Ampleness of the CM line bundle on the moduli space of canonically polarized varieties

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ABSTRACT

We prove that the CM line bundle is ample on the proper moduli space which parametrizes KSBA-stable varieties.

1. Introduction

Throughout this note, we work over a ground field k of characteristic zero. The moduli space M^{can} of canonically polarized manifolds, as well as its natural geometric compactification, has attracted considerable interest over the past few decades. It has been shown that geometric invariant theory (GIT) can be used to construct the moduli space M^{can} (cf. [Vie95, Don01]). However, applying the GIT methods to construct a natural compactification is considerably more challenging in higher dimension than in dimension one (see [WX14] for some new difficulties that arise). On the other hand, an alternative approach which uses the minimal model program (MMP) theory was first outlined in [KSB88] (see also [Ale96]). Based on the recent progress in MMP, this strategy turns out to give a satisfying compactification M^{ksba} . Although the coarse moduli space M^{ksba} first exists only as an algebraic space, [Kol90] has developed a strategy to verify its projectivity, which was later completed in [Fuj12] for the case of varieties and in [KP15] for pairs.

We can apply Knudsen–Mumford’s determinant bundle construction to the sequence of ample line bundles on M^{ksba} constructed by Kollár. The coefficient of the leading term is the Chow–Mumford (CM) line bundle, see Section 2.3, which was first introduced in [Tia97] and later formulated in this way in [PT09]. The curvature calculation on the Weil–Petersson metric of the CM line bundle suggests that it is ample. However, due to the presence of possibly singular fibers, this is only completely worked out for M^{can} (see [Sch12]).

In this note we give a purely algebraic proof of the fact that the CM line bundle is ample on M^{ksba} . Our approach is inspired by the recent interplay of studying M^{ksba} from both the algebraic and differential geometry view points, especially the equivalence of Kollár–Shepherd-Barron–Alexeev (KSBA) stability and K-stability for canonically polarized varieties (see [Oda12]).

THEOREM 1.1. *The CM line bundle is ample on the KSBA moduli space.*

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Using the Nakai–Moishezon criterion and the formula of the CM line bundle for a family of KSBA-stable varieties, we immediately see that this is implied by the following theorem.

THEOREM 1.2. *Let T be a normal variety, and let $(Z, \Delta) \rightarrow T$ be a family of n -dimensional KSBA-stable pairs with finite fiber isomorphism equivalence classes (see Definition 2.5); then $f_*((K_{Z/T} + \Delta)^{n+1})$ is ample on T .*

To conclude, we want to remark that the positivity of the CM line bundle is expected for spaces parametrizing Kähler–Einstein varieties or even polarized varieties with constant scalar curvature (see, for example, [PT09]). But in general, not much is known. In the case of the moduli space of Kähler–Einstein Fano varieties, it is still not known how to show the positivity of the CM line bundle with only algebro-geometric tools. Using the fact that the curvature of the CM line bundle is the Weil–Petersson metric for a smooth family and some deep results in analysis, one can verify that it induces an embedding when restricted to the locus which parametrizes Kähler–Einstein Fano manifolds (see [LWX15]). It remains an interesting and challenging question to prove similar results using only algebraic geometry.

2. Preliminary

2.1 Notation and conventions

We follow the notation in [KM98] and [Kol13b].

When X is a demi-normal (for the definition of demi-normal, see [Kol13b, Definition 5.1]) variety over k , we say that (X, Δ) is a pair if $\Delta = \sum a_i \Delta_i$ is an effective $\mathbb{Q}$ -divisor with $a_i \leq 1$, any component Δ_i is not contained in $\text{Sing}(X)$ and $K_X + \Delta$ is $\mathbb{Q}$ -Cartier.

We refer to [Kol13b, Definition-Lemma 5.10] for the definition of a pair (X, Δ) to be *semi-log canonical* (slc). Over an arbitrary base scheme T , let $f: X \rightarrow T$ be a flat family of slc models. We refer to [Kol13a, Definition 28] for the definition of the m th reflexive power $\omega_{X/T}^{[m]}$.

2.2 KSBA-stable family

In this section, we briefly introduce the concept of KSBA family. See [Kol13a] for more background.

DEFINITION 2.1. A pair (X, Δ) over k is *KSBA stable* if it is proper over k , it has slc singularities and $K_X + \Delta$ is ample.

