolor{red}{(2.6.1)} \downarrow & & \\ \delta^* \delta_* f^*(F_1 \otimes F_2) & \xrightarrow{\mathbf{C}_f(F_1 \otimes F_2)} & f^!(F_1 \otimes F_2). \end{array}$$

Proof. The diagram expands, naturally, to give diagram (4.3.1), where α is induced by the composite

$$p_2^* f^* \xrightarrow{\sim} (fp_2)^* = (fp_1)^* \xrightarrow{\sim} p_1^* f^*$$

and by the base-change isomorphism β .

$$\begin{array}{c}
 \delta^* \delta_* f^* F_1 \otimes f^* F_2 \longrightarrow \delta^* p_1^! f^* F_1 \otimes f^* F_2 \longrightarrow \delta^* p_2^! f^! F_1 \otimes f^* F_2 \longrightarrow f^! F_1 \otimes f^* F_2 \\
 \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \parallel \\
 \delta^*(\delta_* f^* F_1 \otimes p_1^! f^* F_2) \qquad \delta^*(p_1^! f^* F_1 \otimes p_1^* f^* F_2) \xrightarrow{\alpha} \delta^*(p_2^! f^! F_1 \otimes p_2^* f^* F_2) \qquad \delta^*(p_2^! f^! F_1 \otimes p_2^* f^* F_2) \\
 \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \\
 \delta^* \delta_*(f^* F_1 \otimes \delta^* p_1^* f^* F_2) \qquad \delta^* p_1^!(f^* F_1 \otimes f^* F_2) \qquad \delta^* p_2^*(f^! F_1 \otimes f^* F_2) \longrightarrow f^! F_1 \otimes f^* F_2 \\
 \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \\
 \delta^* \delta_*(f^* F_1 \otimes f^* F_2) \qquad \delta^* p_1^!(f^* F_1 \otimes f^* F_2) \qquad \delta^* p_2^* \chi_f \qquad \text{via } \chi_f \\
 \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \\
 \delta^* \delta_* f^*(F_1 \otimes F_2) \longrightarrow \delta^* p_1^! f^*(F_1 \otimes F_2) \longrightarrow \delta^* p_2^! f^!(F_1 \otimes F_2) \longrightarrow f^!(F_1 \otimes F_2).
 \end{array} \tag{4.3.1}$$

The commutativity of ① follows from that of the following diagram of natural isomorphisms, whose subdiagrams commute either clearly or by the pseudofunctoriality of $(-)^*$:

$$\begin{array}{ccccc}
 f^* = (p_1 \delta)^* f^* & \xlongequal{\quad} & (p_2 \delta)^* f^* = f^* \\
 \downarrow & \searrow & \swarrow & \downarrow \\
 & (f p_1 \delta)^* & \xlongequal{\quad} & (f p_2 \delta)^* \\
 & \downarrow & & \downarrow \\
 & \delta^*(f p_1)^* & \xlongequal{\quad} & \delta^*(f p_2)^* \\
 \swarrow & & \searrow & \\
 \delta^* p_1^* f^* & \xlongequal{\quad} & \delta^* p_2^* f^*.
 \end{array}$$

Let $E_1 := f^! F_1$, let $E_2 := f^* F_2$, and let $p := p_2$. Then subdiagram ② of (4.3.1) expands, naturally, to

$$\begin{array}{ccccc}
 \delta^* p^* E_1 \otimes E_2 & \longrightarrow & (p \delta)^* E_1 \otimes E_2 & \xlongequal{\quad} & E_1 \otimes E_2 \\
 \parallel & & \parallel & \nearrow \text{⑤} & \parallel \\
 \delta^* p^* E_1 \otimes (p \delta)^* E_2 & \longrightarrow & (p \delta)^* E_1 \otimes (p \delta)^* E_2 & \xrightarrow{\gamma} & (p \delta)^*(E_1 \otimes E_2) \\
 \downarrow & \nearrow & \text{⑥} & & \downarrow \\
 \delta^* p^* E_1 \otimes \delta^* p^* E_2 & \longrightarrow & \delta^*(p^* E_1 \otimes p^* E_2) & \longrightarrow & \delta^* p^*(E_1 \otimes E_2).
 \end{array}$$

The commutativity of ⑥ is given by the dual of the commutative diagram [Lip09, 3.6.7.2] (see the proof of [Lip09, 3.6.10]). As $p \delta$ is the identity map of X , the same diagram, with $g = f = \text{id}$, yields that the isomorphism γ is idempotent, whence it is the identity map, so that ⑤ commutes.

