Jacobian syzygies, stable reflexive sheaves, and Torelli properties for projective hypersurfaces with isolated singularities

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ABSTRACT

We investigate the relations between the syzygies of the Jacobian ideal of the defining equation for a projective hypersurface V with isolated singularities and the Torelli properties of V (in the sense of Dolgachev–Kapranov). We show, in particular, that hypersurfaces with a small Tjurina number are Torelli in this sense. When V is a plane curve or, more interestingly, a surface in $\mathbb{P}^3$, we discuss the stability of the reflexive sheaf of logarithmic vector fields along V . A new lower bound for the minimal degree of a syzygy associated with a 1-dimensional almost complete intersection is also given.

1. Introduction

Let X be the complex projective space $\mathbb{P}^n$, and consider the associated graded $\mathbb{C}$ -algebra $S = \bigoplus_k S_k$ with $S_k = H^0(X, \mathcal{O}_X(k))$. For a nonzero section $f \in S_N$ with $N > 1$, thought of as a homogeneous polynomial of degree N , we consider the hypersurface $V = V(f)$ in X given by the zero locus of f and let Y denote the singular locus of V , endowed with its natural scheme structure; see [Dim13b]. We assume in this paper that V has only isolated singularities.

Let $\mathcal{I}_Y \subset \mathcal{O}_X$ be the ideal sheaf defining this 0-dimensional subscheme $Y \subset X$, and consider the graded ideal $I = \bigoplus_k I_k$ in S with $I_k = H^0(X, \mathcal{I}_Y(k))$. Let $Z = \text{Spec}(S)$ be the corresponding affine space $\mathbb{C}^{n+1}$, and denote by $\Omega^k = H^0(Z, \Omega_Z^k)$ the S -module of global, regular k -forms on Z . Using a linear coordinate system $x = (x_0, \dots, x_n)$ on X , one sees that there is a natural grading on Ω^k ; see [DSt16] for details if necessary.

There is a well-defined differential 1-form $df \in \Omega^1$, and using it, we define two graded S -submodules in Ω^n , namely

$$\text{AR}(f) = \ker \{df \wedge : \Omega^n \rightarrow \Omega^{n+1}\} \quad (1.1)$$

and

$$\text{KR}(f) = \text{im} \{df \wedge : \Omega^{n-1} \rightarrow \Omega^n\}.$$

If one computes in a coordinate system x , then $\text{AR}(f)_m$ is the vector space of *all relations* of

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the type

$$R_m: a_0 f_{x_0} + \cdots + a_n f_{x_n} = 0,$$

with f_{x_j} the partial derivative of the polynomial f with respect to x_j and $a_j \in S_m$. Moreover, $\text{KR}(f)$ is the module of *Koszul relations* spanned by obvious relations of the type $f_{x_j} f_{x_i} + (-f_{x_i}) f_{x_j} = 0$, and the quotient

$$\text{ER}(f) = \text{AR}(f) / \text{KR}(f) \quad (1.2)$$

is the graded module of *essential relations* (which is, of course, nothing else than the n th cohomology group of the Koszul complex of $f_{x_0}, \dots, f_{x_n}$, maybe up to a shift in grading); see [Dim13b, DSt15b]. Note also that with this notation, the ideal I is just the saturation of the Jacobian ideal $J_f = (f_{x_0}, \dots, f_{x_n}) \subset S = \mathbb{C}[x_0, \dots, x_n]$.

Let α_V be the Arnold exponent of the hypersurface V , which is by definition the minimum of the Arnold exponents of the singular points of V ; see [DSa12, DSa14]. Using Hodge theory, one can prove that

$$\text{ER}(f)_m = 0 \quad \text{for any } m < \alpha_V N - n, \quad (1.3)$$

under the *additional hypothesis that all the singularities of V are weighted homogeneous* [DSa14].

While this inequality is the best possible in general, as one can see by considering hypersurfaces with many singularities, see [DSt12, Dim13a], for situations when the hypersurface V has a small number of singularities this result is far from optimal. Our first result gives the following better bound in this case.

THEOREM 1.1. *Assume that the hypersurface $V: f = 0$ in $\mathbb{P}^n$ has degree N and only isolated singularities. Then*

$$\text{ER}(f)_m = 0 \quad \text{for any } m \leq n(N - 2) - \tau(V),$$

where $\tau(V)$, the Tjurina number of V , is the sum of the Tjurina numbers of all the singularities of V .

See also Theorem 2.4 for a stronger result. The proof of these results is elementary (that is, without Hodge theory), and it does not require the hypothesis of V having weighted homogeneous singularities. In fact, it applies to the following more general situation.

Let $\mathbf{f} = (f_0, \dots, f_n)$ be a collection of $n + 1$ homogeneous polynomials in S_e for some degree $e > 0$. Assume that the associated ideal $J_{\mathbf{f}} = (f_0, \dots, f_n)$ in S is a *1-dimensional almost complete intersection*, that is, $\dim S/J_{\mathbf{f}} = 1$ and $(f_1, \dots, f_n)$ is a regular sequence in S . Replacing the differential df by the 1-form

$$\omega = f_0 dx_0 + \cdots + f_n dx_n,$$

one can define the graded S -modules $\text{AR}(\mathbf{f})$ and $\text{ER}(\mathbf{f})$ by the formulas (1.1) and (1.2), respectively. The ideal $J_{\mathbf{f}}$ defines a 0-dimensional subscheme $Y(\mathbf{f})$ of $\mathbb{P}^n$ and for each point p in the support $|Y(\mathbf{f})|$ of this scheme, we consider the corresponding local ring $\mathcal{O}_{Y(\mathbf{f}), p}$. This is an Artinian local ring, with maximal ideal denoted by m_p , and we denote by $o(m_p)$ the order of this ideal, that is, the smallest integer $k > 0$ such that $m_p^k = 0$ in $\mathcal{O}_{Y(\mathbf{f}), p}$. With this notation we have the following generalization of Theorem 2.4.