We define the notion of a KSBA family in full generality only for the boundary-free case, since that is what we need in Theorem 1.1. In the presence of a boundary, we define the notion of KSBA family only over normal bases, as that is sufficient for the purposes of Theorem 1.2. We want to note that when there is a boundary, the definition of a KSBA family over a general base is subtle (see [Kol13a]).

DEFINITION 2.2. For any scheme T over k , a family $f: X \rightarrow T$ is a KSBA-stable family if f is flat, X_t is KSBA stable for each $t \in T$ and the family satisfies the following Kollár condition: $\omega_{X/T}^{[m]}$ is compatible with base change for each integer m ; in other words, if $S \rightarrow T$ is a morphism, then $\omega_{X_S/S}^{[m]} \cong (\omega_{X/T}^{[m]})_S$.

A proper flat morphism $(X, D) \rightarrow T$ onto a normal variety is a KSBA-stable family, if D avoids the generic and the singular codimension one points of each fiber, (X_t, D_t) is KSBA stable for each $t \in T$ and $K_{X/T} + D$ is $\mathbb{Q}$ -Cartier.

Remark 2.3. Definitions 2.1 and 2.2 are compatible. That is, if $X \rightarrow T$ is a KSBA-stable family in the second sense, then it is automatically a KSBA-stable family in the first sense, that is, it satisfies the Kollár condition, according to [Kol11, (4.4)] and [Kol08, Corollary 25].

We also need the following definition from [KP15, Definition 5.16].

DEFINITION 2.4 (Variation). Consider a KSBA family $f: (X, D) \rightarrow T$ over an irreducible normal variety, with dimension $\dim(X_t) = n$ and volume $(K_{X_t} + D_t)^n = v$ for a general t . Let I be the set of all possible sums equal to at most 1, formed from the coefficients of D . Then, there is an associated moduli map $\mu: Y \rightarrow \mathcal{M}_{n,v,I}$ to the moduli space of KSBA-stable pairs with dimension n , volume v and coefficient set I . The variation $\text{var}(f)$ of f is defined as the dimension of the image of μ .

There is a more intuitive way to define $\text{var}(f)$ that does not use the existence of the moduli space $\mathcal{M}_{n,v,I}$: let $f: (X, D) \rightarrow T$ be a KSBA-stable family over a normal base; then we define the variation $\text{var}(f)$ of the family to be $\dim T - d$, where d is the dimension of a general isomorphism equivalence class of the fibers.

Since the moduli map μ sends t and t' to the same point on $\mathcal{M}_{n,v,I}$ if and only if (X_t, D_t) is isomorphic to $(X_{t'}, D_{t'})$, the closed points of the closed fibers of the moduli map are the equivalence classes under the equivalence relation on the closed points of T given by $t \equiv t'$ if and only if (X_t, D_t) is isomorphic to $(X_{t'}, D_{t'})$. Then the simple addition formula for the dimensions of the total space, the general fiber and the base of a fibration yields that the dimension of the image of the moduli map is $\dim T - d$, where d is the dimension of a general equivalence class as above. This says that the two ways of defining $\text{var}(f)$ are equivalent.

If $\text{var } f = \dim T$, that is, a general fiber is isomorphic to only finitely many others, we say that the family has maximal variation.

DEFINITION 2.5 (Finite fiber isomorphism equivalence classes). Let $f: (X, D) \rightarrow T$ be a KSBA-stable family over a normal base of dimension d . We say that f has finite fiber isomorphism equivalence classes if for each $t \in T$, the set

$$\{u \in T \mid (X_u'', D_u'') \cong (X_t'', D_t'')\}$$

is finite.

Remark 2.6. By the existence of the Isom schemes of KSBA-stable families [KP15, Proposition 5.8] there is an open set $U \subseteq T$ such that for every $u \in U$, the locus $\{t \in T \mid (X_t, D_t) \cong (X_u, D_u)\}$ is a locally closed subset of the same (general) dimension.