Subdiagram ③ of (4.3.1) without δ^* expands, naturally, to the following, with $E_1 := f^* F_1$, $E_2 := f^* F_2$, and $p := p_1$.

$$\begin{array}{ccccc}
 \delta_* E_1 \otimes p^* E_2 & \longrightarrow & \delta_* \delta^! p^! E_1 \otimes p^* E_2 & \longrightarrow & p^! E_1 \otimes p^* E_2 \\
 \downarrow & & \downarrow & \nearrow \textcircled{7} & \downarrow \chi_p \\
 & & \delta_*(\delta^! p^! E_1 \otimes \delta^* p^* E_2) & \xrightarrow{\delta_* \chi_\delta} & \delta_* \delta^! (p^! E_1 \otimes p^* E_2) \\
 \downarrow & \nearrow & \downarrow & \downarrow \delta_* \delta^! \chi_p & \downarrow \\
 \delta_*(E_1 \otimes \delta^* p^* E_2) & \xrightarrow{\delta_*((p\delta)^! E_1 \otimes (p\delta)^* E_2)} & \delta_*((p\delta)^! E_1 \otimes (p\delta)^* E_2) & \xrightarrow{\textcircled{8}} & \delta_* \delta^! p^! (E_1 \otimes E_2) \\
 \downarrow & \nearrow & \downarrow & \downarrow & \downarrow \\
 \delta_*(E_1 \otimes E_2) & \longrightarrow & \delta_* \delta^! p^! (E_1 \otimes E_2) & \longrightarrow & p^! (E_1 \otimes E_2) .
 \end{array}$$

The commutativity of subdiagram $\textcircled{7}$ is immediate from the definition of χ_δ (δ being proper). In [Nee17], there is a blanket convention that the functors and natural transformations strictly respect identities, hence $(p\delta)^* = \text{id}^* = \text{id} = \text{id}^! = (p\delta)^!$. The commutativity of $\textcircled{8}$ follows from [Nee17, Theorem 13.4]. The commutativity of subdiagram $\textcircled{4}$ is given by [Nee17, Proposition 13.7]. The commutativity of all the remaining (unlabeled) subdiagrams is clear. $\square$

4.4. Let $f: X \rightarrow Z$ be a flat $\mathcal{E}$ -map. The map f is *equidimensional of relative dimension d* if for each $x \in X$ that is a generic point of an irreducible component of the fiber $f^{-1}f(x)$, the transcendence degree of the residue field of the local ring $\mathcal{O}_{X,x}$ over that of $\mathcal{O}_{Z,f(x)}$ is d . When f is of finite type, this just means that every irreducible component of every fiber has dimension d .

An essentially smooth $\mathcal{E}$ -map of constant relative dimension d is equidimensional of relative dimension d .

For any flat $\mathcal{E}$ -map f that is equidimensional of relative dimension d , we have $H^e f^! \mathcal{O}_Z = 0$ whenever $e < -d$. Indeed, this assertion is local on Z and X (see the remark right after (1.1.3)). By [EGAIV, 13.3.1], the map f may be assumed to be of the form $\text{Spec } B \rightarrow \text{Spec } A$, where B is a localization of a module-finite quasi-finite algebra B_0 over the polynomial ring $A[T_1, \dots, T_d]$. By Zariski's Main Theorem [EGAIV, 8.12.6], the natural map $\text{Spec } B_0 \rightarrow \text{Spec } A[T_1, \dots, T_d]$ factors as $\text{finite} \circ (\text{open immersion})$. The isomorphism (3.1.2) shows that the relative dualizing complex for the map $\text{Spec } A[T_1, \dots, T_d] \rightarrow \text{Spec } A$ is concentrated in degree $-d$, so (1.2.1) implies that the assertion holds for any map $\text{Spec } B_1 \rightarrow \text{Spec } A$ with B_1 finite over $A[T_1, \dots, T_d]$, and then by property (i) in Section 1.1, it holds for $\text{Spec } B \rightarrow \text{Spec } A$.

So there is a canonical composite map

$$\Omega_f^d[d] \xrightarrow{(3.2.2)} (H^{-d} \delta^* \delta_* \mathcal{O}_X)[d] \xrightarrow{H^{-d} c_f} (H^{-d} f^! \mathcal{O}_Z)[d] \longrightarrow f^! \mathcal{O}_Z, \quad (4.4.1)$$

whence a canonical map

$$c_f: \Omega_f^d[d] \otimes f^* \longrightarrow f^! \mathcal{O}_Z \otimes f^* \xrightarrow{(2.6.6)} f^!.$$

For essentially smooth f , there is then an obvious question, the one that motivated the present paper, and to which the answer is affirmative, as stated in the following theorem.

THEOREM 4.5. For essentially smooth $\mathcal{E}$ -maps $f: X \rightarrow Z$ of relative dimension d and $F \in \mathbf{D}_{\text{qc}}(Z)$, the fundamental class map $c_f(F)$ is inverse to Verdier's isomorphism $v_f(F)$.

