Unirationality of moduli spaces of special cubic fourfolds and K3 surfaces

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ABSTRACT

We provide explicit descriptions of the generic members of Hassett’s divisors $\mathcal{C}_d$ in the moduli space $\mathcal{C}$ of smooth cubic fourfolds for relevant $18 \leq d \leq 38$ and for $d = 44$. In doing so, we prove that $\mathcal{C}_d$ is unirational for these d . As a corollary, we prove that the moduli space $\mathcal{N}_d$ of polarized K3 surfaces of degree d is unirational for $d = 14, 26, 38$. The case $d = 26$ is entirely new, while the other two cases have been previously proven by Mukai.

1. Introduction

In this paper, we systematically describe explicit surfaces that are contained in a generic special cubic fourfold of discriminant $d \leq 44$ with $d \neq 42$, recovering descriptions of the previously known cases for $d = 12, 14, 20$. Recall that a smooth cubic fourfold $X \subset \mathbb{P}^5$ is the vanishing locus of a degree 3 homogeneous polynomial in 6 variables, and X is called special if it contains the class of an algebraic surface S not homologous to a complete intersection. Denoting the square of the hyperplane class by h^2 , one defines the discriminant d to be the discriminant of the saturated sublattice generated by h^2 and S . In [Has00], Hassett defined the sublocus of the moduli space $\mathcal{C}$ of smooth cubic fourfolds consisting of special cubic fourfolds of discriminant d , denoted $\mathcal{C}_d$. He proved that $\mathcal{C}_d$ is an irreducible divisor for any $d > 6$ with $d \equiv 0, 2 \pmod{6}$ and empty otherwise.

For the almost twenty years since Hassett’s work, the only $\mathcal{C}_d$ whose generic member X could be described explicitly were for $d = 8, 12, 14$, and 20 . The surfaces S in these cases were given by planes, cubic scrolls, quintic del Pezzos, and Veronese surfaces, respectively. For some of these choices of d , the generic $X \in \mathcal{C}_d$ admits an alternative description. For example, the generic $X \in \mathcal{C}_8$ can be described as containing an octic K3 surface, and the generic $X \in \mathcal{C}_{14}$ contains a quartic scroll. It is nevertheless notable that for each d above, the surface S can be taken to be a smooth rational surface. By using the deformation theory of Hilbert schemes of flags and appropriately chosen linear systems on rational surfaces, we show that this trend continues for all relevant $d \leq 38$. For $d = 44$, however, we obtain a different kind of description.

THEOREM 1.1. *The generic element of $\mathcal{C}_d$ for $12 \leq d \leq 38$ contains a smooth rational surface obtained as the blow-up of $\mathbb{P}^2$ at p generic points and embedded into $\mathbb{P}^5$ via the very ample linear*

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system $|H| = |aL - (E_1 + \cdots + E_i) - 2(E_{i+1} + \cdots + E_{i+j}) - 3(E_{i+j+1} + \cdots + E_p)|$, where H is given by Table 1. Moreover, the generic $X \in \mathcal{C}_{44}$ contains a Fano embedded Enriques surface.

The problem of concretely describing the generic $X \in \mathcal{C}_d$ is interesting in its own right, and it is only natural to attempt to describe the cubics in $\mathcal{C}_d$ in terms of their geometry. But the problem is also intimately related to the geometry of the $\mathcal{C}_d$ themselves. Indeed, $\mathcal{C}_d$ is known to be unirational for $d = 8, 12, 14, 20$, for example, precisely because the cubics they parametrize contain the specific surfaces mentioned above. The method we use to prove Theorem 1.1 above gives unirationality as a consequence.

THEOREM 1.2. *For $12 \leq d \leq 44$ with $d \neq 42$, the moduli space $\mathcal{C}_d$ is unirational.*

Unirationality of a moduli space is a very useful property. It indicates that the generic element can be written down explicitly in free coordinates.