PROPOSITION 2.7 ([KP15, Corollary 5.20]). *Given a family $f: (X, D) \rightarrow T$ of stable log varieties over a normal variety T , there exist a generically finite proper map $T' \rightarrow T$ from a normal variety, another proper map $T' \rightarrow T''$ to a normal variety and a family of stable log varieties $f'': (X'', D'') \rightarrow T''$ with maximal variation and finite fiber isomorphism equivalence classes such that the pullbacks of the two families over T' are isomorphic.*

LEMMA 2.8. *Let $f: (X, D) \rightarrow T$ be a maximal variation KSBA-stable family over a normal variety T with n -dimensional fibers and H an ample divisor on X ; then $f_*(H^{n+1})$ is $\mathbb{Q}$ -linearly equivalent to an effective cycle with the support S such that $f_S: (X, D) \times_T S \rightarrow S$ has maximal variation.*

Proof. Let us search for the required effective cycle E in the form

$$f_*(H_1 \cap \cdots \cap H_{n+1}),$$

where $H_i \in |mH|$ for some integer $m \gg 0$. We are ready as soon as we make sure that each component of E intersects the set U of Remark 2.6. Let Z_l ($l = 1, \dots, t$) be the components of $T \setminus U$. Then it is enough to guarantee that no component of E is contained in any of the Z_l . For that, just choose the H_i inductively to be general members of $|mH|$, which therefore does not contain any irreducible component of $f^{-1}Z_l \cap (\bigcap_{j=1}^{i-1} H_j)$. This way, $f^{-1}Z_l \cap (\bigcap_{i=1}^{n+1} H_i)$ will have dimension $\dim Z_l - 1$, which shows that no component of E is contained in Z_l . $\square$

2.3 The CM line bundle

For the reader's convenience, in this section we recall some of the basic background on the CM line bundle. For more details see [Tia97, PT09, FR06, PRS08, Wan12, WX14]. The concept of the CM line bundle was first introduced in [Tia97]. Unlike the Chow line bundle, it is not positive on the entire Hilbert scheme (see [FR06]). However, it is expected to be positive on the locus where the fibers are K-polystable.

Let $f: X \rightarrow B$ be a proper flat morphism of schemes of constant relative dimension $n \geq 1$, and let A be a relatively ample line bundle on X . We will assume throughout that B is normal and that X is S_2 and has pure dimension. We also assume that the fibers of f are S_2 and G_1 .

Then Knudsen–Mumford's determinant bundle construction shows that there are line bundles $\lambda_0, \dots, \lambda_{n+1}$ such that the following formula holds:

$$\det f_! (A^{\otimes k}) = \det R^\bullet f (A^{\otimes k}) = \lambda_{n+1}^{\binom{k}{n+1}} \otimes \lambda_n^{\binom{k}{n}} \otimes \dots \otimes \lambda_0.$$

Let $\mu := -(K_{X_t} \cdot A_{|X_t}^{n-1})/A_{|X_t}^n$; then

$$\lambda_{\text{CM}} = \lambda_{\text{CM}}(X/B, A) := \lambda_{n+1}^{n\mu+n(n+1)} \otimes \lambda_n^{-2(n+1)}.$$

A straightforward calculation using the Grothendieck–Riemann–Roch formula (see, for example, [FR06]) shows

$$c_1(\lambda_{n+1}) = f_*(c_1(A)^{n+1}) \quad \text{and} \quad nc_1(\lambda_{n+1}) - 2c_1(\lambda_n) = f_*(c_1(A)^n c_1(K_{X/B})).$$

Hence

$$c_1(\lambda_{\text{CM}}) = f_*(n\mu c_1(A)^{n+1} + (n+1)c_1(K_{X/B})c_1(A)^n).$$

In particular, if $K_{X/B}$ is $\mathbb{Q}$ -Cartier and relatively ample and we take $A = K_{X/B}$, we simply have

$$c_1(\lambda_{\text{CM}}) = f_*((K_{X/B})^{n+1}).$$

Similarly, a log extension as in [WX14] shows that if we consider the log setting, assume $K_{X/B} + D$ to be $\mathbb{Q}$ -Cartier and relatively ample, and take $A = K_{X/B} + D$, then

$$c_1(\lambda_{\text{CM}}((X, D)/B)) = f_*((K_{X/B} + D)^{n+1}).$$

(See [WX14, Definition 2.8 and Proposition 2.9].)