Proof. As in Section 4.1, set $Y := X \times_Z X$, let $\delta: X \rightarrow Y$ be the diagonal map, and let $p_i: Y \rightarrow X$ for $i = 1, 2$ be the projections onto the first and second factors, respectively, so that we have the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\delta} & Y & \xrightarrow{p_2} & X \\ & & p_1 \downarrow & \clubsuit & \downarrow f \\ & & X & \xrightarrow[f]{} & Z, \end{array} \quad (4.5.1)$$

with fiber square $\clubsuit$. Resolving $\delta_* \mathcal{O}_X$ locally by a Koszul complex, one sees that $H^e \delta^* \delta_* \mathcal{O}_X = 0$ if $e < -d$; so one has natural maps

$$\omega := \Omega_f^d[d] \xrightarrow{(3.2.2)} (H^{-d} \delta^* \delta_* \mathcal{O}_X)[d] \longrightarrow \delta^* \delta_* \mathcal{O}_X, \quad (4.5.2)$$

and c_f is the composite

$$\omega \otimes f^* \xrightarrow{(4.5.2) \otimes \text{id}} \delta^* \delta_* \mathcal{O}_X \otimes f^* \xrightarrow{c_f(\mathcal{O}_Z) \otimes \text{id}} f^! \mathcal{O}_Z \otimes f^* \xrightarrow[(2.6.6)]{\chi_f} f^!.$$

In view of Proposition 3.4.5, the theorem then says that the composite

$$\omega \otimes f^* \xrightarrow{(4.5.2) \otimes \text{id}} \delta^* \delta_* \mathcal{O}_X \otimes f^* \xrightarrow{c_f(\mathcal{O}_Z) \otimes \text{id}} f^! \mathcal{O}_Z \otimes f^* \xrightarrow[v_f(\mathcal{O}_Z) \otimes \text{id}]{} \omega \otimes f^*$$

is the identity map.

Here we implicitly used the commutativity of the diagram of natural isomorphisms

$$\begin{array}{ccc} f^* \mathcal{O}_Z \otimes f^* & \xrightarrow{\sim} & \mathcal{O}_X \otimes f^* \\ \simeq \downarrow & & \downarrow \simeq \\ f^*(\mathcal{O}_X \otimes \text{id}) & \xrightarrow{\sim} & f^*, \end{array} \quad (4.5.3)$$

which holds since, mutatis mutandis, this diagram is *dual* [Lip09, 3.4.5] to the commutative subdiagram ② in the proof of [Lip09, 3.4.7(iii)].

It therefore suffices to prove Theorem 4.5 when $F = \mathcal{O}_Z$.

Let $(-)'$ be the endofunctor $\mathcal{H}\text{om}_X(-, \mathcal{O}_X)$ of $\mathbf{D}(X)$. The “mirror image” of the evaluation map $\text{ev}(X, E, \mathcal{O}_X)$ (see (2.2.2)) is the natural composite

$$E \otimes E' \xrightarrow{(2.2.2)} E' \otimes E \longrightarrow \mathcal{O}_X, \quad E \in \mathbf{D}(X). \quad (4.5.4)$$

Now, after unwinding the relevant definitions, one sees that Theorem 4.5 for $F = \mathcal{O}_Z$ states that the border of the following natural diagram (4.5.5) commutes. (Going around clockwise from the upper left corner to the bottom right one gives $c_f(\mathcal{O}_Z)$, while going around counterclockwise gives $v_f(\mathcal{O}_Z)^{-1}$.)

In this diagram, $\otimes := \otimes_X$ and $\psi_2^0 := \psi(\delta, p_2, \mathcal{O}_X, \mathcal{O}_X)$, which is the identity map of $\delta^* \delta_* \mathcal{O}_X$ (see Corollary 2.9.5), and $\psi_2 := \psi(\delta, p_2, \mathcal{O}_X, \delta^! p_1^! \mathcal{O}_X)$. The commutativity of the unlabeled subdiagrams is straightforward to check. The problem is to show the commutativity of ①.

We will need another description of the map (3.1.6) (Lemma 4.5.10 below).

For this, first check that the counit map $\delta^* \delta_* G \rightarrow G$ ($G \in \mathbf{D}(X)$) has right inverses $\tau_p(G)$, where $p: Y \rightarrow X$ is any map such that $p\delta = \text{id}_X$ (for example, $p = p_1$ or p_2) and $\tau_p(G)$ is the