One expects that $\mathcal{C}_d$ ceases to be unirational as d grows, and it is natural to ask what is the smallest d such that $\mathcal{C}_d$ is not unirational, and at the other extreme, one can ask if there is a minimal d after which $\mathcal{C}_d$ is of general type. Questions of this nature have been previously investigated by Gritsenko, Hulek, and Sankaran in the cases of polarized K3 surfaces and certain families of holomorphic symplectic manifolds (see [GHS13] for a good account). They prove, for example, that the moduli space $\mathcal{N}_{2d}$ of polarized K3 surfaces of degree $2d$ has non-negative Kodaira dimension for $d \geq 40$ with $d \neq 41, 44, 45, 47$ and is of general type for $d > 61$, as well as for $d = 46, 50, 52, 54, 57, 58, 60$. By using a surprising and beautiful connection between the period domains of cubic fourfolds and polarized K3 surfaces [Has00], one can translate this result about $\mathcal{N}_d$ to get the following.

PROPOSITION 1.3. *Let $d > 80$ with $d \equiv 2 \pmod{6}$ and $4 \nmid d$ be such that for any odd prime p , we have that $p \mid d$ implies $p \equiv 1 \pmod{3}$. Then the Kodaira dimension of $\mathcal{C}_d$ is non-negative. If moreover $d > 122$, then $\mathcal{C}_d$ is of general type.*

Proposition 1.3 thus provides an infinite number of large d such that $\mathcal{C}_d$ is of general type, and one expects that the gaps can be filled in using automorphic form techniques as in the K3 case. This has recently been shown to be the case by Várilly-Alvarado and Tanimoto [TV15].¹ The results of [TV15] bound from above the minimum discriminant required for $\mathcal{C}_d$ to have non-negative Kodaira dimension, and Theorem 1.2 gives 44 as a lower bound.

Through the isomorphism of period domains associated with $\mathcal{N}_d$ and $\mathcal{C}_d$ for special values of d [Has00], Theorem 1.2 implies the following result.

THEOREM 1.4. *The moduli space $\mathcal{N}_d$ parametrizing polarized K3 surfaces of degree d is unirational for $d = 14, 26, 38$.*

The cases $d = 14, 38$ were already known due to the work of Mukai. He obtains his results by presenting the generic $T \in \mathcal{N}_d$ as a complete intersection in a certain homogeneous space. The case $d = 26$ is entirely new as far as I know and fills in a long-standing gap in known unirationality results ($\mathcal{N}_d$ was already known to be unirational for $2 \leq d \leq 24$ and $d = 30, 32, 34, 38$; for $d = 26$, however, $\mathcal{N}_{26}$ was shown to have negative Kodaira dimension by Peterson [Pet15]). While this result proves the unirationality of $\mathcal{N}_{26}$, it does not provide a geometric construction of the generic K3 surface it parametrizes. Such a construction remains an interesting open problem.

¹In particular, they have shown that $\mathcal{C}_{6n+2}$ is of general type for $n > 18$ and $n \neq 20, 21, 25$ and has non-negative Kodaira dimension for $n > 13$ and $n \neq 15$. Moreover, $\mathcal{C}_{6n}$ is of general type for $n > 18$ and $n \neq 20, 22, 23, 25, 30, 32$, and it has non-negative Kodaira dimension for $n > 16$ and $n \neq 18, 20, 22, 30$.

2. Review: cubic fourfolds and the divisors $\mathcal{C}_d \subset \mathcal{C}$

Let $X \subset \mathbb{P}^5$ be a smooth cubic fourfold, and define $A(X) := H^{2,2}(X) \cap H^4(X, \mathbb{Z})$. In [Has00], Hassett studied special cubic fourfolds, where a cubic fourfold X is *special* if $\text{rk } A(X) \geq 2$. Inside the coarse moduli space $\mathcal{C}$ of smooth cubic fourfolds, which is defined as the GIT quotient $\mathcal{C} := \mathbb{P}H^0(\mathbb{P}^5, \mathcal{O}(3))_{\text{smooth}} // \text{PGL}_6$, a very general cubic fourfold X satisfies $\text{rk } A(X) = 1$. Define the rank 2 lattice K_d to be

$$K_d := \begin{cases} \begin{array}{c|cc} & h^2 & S \\ \hline h^2 & 3 & 1 \\ S & 1 & 2n+1 \end{array} & \text{if } d = 6n + 2, n \geq 1, \\ \begin{array}{c|cc} & h^2 & S \\ \hline h^2 & 3 & 0 \\ S & 0 & 2n \end{array} & \text{if } d = 6n, n \geq 2, \end{cases}$$

and let $\mathcal{C}_d$ be the locus of $X \in \mathcal{C}$ for which K_d is a saturated sublattice of $A(X)$ containing h^2 . Then the main theorem we need is the following.

