Representation theory and the cycle map of a classifying space

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ABSTRACT

We compute the Chern subgroup of the fourth integral cohomology group of a certain classifying space and show that it is a proper subgroup. Such a classifying space gives us new counterexamples for the integral Hodge and Tate conjectures modulo torsion.

1. Introduction

Let G be a reductive complex linear algebraic group. In [Tot97], Totaro defined the Chow group $\mathrm{CH}^i \mathrm{BG}$ of its classifying space BG and studied the cycle map to the integral cohomology $H^{2i}(\mathrm{BG}; \mathbb{Z})$. Approximating the classifying space with smooth projective varieties, we obtain interesting examples. For instance, counterexamples for the integral Hodge conjecture due to Atiyah and Hirzebruch [AH62] are obtained by studying the cycle map

$$\mathrm{cl}: \mathrm{CH}^i \mathrm{BG} \rightarrow H^{2i}(\mathrm{BG}; \mathbb{Z})$$

for $i = 2$ and $G = (\mathbb{Z}/p)^3 \times \mathbb{G}_m$, where p is an odd prime number, $(\mathbb{Z}/p)^3$ is the elementary abelian p -group of rank 3 and $\mathbb{G}_m$ is the multiplicative group of non-zero complex numbers. Changing the base field to the algebraic closure of a finite field whose characteristic is prime to p , Colliot-Thélène and Szamuely obtained counterexamples for the integral Tate conjecture in [CS10]. However, the integral Tate conjecture is known to hold for certain classes of varieties, and it is also an important subject of study; see, for example, [CP15] by Charles and Pirutka. We refer the reader to the introduction of Antieau’s paper [Ant16] for a detailed account of the integral Hodge and Tate conjectures. Since a reductive complex linear algebraic group is homotopy equivalent to its maximal compact subgroup, as far as its cohomology is concerned, we may consider the corresponding compact Lie group instead of the reductive complex linear algebraic group. In this paper, we consider the special unitary group $\mathrm{SU}(p^2)$ instead of the special linear group SL_{p^2} and, by abuse of notation, we write $\mathrm{CH}^i \mathrm{BSU}(p^2)$ for $\mathrm{CH}^i \mathrm{BSL}_{p^2}$ and so on.

Recently, using the exceptional groups G_2 , F_4 , E_8 for $p = 2, 3, 5$, respectively, Pirutka and Yagita constructed counterexamples for the integral Hodge and Tate conjectures modulo torsion in [PY15]. In [Kam15], the author generalized the result of Pirutka and Yagita using compact Lie groups $\mathrm{SU}(p) \times \mathrm{SU}(p)/\mu_p$ to all prime numbers p , where μ_p is the cyclic group of order p which acts diagonally on $\mathrm{SU}(p) \times \mathrm{SU}(p)$. Very recently, in [Ant16], Antieau gave rather different and

Received 2 February 2016, accepted in final form 2 July 2016.

2010 Mathematics Subject Classification 14C15 (primary), 55R40, 55R35 (secondary).

Keywords: classifying space, Chow ring, cycle map, Chern class.

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The author is partially supported by the Japan Society for the Promotion of Science, Grant-in-Aid for Scientific Research (C) 25400097.

very interesting counterexamples for the integral Hodge and Tate conjectures modulo torsion for $p = 2, 3$ by studying Chern classes of $\mathrm{SU}(8)/\mu_2$ and $\mathrm{SU}(9)/\mu_3$, where μ_p is the subgroup of order p of the center. By the Riemann–Roch formula without denominators, the Chow group $\mathrm{CH}^2 \mathrm{BG}$ is generated by Chern classes of complex representations. So, the question regarding the surjectivity of the cycle map

$$\mathrm{cl}: \mathrm{CH}^2 \mathrm{BG} \rightarrow H^4(\mathrm{BG}, \mathbb{Z})$$

is equivalent to the question regarding the integral cohomology group $H^4(\mathrm{BG}; \mathbb{Z})$ being generated by Chern classes of complex representations. We refer the reader to Totaro’s paper [Tot99] and his book [Tot14] for details of these facts. In [Ant16], Antieau calculated second Chern classes for all irreducible complex representations of $\mathrm{SU}(8)/\mu_2$ and $\mathrm{SU}(9)/\mu_3$ using a computer algebra system and showed that these second Chern classes do not generate the fourth integral cohomology groups. Also, Antieau conjectured that similar results hold for all odd prime numbers, that is, Theorem 1.1 [Ant16, Conjecture 2.10] below.