2.4 Divisorial log terminal blow-up

PROPOSITION 2.9. *Let $g: (Z, \Delta) \rightarrow T$ be a KSBA-stable family over a smooth variety T . We further assume that the generic fiber (Z_t, Δ_t) is log canonical. Then for each $0 < \epsilon \ll 1$, there exist a pair (X, D_ϵ) and a divisor $0 \leq D$ on X with a morphism $p: X \rightarrow Z$, such that*

- (i) $K_X + D = p^*(K_Z + \Delta)$,
- (ii) (X, D_ϵ) is Kawamata log terminal (klt),

- (iii) $f: (X, D_\epsilon) \rightarrow T$ is a KSBA-stable family,
- (iv) $D - D_\epsilon$ is effective and its support is contained in $\text{Ex}(p) \cap \text{Supp}(p_*^{-1}\Delta^{=1})$, and furthermore,
- (v) if the variation of $(Z, \Delta) \rightarrow T$ is maximal, then so is the variation of (X, D_ϵ) .

Proof. Let $\tilde{p}: \tilde{X} \rightarrow Z$ be a $\mathbb{Q}$ -factorial divisorial log terminal (dlt) modification of (Z, Δ) [Kol13b, Corollary 1.36], and write

$$\tilde{p}^*(K_Z + \Delta) = K_{\tilde{X}} + \tilde{D}.$$

Set $\tilde{D}^{=1} = [\tilde{D}]$ and $\tilde{D}^{<1} = \tilde{D} - \tilde{D}^{=1}$. By [BCHM10], we may take $p: X \rightarrow Z$ to be the relative log canonical model of $(\tilde{X}, (1-\epsilon)\tilde{D}^{=1} + \tilde{D}^{<1})$ over Z . Let $q: \tilde{X} \dashrightarrow X$ be the induced morphism and $D, D^{=1}$ and $D^{<1}$ the corresponding pushforwards. Define $D_\epsilon := (1-\epsilon)D^{=1} + D^{<1}$; then properties (ii) and (iv) follow. Note that by [BCHM10, Section 2, Theorem E], X is the same for all $0 < \epsilon \ll 1$.

Since $-\epsilon\tilde{D}^{=1} \equiv_Z K_{\tilde{X}} + (1-\epsilon)\tilde{D}^{=1} + \tilde{D}^{<1}$, we have that $-\epsilon D^{=1} \equiv_Z K_X + D_\epsilon$ is ample over Z . Furthermore,

$$\begin{aligned} K_X + D_\epsilon &= q_*(K_{\tilde{X}} + (1-\epsilon)\tilde{D}^{=1} + \tilde{D}^{<1}) \\ &= q_*(q^*p^*(K_Z + \Delta) - \epsilon\tilde{D}^{=1}) = p^*(K_Z + \Delta) - \epsilon D^{=1}. \end{aligned} \quad (2.1)$$

So, since $K_Z + \Delta$ is ample over T , the divisor $K_X + D_\epsilon$ is ample over T as well for $0 < \epsilon \ll 1$. A computation similar to (2.1), but with D instead of D_ϵ , shows property (i).

Let n be the relative dimension of Z over Y . To obtain property (iii), we need to show that X is flat over T , that D_ϵ does not contain any component of X_t or any divisor in the singular locus of X_t for any $t \in T$ and that $(X_t, D_{\epsilon,t})$ is slc of dimension n for all $t \in T$. We work on a neighborhood of an arbitrary point $t \in T$.

To see the above statements, first note that if $W \subset X_t$ is a component, then $\dim W \geq n$. Let $H_1, \dots, H_d$ be d general hypersurfaces passing through t , where $d = \dim(T)$. Then as $(Z, \Delta + g^*(\sum_{i=1}^d H_i))$ is crepant birational to $(X, D + f^*(\sum_{i=1}^d H_i))$, the latter is log canonical. As $(X, D + f^*(\sum_{i=1}^d H_i))$ is log canonical at W , the divisors f^*H_i are Cartier and $W \subset \bigcap_{i=1}^d f^*H_i$, by [dFKX12, Proposition 34] we know that $(X, f^*(\sum_{i=1}^d H_i))$ is snc at the generic point of W and W is not contained in D_t , which has the same support as $D_{\epsilon,t}$. Therefore $\dim W = n$ and X_t is smooth at the generic point of W . In particular, X_t is reduced and equidimensional.