THEOREM 2.1 ([Has00, Theorem 1.0.1]). *If a cubic fourfold X is special, then $[X] \in \mathcal{C}_d$ for some d . Moreover, $\mathcal{C}_d \neq \emptyset$ if and only if $d > 6$ is an integer with $d \equiv 0, 2 \pmod{6}$, in which case $\mathcal{C}_d$ is an irreducible algebraic divisor of $\mathcal{C}$.*

3. Explicit descriptions of the divisors $\mathcal{C}_d$ for $d \leq 38$ and $d = 44$

Hassett provided an explicit description of the generic members of the divisors $\mathcal{C}_d$ for $d \leq 20$ with $d \neq 18$. In this section, we find explicit descriptions of the generic members of the divisors $\mathcal{C}_d$ for $18 \leq d \leq 38$ and $d = 44$. In fact, our methods recover all of the previously known results. Our main tools are the deformation theory of Hilbert schemes of flags, semicontinuity arguments, and explicit Macaulay2 calculations.²

Consider the Flag Hilbert scheme

$$\mathbf{FH} = \{(S, X) \mid S \subset X\} \subset \text{Hilb}_{\mathbb{P}^5}^{\chi(\mathcal{O}_S(n))} \times V,$$

where S a smooth surface and $V \subset \mathbb{P}^{55}$ parametrizes smooth cubic fourfolds, and its two projections

$$\begin{array}{ccc} & \mathbf{FH} & \\ \pi_2 \swarrow & & \searrow \pi_1 \\ \text{Hilb}_{\mathbb{P}^5}^{\chi(\mathcal{O}_S(n))} & & V \subset \mathbb{P}^{55}. \end{array}$$

The first projection $\pi_1: \mathbf{FH} \rightarrow \text{Hilb}_{\mathbb{P}^5}^{\chi(\mathcal{O}_S(n))}$ sends a flag $(S \subset X)$ to the corresponding point $[S]$ in the Hilbert scheme $\text{Hilb}_{\mathbb{P}^5}^{\chi(\mathcal{O}_S(n))}$. The fiber of this morphism above $[S]$ is $\mathbb{P}(H^0(\mathcal{I}_{S/\mathbb{P}^5}(3)))$. By the semicontinuity theorem, $h^0(\mathcal{I}_{S/\mathbb{P}^5}(3))$ achieves its minimum value on an open subset

²Scripts for duplicating the computer calculations performed in this paper are available at the author's website <http://www.math.rutgers.edu/~hjn11/publications/index.html>.

$W \subset \text{Hilb}_{\mathbb{P}^5}^{\chi(\mathcal{O}_S(n))}$. If we assume that $\text{Hilb}_{\mathbb{P}^5}^{\chi(\mathcal{O}_S(n))}$ is smooth at $[S] \in W$, so that

$$\dim_{[S]} \text{Hilb}_{\mathbb{P}^5}^{\chi(\mathcal{O}_S(n))} = \dim T_{[S]} \text{Hilb}_{\mathbb{P}^5}^{\chi(\mathcal{O}_S(n))} = h^0(S, N_{S/\mathbb{P}^5}) \quad (3.1)$$

by standard deformation theory, the open subset $\pi_1^{-1}(W) \subset \mathbf{FH}$ is a projective bundle over W , with fibers isomorphic to $\mathbb{P}^{h^0(\mathcal{I}_{S/\mathbb{P}^5}(3)) - 1}$, and is smooth around the fiber $\pi_1^{-1}([S])$. Thus for any cubic fourfold $X \supset S$, we have by (3.1) that

$$\dim_{(S \subset X)} \mathbf{FH} = h^0(S, N_{S/\mathbb{P}^5}) + h^0(\mathcal{I}_{S/\mathbb{P}^5}(3)) - 1. \quad (3.2)$$