In this paper, we call the subgroup of the integral cohomology group $H^i(\mathrm{BG}; \mathbb{Z})$ generated by all complex representations the *Chern subgroup* of $H^i(\mathrm{BG}; \mathbb{Z})$. The purpose of this paper is to prove Antieau’s conjecture. We not only prove Antieau’s conjecture but also give a simpler proof for the case $p = 3$ without using a computer algebra system.

THEOREM 1.1. *Let p be an odd prime number. Let μ_p be the subgroup of the center of $\mathrm{SU}(p^2)$ of order p . Then, the Chern subgroup of $H^4(\mathrm{BSU}(p^2)/\mu_p; \mathbb{Z}) = \mathbb{Z}$ is*

$$p \cdot H^4(\mathrm{BSU}(p^2)/\mu_p; \mathbb{Z}) = p \cdot \mathbb{Z}.$$

As a corollary, we have the following result on the cycle map.

COROLLARY 1.2. *Let p be an odd prime number. Let μ_p be the subgroup of order p of the center of $\mathrm{SU}(p^2)$. Then, the cycle map*

$$\mathrm{cl}: \mathrm{CH}^2 \mathrm{BSU}(p^2)/\mu_p \rightarrow H^4(\mathrm{BSU}(p^2)/\mu_p; \mathbb{Z}) = \mathbb{Z}$$

is not surjective.

One of the key ingredients in Antieau’s paper is the fact [Ant16, Proposition 2.7] that the induced homomorphism

$$\pi^*: H^4(\mathrm{BSU}(p^2)/\mu_p; \mathbb{Z}) \rightarrow H^4(\mathrm{BSU}(p^2); \mathbb{Z})$$

induced by the obvious projection $\pi: \mathrm{SU}(p^2) \rightarrow \mathrm{SU}(p^2)/\mu_p$ is an isomorphism for all odd prime numbers p , where, by abuse of notation, we denote the map between classifying spaces $\mathrm{BSU}(p^2) \rightarrow \mathrm{BSU}(p^2)/\mu_p$ induced by the projection $\pi: \mathrm{SU}(p^2) \rightarrow \mathrm{SU}(p^2)/\mu_p$ by the same symbol π . Antieau used $\mathrm{SU}(p^2)$ to detect the integral cohomology group $H^4(\mathrm{BSU}(p^2)/\mu_p; \mathbb{Z})$. In this paper, we replace $\mathrm{SU}(p^2)$ by

$$\mathrm{S}^1 = \{x \in \mathbb{C} \mid |x| = 1\}.$$

The complex representation ring $R(\mathrm{S}^1)$ is $\mathbb{Z}[z, z^{-1}]$, and the integral cohomology ring $H^*(\mathrm{BS}^1; \mathbb{Z})$ is $\mathbb{Z}[t]$, where t is a generator of $H^2(\mathrm{BS}^1; \mathbb{Z}) = \mathbb{Z}$ and z is represented by the canonical line bundle over the infinite-dimensional complex projective space $\mathbb{CP}^\infty = \mathrm{BS}^1$. Both the complex representation ring of S^1 and the integral cohomology ring of BS^1 are much simpler than those of $\mathrm{SU}(p^2)$ and $\mathrm{BSU}(p^2)$, respectively, yet there exists a map

$$\varphi_1: \mathrm{S}^1 \rightarrow \mathrm{SU}(p^2)$$

such that the induced homomorphism $(\pi \circ \varphi_1)^*$ detects the fourth integral cohomology group of $\mathrm{BSU}(p^2)/\mu_p$. Because of the simplicity of the complex representation ring of S^1 and the integral cohomology ring of BS^1 , we can compute complex representations and their Chern classes without using a computer algebra system. Also, in this paper, we consider algebra generators of the complex representation ring $R(\mathrm{SU}(p^2))$ rather than irreducible representations. This also reduces the number of computations we need.