THEOREM 1.2. *Assume that the ideal $J_{\mathbf{f}} = (f_0, \dots, f_n)$ in S is a 1-dimensional almost complete intersection with $f_j \in S_e$ for all $j = 0, \dots, n$. Then*

$$\text{ER}(\mathbf{f})_m = 0 \quad \text{for any } m \leq n(e - 1) - \sum_{p \in |Y(\mathbf{f})|} o(m_p).$$

Let us come back to the case of a projective hypersurface V with isolated singularities. The exact sequence of coherent sheaves on X given by

$$0 \rightarrow T\langle V \rangle \rightarrow \mathcal{O}_X(1)^{n+1} \rightarrow \mathcal{I}_Y(N) \rightarrow 0, \quad (1.4)$$

with the last nonzero morphism induced by $(a_0, \dots, a_n) \mapsto a_0 f_{x_0} + \dots + a_n f_{x_n}$, can be used to define the sheaf $T\langle V \rangle$ of logarithmic vector fields along V ; see [Ser14]. This is a reflexive sheaf which is locally free when $n = 2$; see, for instance, [OSS80]. The above exact sequence clearly yields

$$\mathrm{AR}(f)_m = H^0(X, T\langle V \rangle(m-1)) \quad (1.5)$$

for any integer m . This equality can be used to show that the reflexive sheaf $T\langle V \rangle$ is stable in many cases. This was done in the case $n = 2$ in [DSe14] and the corresponding result is stated below in Corollary 3.1 and Example 3.2 without the hypothesis of weighted homogeneous singularities. But now we can go to higher dimension, and as an illustration we prove the following result.

THEOREM 1.3. *Assume that the surface $V: f = 0$ in $\mathbb{P}^3$ has degree $N = 3N_1 + \epsilon > 1$ with $N_1 \in \mathbb{N}$ and $\epsilon \in \{1, 2, 3\}$ and that V has only isolated singularities such that*

$$\tau(V) \leq 8N_1 + 3(\epsilon - 2).$$

Then $F = T\langle V \rangle(N_1 - 1)$ is a normalized stable rank 3 reflexive sheaf on $\mathbb{P}^3$ with first Chern class $c_1(F) = 1 - \epsilon \in \{0, -1, -2\}$. This reflexive sheaf F is locally free if and only if V is smooth.

The other Chern classes of F can be easily computed; see Remark 3.4. Recall the following notion.

DEFINITION 1.4. A reduced hypersurface $V \subset X = \mathbb{P}^n$ is called *DK–Torelli* (where DK stands for Dolgachev–Kapranov) if the hypersurface V can be reconstructed as a subset of X from the sheaf $T\langle V \rangle$.

For a discussion of this notion and various examples, we refer to [DK93, UY09, WY08, FMV13, DSe14]. In particular, E. Sernesi and the author have shown in [DSe14] that the nodal curves with a small number of nodes are DK–Torelli. In the proof, which follows the line of the proof for smooth hypersurfaces outlined by K. Ueda and M. Yoshinaga in [UY09], we have used the inequality (1.3) for $n = 2$. Since in the case of a small number of singularities the bound obtained in Theorem 1.1 is better, it is natural to see if this new bound gives a stronger result. Theorem 1.1 is hence applied to prove the following result, which slightly improves the results on the Torelli properties of nodal (or nodal and cuspidal) curves obtained in [DSe14]. The following result also extends the result by K. Ueda and M. Yoshinaga concerning smooth hypersurfaces in [UY09] to hypersurfaces having a small Tjurina number.

THEOREM 1.5. *Let $V: f = 0$ be a degree $N \geq 4$ hypersurface in $\mathbb{P}^n$ having only isolated singularities. If*

$$\tau(V) \leq \frac{(n-1)(N-4)}{2} + 1,$$

then one of the following holds:

- (i) *The hypersurface V is DK–Torelli.*
- (ii) *The hypersurface V is of Sebastiani–Thom type; that is, in some linear coordinate system $(x_0, \dots, x_n)$ on $\mathbb{P}^n$, the defining polynomial f for V is written as a sum $f = g + h$, with g and h*

polynomials involving only $x_0, \dots, x_r$ and $x_{r+1}, \dots, x_n$, respectively, for some integer r satisfying $0 \leq r < n$.

The interest in having both Theorems 1.3 and 1.5 is that this allows the construction of injective mappings from the varieties parametrizing surfaces with a fixed type of singularities into the moduli spaces of rank 3 stable reflexive sheaves on $\mathbb{P}^3$, exactly as in the case $n = 2$ discussed in [DSe14]. The case of rank 2 stable bundles on surfaces is rather well understood, see for instance [HL97], but the case of rank 3 reflexive sheaves on $\mathbb{P}^3$ seems to be still very mysterious. Our construction may bring new light in the study of such moduli spaces, a problem which has attracted much attention in the past; see [Coa86, Har80, Mar77, Mar78, Mir87, OS83, OSS80].