Since (X, D_ϵ) is klt, X is Cohen–Maucaulay (see [KM98, Theorem 5.22]). Since $f: X \rightarrow T$ is an equidimensional morphism and T is smooth, X_t is cut out by a regular sequence. In particular, X_t is CM and f is flat. Consider $C = \bigcap_{i=1}^{d-1} H_i$, which is a smooth curve passing through t . Then $X_C := X \times_T C$ is normal and

$$(X_C, D_{\epsilon,C} := D_\epsilon \times_T C)$$

satisfies that $(X_C, D_{\epsilon,C} + X_t)$ is log canonical by adjunction. This implies that $(X_C, D_{\epsilon,C}) \rightarrow C$ is a KSBA family. Therefore, D_ϵ has to avoid the general point η of any codimensional one component of the singular locus of X_t . Thus $(X_t, D_{\epsilon,t})$ is slc by adjunction and we conclude that $(X, D_\epsilon) \rightarrow T$ is a KSBA family over T .

To prove property (v), just note that for a general $t \in T$, the pair (Z_t, Δ_t) is the log canonical model of (X_t, D_t) . So, for general $t, u \in T$ and for $0 < \epsilon \ll 1$, we have $(X_t, D_{\epsilon,t}) \cong (X_u, D_{\epsilon,u})$ if and only if $(X_t, D_t) \cong (X_u, D_u)$, from which it follows that $(Z_t, \Delta_t) \cong (Z_u, \Delta_u)$. This shows property (v). $\square$

2.5 Bigness and nefness of the relative canonical bundle

We first collect some results about the pushforwards of the powers of the relative canonical bundle.

DEFINITION 2.10. A torsion-free coherent sheaf $\mathcal{F}$ on a normal variety X is big if for an ample line bundle H on X , there exist an integer $a > 0$ and a generically surjective homomorphism $\bigoplus H \rightarrow S^{[a]}(\mathcal{F}) := (S^a(\mathcal{F}))^{**}$.

THEOREM 2.11. *If $f: (X, D) \rightarrow T$ is a maximal variation KSBA-stable family over a normal projective variety T with klt general fibers, then $f_*\mathcal{O}_X(r(K_{X/T} + D))$ is big for every sufficiently divisible integer $r > 0$.*

Proof. This follows from [KP15, Theorem 7.1]. $\square$

Remark 2.12. Note that the klt assumption in Theorem 2.11 cannot be weakened to log canonical according to [KP15, Examples 7.5–7.7].

THEOREM 2.13. *If $f: (X, D) \rightarrow T$ is a KSBA-stable family over a normal projective variety T , then $f_*\mathcal{O}_X(r(K_{X/T} + D))$ is nef for every sufficiently divisible integer $r > 0$.*

Proof. This follows from [Fuj12, Theorem 1.13]. $\square$

COROLLARY 2.14. *If $f: (X, D) \rightarrow T$ is a KSBA-stable family over a normal projective variety T , then*

$$f_*((K_{X/T} + D)^{n+1})$$

is nef.

Proof. Since $f_*((K_{X/T} + D)^{n+1})$ is compatible with base change, and nefness is decided on curves, we may assume that T is a curve. However, we then only need to prove $0 \leq \deg f_*((K_{X/T} + D)^{n+1})$, which follows if we show that $(K_{X/T} + D)^{n+1}$ is the limit of effective cycles. The latter statement follows from the nefness of $K_{X/T} + D$.

Thus it suffices to show that $K_{X/T} + D$ is nef. Since there is an embedding

$$X \subset \mathbb{P}_T(f_*\mathcal{O}_X(r(K_{X/T} + D)))$$

for r sufficiently large and

$$\mathcal{O}(1)|_X \cong r(K_{X/T} + D),$$

the nefness of $K_{X/T} + D$ is a straightforward consequence of Theorem 2.13. $\square$

PROPOSITION 2.15. *If $f: (X, D) \rightarrow T$ is a maximal variation KSBA-stable family over a smooth projective variety T such that the generic fiber (X_t, D_t) is log canonical, then $K_{X/T} + D$ is big and nef for every sufficiently divisible integer $r > 0$.*

Proof. We have shown the nefness of $K_{X/T} + D$ in the last proof.