This paper is organized as follows: In Section 2, we give a necessary condition for an element x in the complex representation ring $R(\mathrm{SU}(p^2))$ of $\mathrm{SU}(p^2)$ to be in the image of the induced homomorphism

$$\pi^*: R(\mathrm{SU}(p^2)/\mu_p) \rightarrow R(\mathrm{SU}(p^2)).$$

In Section 3, we define the map $\varphi_1: S^1 \rightarrow \mathrm{SU}(p^2)$ above. Then, we compute the second Chern class of $\varphi_1^*(x) \in R(S^1)$ modulo p and complete the proof of Theorem 1.1.

The author was informed by Benjamin Antieau that Arnav Tripathy also proved Theorem 1.1 independently [Tri16].

2. Representations

In this section, we consider a necessary condition for an element x in the complex representation ring $R(\mathrm{SU}(p^2))$ to be in the image of the induced homomorphism

$$\pi^*: R(\mathrm{SU}(p^2)/\mu_p) \rightarrow R(\mathrm{SU}(p^2)).$$

We use the following commutative diagram:

$$\begin{array}{ccc} \mu_p & \xrightarrow{\Delta_0} & \mathrm{T} \\ \Delta_1 \downarrow & & \downarrow \iota_0 \\ \mathrm{SU}(p^2) & \xrightarrow{\iota_1} & \mathrm{U}(p^2) \\ \pi \downarrow & & \\ \mathrm{SU}(p^2)/\mu_p & & \end{array}$$

where $\mathrm{T} = S^1 \times \cdots \times S^1$ (the product of p^2 copies of S^1) is the maximal torus of the unitary group $\mathrm{U}(p^2)$ consisting of diagonal matrices, μ_p is the cyclic group of order p consisting of the p th roots of unity in $S^1 \subset \mathbb{C}$, the maps ι_0 and ι_1 are obvious inclusion maps, and Δ_0 is the diagonal map

$$\Delta_0(x) = (x, x, \dots, x).$$

The complex representation rings of μ_p , T , $\mathrm{SU}(p^2)$ and $\mathrm{U}(p^2)$ are well known. See, for instance, Bröcker and tom Dieck's book [BtD95, II, Section 8] and Husemoller's book [Hus94, Chapter 14].

We recall that

$$\begin{aligned} R(\mu_p) &= \mathbb{Z}[\zeta]/(\zeta^p - 1) = \mathbb{Z}\{1, \zeta, \dots, \zeta^{p-1}\}, \\ R(T) &= \mathbb{Z}[z_1, z_1^{-1}, \dots, z_{p^2}, z_{p^2}^{-1}] = \mathbb{Z}[z_1, \dots, z_{p^2-1}, z_{p^2}, \lambda_{p^2}^{-1}], \\ R(U(p^2)) &= \mathbb{Z}[\lambda_1, \dots, \lambda_{p^2-1}, \lambda_{p^2}, \lambda_{p^2}^{-1}], \\ R(SU(p^2)) &= \mathbb{Z}[\lambda_1, \dots, \lambda_{p^2-1}], \end{aligned}$$

where ζ is represented by the inclusion of μ_p to S^1 , z_i is $p_i^*(z)$, $p_i: T \rightarrow S^1$ is the projection onto the i th factor for $i = 1, \dots, p^2$, λ_ℓ is the ℓ th symmetric function of $z_1, \dots, z_{p^2}$ and λ_ℓ is represented by the ℓ th exterior power representation. It is clear from the definition of Δ_0 that $\Delta_0^*(z_i) = \zeta$ for $i = 1, \dots, p^2$. Hence, we have

$$\Delta_1^*(\lambda_\ell) = \binom{p^2}{\ell} \zeta^\ell$$

and

$$\Delta_1^*(\lambda_{\ell_1} \cdots \lambda_{\ell_r}) = \binom{p^2}{\ell_1} \cdots \binom{p^2}{\ell_r} \zeta^{\ell_1 + \cdots + \ell_r}.$$