2. A new bound on the minimal degree of a syzygy

Let $\mathcal{O}_n$ denote the ring of holomorphic function germs at the origin of $\mathbb{C}^n$, and let $m_n \subset \mathcal{O}_n$ be its unique maximal ideal. For a function germ $g \in \mathcal{O}_n$ defining an isolated hypersurface singularity at the origin of $\mathbb{C}^n$, we introduce an invariant

$$a(g) = \min \{a \in \mathbb{N} : m_n^a \subset J_g + (g)\},$$

where J_g is the Jacobian ideal of g in $\mathcal{O}_n$ and (g) is the principal ideal spanned by g in $\mathcal{O}_n$. We let $M(g) = \mathcal{O}_n/J_g$ denote the *Milnor algebra* of the singularity g and $T(g) = \mathcal{O}_n/(J_g + (g))$ the corresponding *Tjurina algebra*. Note that $a(g) = o(\tilde{m}_0)$, where $\tilde{m}_0$ denotes the maximal ideal of the Tjurina algebra $T(g)$. This invariant $a(g)$ clearly depends only on the contact class of the germ g (that is, the isomorphism class of the analytic germ $(\{g = 0\}, 0)$), which is why we keep this notation in addition to the notation $o(\tilde{m}_0)$. This invariance is crucial in the computations given in the following example.

EXAMPLE 2.1. (i) If $g = 0$ is a node, that is, an A_1 -singularity, then $a(g) = 1$.

(ii) If $g = 0$ is a cusp, that is, an A_2 -singularity, then $a(g) = 2$.

(iii) If $g = 0$ is a D_4 -singularity, for example an ordinary 3-tuple point when $n = 2$, then $a(g) = 3$.

(iv) One always has $a(g) \leq \tau(g)$, where $\tau(g) = \dim T(g)$ is the Tjurina number of g . Usually this inequality is strict. For instance, when $g = x^d + y^d$ is an ordinary point of multiplicity d and $n = 2$, one has $a(g) = 2d - 3 < (d - 1)^2$ for $d \geq 3$. The case $d = 3$ corresponds to the D_4 -singularity in dimension 2 mentioned above.

One has a natural morphism $\mathcal{O}_X(k) \rightarrow \mathcal{O}_X(k)/\mathcal{I}_Y(k)$ for any integer k , inducing an evaluation morphism

$$\mathrm{ev}_k : S_k = H^0(X, \mathcal{O}_X(k)) \rightarrow H^0(X, \mathcal{O}_X(k)/\mathcal{I}_Y(k)) = H^0(Y, \mathcal{O}_Y),$$

where the last equality comes from the fact that Y is 0-dimensional; that is, its support $|Y| = V_{\mathrm{sing}}$, the singular set of V , consists of finitely many points $p_1, \dots, p_s$. This fact also implies

$$H^0(Y, \mathcal{O}_Y) = \bigoplus_{p \in V_{\mathrm{sing}}} \mathcal{O}_{Y,p}.$$

On the other hand, if $g_p = 0$ is a local analytic equation for the hypersurface singularity (V, p) , one has an isomorphism $\mathcal{O}_{Y,p} = T(g_p)$ of local $\mathbb{C}$ -algebras. The following result is elementary and has appeared in various forms; see, for instance, [BS93, Corollary 2], [Dim92, Proposition 1.3.9], or [Még12, Section 3].

LEMMA 2.2. *For any $k \geq \sum_{p \in V_{\text{sing}}} a(g_p) - 1$, the evaluation morphism $\text{ev}_k: S_k \rightarrow \bigoplus_{p \in V_{\text{sing}}} \mathcal{O}_{Y,p}$ is surjective. In other words, if one defines the k th defect of the singular locus subscheme Y by*

$$\text{def}_k Y = \dim \text{coker } \text{ev}_k ,$$

then $\text{def}_k Y = 0$ for $k \geq \sum_{p \in V_{\text{sing}}} a(g_p) - 1$.

Proof. One considers the following decomposition of the evaluation map ev_k :

$$S_k \rightarrow \bigoplus_{p \in |Y|} \mathcal{O}_{X,p} / m_{n,p}^{a(g_p)} \rightarrow \bigoplus_{p \in |Y|} \mathcal{O}_{Y,p} ,$$

with $m_{n,p}$ the maximal ideal of $\mathcal{O}_{X,p}$. Then one notices that the first morphism is surjective by [BS93, Corollary 2.1], and the second morphism is surjective by the definition of the invariants $a(g_p)$. $\square$

When $J_{\mathbf{f}} = (f_0, \dots, f_n)$ in S is a 1-dimensional almost complete intersection as in the introduction, we have the following similar result (with an identical proof).

LEMMA 2.3. *The evaluation morphism $\text{ev}_k: S_k \rightarrow \bigoplus_{p \in |Y(\mathbf{f})|} \mathcal{O}_{Y(\mathbf{f}),p}$ is surjective for any $k \geq \sum_{p \in |Y(\mathbf{f})|} o(m_p) - 1$. In other words, if one defines the k th defect of the subscheme $Y(\mathbf{f})$ by*

$$\text{def}_k Y(\mathbf{f}) = \dim \text{coker } \text{ev}_k ,$$

then $\text{def}_k Y(\mathbf{f}) = 0$ for $k \geq \sum_{p \in |Y(\mathbf{f})|} o(m_p) - 1$.

The main result of this section in the hypersurface case is the following.