$$\begin{array}{ccccccc}
 \mathcal{O}_X \otimes \omega & \xlongequal{\quad} & \omega \otimes \mathcal{O}_X & \xrightarrow{(4.5.2)} & \delta^* \delta_* \mathcal{O}_X \otimes \mathcal{O}_X & \xlongequal{\psi_2^0} & \delta^* \delta_* \mathcal{O}_X \\
 \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
 \delta^! p_1^! \mathcal{O}_X \otimes \omega & \xleftarrow{\sim} & \omega \otimes \delta^! p_1^! \mathcal{O}_X & \xrightarrow{(4.5.2)} & \delta^* \delta_* \mathcal{O}_X \otimes \delta^! p_1^! \mathcal{O}_X & \xrightarrow{\psi_2} & \delta^* \delta_* \delta^! p_1^! \mathcal{O}_X \\
 \downarrow \simeq (2.6.7) & & \downarrow \simeq (2.6.6) & & \textcircled{1} & & \downarrow \\
 \delta^! p_2^* f^! \mathcal{O}_Z \otimes \omega & \xrightarrow{\quad} & \omega \otimes \delta^! \mathcal{O}_Y \otimes \delta^* p_1^! \mathcal{O}_X & \xrightarrow{(3.1.6)} & \omega \otimes \omega' \otimes \delta^* p_1^! \mathcal{O}_X & \xrightarrow{(4.5.4)} & \delta^* p_1^! \mathcal{O}_X \\
 \downarrow \simeq \chi_\delta^{-1} & & \downarrow & & \downarrow & & \downarrow \simeq (2.6.7) \\
 \delta^! \mathcal{O}_Y \otimes \delta^* p_2^* f^! \mathcal{O}_Z \otimes \omega & \xrightarrow{\quad} & \omega \otimes \delta^! \mathcal{O}_Y \otimes \delta^* p_2^* f^! \mathcal{O}_Z & \xrightarrow{(3.1.6)} & \omega \otimes \omega' \otimes \delta^* p_2^* f^! \mathcal{O}_Z & \xrightarrow{(4.5.4)} & \delta^* p_2^* f^! \mathcal{O}_Z \\
 \downarrow \simeq & \searrow \simeq & \downarrow \simeq (2.6.7) & & \downarrow \simeq (2.6.7) & & \downarrow \simeq \\
 \delta^! \mathcal{O}_Y \otimes f^! \mathcal{O}_Z \otimes \omega & \xrightarrow{(3.1.6)} & \omega' \otimes f^! \mathcal{O}_Z \otimes \omega & \xrightarrow{\quad} & \omega \otimes \omega' \otimes f^! \mathcal{O}_Z & \xrightarrow{(4.5.4)} & f^! \mathcal{O}_Z .
 \end{array} \tag{4.5.5}$$

natural composite

$$G \xrightarrow{\sim} \delta^* p^* G \xrightarrow{\sim} \delta^* p^* p_* \delta_* G \longrightarrow \delta^* \delta_* G, \tag{4.5.6}$$

that is, the composite $G \xrightarrow{\sim} \delta^* p^* G \xrightarrow{\delta^* v(G)} \delta^* \delta_* G$, where

$$v: p^* \rightarrow \delta_* \tag{4.5.7}$$

is the map adjoint to $\text{id} \xrightarrow{\sim} p_* \delta_*$.

Let ρ_0 be the natural composite (of isomorphisms, since $\delta_* \mathcal{O}_X$ is perfect)

$$\delta^* \delta_* \delta^! \mathcal{O}_Y \xrightarrow{(1.2.1)} \delta^* \mathcal{H}om_Y(\delta_* \mathcal{O}_X, \mathcal{O}_Y) \xrightarrow{(2.6.4)} (\delta^* \delta_* \mathcal{O}_X)',$$

and set

$$\rho = \rho_p := \rho_0 \tau_p(\delta^! \mathcal{O}_Y): \delta^! \mathcal{O}_Y \longrightarrow (\delta^* \delta_* \mathcal{O}_X)'. \tag{4.5.8}$$

As noted just before (3.2.7), one has $H^e(\delta^! \mathcal{O}_Y) = 0$ for any $e \neq d$; hence the counit map $\delta^* \delta_* \delta^! \mathcal{O}_Y \rightarrow \delta^! \mathcal{O}_Y$ is taken to an isomorphism by the functor H^d . The inverse of this isomorphism is $H^d \tau_p(\delta^! \mathcal{O}_Y)$ because $\tau_p(\delta^! \mathcal{O}_Y)$ is right-inverse to $\delta^* \delta_* \delta^! \mathcal{O}_Y \rightarrow \delta^! \mathcal{O}_Y$. It then follows from its description via (3.2.6) that (3.2.8), with $U = X$, $W = Y$ and $n = d$, is the natural composite

$$\delta^! \mathcal{O}_Y \cong (H^d(\delta^! \mathcal{O}_Y))[-d] \xrightarrow{H^d(\rho)[-d]} (H^d((\delta^* \delta_* \mathcal{O}_X)'))[-d] \xrightarrow{\sim} (\Omega_f^d)'[-d], \tag{4.5.9}$$

the last isomorphism arising via (3.2.5) (with $n = d$) and (4.5.2).

