Automorphisms of elliptic K3 surfaces and Salem numbers of maximal degree

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Dedicated to Professor Shing-Tung Yau on the occasion of his sixty-fifth birthday

ABSTRACT

Using elliptic structures, we show that any supersingular K3 surface of Artin invariant 1 in characteristic $p \neq 5, 7, 13$ has an automorphism the entropy of which is the natural logarithm of a Salem number of degree 22.

1. Introduction

If p is a prime number, there is an Artin invariant 1 supersingular K3 surface $X(p)$, defined over the congruence field $\mathbb{F}_p$. It is unique up to isomorphism (see Section 2). Since Ogus' seminal work [Ogu79, Ogu83], this surface has been studied from various viewpoints.

In [ES13] it is shown that, as in characteristic 0, in positive characteristic the maximum of the absolute values of the algebraic integers which are eigenvalues of an automorphism of a smooth projective surface acting on its ℓ -adic cohomology is taken on the Néron–Severi group. Thus by analogy with complex geometry, the logarithm of this maximum is called the entropy. One knows that the entropy of an automorphism of a K3 surface is either 0 or the logarithm of a Salem number (see Section 3), hence of degree at most the rank of the Néron–Severi group, thus at most 20 for projective K3 surfaces in characteristic 0.

Over $k = \overline{\mathbb{F}}_p$, Jang [Jan14] showed that the image of the canonical representation

$$\mathrm{Aut}(X(p)) \rightarrow \mathrm{GL}(H^0(X(p), \omega_{X(p)})) \simeq k^\times$$

is isomorphic to the cyclic group of order $p+1$. In particular, for p large, there are automorphisms which are not geometrically liftable to characteristic 0. Jang's proof relies on Ogus' Torelli theorem [Ogu83]. From this, and from Shioda's study of Mordell–Weil lattices [Shi90], it is deduced in [EO15] that for p very large, there are automorphisms of $X(p)$ of positive entropy which are not geometrically liftable to characteristic 0 (see Section 2.3 for the definition of this liftability notion).

The main result of this note is the following.

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THEOREM 1.1. *Let $p \neq 5, 7, 13$. Then there is an automorphism $f \in \text{Aut}(X(p))$, defined over $\bar{\mathbb{F}}_p$, the entropy of which is the logarithm of a Salem number of degree 22.*

In particular, those automorphisms are not geometrically liftable to characteristic 0. The result is known over $k = \bar{\mathbb{F}}_p$ for $p = 2$ [BC13] and $p = 3$ [EO15].

While for $p = 2$ the proof relies on a very detailed study of $X(2)$ in [DK03], and for $p = 3$ it is computer aided and relies on the explicit study of $\text{Aut}(X(3))$ in [KS14], our proof of Theorem 1.1 is abstract and based on the theory of the Mordell–Weil group of $X(p)$ by Shioda [Shi13]. We show the following strengthening of Theorem 1.1.

THEOREM 1.2. *Let k be an algebraically closed field of characteristic $p \geq 0$. Let X be a K3 surface over k of Picard number $\rho = 2d \geq 4$. Assume that X has two non-isomorphic elliptic fibrations $\varphi_i: X \rightarrow \mathbb{P}^1$ ($i = 1, 2$) such that the Mordell–Weil group $\text{MW}(\varphi_1)$ of φ_1 is of maximal rank, that is, $2d - 2$, and $\text{MW}(\varphi_2)$ is of positive rank. Then X has an automorphism f , the entropy of which is the logarithm of a Salem number of degree $2d$.*

One deduces Theorem 1.1 from Theorem 1.2 using Shioda’s theorem [Shi13].

THEOREM 1.3. *Let $p = 11$ or $p > 13$. Then $X(p)$ admits an elliptic fibration of Mordell–Weil rank $20 = 22 - 2$.*

This also explains the restriction on p in Theorem 1.1. It is not unlikely that this restriction in Theorem 1.1 is unnecessary. More generally, it is likely that Theorem 1.2, Theorem 4.1 (mimicking [BC13]), and Theorem 4.6 could have a larger range of applications.