THEOREM 2.4. *Assume that the hypersurface $V: f = 0$ in $\mathbb{P}^n$ has degree N and only isolated singularities, with local equations $g_p = 0$ for $p \in V_{\text{sing}}$. Then*

$$\text{ER}(f)_m = 0 \quad \text{for any } m \leq n(N-2) - \sum_{p \in V_{\text{sing}}} a(g_p) .$$

Proof. Using Theorem 1 in [Dim13b], we see that

$$\dim \text{ER}(f)_m = \text{def}_{nN-2n-1-m} Y . \tag{2.1}$$

The claim then follows from Lemma 2.2. $\square$

Recall that for a homogeneous polynomial $f \in S$, we define its Milnor (or Jacobian) graded algebra to be the quotient $M(f) = S/J_f$. Then the *coincidence threshold* $\text{ct}(V)$ is defined as

$$\text{ct}(V) = \max\{q: \dim_K M(f)_k = \dim M(f_s)_k \text{ for all } k \leq q\} ,$$

with f_s a homogeneous polynomial in S of degree N such that $V_s: f_s = 0$ is a smooth hypersurface in $\mathbb{P}^n$. Finally, the *minimal degree* $\text{mdr}(V)$ of a nontrivial relation is defined as

$$\text{mdr}(V) = \min\{q: \text{ER}(f)_q \neq 0\} .$$

It is known that one has the equality

$$\text{ct}(V) = \text{mdr}(V) + N - 2 ; \tag{2.2}$$

see [DSt15b, formula (1.3)]. Theorem 2.4 clearly implies the following.

COROLLARY 2.5. *Assume that the hypersurface $V: f = 0$ in $\mathbb{P}^n$ has degree N and only isolated singularities, with local equations $g_p = 0$ for $p \in V_{\text{sing}}$. Then*

$$\text{mdr}(V) \geq n(N-2) - \sum_{p \in V_{\text{sing}}} a(g_p) + 1$$

and

$$\mathrm{ct}(V) \geq T - \sum_{p \in V_{\mathrm{sing}}} a(g_p) + 1$$

with $T = (n+1)(N-2)$.

EXAMPLE 2.6. Consider a nodal hypersurface V in $\mathbb{P}^n$ having $\sharp A_1$ singularities A_1 . In this case $\alpha_V = n/2$, see [DSa14], hence the inequality (1.3) yields

$$\mathrm{ER}(f)_m = 0 \quad \text{for any } m < n(N-2)/2.$$

On the other hand, Theorem 2.4 yields

$$\mathrm{ER}(f)_m = 0 \quad \text{for any } m \leq n(N-2) - \sharp A_1.$$

The second vanishing result is stronger than the first one exactly when $n(N-2)/2 \leq n(N-2) - \sharp A_1$, that is, if and only if

$$\sharp A_1 \leq n(N-2)/2.$$

For $\sharp A_1 = 1$, this implies $\mathrm{ct}(V) \leq n(N-2) + N - 2 = (n+1)(N-2) = T$ and we know that this is in fact an equality by [DSt15b, Example 4.3(i)]. Similarly, [DSt15b, Example 4.3(ii)] shows that $\mathrm{ct}(V) = T - 1$ when $\sharp A_1 = 2$. Hence in these two cases, the inequality in Theorem 2.4 is in fact an equality. Example 4.3(iii) in [DSt15b] shows that $\mathrm{ct}(V) = T - 1$ or $\mathrm{ct}(V) = T - 2$ when $\sharp A_1 = 3$, depending on whether the three nodes are collinear or not. It follows that the bound given by Theorem 2.4 is optimal for $\sharp A_1 \leq 3$.

EXAMPLE 2.7. Consider a reduced plane curve $V: f = 0$ in $\mathbb{P}^2$ having n_k ordinary singularities of multiplicity k for $k = 2, 3, 4$ and no other singularities. Theorem 2.4 and Example 2.1 yield

$$\mathrm{ER}(f)_m = 0 \quad \text{for any } m \leq 2(N-2) - n_2 - 3n_3 - 5n_4.$$

In the nodal case, that is, when $n_3 = n_4 = 0$, this bound can be better than the one given by the inequality (1.3), but only when V is irreducible (indeed, otherwise $\mathrm{ER}(f)_{N-2} \neq 0$, as shown in [DSt15b] via Hodge theory and in [EU13] without Hodge theory and in a more general setting).

Now, we consider the case of a 1-dimensional almost complete intersection and give *the proof of Theorem 1.2*. The proof is very similar to that of Theorem 1.1, but we choose to present these results separately in order to preserve the additional geometric flavor of Theorem 1.1. Note that the key equation (2.1) holds in the more general case of a 1-dimensional almost complete intersection, as noted at the end of [Dim13b, Remark 4]. Namely, we have

$$\dim \mathrm{ER}(\mathbf{f})_m = \mathrm{def}_{n(e-1)-1-m} Y(\mathbf{f}). \quad (2.3)$$

It remains to apply Lemma 2.3 instead of Lemma 2.2 to complete the proof of Theorem 1.2. Here is one easy consequence.