LEMMA 4.5.10. For any $\rho = \rho_p$ as in (4.5.8), the map (3.1.6) factors as

$$\delta^! \mathcal{O}_Y \xrightarrow{\rho} (\delta^* \delta_* \mathcal{O}_X)' \xrightarrow{\text{via (4.5.2)}} \omega'.$$

Proof. For any complex $E \in \mathbf{D}(X)$, set $E^{\geq d} := t_{\geq d} E$, with $t_{\geq d}$ the truncation functor (denoted by $\tau_{\geq d}$ in [Lip09, § 10.1]). The natural map $H^d(E)[-d] \rightarrow E^{\geq d}$ is an isomorphism if $H^e E = 0$ for all $e > d$, a condition satisfied when $E = (\delta^* \delta_* \mathcal{O}_X)'$ or ω' .

Accordingly, and in view of the description of (3.1.6) via (3.2.8) = (4.5.9), the lemma asserts that the border of the following diagram, whose top row is (4.5.9), commutes:

$$\begin{array}{ccccccc} \delta^! \mathcal{O}_Y & \longrightarrow & (\delta^! \mathcal{O}_Y)^{\geq d} & \longrightarrow & ((\delta^* \delta_* \mathcal{O}_X)')^{\geq d} & \longrightarrow & (\Omega_f^d)'[-d] \\ \downarrow \rho & & & \nearrow & \downarrow \text{via (4.5.2)} & \textcircled{2} & \downarrow (3.1.5) \\ & & & & (\omega')^{\geq d} & & \omega' \\ & & & \searrow & & & \\ & & & & & \text{via (4.5.2)} & \end{array}$$

The unlabeled subdiagrams clearly commute. Subdiagram ② expands, naturally, as follows, with $\mathcal{H}\text{om} = \mathcal{H}\text{om}_X$:

$$\begin{array}{ccc} H^d((\delta^* \delta_* \mathcal{O}_X)')[-d] & \xrightarrow{(3.2.5)} & ((H^{-d} \delta^* \delta_* \mathcal{O}_X)')[-d] \\ \uparrow \simeq & & \downarrow \text{cf. (4.5.2)} \\ ((\delta^* \delta_* \mathcal{O}_X)')^{\geq d} & & ((\Omega_f^d)'[-d]) \\ \downarrow & & \downarrow (3.1.5) \\ ((H^{-d} \delta^* \delta_* \mathcal{O}_X[d])')^{\geq d} & & \omega' \\ \downarrow \text{via (4.5.2)} & & \\ (\omega')^{\geq d} & \xlongequal{\quad} & \omega' \end{array}$$

The vertices of this diagram can all be identified with the complex G that is $H^0 \mathcal{H}\text{om}(\Omega_f^d, \mathcal{O}_X)$ in degree d and 0 elsewhere. After this identification, all the maps in the diagram are identity maps except for the two labeled (3.2.5) and (3.1.5), which are both multiplication in G by $(-1)^{d(d+1)/2}$. (See the remarks following equations (3.2.5) and (3.1.5).) Hence subdiagram ② commutes, and Lemma 4.5.10 is proved. $\square$

One now has that subdiagram ① without $p_1^! \mathcal{O}_X$ expands, naturally, to (4.5.11), in which χ is as in (2.6.6), id is the identity functor, and

$$\begin{aligned} \psi_3 &:= \psi(\delta, p_2, \mathcal{O}_X, \delta^! \mathcal{O}_Y \otimes_Y \delta^*(-)), \\ \psi_4 &:= \psi(\delta, p_2, \delta^! \mathcal{O}_Y, \delta^*(-)), \\ \psi_7 &:= \psi(\delta, p_2, \mathcal{O}_X, \delta^! \mathcal{O}_Y). \end{aligned}$$

The commutativity of the unlabeled subdiagrams is clear. That of ③ is given by Lemma 2.9.4. That of ⑤ (without δ^*) is the definition of $\chi_\delta = \chi_\delta(\mathcal{O}_Y, -)$. The commutativity of ⑥ is given by that of the first diagram in [Lip09, 3.5.3(h)], with $(A, B, C) := (\delta^* \delta_* \mathcal{O}_X, \omega, \mathcal{O}_X)$. For that of ⑧, compare with (4.5.3).