For the bigness, when the general fiber is klt, by Theorem 2.11, there is a sequence of generically surjective morphisms

$$f^*(\bigoplus H) \rightarrow f^*S^a f_*\mathcal{O}_X(r(K_{X/T} + D)) \rightarrow \mathcal{O}_X(ar(K_{X/T} + D)),$$

for some a and sufficiently divisible r . After replacing a by its multiple, and tensoring by $A := r(K_{X/T} + D)$, we know that there is a nontrivial morphism

$$f^*(\bigoplus mH) \otimes \mathcal{O}_X(r(K_{X/T} + D)) \rightarrow \mathcal{O}_X((ma + 1)r(K_{X/T} + D)),$$

which implies that $K_{X/T} + D$ is big.

In general, applying Proposition 2.9, we know that we can find a birational model $h: Y \rightarrow X$ such that if we define D_Y in such a way that that

$$h^*(K_X + D) = K_Y + D_Y$$

holds, then there exists a divisor $D'_Y \leq D_Y$ for which $(Y, D'_Y) \rightarrow T$ is a maximal variation KSBA-stable family with generic klt fibers. Thus $K_{Y/T} + D'_Y$ is big, which implies that $K_{X/T} + D$ is big. $\square$

3. Proof of the main theorems

In this section, we prove Theorem 1.2 and hence Theorem 1.1. By induction, we may assume that the base is of dimension d and that Theorem 1.2 already holds when the base is of dimension at most $d - 1$. Note that the $d = 0$ case is tautologically true, hence the starting point of the induction is fine.

3.1 Log canonical case

LEMMA 3.1. *Theorem 1.2 is true if a general fiber (X_t, D_t) is log canonical.*

Proof. Set $A := K_{X/T} + D$ and $d := \dim(T)$. According to Corollary 2.14, by the Nakai–Moishezon criterion and resolution of singularities, it is enough to show that when T is a d -dimensional smooth projective variety and $(X, D)/T$ is a KSBA family of maximal variation, we have $(f_*(A^{n+1}))^d > 0$.

Write $A = H + F$ for some ample $\mathbb{Q}$ -divisor H and effective $\mathbb{Q}$ -divisor F . According to Lemma 2.8, the cycle $f_*(H^{n+1})$ is linearly equivalent to a $(d - 1)$ -dimensional effective $\mathbb{Q}$ -cycle over whose support (X, D) has maximal variation. Hence by induction we have

$$(f_*(A^{n+1}))^{d-1} \cdot f_*(H^{n+1}) > 0.$$

So we only need to show that for any $0 \leq k \leq n$,

$$(f_*(A^{n+1}))^{d-1} \cdot f_*(H^k \cdot F \cdot A^{n-k}) \geq 0.$$

This again follows from the induction since $f_*(H^k \cdot F \cdot A^{n-k})$ is a limit of effective $(d - 1)$ -dimensional $\mathbb{Q}$ -cycles. For any $(d - 1)$ -dimensional smooth projective variety $P \rightarrow T$, by the projection formula

$$(g_*(A|_P^{n+1}))^{d-1} = (f_*(A^{n+1}))^{d-1} \cdot P,$$

where $A|_P$ denotes the restriction of A on $X \times_T P$ and $g: X \times_T P \rightarrow P$ is the natural morphism. Then by the induction we have

$$(f_*(A^{n+1}))^{d-1} \cdot P \geq 0,$$

and this implies what we need by resolution of singularities. $\square$

3.2 Semi-log canonical case

Let $f: (Z, \Delta) \rightarrow S$ be a KSBA-stable family over a normal variety T . Taking a normalization $f: X \rightarrow Z$, we get

$$(X, D) = \sqcup_{i=1}^m (X_i, D_i) \rightarrow T$$

with conductor divisor E , where D_i is the sum of the conductor divisor and the pullback Δ_{X_i} of Δ . Furthermore, there is an involution $\tau: E^n \rightarrow E^n$ on the normalization E^n of E which preserves

the difference divisor $\text{Diff}_{E^n}(\Delta_X)$. In fact, we know that there is a one-to-one correspondence between $(Z, \Delta)/S$ and $(X, D, E, \tau)/S$ (see [Kol13b, Theorem 5.13]).