COROLLARY 2.8. *Let $\mathbf{g} = (g_1, \dots, g_n)$ be a collection of n homogeneous polynomials in S_e for some degree $e > 0$ such that $(g_1, \dots, g_n)$ is a regular sequence in S . The ideal spanned by the g_i define a 0-dimensional subscheme $Y(\mathbf{g})$. For each point q in the support $|Y(\mathbf{g})|$ of this scheme, let m'_q be the maximal ideal in the local ring $\mathcal{O}_{Y(\mathbf{g}),q}$ and let $o(m'_q)$ denote its order. Then*

$$\sum_{q \in |Y(\mathbf{g})|} o(m'_q) \geq n(e-1) + 1.$$

A. DIMCA

Proof. Apply Theorem 1.2 to the collection $\mathbf{f} = (f_0, \dots, f_n)$, with $f_0 = 0$ and $f_i = g_i$ for $i > 0$. The relation $1 \cdot f_0 = 0$ implies $\text{ER}(\mathbf{f})_0 \neq 0$, and this yields the result. $\square$

Remark 2.9. (i) The example $g_i = x_i^e$ for $i = 1, \dots, n$ shows that the inequality in Corollary 2.8 is sharp.

(ii) It is clear that $o(m'_q) \leq \dim_{\mathbb{C}} \mathcal{O}_{Y(\mathbf{g}),q}$, where the equality holds if and only if the 0-dimensional singularity $(Y(\mathbf{g}), q)$ has embedding dimension at most 1. In particular, we get

$$\sum_{q \in |Y(\mathbf{g})|} o(m'_q) \leq \sum_{q \in |Y(\mathbf{g})|} \dim_{\mathbb{C}} \mathcal{O}_{Y(\mathbf{g}),q} = e^n.$$

(iii) The reader interested in the case of a 1-dimensional complete intersection $\mathbf{f} = (f_0, \dots, f_n)$, where the degrees of the f_i are different, can also obtain bounds on the degree of syzygies following the above approach. Indeed, the relation (2.3) essentially continues to hold, as noted at the end of [Dim13b, Remark 4] and the references given there, but additional care is needed to get the right homogeneous components in $\text{ER}(\mathbf{f})$ in this case.

3. Stability of $T\langle V \rangle$ for $n = 2$ and $n = 3$

First, we discuss the simpler case $n = 2$. Using [Ser14, Proposition 2.4] which says that $T\langle V \rangle$ is stable if and only if $\text{AR}(f)_m = 0$ for all $m \leq (N-1)/2$, we get the following consequence of our Theorem 2.4.

COROLLARY 3.1. *Assume that the curve $V: f = 0$ in $X = \mathbb{P}^2$ has degree N and only isolated singularities, with local equations $g_p = 0$ for $p \in V_{\text{sing}}$. Then the vector bundle $T\langle V \rangle$ is stable if*

$$[(N-1)/2] \leq 2(N-2) - \sum_{p \in V_{\text{sing}}} a(g_p), \quad (3.1)$$

where $[y]$ denotes the largest integer satisfying $[y] \leq y$. In particular, since a rank 2 stable vector bundle is not splittable, it follows that V is not a free divisor when the inequality (3.1) holds.

EXAMPLE 3.2. For $N \geq 5$, consider the family of plane curves $V_N: f_N = 0$ in $\mathbb{P}^2$ given by the equation

$$f_N = x^2 y^2 z^{N-4} + x^5 z^{N-5} + y^5 z^{N-5} + x^N + y^N = 0.$$

Then V_N has a unique singularity at $p_1 = (0, 0, 1)$ which is isomorphic to the singularity $g(u, v) = u^2 v^2 + u^5 + v^5$. It follows that $\tau(V_N) = \tau(g) = 10$ and $a(g) = 5$; see, for instance, [Dim87, Example (6.56)]. Moreover, this singularity, usually denoted by $T_{2,5,5}$ in Arnold's classification, is not weighted homogeneous since $11 = \mu(g) > \tau(g) = 10$.

Theorem 2.4 yields $\text{mdr}(V_N) \geq 2N-8$, while Theorem 1.1 yields the weaker bound $\text{mdr}(V_N) \geq 2N-13$. A direct computation of the Jacobian syzygies in the case $5 \leq N \leq 10$ using SINGULAR shows that $\text{mdr}(V_N) = 2N-7$. Therefore, Theorem 2.4 is almost sharp in these cases.

Using Corollary 3.1, this computation also implies that the curves V_N have the property that the associated bundle $T\langle V_N \rangle$ is stable for any $N \geq 5$.

EXAMPLE 3.3. If V is an irreducible free divisor in $X = \mathbb{P}^2$ with degree N and only isolated singularities, with local equations $g_p = 0$ for $p \in V_{\text{sing}}$, it follows from Corollary 3.1 that one has

$$\sum_{p \in V_{\text{sing}}} a(g_p) > 2(N-2) - [(N-1)/2].$$

In other words, such a curve should have many singularities (or singularities with large invariants $a(g_i)$) and this explains the difficulty and the interest in constructing such examples; see, for instance, [BC13, ST14, DSt15a]. The example V_5 above shows that the inequality (3.3) is not sufficient to imply the freeness of the divisor.