Now, consider the second projection $\pi_2: \mathbf{FH} \rightarrow V$. As those cubic fourfolds X lying in the image of π_2 are special, the image of π_2 must be contained in the union of the preimages of the $\mathcal{C}_d$ under the quotient map $V \rightarrow V//\text{PGL}_6 = \mathcal{C}$ and thus has dimension at most $54 = \dim \mathcal{C}_d + \dim \text{PGL}_6$. It follows from the theorem on the dimension of fibers that the fibers of $\pi_2|_{\pi_1^{-1}(W)}$ must satisfy

$$\dim \pi_2^{-1}(X) \geq \dim \mathbf{FH} - \dim \pi_2(\mathbf{FH}) \geq \dim \mathbf{FH} - 54 = h^0(S, N_{S/\mathbb{P}^5}) + h^0(\mathcal{I}_{S/\mathbb{P}^5}(3)) - 55, \quad (3.3)$$

where the last equality follows from (3.2). By this inequality, we get

$$h^0(S, N_{S/X}) \geq \dim \pi_2^{-1}(X) \geq h^0(S, N_{S/\mathbb{P}^5}) + h^0(\mathcal{I}_{S/\mathbb{P}^5}(3)) - 55$$

because $\pi_2^{-1}(X) = \text{Hilb}_X^{\chi(\mathcal{O}_S(n))}$, which has tangent space $H^0(S, N_{S/X})$.

Thus, if we can exhibit a specific flag $(S \subset X)$ such that $h^0(\mathcal{I}_{S/\mathbb{P}^5}(3))$ is minimal, $\text{Hilb}_{\mathbb{P}^5}^{\chi(\mathcal{O}_S(n))}$ is smooth at $[S]$, and $h^0(S, N_{S/X}) = h^0(S, N_{S/\mathbb{P}^5}) + h^0(\mathcal{I}_{S/\mathbb{P}^5}(3)) - 55$, then we must have equality everywhere in (3.3) and (3.2). For the irreducible component $Z \subset W$ containing $[S]$, it follows that the closure $\overline{\pi_2(\pi_1^{-1}(Z))}$ is an irreducible divisor. By computing the self-intersection of $S \subset X$ using the formula from [Has00, Section 4.1],

$$S^2 = c_2(N_{S/X}) = 6H^2 + 3H.K_S + K_S^2 - \chi_S \quad (3.4)$$

in each of our cases, we can show that $\overline{\pi_2(\pi_1^{-1}(Z))} = \mathcal{C}_d$ for $d = 3S^2 - (H^2)^2$. Note that here $H = c_1(\mathcal{O}_S(1))$.

We use **Macaulay2** to exhibit such flags for each $18 \leq d \leq 38$ and $d = 44$. The case $d = 44$ is exceptional, and we deal with it separately. First, let S be the blow-up of $\mathbb{P}^2$ at p points in general position, and denote by E_m for $m = 1, \dots, p$ the exceptional divisors. Then $K_S^2 = 9 - p$ and $\chi(S) = 3 + p$. Consider divisors of the form $H = aL - (E_1 + \dots + E_i) - 2(E_{i+1} + \dots + E_{i+j}) - 3(E_{i+j+1} + \dots + E_p)$ with $\deg S = H^2$, where L is the pull-back of the hyperplane class on $\mathbb{P}^2$. For each $12 \leq d \leq 38$ such that $\mathcal{C}_d \neq \emptyset$, we choose the polarization H according to Table 1. For each d , we present the very ample divisor with lowest value of a on a given surface S (that is, for a fixed p).³

By choosing p random points in $\mathbb{P}^2(\mathbb{F}_q)$ for a large prime q , one can verify in **Macaulay2** that the linear system $|H| = |aL - (E_1 + \dots + E_i) - 2(E_{i+1} + \dots + E_{i+j}) - 3(E_{i+j+1} + \dots + E_p)|$ embeds the blow-up S as a linearly normal smooth surface in $\mathbb{P}^5$ of degree H^2 and thus is very ample. It is easy to see that the blow-up S satisfies $h^1(N_{S/\mathbb{P}^5}) = 0$, so the Hilbert scheme $\text{Hilb}_{\mathbb{P}^5}^{\chi(\mathcal{O}_S(n))}$ is smooth at $[S]$, of dimension $h^0(N_{S/\mathbb{P}^5}) = 2p + 27$. Furthermore, we find that for a random choice of these p points and for a random choice of cubic fourfold $X \in \mathbb{P}H^0(\mathcal{I}_{S/\mathbb{P}^5}(3))$, the dimensions $h^0(N_{S/X})$ and $h^0(\mathcal{I}_{S/\mathbb{P}^5}(3))$ achieve the values reported in Table 1.