In this note, we do not address some questions of more arithmetical flavor. We know that the Néron–Severi group of $X(p)$ is defined over $\mathbb{F}_{p^2}$ [Sch12]. Over which field are those automorphisms of Theorem 1.1 defined? This depends on the field of definition of the Mordell–Weil groups in Theorem 1.3. Further, we know that Salem numbers of bounded degree are discrete. This raises the question whether or not the minimal Salem number of degree 22 arises as the logarithm of the entropy of an automorphism on a supersingular K3 surface (see [McM16] and references there). We also know that powers of Salem numbers are Salem numbers. Given a Salem number of degree 22 which is the power of another Salem number of degree 22, and such that its logarithm is the entropy of an automorphism f on a supersingular K3 surface, when is f itself the power of an automorphism? Finally, it would be interesting to relate this work to [GM02, McM02], in which the authors show that any unramified degree 22 Salem number is the logarithm of the entropy of an automorphism of a non-projective complex K3 surface.

2. Preliminaries on K3 surfaces and liftings

In this section, we fix the notation and recall basic facts on K3 surfaces and liftings from [EO15] and references therein.

2.1 K3 surfaces

Let X be a K3 surface defined over an algebraically closed field k of characteristic $p \geq 0$; that is, X is a smooth projective surface defined over k such that $H^1(X, \mathcal{O}_X) = 0$ and the dualizing sheaf is trivial: $\omega_X \simeq \mathcal{O}_X$. We denote by $\text{NS}(X)$ the Néron–Severi group of X . Then the Picard group $\text{Pic}(X)$ is isomorphic to the Néron–Severi group $\text{NS}(X)$, which is a free $\mathbb{Z}$ -module of finite rank. The rank of $\text{NS}(X)$ is called the Picard number of X and is denoted by $\rho(X)$. It is at

least 1 as X is assumed to be projective, and at most 22, the second ℓ -adic Betti number. In characteristic 0, it is at most 20 by Hodge theory. The intersection form $(*, **)$ on $\mathrm{NS}(X)$ is of signature $(1, \rho(X) - 1)$, and $(\mathrm{NS}(X), (*, **))$ is then an even hyperbolic lattice. The dual $\mathbb{Z}$ -module $\mathrm{NS}(X)^* := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{NS}(X), \mathbb{Z})$ is regarded as a $\mathbb{Z}$ -submodule of $\mathrm{NS}(X) \otimes \mathbb{Q}$, containing $\mathrm{NS}(X)$ through the intersection form $(*, **)$, which is non-degenerate. The quotient module $\mathrm{NS}(X)^*/\mathrm{NS}(X)$ is called the discriminant group of X .

The surface X is called supersingular if $\rho(X) = 22$, the maximum possible value. (As the Tate conjecture is not yet proven for $p = 2$, one should rather say Shioda supersingular in this case, but we consider only supersingular K3 surfaces in odd characteristic in this note).

Artin [Art74] proved that the discriminant group $\mathrm{NS}(X)^*/\mathrm{NS}(X)$ of a supersingular K3 surface X is p -elementary; more precisely, as an abelian group

$$\mathrm{NS}(X)^*/\mathrm{NS}(X) \simeq (\mathbb{Z}/p)^{2\sigma(X)},$$

where $\sigma(X)$ is an integer such that $1 \leq \sigma(X) \leq 10$. The integer $\sigma(X)$ is called the Artin invariant of X . Let σ be an integer such that $1 \leq \sigma \leq 10$. Then the supersingular K3 surfaces over k with $\sigma(X) \leq \sigma$ form a $(\sigma - 1)$ -dimensional family over k . There is, up to isomorphism, a unique Artin invariant 1 supersingular K3 surface $X(p)$ in each positive characteristic $p > 0$. It is defined over $\mathbb{F}_p$ and its Néron–Severi group is defined over $\mathbb{F}_{p^2}$, not over $\mathbb{F}_p$ [Ogu79, Sch12]. In a wide sense, the $X(p)$ are the most special K3 surfaces. The uniqueness of $X(p)$ in particular shows that $X(p) \simeq \mathrm{Km}(E \times_{\mathbb{F}_p} E)$ for any supersingular elliptic curve E over $\mathbb{F}_p$ [Ogu79, Shi75].

2.2 Lifting of K3 surfaces

Let X be a K3 surface defined over a perfect field k of positive characteristic $p > 0$, and let R be a discrete valuation ring with residue field k and field of fractions $K = \mathrm{Frac}(R)$ of characteristic 0. We call a proper flat (thus smooth) morphism of schemes $X_R \rightarrow \mathrm{Spec} R$ which restricts to $X \rightarrow \mathrm{Spec}(k)$ a characteristic 0 model of X/k . The generic fiber $X_K = X_R \otimes_R K$ is a K3 surface. For any field L containing K , the K3 surface $X_L = X_K \otimes_K L$ is called a lift of X over L . Characteristic 0 lifts of X are lifts over some L as above. They are unobstructed [Del81].