Now, we pass to the case $n = 3$ and prove Theorem 1.3. First, we compute the Chern classes of the sheaf $T\langle V \rangle$. The exact sequence (1.4) yields

$$c(T\langle V \rangle) \cdot c(\mathcal{I}_Y(N)) = (1 + \alpha)^4,$$

where $\alpha \in H^2(\mathbb{P}^3)$ is the standard generator and $\alpha^4 = 0$. To compute $c(\mathcal{I}_Y(N))$, we use the exact sequence

$$0 \rightarrow \mathcal{I}_Y(N) \rightarrow \mathcal{O}_X(N) \rightarrow \mathcal{O}_Y \rightarrow 0, \quad (3.2)$$

which gives

$$c(\mathcal{I}_Y(N)) \cdot (1 + \tau(V)\alpha^3) = 1 + N\alpha.$$

Using this, by a direct computation, we finally get the following Chern classes (identified to the corresponding Chern numbers): $c_1(T\langle V \rangle) = -N + 4$,

$$c_2(T\langle V \rangle) = N^2 - 4N + 6, \quad \text{and} \quad c_3(T\langle V \rangle) = -N^3 + 4N^2 - 6N + 4 + \tau(V). \quad (3.3)$$

To study the stability of $T\langle V \rangle$, we first have to normalize it, that is, find an integer m such that $c_1(T\langle V \rangle(m)) \in \{0, -1, -2\}$. Now,

$$c_1(T\langle V \rangle(m)) = -N + 4 + 3m = -3N_1 - \epsilon + 4 + 3m,$$

and hence $m = N_1 - 1$ is the right choice. Indeed, $F = T\langle V \rangle(m)$ satisfies $c_1(F) = 1 - \epsilon$.

Consider first the case $\epsilon = 1$, hence $c_1(F) = 0$. Then, using [OSS80, II, Section 1.2, p. 85], we see that F is stable if and only if one has

$$H^0(X, F) = H^0(X, F^*) = 0, \quad (3.4)$$

where $X = \mathbb{P}^3$ and F^* is the dual of F . The first condition is easy to check. Indeed, using (1.5), we have $H^0(X, F) = \text{AR}(f)_{m+1}$. On the other hand, $\text{AR}(f)_{m+1} = 0$ since the condition $\tau(V) \leq 8N_1 - 3$ is exactly what we need to apply Theorem 1.1. Also note that $m+1 = (N-1)/3 < N-1$ and hence $\text{AR}(f)_{m+1} = \text{ER}(f)_{m+1}$. The second vanishing $H^0(X, F^*) = 0$ requires more work. If we dualize the exact sequence (1.4), we get

$$0 \rightarrow \mathcal{I}_Y(N)^* \rightarrow \mathcal{O}_X(-1)^4 \rightarrow T\langle V \rangle^* \rightarrow \mathcal{E}\text{xt}^1(\mathcal{I}_Y(N), \mathcal{O}_X) \rightarrow \cdots. \quad (3.5)$$

We show that $\mathcal{E}\text{xt}^1(\mathcal{I}_Y(N), \mathcal{O}_X) = 0$ by proving the corresponding vanishing at the stalk level. There are three cases to discuss.

Case 1. If $x \in X$ is not a singular point in Y , then $\mathcal{I}_Y(N)_x = \mathcal{O}_{X,x}$ and hence clearly $\text{Ext}^1(\mathcal{I}_Y(N)_x, \mathcal{O}_{X,x}) = 0$.

Case 2. Assume that $x \in X$ is a weighted homogeneous singular point of V , with local equation $g = 0$. Then $g \in J_g$ and hence $\mathcal{I}_Y(N)_x = J_g$, which is a 0-dimensional complete intersection. The exact sequence

$$0 \rightarrow J_g \rightarrow \mathcal{O}_{X,x} \rightarrow M(g) \rightarrow 0$$

yields a long exact sequence containing the sequence

$$0 = \text{Ext}^1(\mathcal{O}_{X,x}, \mathcal{O}_{X,x}) \rightarrow \text{Ext}^1(J_g, \mathcal{O}_{X,x}) \rightarrow \text{Ext}^2(M(g), \mathcal{O}_{X,x}) = 0.$$

The last vanishing is a well-known property of complete intersections, see for instance [GH78, 5.3, Subsection “Intrinsic Form of Local Duality”, p. 690], since $\dim X = 3 > 2$.

Case 3. Now, consider the case of a singularity (V, x) given by $g = 0$ which is not weighted homogeneous, so that $\mathcal{I}_Y(N)_x = (g) + J_g$. The exact sequence

$$0 \rightarrow J_g \rightarrow (g) + J_g \rightarrow ((g) + J_g)/J_g \rightarrow 0$$

and case 2 show that it is enough to prove that $\text{Ext}^1(((g) + J_g)/J_g, \mathcal{O}_{X,x}) = 0$. Let K_g be the kernel of the morphism $g: M(g) \rightarrow M(g)$ given by multiplication. Then we have a short exact sequence

$$0 \rightarrow K_g \rightarrow M(g) \rightarrow ((g) + J_g)/J_g \rightarrow 0,$$

which gives the result since

$$0 = \text{Hom}(K_g, \mathcal{O}_{X,x}) \rightarrow \text{Ext}^1(((g) + J_g)/J_g, \mathcal{O}_{X,x}) \rightarrow \text{Ext}^1(M(g), \mathcal{O}_{X,x}) = 0.$$

In conclusion, (3.5) is a short exact sequence. Twisting by $(-m)$, we get

$$0 \rightarrow \mathcal{I}_Y(N+m)^* \rightarrow \mathcal{O}_X^4(-1-m) \rightarrow F^* \rightarrow 0. \quad (3.6)$$

Hence, to show that $H^0(X, F^*) = 0$, it is enough to show that $H^1(X, \mathcal{I}_Y(N+m)^*) = 0$ since $m \geq 0$. The sheaf $\mathcal{I}_Y(N+m)^*$ is a reflexive sheaf of rank 1, and hence it is an invertible sheaf; see [OSS80, II, Section 1.1, p. 78]. Since $X = \mathbb{P}^3$, it follows that $\mathcal{I}_Y(N+m)^* = \mathcal{O}_X(k)$ for some integer k , and then the vanishing

$$H^1(X, \mathcal{I}_Y(N+m)^*) = H^1(X, \mathcal{O}_X(k)) = 0$$

is well known.