³For some choices of d and p , we found multiple very ample polarizations giving the same numerical and cohomological invariants. Upon inspection, these were usually found to be equivalent by thinking of S as a blow-up of $\mathbb{P}^2$ at p points in a different way.

From the long exact sequence on cohomology associated with the short exact sequence

$$0 \rightarrow \mathcal{I}_{S/\mathbb{P}^5}(3) \rightarrow \mathcal{O}_{\mathbb{P}^5}(3) \rightarrow \mathcal{O}_S(3) \rightarrow 0$$

and a Riemann–Roch calculation, these values of $h^0(\mathcal{I}_{S/\mathbb{P}^5}(3))$ can be seen to be the smallest possible. Furthermore, in each case we see that $h^0(S, N_{S/X}) = h^0(S, N_{S/\mathbb{P}^5}) + h^0(\mathcal{I}_{S/\mathbb{P}^5}(3)) - 55$, as required. By the openness of smoothness and very ampleness, it follows that all of the above holds true for a generic choice of p points in $\mathbb{P}^2$ over $\mathbb{C}$ and a generic choice of cubic in $H^0(\mathcal{I}_{S/\mathbb{P}^5}(3))$. For more on this style of proof in the case of moduli of curves, see [Sch13].

We have thus proven the following theorem.

THEOREM 3.1. *The generic element of $\mathcal{C}_d$ for $12 \leq d \leq 38$ contains a smooth rational surface obtained as the blow-up of $\mathbb{P}^2$ at p generic points and embedded into $\mathbb{P}^5$ via the very ample linear system $|H| = |aL - (E_1 + \cdots + E_i) - 2(E_{i+1} + \cdots + E_{i+j}) - 3(E_{i+j+1} + \cdots + E_p)|$, where H is given by Table 1.*

Now, let us treat the case $d = 44$. We consider Fano models of Enriques surfaces. These are given by very ample polarizations Δ with $\Delta^2 = 10$ and $\Delta \cdot F \geq 3$ for every effective F with $F^2 = 0$. Every Enriques surface S admits such a polarization, and these polarizations embed S into $\mathbb{P}^5$ as a surface of degree 10 whose homogeneous ideal is generated by ten cubics. It is well known that the Hilbert scheme $\text{Hilb}_{\mathbb{P}^5}^{5n^2+1}$ contains a smooth open subset of dimension 45 parametrizing Fano models of Enriques surfaces [CD85]. It follows that we get a smooth open subset of $\mathbf{FH}$ of dimension 54. By again taking a random Fano Enriques S and cubic fourfold X containing it, one finds that $h^0(N_{S/X}) = 0$. This proves the following.

THEOREM 3.2. *The generic element of $\mathcal{C}_{44}$ contains a Fano model of an Enriques surface.*