$$\begin{array}{ccccc}
 \omega \otimes \delta^! & \xrightarrow{(4.5.2)} & \delta^* \delta_* \mathcal{O}_X \otimes \delta^! & \xrightarrow{\psi_2} & \delta^* \delta_* \delta^! \\
 \downarrow \chi_\delta^{-1} & & \downarrow \chi_\delta^{-1} & & \downarrow \chi_\delta^{-1} \\
 \omega \otimes \delta^! \mathcal{O}_Y \otimes \delta^* & \xrightarrow{(4.5.2)} & \delta^* \delta_* \mathcal{O}_X \otimes \delta^! \mathcal{O}_Y \otimes \delta^* & \xrightarrow{\psi_3} & \delta^* \delta_* (\delta^! \mathcal{O}_Y \otimes_Y \delta^*) \\
 \downarrow \text{via } \rho & & \downarrow \text{via } \rho & & \downarrow \text{via } \rho \\
 \omega \otimes (\delta^* \delta_* \mathcal{O}_X)' \otimes \delta^* & \xrightarrow{(4.5.2)} & \delta^* \delta_* \mathcal{O}_X \otimes (\delta^* \delta_* \mathcal{O}_X)' \otimes \delta^* & \xrightarrow{(4.5.4)} & \omega \otimes \omega' \otimes \delta^* \\
 \downarrow (4.5.2) & & \downarrow (4.5.2) & & \downarrow (4.5.2) \\
 \omega \otimes \omega' \otimes \delta^* & \xrightarrow{(4.5.4)} & \mathcal{O}_X \otimes \delta^* & \xrightarrow{\sim} & \delta^*
 \end{array}
 \tag{4.5.11}$$

After restoring the term $p_1^! \mathcal{O}_X \cong p_2^* f^! \mathcal{O}_Z$ omitted above, one obtains the commutativity of ④ from Lemma 2.9.2, with $E := \delta^! \mathcal{O}_Y$ and $F := f^! \mathcal{O}_Z$. That leaves subdiagram ⑦, which without $\otimes \delta^*$ expands, naturally, to the following diagram, where $p := p_2$ and $F'' := \mathcal{H}om_Y(F, \mathcal{O}_Y)$ ($F \in \mathbf{D}(Y)$):

$$\begin{array}{ccccc}
 \delta^* \delta_* \mathcal{O}_X \otimes \delta^! \mathcal{O}_Y & \xrightarrow{(2.6.1)} & \delta^* (\delta_* \mathcal{O}_X \otimes p^* \delta^! \mathcal{O}_Y) & \xrightarrow{(2.6.3)} & \delta^* \delta_* (\mathcal{O}_X \otimes \delta^* p^* \delta^! \mathcal{O}_Y) \\
 \downarrow \text{via } \tau_p(\delta^! \mathcal{O}_Y) & \text{⑦}_1 & \downarrow \text{via (4.5.7)} & & \downarrow \simeq \\
 \delta^* \delta_* \mathcal{O}_X \otimes \delta^* \delta_* \delta^! \mathcal{O}_Y & \xrightarrow{(2.6.1)} & \delta^* (\delta_* \mathcal{O}_X \otimes \delta_* \delta^! \mathcal{O}_Y) & & \delta^* \delta_* \delta^* p^* \delta^! \mathcal{O}_Y \\
 \downarrow \text{via (1.2.1)} & \text{⑦}_2 & \downarrow \text{via (1.2.1)} & \text{⑦}_4 & \downarrow \simeq \\
 \delta^* \delta_* \mathcal{O}_X \otimes \delta^* (\delta_* \mathcal{O}_X)'' & \xrightarrow{(2.6.1)} & \delta^* (\delta_* \mathcal{O}_X \otimes (\delta_* \mathcal{O}_X)'') & & \delta^* \delta_* \delta^! \mathcal{O}_Y \\
 \downarrow \text{via (2.6.4)} & \text{⑦}_3 & \searrow \text{cf. (4.5.4)} & & \downarrow \\
 \delta^* \delta_* \mathcal{O}_X \otimes (\delta^* \delta_* \mathcal{O}_X)' & \xrightarrow{(4.5.4)} & \mathcal{O}_X & \xleftarrow{\sim} & \delta^* \mathcal{O}_Y
 \end{array}$$

The commutativity of subdiagram ⑦₁ results from the definition of τ_p . That of ⑦₂ is clear. That of ⑦₃ follows from [Lip09, 3.5.6(a)] (with $f := \delta$, $A := \delta_* \mathcal{O}_X$ and $B := \mathcal{O}_Y$).