Now, we treat the remaining cases, that is, the cases where $c_1(F) = 1 - \epsilon \neq 0$. Remark 1.2.6(b) in [OSS80, II, p. 85] implies that $F = T\langle V \rangle(m)$ is stable in such cases if and only if one has

$$H^0(X, F) = H^0(X, F^*(-1)) = 0.$$

These vanishings are proved exactly as above. Namely, in the case $N = 3N_1 + 3$, the sufficient condition to get the vanishing $H^0(X, F) = 0$ via Theorem 1.1 is

$$\tau(V) \leq 8N_1 + 3.$$

In the case $N = 3N_1 + 2$, the sufficient condition for $H^0(X, F) = 0$ is $\tau(V) \leq 8N_1$. Moreover, the vanishing $H^0(X, F^*(-1)) = 0$ follows from the short exact sequence (3.6), by twisting by (-1) and using the fact that $m \geq -1$.

Finally, we show that the sheaf F is locally free only when V is smooth. For $n \geq 2$, we have the following fact attributed to R. Hartshorne, see [AY08, Lemma 3.2] (the proof can be found in the arXiv version or [BS13, Exercise 20.4.22(ii)]: a coherent sheaf G on $X = \mathbb{P}^n$ is locally free if and only if $H^j(X, G(k)) = 0$ for all $j < n$ and $k \ll 0$. These vanishings for a locally free sheaf follow, in fact, from Serre’s vanishing theorem [Har80, Theorem III.5.2] combined with Serre’s duality theorem [Har80, Theorem III.7.6].

The exact sequence (3.2) tensored by $\mathcal{O}_X(k)$ shows that $H^1(X, \mathcal{I}_Y(N+k)) = H^0(X, \mathcal{O}_Y) \neq 0$ when $k \ll 0$ exactly when Y , the singular locus of V , is nonempty. And then the exact sequence (1.4) implies that

$$H^2(X, T\langle V \rangle(k)) = H^1(X, \mathcal{I}_Y(N+k)) \neq 0$$

for $n \geq 3$ and $k \ll 0$. This completes the proof of Theorem 1.3.

Remark 3.4. To compute the higher Chern classes of the sheaf F , it suffices to use the formulas for the Chen classes of a tensor product; see [OSS80, I, Section 1.2, p. 16]. In this way, we get

$$c_2(T\langle V \rangle(m)) = c_2(T\langle V \rangle) + 2mc_1(T\langle V \rangle) + 3m^2$$

and

$$c_3(T\langle V \rangle(m)) = c_3(T\langle V \rangle) + mc_2(T\langle V \rangle) + m^2c_1(T\langle V \rangle) + m^3.$$

In the case $\epsilon = 1$, using the formulas in (3.3) for $c_i(T\langle V \rangle)$ and replacing m by $N_1 - 1$ yields

$$c_2(F) = 6N_1^2 \quad \text{and} \quad c_3(F) = -20N_1^3 - 3(N_1 - 1)^3 + \tau(V).$$

4. DK–Torelli-type properties for singular hypersurfaces

Now, we turn to the proof of Theorem 1.5 stated in the introduction. This proof follows closely the proof of the corresponding result in [DSe14]. We repeat below the main steps, for the reader's convenience and also to point out the new facts necessary to treat the n -dimensional case.

LEMMA 4.1. *With the above notation and hypothesis, the sheaf $T\langle V \rangle$ determines the vector subspace $J_{f,N-1} \subset S_{N-1}$.*

To prove this lemma, let $E : g = 0$ be a (possibly nonreduced) hypersurface in $X = \mathbb{P}^n$ of degree $N - 1$. For any $k \in \mathbb{Z}$, consider the exact sequence

$$0 \rightarrow \mathcal{O}_X(k - N + 1) \rightarrow \mathcal{O}_X(k) \rightarrow \mathcal{O}_E(k) \rightarrow 0,$$

where the first morphism is induced by the multiplication by g . Tensoring this sequence of sheaves by $T\langle V \rangle$, we get a new short exact sequence

$$0 \rightarrow T\langle V \rangle(k - N + 1) \rightarrow T\langle V \rangle(k) \rightarrow T\langle V \rangle(k) \otimes \mathcal{O}_E \rightarrow 0.$$

The injectivity of the first morphism comes from the fact that $T\langle V \rangle(k - N + 1)$ is a torsion-free sheaf. The associated long exact sequence of cohomology groups looks like

$$\begin{aligned} 0 \rightarrow H^0(T\langle V \rangle(k - N + 1)) \rightarrow H^0(T\langle V \rangle(k)) \rightarrow H^0(T\langle V \rangle(k) \otimes \mathcal{O}_E) \rightarrow \\ \rightarrow H^1(T\langle V \rangle(k - N + 1)) \rightarrow H^1(T\langle V \rangle(k)) \rightarrow \dots \end{aligned}$$

Then, using the formula (1.5), we see that

$$\delta_k = \dim H^0(T\langle V \rangle(k)) - \dim H^0(T\langle V \rangle(k - N + 1)) = \dim \text{AR}(f)_{k+1} - \dim \text{AR}(f)_{k-N+2}$$

depends only on f and not on g . Next, note that the morphism

$$H^1(T\langle V \rangle(k - N + 1)) \rightarrow H^1(T\langle V \rangle(k))$$

in the above exact sequence can be identified, using [Ser14, formulas (5) and (9)] with the morphism

$$g_{k+1}^* : (I/J_f)_{k+1} \rightarrow (I/J_f)_{k+N}$$

induced by the multiplication by g (we recall that I is the saturation of the Jacobian ideal J_f). The above proves the following equality:

$$\dim H^0(T\langle V \rangle(k) \otimes \mathcal{O}_E) = \delta_k + \dim \ker g_{k+1}^*.$$

Let m be the largest integer such that $2m \leq N - 2$. Since clearly $m < N - 1$, it follows that $J_{f,m} = 0$ and hence g_m^* is defined on I_m . If $g \in J_f$, then clearly $g_m^* = 0$, and hence the kernel of g has maximal possible dimension.