The cases $(d, p) = (30, 10)$ and $(d, p) = (38, 10)$ are intimately related to each other and to the case $d = 44$ and have their origin in some very classical algebraic geometry. If we allow the K3 cover of an Enriques surface to develop an ordinary double point (ODP) fixed by the involution, then the resulting quotient surface has a quartic singularity whose resolution is an irreducible smooth rational curve D_0 with self-intersection -4 . The resulting surface S , known as a Coble surface, can be realized as the blow-up of $\mathbb{P}^2$ at the ten nodes of a rational planar sextic, the proper transform of which is $D_0 \in |-2K_S|$. These nodal curves, called Coble curves, were studied extensively at the beginning of the twentieth century by Coble. Analogous to the theory of Fano polarizations for Enriques surfaces, one can define Fano polarizations for Coble surfaces, and the divisor $H = 10L - 3(E_1 + \cdots + E_{10})$ is the standard one. The linear system $|H|$ maps the Coble surface to $\mathbb{P}^5$ as a degree 10 surface while contracting D_0 . Deforming the ten points into general position ensures that $|-mK_S| = \emptyset$ for any $m \geq 1$, so that $|H|$ contracts no curves and thus is ample. We proved in Theorem 3.1 that $|H|$ is in fact very ample, and by considering the cubics containing these “generalized” Coble surfaces, we obtained the first description of $\mathcal{C}_{38}$ above. The cubic fourfolds which contain the image of the genuine Coble surfaces form an irreducible divisor contained in the boundary of $\mathcal{C}_{38}$, as they generically admit a single ODP (and thus are rational). The adjoint Fano embedding, $|H + K_S| = |7L - 2(E_1 + \cdots + E_{10})|$, gives the above description of $\mathcal{C}_{30}$. The locus of cubic fourfolds containing the genuine Coble surfaces is again a divisor in this case, but this divisor is no longer only contained in the boundary and in fact parametrizes smooth cubic fourfolds. Viewed together as subvarieties of $\mathbb{P}^5$, these three cases form part of a larger story involving the irreducible components of the same Hilbert scheme, which we hope to come back to elsewhere.

TABLE 1. Smooth rational surfaces

d	p	H	H^2	$H \cdot K_S$	$h^0(\mathcal{I}_{S/\mathbb{P}^5}(3))$	$h^0(N_{S/\mathbb{P}^5})$	$h^0(N_{S/X})$
12	7	$4L - (E_1 + \cdots + E_6) - 3E_7$	6	-4	22	41	8
12	13	$5L - (E_1 + \cdots + E_{12}) - 2E_{13}$	9	-1	13	53	11
12	16	$7L - (E_1 + \cdots + E_9) - 2(E_{10} + \cdots + E_{16})$	12	2	4	59	8
14	4	$3L - E_1 - E_2 - E_3 - E_4$	5	-5	25	35	5
14	9	$4L - E_1 - \cdots - E_9$	7	-3	19	45	9
14	11	$5L - (E_1 + \cdots + E_9) - 2(E_{10} + E_{11})$	8	-2	16	49	10
14	14	$6L - (E_1 + \cdots + E_{10}) - 2(E_{11} + \cdots + E_{14})$	10	0	10	55	10
14	15	$7L - (E_1 + \cdots + E_9) - 2(E_{10} + \cdots + E_{14}) - 3E_{15}$	11	1	7	57	9
14	16	$8L - (E_1 + \cdots + E_6) - 2(E_7 + \cdots + E_{15}) - 3E_{16}$	13	3	1	59	5
18	12	$6L - (E_1 + \cdots + E_7) - 2(E_8 + \cdots + E_{12})$	9	-1	13	51	9
18	15	$8L - (E_1 + \cdots + E_6) - 2(E_7 + \cdots + E_{13}) - 3(E_{14} + E_{15})$	12	2	4	57	6
20	0	$2L$	4	-6	28	27	0
20	10	$6L - (E_1 + \cdots + E_4) - 2(E_5 + \cdots + E_{10})$	8	-2	16	47	8
20	13	$7L - (E_1 + \cdots + E_6) - 2(E_7 + \cdots + E_{12}) - 3E_{13}$	10	0	10	53	8
20	14	$7L - (E_1 + \cdots + E_6) - 2(E_7 + \cdots + E_{14})$	11	1	7	55	7
20	15	$8L - (E_1 + E_2 + E_3) - 2(E_4 + \cdots + E_{15})$	13	3	1	57	3
24	11	$7L - (E_1 + \cdots + E_3) - 2(E_4 + \cdots + E_{10}) - 3E_{11}$	9	-1	13	49	7
24	14	$8L - (E_1 + E_2 + E_3) - 2(E_4 + \cdots + E_{13}) - 3E_{14}$	12	2	4	55	4
26	12	$7L - (E_1 + E_2 + E_3) - 2(E_4 + \cdots + E_{12})$	10	0	10	51	6
26	13	$8L - (E_1 + E_2 + E_3) - 2(E_4 + \cdots + E_{11}) - 3(E_{12} + E_{13})$	11	1	7	53	5
30	10	$7L - 2(E_1 + \cdots + E_{10})$	9	-1	13	47	5
32	11	$9L - E_1 - 2(E_2 + \cdots + E_5) - 3(E_6 + \cdots + E_{11})$	10	0	10	49	4
36	12	$10L - 2(E_1 + \cdots + E_4) - 3(E_5 + \cdots + E_{12})$	12	2	4	51	0
38	10	$10L - 3(E_1 + \cdots + E_{10})$	10	0	10	47	2
38	11	$10L - 2(E_1 + E_2) - 3(E_3 + \cdots + E_{11})$	11	1	7	49	1