If $x \in R(SU(p^2))$ is in the image of the induced homomorphism

$$\pi^*: R(SU(p^2)/\mu_p) \rightarrow R(SU(p^2)),$$

then $\Delta_1^*(x)$ is in

$$\mathbb{Z}\{1\} \subset \mathbb{Z}\{1, \zeta, \dots, \zeta^{p-1}\} = R(\mu_p).$$

Remark 2.1. Since

$$\Delta_1^*(\lambda_p) = \binom{p^2}{p} \zeta^p = \binom{p^2}{p} \cdot 1 \in \mathbb{Z}\{1\}$$

and since

$$\Delta_1^*(\lambda_1^p) = \binom{p^2}{1}^p \zeta^p = \binom{p^2}{1}^p \cdot 1 \in \mathbb{Z}\{1\},$$

there exist complex representations y_1, y_2 of $SU(p^2)/\mu_p$ such that $\pi^*(y_1) = \lambda_p$ and $\pi^*(y_2) = \lambda_1^p$.

Now, we consider a basis for $R(SU(p^2))$. Let $L = (\ell_1, \dots, \ell_r)$ be a non-decreasing finite sequence of integers in $\{1, \dots, p^2 - 1\}$. We define a monomial λ_L to be

$$\lambda_\emptyset = 1, \quad \lambda_{(\ell)} = \lambda_\ell, \quad \lambda_{(\ell_1, \dots, \ell_r)} = \lambda_{\ell_1} \cdots \lambda_{\ell_r} \quad \text{for } r \geq 2.$$

Then, any element x in $R(SU(p^2))$ is a linear combination of λ_L , and we have

$$x = \sum_L \alpha_L(x) \lambda_L, \quad \alpha_L(x) \in \mathbb{Z},$$

where L ranges over the set of non-decreasing finite sequences of positive integers less than p^2 and the $\alpha_L(x)$ are 0 except for a finite number of L . In this section, we prove the following proposition, which gives us a necessary condition for $x \in R(SU(p^2))$ to be in the image of π^* . The use of this proposition in the proof of Theorem 1.1 was suggested by Antieau to the author. Also, the author learned its proof from Antieau.

PROPOSITION 2.2. *Let $L = (\ell_1, \dots, \ell_r)$. Suppose $x \in \text{Im } \pi^*$. If $\ell_1 + \cdots + \ell_r \not\equiv 0 \pmod{p}$, then $\alpha_L(x) = 0$.*

Proof. Let V' be a complex representation of $\mathrm{SU}(p^2)/\mu_p$. Then $\pi^*(V')$ is a direct sum of irreducible representations, say $\pi^*(V') = V_1 \oplus \cdots \oplus V_s$. Since μ_p acts on $\pi^*(V)$ trivially, it acts trivially on each V_i . Therefore, for each i , there exists a V'_i in $R(\mathrm{SU}(p^2)/\mu_p)$ such that $\pi^*(V'_i) = V_i$. Since a complex representation ring is additively generated by irreducible representations, to prove the proposition it suffices to prove it for irreducible representations. Let V be an irreducible representation of $\mathrm{SU}(p^2)$. It is a direct summand of an n -fold tensor product of the standard representation λ_1 for some n . The irreducible representation V corresponds to a sum of $\dim(V)$ homogeneous monomials of degree n in $R(T')$, where $T' = T \cap \mathrm{SU}(p^2)$ is the maximal torus of $\mathrm{SU}(p^2)$, so that it is a linear combination of products $\lambda_{\ell_1} \cdots \lambda_{\ell_r}$ with $\ell_1 + \cdots + \ell_r = n$, where $\ell_i \in \{1, \dots, p^2\}$. Since

$$\Delta_1^*(V) = (\dim V)\zeta^n,$$

$\Delta_1^*(V)$ is in

$$\mathbb{Z}\{1\} \subset \mathbb{Z}\{1, \zeta, \dots, \zeta^{p-1}\} = R(\mu_p)$$