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To complete the proof of Lemma 4.1, it is enough to show that the converse also holds. To do this, we first show that there are two elements $h_1, h_2 \in I_m$ having no irreducible factor in common. Otherwise, all the elements in I_m are divisible by a homogeneous polynomial of degree at least 1, and hence, in particular, one has $\dim I_m \leq \dim S_{m-1}$, which implies

$$\tau(V) \geq \dim S_m/I_m \geq \binom{m+n}{n} - \binom{m+n-1}{n} = \binom{m+n-1}{n-1}.$$

One also has the (perhaps well-known) inequality

$$\binom{m+n-1}{n-1} \geq (n-1)m+1, \quad (4.1)$$

which can be proved by looking at the subsets E' of cardinal $n-1$ of a set E which is a disjoint union $E = E_1 \cup E_2$, with $\#E_1 = m$ and $\#E_2 = n-1$, and counting how many subsets E' satisfy $\#(E' \cap E_1) \leq 1$. It follows that

$$2m \leq 2 \cdot \frac{\tau(V)-1}{n-1} \leq N-4,$$

giving a contradiction with the choice of m . This shows that there are two elements $h_1, h_2 \in I_m$ having no irreducible factor in common.

Then $g_m^* = 0$ implies $gh_1 = \sum_{j=0,n} a_j f_{x_j}$ and $gh_2 = \sum_{j=0,n} b_j f_{x_j}$ for some polynomials $a_j, b_j \in S_m$. It follows that

$$\sum_{j=0,n} (a_j h_2 - b_j h_1) f_{x_j} = 0.$$

Since

$$\sum_{i=1,s} a(g_i) \leq \tau(V) = \sum_{i=1,s} \tau(g_i) \leq \frac{(n-1)(N-4)}{2} + 1 \leq (n-1)(N-2),$$

it follows that

$$2m \leq N-2 \leq n(N-2) - \sum_{i=1,s} a(g_i).$$

Theorem 2.4 implies that the only syzygy of degree $2m$ is the trivial one, that is, $a_j h_2 = b_j h_1$ for any j . These relations imply that the polynomials a_j are all divisible by h_1 in S , and hence $g \in J_f$.

It follows that $g \in J_{f,N-1}$ if and only if

$$\dim H^0(T\langle V \rangle(m-1) \otimes \mathcal{O}_E) = \delta_{m-1} + \dim I_m,$$

that is, the sheaf $T\langle V \rangle$ determines the homogeneous component $J_{f,N-1}$ of the Jacobian ideal J_f , and this completes the proof of Lemma 4.1.

To finish the proof of Theorem 1.5, it is enough to use Theorem 1.1 of Z. Wang's paper [Wan15], which generalizes a lemma in [DSe14] covering the case $n = 2$. Indeed, this theorem says that we can have the following situations:

(A) The Jacobian ideal J_f (or its homogeneous component $J_{f,N-1}$) determines f up to a multiplicative nonzero constant. In this case, V is DK-Torelli.

(B) The hypersurface V is of Sebastiani-Thom type.

(C) The hypersurface V has at least one singular point p_i with multiplicity $N-1$; that is, there is a local equation g_i such that $g_i \in m_{n,p_i}^{N-1}$. However, this is impossible in our conditions, as we

now show. If $g_i \in m_{n,p_i}^{N-1}$, then the monomials in the corresponding local coordinates $u_1, \dots, u_n$ of degree at most $N - 3$ are linearly independent in $\mathcal{O}_n/(J_{g_i} + (g_i))$. This implies that

$$\tau(V) \geq \tau(g_i) \geq \dim S_{N-3} = \binom{N-3+n}{n} \geq n(N-3) + 1,$$

as in (4.1). But this is in contradiction with the hypothesis

$$\tau(V) \leq \frac{(n-1)(N-4)}{2} + 1,$$

so the proof of Theorem 1.5 is complete.

COROLLARY 4.2. *Assume that the curve $V: f = 0$ in $\mathbb{P}^2$ has degree $N \geq 4$ and has ν nodes, κ cusps, and no other singularities. If*

$$\nu + 2\kappa \leq \frac{N-2}{2},$$

then the curve V is DK–Torelli.

This is a direct consequence of Theorem 1.5, since a curve with only nodes and cusps and of degree at least 4 cannot satisfy property (ii); see also [DSe14]. For V irreducible and $\kappa = 0$, this result coincides with the result given in [DSe14]. For the remaining cases, Corollary 4.2 is a slight improvement over the corresponding results given in [DSe14]. In particular, Corollary 4.2 shows that a curve with $\nu = 0$ and $\kappa = 1$ is DK–Torelli as soon as $N \geq 6$, while the bound given in [DSe14] for the same result was $N \geq 8$.

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