4. Unirationality of some $\mathcal{C}_d$

From the proof of Theorem 3.1 above, we see that for $12 \leq d \leq 38$, there is an open subset $U_d \subset (\mathbb{P}^2)^p$ parametrizing generic p -tuples of distinct points giving the cohomological invariants in Table 1. Moreover, there is a vector bundle $V_d \rightarrow U_d$ whose fiber over $(x_1, \dots, x_p)$ is the vector space $H^0(\mathcal{I}_{S/\mathbb{P}^5}(3))$, where S is the blow-up of $\mathbb{P}^2$ at the p points $x_1, \dots, x_p$ and the embedding into $\mathbb{P}^5$ is given by $|aL - (E_1 + \dots + E_i) - 2(E_{i+1} + \dots + E_{i+j}) - 3(E_{i+j+1} + \dots + E_p)|$. Finally, Theorem 3.1 shows that the natural morphism $\mathbb{P}(V_d) \rightarrow \mathcal{C}_d$ is dominant. From the rationality of $\mathbb{P}(V_d)$, we get the following result.

COROLLARY 4.1. *For $12 \leq d \leq 38$, the moduli space $\mathcal{C}_d$ is unirational.*

Similarly, for $d = 44$, we found a vector bundle V_{44} over the moduli space of Fano polarized Enriques surfaces with fibers $H^0(\mathcal{I}_{S/\mathbb{P}^5}(3))$ such that the natural moduli morphism $\mathbb{P}(V_{44}) \rightarrow \mathcal{C}_{44}$ is generically finite and dominant. Verra proved in [Ver84] that the moduli space of Fano polarized Enriques surfaces is unirational. From the generic finiteness and dominance of the morphism $\mathbb{P}(V_{44}) \rightarrow \mathcal{C}_{44}$, the next statement follows as in [Uen75, Theorem 6.10].

COROLLARY 4.2. *The moduli space $\mathcal{C}_{44}$ is unirational.*

5. Unirationality of $\mathcal{N}_d$ for $d = 14, 26, 38$

In addition to describing the $\mathcal{C}_d$ and providing conditions for their non-emptiness, Hassett determined in [Has00] for which d there exists a degree d polarized K3 surface (T, f) such that the primitive cohomology $H^2(T, \mathbb{C})^0 := (f)^\perp \subset H^2(T, \mathbb{C})$ is equivalent to the non-special cohomology $W_{X, K_d} := K_d^\perp$, where $X \in \mathcal{C}_d$ and K_d is the corresponding sublattice of discriminant d . In this case, we say that X has an associated K3 surface, namely (T, f) . Hassett proved the following important result.

THEOREM 5.1 ([Has00, Theorem 5.2.1]). *A cubic fourfold $X \in \mathcal{C}_d$ has an associated K3 surface if and only if $4 \nmid d$ and $9 \nmid d$, and any odd prime p such that $p \mid d$ satisfies $p \equiv 0, 1 \pmod{3}$.*

In fact, Hassett proved a bit more. For those d as in Theorem 5.1, Hassett proved that the moduli spaces $\mathcal{C}_d$ and $\mathcal{N}_d$ are related. In particular, he showed that there is a rational map $\mathcal{N}_d \dashrightarrow \mathcal{C}_d$ which is birational if $d \equiv 2 \pmod{6}$ and a double cover if $d \equiv 0 \pmod{6}$ [Has00, Section 5.3]. For $d = 14, 26, 38$, every $X \in \mathcal{C}_d$ has an associated K3 surface, and the corresponding moduli spaces are birational. Using Corollary 4.1, we prove the unirationality of the corresponding moduli spaces of K3 surfaces.