LEMMA 3.2. *Let $f: (Z, \Delta) \rightarrow S$ be a KSBA-stable family over a normal variety T . Taking a normalization $f: X \rightarrow Z$, we get*

$$(X, D) = \sqcup_{i=1}^m (X_i, D_i) \rightarrow T.$$

Then $f_i: (X_i, D_i) \rightarrow T$ is a KSBA-stable family over T .

Proof. Let $0 \in T$ be a closed point, $(\bar{X}_{i,0}, \bar{D}_{i,0}) \rightarrow (X_{i,0}, D_{i,0})$ the normalization and $\bar{D}_{i,0}$ the sum of the pullback of $D_{i,0}$ and the conductor divisor. Then $(\bar{X}_{i,0}, \bar{D}_{i,0})$ is a KSBA-stable pair.

We first want to check that $X \rightarrow T$ is flat at the codimension one point of X_0 . In fact, to see this, by cutting the fiber using general hypersurfaces, we can assume that $Z \rightarrow T$ has relative dimension one, in other words, that the fibers are nodal curves. Let E be the conductor divisor of the normalization $g: X \rightarrow Z$. By definition, g is an isomorphism outside E . Furthermore, if a codimension one point P of the fiber is contained in E , then $g(P)$ is a nodal point of the fiber, and analytically locally around $g(P)$, we can write it as $\hat{\mathcal{O}}_T[[x, y]]/(xy - a)$ for some element $a \in \hat{\mathcal{O}}_T$. However, since $P \in E$, we conclude $a = 0$, which implies that $X \rightarrow T$ is smooth along E .

Thus we can apply the numerical stability in [Kol16, Theorem 6] and conclude that for a general point $s \in T$,

$$(K_{\bar{X}_{i,0}} + \bar{D}_{i,0})^n \geq (K_{\bar{X}_{i,s}} + \bar{D}_{i,s})^n.$$

Furthermore, equality holds if and only if $(X_i, D_i) \rightarrow T$ is a KSBA-stable family over an open neighborhood of $0 \in T$.

On the other hand, we know

$$\begin{aligned} \sum_{i=1}^m (K_{\bar{X}_{i,s}} + \bar{D}_{i,s})^n &= (K_{Z_s} + \Delta_s)^n \\ &= (K_{Z_0} + \Delta_0)^n \\ &= \sum_{i=1}^m (K_{\bar{X}_{i,0}} + \bar{D}_{i,0})^n, \end{aligned}$$

where the second equality follows from the fact that (Z, Δ) is a stable family over T . Thus we conclude that for each i ,

$$(K_{\bar{X}_{i,0}} + \bar{D}_{i,0})^n = (K_{\bar{X}_{i,s}} + \bar{D}_{i,s})^n. \quad \square$$

Remark 3.3. We give a sketch of a more straightforward argument for a weaker statement than Lemma 3.2, which says that there exists a proper, dominant, generically finite morphism $T' \rightarrow T$ such that the normalization (X'_i, D'_i) of $(Z, \Delta) \times_T T'$ is a KSBA-stable family over T' . This is enough for our calculation in the proof of Theorem 1.2 for the general case.

By generic flatness, there is an open set T^0 such that $(X_i^0, D_i^0) := (X_i, D_i) \times_T T^0 \rightarrow T^0$ is a KSBA family over T^0 . Applying [AK00], we can assume that there is a proper, dominant, generically finite base change $g: T' \rightarrow T$ such that

$$(X_i \times_T T', D_i \times_T T')$$

admits a weak semistable reduction. It follows from [HX13, Theorem 1] that by running a relative MMP of $(X_i \times_T T', D_i \times_T T')$ over T' , we obtain a relative good minimal model (X_i^m, Δ_i^m) . An argument similar to that in the proof of Proposition 2.9 shows that (X_i^m, Δ_i^m) is flat over T' ,

that for any $t \in T'$, the divisor $\Delta_{i,t}^m$ does not contain any component or codimension one singular point of $X_{i,t}^m$ and that $(X_{i,t}^m, \Delta_{i,t}^m)$ is slc.

Then the injectivity theorem (see [Fuj13, Theorem 6.4]) implies that, fiberwise, the relative log canonical models (X_i^c, Δ_i^c) of (X_i^m, Δ_i^m) over T' gives the log canonical model of $(X_{i,t}^m, \Delta_{i,t}^m)$. Furthermore, (X_i^c, Δ_i^c) is flat over T' by Grauert's criterion. Thus (X_i^c, Δ_i^c) is a KSBA-stable family over T' .