if and only if n is divisible by p . This completes the proof. $\square$

3. Chern classes

In this section, we prove Theorem 1.1. We define a map $\varphi_0: S^1 \rightarrow T$ by

$$\varphi_0(x) = (x, x^{-1}, 1, \dots, 1).$$

The map $\varphi_1: S^1 \rightarrow \mathrm{SU}(p^2)$ mentioned in Section 1 is defined by the following commutative diagram:

$$\begin{array}{ccc} S^1 & \xrightarrow{\varphi_0} & T \\ \varphi_1 \downarrow & & \downarrow \iota_0 \\ \mathrm{SU}(p^2) & \xrightarrow{\iota_1} & \mathrm{U}(p^2). \end{array}$$

As in Section 2, we denote the algebra generators of $R(T)$ by $z_1, \dots, z_{p^2}$, those of $R(\mathrm{SU}(p^2))$ by $\lambda_1, \dots, \lambda_{p^2-1}$ and those of $R(\mathrm{U}(p^2))$ by $\lambda_1, \dots, \lambda_{p^2}, \lambda_{p^2}^{-1}$, where $\iota_0^*(\lambda_i)$ is the i th symmetric polynomial in $z_1, \dots, z_{p^2}$ and $\iota_1^*(\lambda_\ell) = \lambda_\ell$ ($\ell = 1, \dots, p^2 - 1$) and $\iota_1^*(\lambda_{p^2}) = 1$. Let z be the generator of $R(S^1) = \mathbb{Z}[z, z^{-1}]$ represented by the canonical line bundle, and let t be the generator of $H^*(BS^1; \mathbb{Z})$, as in Section 1. It is clear that $\varphi_0^*(z_1) = z$ and $\varphi_0^*(z_2) = z^{-1}$, while $\varphi_0^*(z_i) = 1$ for $i = 3, \dots, p^2$. For a vector bundle ξ over a space X , we denote by

$$c(\xi) = 1 + c_1(\xi) + c_2(\xi) + \cdots \in H^*(X; \mathbb{Z})$$

the total Chern class of ξ , where $c_i(\xi)$ denotes the i th Chern class in $H^{2i}(X; \mathbb{Z})$.

PROPOSITION 3.1. *For $\ell = 1, \dots, p^2 - 1$, we have $c_1(\varphi_1^*(\lambda_\ell)) = 0$ and $c_2(\varphi_1^*(\lambda_{pk})) \equiv 0 \pmod{p}$, where $0 < k < p$.*

Proof. The fact that $c_1(\varphi_1^*(\lambda_\ell)) = 0$ is clear from the fact that $H^2(\mathrm{BSU}(p^2); \mathbb{Z}) = \{0\}$. Let us use the convention that $\binom{a}{b} = 0$ for $b < 0$ and $b > a$. Then, for $\ell = 1, \dots, p^2 - 1$, the number of monomials of the form $z_1 z_2 z_{i_1} \cdots z_{i_{\ell-2}}$ ($3 \leq i_1 < \cdots < i_{\ell-2} \leq p^2$) is $\binom{p^2-2}{\ell-2}$, both the number of monomials of the form $z_1 z_{i_1} \cdots z_{i_{\ell-1}}$ ($3 \leq i_1 < \cdots < i_{\ell-1} \leq p^2$) and the number of monomials of

the form $z_2 z_{i_1} \cdots z_{i_{\ell-1}}$ ($3 \leq i_1 < \cdots < i_{\ell-1} \leq p^2$) are $\binom{p^2-2}{\ell-1}$ and the number of monomials of the form $z_{i_1} z_{i_2} \cdots z_{i_\ell}$ ($3 \leq i_1 < \cdots < i_\ell \leq p^2$) is $\binom{p^2-2}{\ell}$. Hence, we have

$$\varphi_1^*(\lambda_\ell) = \binom{p^2-2}{\ell-2} + \binom{p^2-2}{\ell-1}(z + z^{-1}) + \binom{p^2-2}{\ell}$$