THEOREM 5.2. *For $d = 14, 26, 38$, the moduli space of polarized K3 surfaces of degree d is unirational.*

6. Final comments and open questions

6.1 What makes special cubic fourfolds special?

It is worth noting that Voisin has shown that for a special cubic fourfold X , the lattice $A(X)$ can be generated either by smooth surfaces [Voi14, proof of Theorem 5.6] or by possibly singular rational surfaces [Voi07], and many have wondered if $A(X)$ is in fact generated by smooth *rational* surfaces. This has certainly been true for the few low-discriminant cases previously known, and the work here provides a lot more evidence for this possibility.

Nevertheless, experimentation with $\mathcal{C}_{44}$ by generating surfaces on a cubic fourfold $X \in \mathcal{C}_{44}$ via residuation with the Fano Enriques surface S has only given (smooth) surfaces of general type with fairly high degree. So it seems that the generic $X \in \mathcal{C}_{44}$ does not contain a smooth rational surface. Therefore, we are lead to ask the following.

QUESTION 6.1. *What does the condition that the generic $X \in \mathcal{C}_d$ contains a smooth rational surface not homologous to a complete intersection say, if anything, about the geometry of X ? and about the geometry of $\mathcal{C}_d$?*

6.2 A note about the choices of H in Table 1

It seems fitting to end this section with a comment about how we came about the surfaces and embeddings given in Table 1. Completing the square in (3.4), we get

$$(H^2 - 9)^2 = 81 - d + 3(3H \cdot K_S + K_S^2 - \chi_S),$$

and from the assumption that S is the blow-up of $\mathbb{P}^2$ at p points and $H = aL - (E_1 + \cdots + E_i) - 2(E_{i+1} + \cdots + E_{i+j}) - 3(E_{i+j+1} + \cdots + E_p)$, we see that

$$\begin{aligned} H^2 &= a^2 - i - 4j - 9(p - i - j), \\ H \cdot K_S &= -3a + i + 2j + 3k. \end{aligned} \tag{6.1}$$

Combining (6.1) with the facts that $K_S^2 = 9 - p$ and $\chi_S = 3 + p$, we see that

$$(a^2 - i - 4j - 9(p - i - j) - 9)^2 = 99 - d + 21p - 9(3a + 2i + j). \tag{6.2}$$

This equation puts serious constraints on the non-negative integers a, p, i, j and allows us to solve for a in terms of the other three values. We note that automatically

$$H^2(\mathcal{O}_S(H))^\vee = H^0(\mathcal{O}_S(K_S - H)) = 0,$$

so $h^0(\mathcal{O}_S(H)) = \chi(H) + h^1(\mathcal{O}_S(H))$. While not a necessary condition, it is reasonable to assume that the embedding $S \hookrightarrow \mathbb{P}^5$ is linearly normal and given by the complete linear system $|H|$, so that $h^0(\mathcal{O}_S(H)) = 6$. We found the values in Table 1 by searching for non-negative integral solutions to (6.2) for each d under the further assumption that $h^1(\mathcal{O}_S(H)) = 0$ or, equivalently, that

$$6 = \chi(\mathcal{O}_S(H)) = \frac{1}{2}(H^2 - H \cdot K_S) + 1,$$

which reduces the search to only two parameters. It is worth mentioning that this method does not seem to yield results for $d = 42$ even upon relaxing these restrictions. Indeed, if we make mild assumptions on such a surface S (such as $h^0(\mathcal{O}_S(1)) \geq 6$, $h^0(\mathcal{O}_S(2)) \geq 19$, $h^0(\mathcal{O}_S(3)) \leq 55$), then the degree of the embedded surface is bounded above by 15. Even allowing more complicated divisors, with coefficients on the E_i larger than 3, produced either no non-negative integral solutions or solutions for which $\pi_2(\mathbf{FH})$ did not fill out $\mathcal{C}_d$. This certainly indicates that the method presented here has been exhausted for the most part.

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