Subdiagram $\textcircled{7}_4$, with the initial δ^* in each term dropped, expands, naturally, to

$$\begin{array}{ccccc}
 \delta_* \mathcal{O}_X \otimes p^* \delta^! \mathcal{O}_Y & \xrightarrow{(2.6.3)} & \delta_*(\mathcal{O}_X \otimes \delta^* p^* \delta^! \mathcal{O}_Y) & & \\
 \downarrow \text{via (4.5.7)} & & \downarrow \text{via (4.5.7)} & & \downarrow \simeq \\
 \delta_* \mathcal{O}_X \otimes \delta_* \delta^! \mathcal{O}_Y & \xrightarrow{(2.6.3)} & \delta_*(\mathcal{O}_X \otimes \delta^* \delta_* \delta^! \mathcal{O}_Y) & & \delta_* \delta^* p^* \delta^! \mathcal{O}_Y \\
 \parallel & \searrow \textcircled{7}_{41} & \downarrow \textcircled{7}_{42} & \nearrow \simeq & \downarrow \simeq \\
 & \delta_* \mathcal{O}_X \otimes \delta_* \delta^* \delta_* \delta^! \mathcal{O}_Y & & & \delta_* \delta^* p^* \delta^! \mathcal{O}_Y \\
 \delta_* \mathcal{O}_X \otimes \delta_* \delta^! \mathcal{O}_Y & \xrightarrow{(2.2.1)} & \delta_*(\mathcal{O}_X \otimes \delta^! \mathcal{O}_Y) & \xrightarrow{\sim} & \delta_* \delta^! \mathcal{O}_Y \\
 \downarrow \text{via (1.2.1)} & \searrow \textcircled{7}_{43} & \downarrow & \nearrow & \downarrow \\
 & \delta_* \delta^! \mathcal{O}_Y \otimes \delta_* \mathcal{O}_X & \xrightarrow{(2.2.1)} & \delta_*(\delta^! \mathcal{O}_Y \otimes \mathcal{O}_X) & \\
 & \downarrow \zeta(\mathcal{O}_X, \mathcal{O}_Y) \otimes \text{id} & \searrow \text{see (2.6.5)} & \textcircled{7}_{44} & \\
 \delta_* \mathcal{O}_X \otimes (\delta_* \mathcal{O}_X)'' & \xrightarrow{\sim} & (\delta_* \mathcal{O}_X)'' \otimes \delta_* \mathcal{O}_X & \xrightarrow{(2.2.2)} & \mathcal{O}_Y .
 \end{array}$$

Here, the commutativity of the unlabeled subdiagrams is clear. The commutativity of subdiagram $\textcircled{7}_{41}$ is given by the definition of the projection isomorphism (2.6.3). The commutativity of subdiagram $\textcircled{7}_{42}$ (which says, incidentally, that the map v in (4.5.7) is adjoint to the natural isomorphism $\delta^* p^* \xrightarrow{\sim} \text{id}$) follows easily from τ_p in (4.5.6) being right-inverse to the counit map $\delta^* \delta_* G \rightarrow G$. Subdiagram $\textcircled{7}_{43}$ commutes because δ_* is a *symmetric* monoidal functor. (See Section 2.2.) Finally, the commutativity of subdiagram $\textcircled{7}_{44}$ means that the map $\zeta(\mathcal{O}_X, \mathcal{O}_Y)$ is adjoint to the natural composite

$$\delta_* \delta^! \mathcal{O}_Y \otimes_Y \delta_* \mathcal{O}_X \xrightarrow{(2.2.1)} \delta_*(\delta^! \mathcal{O}_Y \otimes_X \mathcal{O}_X) = \delta_* \delta^! \mathcal{O}_Y \longrightarrow \mathcal{O}_Y . \quad (4.5.12)$$

But for any $F \in \mathbf{D}_{\text{qc}}(Y)$, the map $\zeta(\mathcal{O}_X, F)$ is, by definition, right-conjugate to the projection isomorphism

$$\delta_* \delta^* G = \delta_*(\delta^* G \otimes_X \mathcal{O}_X) \xleftarrow[(2.6.3)]{\sim} G \otimes_Y \delta_* \mathcal{O}_X ; \quad (4.5.13)$$

that is (see [Lip09, 3.3.5]), the map $\zeta(\mathcal{O}_X, F)$ is adjoint to the natural composite

$$\delta_* \delta^! F \otimes_Y \delta_* \mathcal{O}_X \xrightarrow[(4.5.13)]{\sim} \delta_* \delta^* \delta_* \delta^! F \longrightarrow \delta_* \delta^! F \longrightarrow F ,$$

which for $F = \mathcal{O}_Y$ is (4.5.12), since for any A and $B \in \mathbf{D}(X)$ (for example, $A = \delta^! \mathcal{O}_Y$ and $B = \mathcal{O}_X$), the following natural diagram commutes (as follows from the definition of (2.6.3)):

$$\begin{array}{ccc}
 \delta_* A \otimes_Y \delta_* B & \xrightarrow{(2.2.1)} & \delta_*(A \otimes_X B) \\
 \updownarrow & \searrow (2.6.3) & \uparrow \\
 \delta_* \delta^* \delta_* A \otimes_Y \delta_* B & \xrightarrow[(2.2.1)]{} & \delta_*(\delta^* \delta_* A \otimes_X B) .
 \end{array}$$

So subdiagram $\textcircled{7}_{44}$ commutes. Thus, $\textcircled{7}$ commutes, whence so does $\textcircled{1}$. $\square$