Let

$$(X', D') = \sqcup_{i=1}^m (X_i^c, D_i^c) \rightarrow T'.$$

Over $g^{-1}(T_0)$, this (X', D') extends the family $(X, D) \times_T g^{-1}(T^0)$. Also, we easily see that both $E \times_T g^{-1}(T^0)$ and the involution condition

$$\tau \times_T g^{-1}(T^0): (E \times_{T^0} g^{-1}(T^0))^{\text{norm}} \rightarrow (E \times_{T^0} g^{-1}(T^0))^{\text{norm}}$$

extend to corresponding data E' and τ' over T' .

Thus $(X', D', E', \tau')/T'$ induces a KSBA-stable family $(Z', \Delta')/T'$ over T' by [Kol13b, Theorem 5.13]. It satisfies

$$(Z', \Delta') \times_{T'} g^{-1}(T^0) = (Z, \Delta) \times_T g^{-1}(T^0).$$

By the separateness of the functor of KSBA-stable families, we conclude that

$$(Z', \Delta') = (Z, \Delta) \times_T T',$$

as both of these give KSBA-stable families over T which are isomorphic over the generic point.

Proof of Theorem 1.2. We use the notation of Lemma 3.2. Applying Proposition 2.7, there is a smooth projective variety T' with a generically finite morphism T' with m dominant morphisms $h_i: T' \rightarrow T_i$ to smooth projective varieties such that if we set $X'_i := X_i \times_T T'$ and $D'_i := D_i \times_T T'$, then

$$(X'_i, D'_i) \cong (Y_i, E_i) \times_{T_i} T',$$

where the $g_i: (Y_i, E_i) \rightarrow T_i$ are KSBA-stable families over T_i with finite fiber isomorphism equivalence classes and log canonical generic fibers. Furthermore,

$$h: T' \rightarrow T_1 \times \cdots \times T_m$$

is a generically finite morphism.

According to Lemma 3.1 and the induction on dimension, we know that $(g_i)_*((K_{Y_i/T_i} + E_i)^{n+1})$ is ample on T_i . Denote the induced morphisms by $f'_i: X'_i \rightarrow T'$ and $p_i: T_1 \times \cdots \times T_m \rightarrow T_i$. Now h is generically finite and

$$\begin{aligned} (f_*((K_{X/T} + \Delta)^{n+1}))_{T'} &= \sum_i (f'_i)_*((K_{X'_i/T'} + D'_i)^{n+1}) \\ &= \sum_i h_i^*(g_{i*}((K_{Y_i/T_i} + E_i)^{n+1})) = h^* \left(\underbrace{\sum_i p_i^*(g_i)_*((K_{Y_i/T_i} + E_i)^{n+1})}_{\text{ample over } T_1 \times \cdots \times T_m} \right); \end{aligned}$$

hence $f_*((K_{X/T} + \Delta)^{n+1})$ is big and nef on T' . Thus the above computation concludes the proof. $\square$

Proof of Theorem 1.1. We apply the Nakai–Moishezon criterion (cf. [Kol90]). So it suffices to check that for any d -dimensional irreducible subspace $B \subset M^{\text{ksba}}$, the top intersection $\lambda_{\text{CM}}^d \cdot B$ is strictly positive.

By [Kol90, Proposition 2.7], we can replace B by a finite surjective base change $\pi: B' \rightarrow B$ such that B' is normal and $B' \rightarrow M^{\text{ksba}}$ lifts to $B' \rightarrow \mathcal{M}^{\text{ksba}}$, where $\mathcal{M}^{\text{ksba}}$ is the fine moduli Deligne–Mumford stack which parametrizes families of KSBA-stable varieties. Thus there is a KSBA family of finite fiber isomorphism equivalence classes X/B' . In particular,

$$\lambda_{\text{CM}}^d \cdot B = \frac{1}{\deg(\pi)} \lambda_{\text{CM}}^d \cdot B' > 0$$

by Theorem 1.2. □

Remark 3.4. A large part of our argument works in the log setting, that is, for KSBA families of log pairs. However, due to the subtlety of the definition of the KSBA functor itself, we will not discuss it here.

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