for $\ell = 1, 2, \dots, p^2 - 1$. Therefore, the total Chern classes are given by

$$\begin{aligned} c(\varphi_1^*(\lambda_\ell)) &= \{(1-t)(1+t)\}^{\binom{p^2-2}{\ell-1}} \\ &= 1 - \binom{p^2-2}{\ell-1} t^2 + \text{higher terms}, \end{aligned}$$

where $\ell = 1, \dots, p^2 - 1$ and higher terms mean terms of topological degree greater than or equal to 6. For $\ell = pk$ with $0 < k < p$, by Lucas's theorem, since $p^2 - 2 = (p-1)p + (p-2)$ and $pk - 1 = (k-1)p + (p-1)$, we have

$$\binom{p^2-2}{pk-1} \equiv \binom{p-1}{k-1} \binom{p-2}{p-1} \equiv 0 \pmod{p}.$$

Therefore, we have $c_2(\varphi_1^*(\lambda_{pk})) \equiv 0 \pmod{p}$. $\square$

To compute second Chern classes of λ_L for $r \geq 2$, we use the following lemma.

LEMMA 3.2. *Let ξ and η be finite-dimensional complex vector bundles over a CW complex of finite type. We assume that $H^*(X; \mathbb{Z})$ has no torsion and that the first Chern classes vanish, that is, $c_1(\xi) = c_1(\eta) = 0$. Then, we have*

$$\dim \xi \eta = (\dim \xi)(\dim \eta), \quad c_1(\xi \eta) = 0, \quad c_2(\xi \eta) = (\dim \eta)c_2(\xi) + (\dim \xi)c_2(\eta).$$

Proof. We consider the Chern character $\text{ch}(\xi)$ of a finite-dimensional complex vector bundle ξ over X . By the assumption of the lemma, the Chern characters $\text{ch}(\xi)$ and $\text{ch}(\eta)$ are given by

$$\begin{aligned} \text{ch}(\xi) &= \dim \xi - c_2(\xi) + \text{higher terms}, \\ \text{ch}(\eta) &= \dim \eta - c_2(\eta) + \text{higher terms}, \end{aligned}$$

where higher terms mean terms of topological degree greater than or equal to 6 as in the proof of Proposition 3.1. The Chern character satisfies

$$\text{ch}(\xi \eta) = \text{ch}(\xi) \text{ch}(\eta)$$

for any finite-dimensional complex representations ξ and η over X . Hence, we have

$$\text{ch}(\xi) \text{ch}(\eta) = (\dim \xi)(\dim \eta) - \{(\dim \eta)c_2(\xi) + (\dim \xi)c_2(\eta)\} + \text{higher terms},$$

and we have the desired result. $\square$

Proof of Theorem 1.1. As we mentioned in Section 1, Antieau proved that the induced homomorphism

$$\pi^*: H^4(\text{BSU}(p^2)/\mu_p; \mathbb{Z}) \rightarrow H^4(\text{BSU}(p^2); \mathbb{Z})$$

is an isomorphism. Since we proved

$$c_2(\varphi_1^*(\lambda_1)) = (-1) \cdot t^2$$

in the proof of Proposition 3.1, it is clear that the induced homomorphism

$$\varphi_1^*: H^4(\text{BSU}(p^2); \mathbb{Z}) \rightarrow H^4(\text{BS}^1; \mathbb{Z})$$

is also an isomorphism. Hence, to prove Theorem 1.1, it suffices to show the following:

- (1) The class $c_2(\varphi_1^*(x))$ is divisible by p for all $x \in \text{Im } \pi^* \subset R(\text{SU}(p^2))$.
- (2) There exists an element $y \in R(\text{SU}(p^2)/\mu_p)$ such that $c_2(\varphi_1^*(\pi^*(y))) = p \cdot t^2$.