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REFERENCES

- AJL11 L. Alonso Tarrío, A. Jeremías López and J. Lipman, *Bivariance, Grothendieck duality and Hochschild homology I: Construction of a bivariate theory*, Asian J. Math. **15** (2011), no. 3, 451–497; <https://doi.org/10.4310/AJM.2011.v15.n3.a7>.
- AJL14 ———, *Bivariance, Grothendieck duality and Hochschild homology II: The fundamental class of a flat scheme-map*, Adv. Math. **257** (2014), 365–461; <https://doi.org/10.1016/j.aim.2014.02.017>.
- Bou07 N. Bourbaki, *Éléments de mathématique. Algèbre. Chapitre 10. Algèbre homologique* (Springer-Verlag, Berlin, 2007); <https://doi.org/10.1007/978-3-540-34493-3>.
- Con00 B. Conrad, *Grothendieck duality and base change*, Lecture Notes in Math., vol. 1750 (Springer-Verlag, Berlin, 2000); <https://doi.org/10.1007/b75857>.
- Con01 ———, *Clarifications and corrections for Grothendieck duality and base change*, 2001, available at <http://math.stanford.edu/~conrad/>.
- EGAIV A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique IV, Étude locale des schémas et des morphismes of schémas*, Publ. Math. Inst. Hautes Études Sci. **20** (Chapter 0, 14–23, and IV, 1), **24** (IV, 2–7), **28** (IV, 8–15), and **32** (IV, 16–21), 1961–1967; available at <http://www.numdam.org>.
- Gro60 A. Grothendieck, *The cohomology theory of abstract algebraic varieties*, Proc. Internat. Congress Math. (Edinburgh, 1958) (Cambridge Univ. Press, New York, 1960), 103–118.
- Har66 R. Hartshorne, *Residues and duality*, Lecture Notes in Math., vol. 20 (Springer-Verlag, Berlin – New York, 1966); <https://doi.org/10.1007/BFb0080482>.
- HS93 R. Hübl and P. Sastry, *Regular differential forms and relative duality*, Amer. J. Math. **115** (1993), no. 4, 749–787; <https://doi.org/10.2307/2375012>.
- Ill71 L. Illusie, *Conditions de finitude relative*, Théorie des Intersections et Théorème de Riemann–Roch (SGA 6), Lecture Notes in Math., vol. 225 (Springer-Verlag, Berlin – New York, 1971), 222–273; <https://doi.org/10.1007/BFb0066287>.
- ILN15 S. B. Iyengar, J. Lipman and A. Neeman, *Relation between two twisted inverse image pseudo-functors in duality theory*, Compos. Math. **151** (2015), no. 4, 735–764; <https://doi.org/10.1112/S0010437X14007672>.
- KM71 G. M. Kelly and S. Mac Lane, *Coherence in closed categories*, J. Pure Appl. Algebra **1** (1971), no. 1, 97–140; [https://doi.org/10.1016/0022-4049\(71\)90013-2](https://doi.org/10.1016/0022-4049(71)90013-2).
- Lip84 J. Lipman, *Dualizing sheaves, differentials and residues on algebraic varieties*, Astérisque, no. 117 (Soc. Math. France, Paris, 1984).
- Lip02 ———, *Lectures on local cohomology and duality*, Local Cohomology and its Applications (G. Lyubeznik, ed.), Lect. Notes Pure Appl. Math., vol. 226 (Marcel Dekker, Inc., New York, 2002), 39–89.
- Lip09 ———, *Notes on derived functors and Grothendieck duality*, Foundations of Grothendieck Duality for Diagrams of Schemes, Lecture Notes in Math., vol. 1960 (Springer, Berlin, 2009), 1–259; <https://doi.org/10.1007/978-3-540-85420-3>.
- LS81 J. Lipman and A. Sathaye, *Jacobian ideals and a theorem of Briançon–Skoda*, Michigan Math. J. **28** (1981), no. 2, 199–222; <https://doi.org/10.1307/mmj/1029002510>.
- Nay09 S. Nayak, *Compactification for essentially finite-type maps*, Adv. Math. **222** (2009), no. 2, 527–546; <https://doi.org/10.1016/j.aim.2009.05.002>.

- Nee17   A. Neeman, *An improvement on the base-change theorem and the functor $f^!$* , 2017, [arXiv:1406.7599v2](#).
- Sas04   P. Sastry, *Base change and Grothendieck duality for Cohen–Macaulay maps*, *Compos. Math.* **140** (2004), no. 3, 729–777; <https://doi.org/10.1112/S0010437X03000654>.
- Ver69   J.-L. Verdier, *Base change for twisted inverse image of coherent sheaves*, *Algebraic Geometry* (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968) (Oxford University Press, London, 1969), 393–408.

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