First, we prove statement (1). Let $L = (\ell_1, \dots, \ell_r)$ be a non-decreasing sequence of positive integers less than p^2 . Let

$$\delta_L = \prod_{i=1}^r \dim \varphi_1^*(\lambda_{\ell_i}) \quad \text{and} \quad \delta_{L,i} = \frac{\delta_L}{\dim \varphi_1^*(\lambda_i)}.$$

By definition, we have

$$\dim \varphi_1^*(\lambda_\ell) = \dim \lambda_\ell = \binom{p^2}{\ell}.$$

This dimension is divisible by p . If $r \geq 2$, then $\delta_{L,i}$ is the product of $\dim \varphi_1^*(\lambda_{\ell_1}), \dots, \dim \varphi_1^*(\lambda_{\ell_r})$ except for $\dim \varphi_1^*(\lambda_{\ell_i})$. Therefore, it is divisible by p for $i = 1, \dots, r$; that is,

$$\delta_{L,i} \equiv 0 \pmod{p}$$

for $r \geq 2$. On the other hand, by Proposition 3.1, the bundles $\varphi_1^*(\lambda_\ell)$ satisfy the assumption of Lemma 3.2. Therefore, by Lemma 3.2, we have

$$c_2(\varphi_1^*(\lambda_L)) = \sum_{i=1}^r \delta_{L,i} c_2(\varphi_1^*(\lambda_{\ell_i})).$$

Hence, for $r \geq 2$, we have $c_2(\varphi_1^*(\lambda_L)) \equiv 0 \pmod{p}$. Thus, modulo p , the class $c_2(\varphi_1^*(x))$ is a linear combination of $c_2(\varphi_1^*(\lambda_\ell))$, where $\ell = 1, \dots, p^2 - 1$. Since $x \in \text{Im } \pi^*$, by Proposition 2.2 for ℓ not divisible by p , we have $\alpha_{(\ell)}(x) = 0$. Therefore, we have

$$c_2(\varphi_1^*(x)) \equiv \sum_{k=1}^{p-1} \alpha_{(pk)}(x) c_2(\varphi_1^*(\lambda_{pk})) \equiv 0 \pmod{p},$$

by Proposition 3.1.

Next, we prove statement (2). Recall Remark 2.1. There exist complex representations y_1 and y_2 such that $\pi^*(y_1) = \lambda_p$ and $\pi^*(y_2) = \lambda_1^p$. Let $L = (1, \dots, 1)$ be the sequence of p copies of 1, so that $\ell_1 = \dots = \ell_p = 1$. Since $\dim \lambda_{\ell_i} = p^2$ for $i = 1, \dots, p$, we have $\delta_L = p^{2p}$ and

$$\delta_{L,i} = \frac{\delta_L}{\dim \varphi_1^*(\lambda_{\ell_i})} = p^{2p-2}$$

for $i = 1, \dots, p$. Recall that we proved

$$c_2(\varphi_1^*(\lambda_1)) = (-1) \cdot t^2$$

in the proof of Proposition 3.1. By Lemma 3.2, we have

$$c_2(\varphi_1^*(\lambda_1^p)) = \sum_{i=1}^p -p^{2p-2} \cdot t^2 = -p^{2p-1} \cdot t^2.$$

On the other hand, in the proof of Proposition 3.1, we proved

$$c_2(\varphi_1^*(\lambda_p)) = -\binom{p^2-2}{p-1} \cdot t^2.$$

Since

$$\binom{p^2-2}{p-1} = \frac{p^2-p}{p-1} \left(\prod_{i=2}^{p-1} \frac{p^2-i}{p-i} \right) \equiv p \pmod{p^2},$$

the greatest common divisor of $\binom{p^2-2}{p-1}$ and p^{2p-1} is p , so that there exist integers β_1 and β_2 such that

$$\beta_1 \binom{p^2-2}{p-1} + \beta_2 p^{2p-1} = -p.$$

Let $y = \beta_1 y_1 + \beta_2 y_2$. Then, y satisfies the required conditions and statement (2) holds. $\square$

ACKNOWLEDGEMENTS

The author would like to thank Benjamin Antieau for sending an early draft of his preprint, giving comments and suggestions for the draft of this paper and informing the author of Tripathy's work. The author would also like to thank the anonymous referee for their report, especially for informing the author of the recent work of Charles and Pirutka, and for comments which improved the presentation of this paper.

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