2.3 Geometric lift of an automorphism of a K3 surface

(See [EO15, Section 2.4].) Let X be a K3 surface defined over a perfect field k of positive characteristic $p > 0$, and let $X_R \rightarrow \mathrm{Spec} R$ be a characteristic 0 model. Recall [BGI71, Chapter X, Appendix] that one has a *specialization homomorphism* $\mathrm{sp}: \mathrm{Pic}(X_{\bar{K}}) \rightarrow \mathrm{Pic}(X)$ on the Picard group, which is defined by spreading out and restriction. It is injective as detected in ℓ -adic cohomology, on which the specialization is an isomorphism [Del77, Chapter V, Theorem 3.1].

One has a *restriction homomorphism* $\mathrm{Aut}(X_R/R) \rightarrow \mathrm{Aut}(X)$. One defines the subgroup $\mathrm{Aut}^e(X_{\bar{K}}/\bar{K}) \subset \mathrm{Aut}(X_{\bar{K}}/\bar{K})$ consisting of those automorphisms which lift to some model $X_R \rightarrow \mathrm{Spec} R$. (Here e stands for extendable.) The group law is defined by base change and the composition of automorphisms. Then the restriction homomorphism yields a *specialization homomorphism* $\iota: \mathrm{Aut}^e(X_{\bar{K}}/\bar{K}) \rightarrow \mathrm{Aut}(X/k)$. Moreover, sp is equivariant under ι . In addition, as automorphisms are detected on the associated formal scheme and $H^0(X, T_{X/k}) = 0$, the specialization homomorphism ι is injective (see [LM11, Lemma 2.3]).

An automorphism f in $\mathrm{Aut}(X)$ is *geometrically liftable* to characteristic 0 if it is in the image of the specialization homomorphism ι for some model X_R/R .

THEOREM 2.1 (See [EO15, Proof of Theorem 6.4]). *Let X be a supersingular K3 surface, and let $f \in \text{Aut}(X)$. Assume that the characteristic polynomial $f^*| \text{NS}(X)$ is irreducible in $\mathbb{Z}[t]$. Then f is never geometrically liftable to characteristic 0.*

Proof. If f lifted geometrically to characteristic 0, say to $g \in \text{Aut}(X_{\bar{K}})$ under $X_R \rightarrow \text{Spec } R$, then, as explained above, the specialization map $\iota: \text{NS}(X_{\bar{K}}) \hookrightarrow \text{NS}(X)$ would be equivariant with respect to g^* and f^* . In particular, the minimal monic polynomial $m_g(t) \in \mathbb{Z}[t]$ of $g^*| \text{NS}(X_{\bar{K}})$, which has degree at most 20, would divide the minimal monic polynomial $m_f(t) \in \mathbb{Z}[t]$ of $f^*| \text{NS}(X)$ in $\mathbb{Z}[t]$, which is irreducible and of degree 22 by assumption. This gives a contradiction. $\square$

3. Preliminaries on Salem numbers and entropy

In this section, we recall the definition of entropy and Salem numbers, again from [EO15] and the references therein.

In what follows, $L = (\mathbb{Z}^{1+t}, (*, **))$ is a hyperbolic lattice, that is, a pair consisting of a free $\mathbb{Z}$ -module of rank $1 + t$ and a $\mathbb{Z}$ -valued symmetric bilinear form $(\ , \)$ of $\mathbb{Z}^{1+t}$ of signature $(1, t)$ with $t > 0$. For any ring K , one denotes the scalar extension of L to K by L_K .

We denote by $O(L)$ the orthogonal group of the quadratic lattice L . It is an algebraic group defined over $\mathbb{Z}$. The determinant

$$\det: O(L) \rightarrow \{\pm 1\}$$

is a surjective homomorphism of algebraic groups. Its kernel $SO(L) \subset O(L)$ is a closed index 2 algebraic subgroup and is the identity component of $O(L)$. As an algebraic group, $SO(L)$ is geometrically connected. Moreover, $SO(L)(\mathbb{R})$ is a connected real Lie group and has index 2 in the real Lie group $O(L)(\mathbb{R})$, thus is its identity component in the real topology.

The subspace

$$P := \{x \in L_{\mathbb{R}} \mid (x^2) > 0\} \subset L_{\mathbb{R}}$$

consists of two connected components $\pm C$ in the real topology. By continuity, for $g \in O(L)(\mathbb{R})$, one has $g(C) = C$ or $g(C) = -C$, where the second case also occurs. Thus

$$O^+(L_{\mathbb{R}}) := \{g \in O(L)(\mathbb{R}) \mid g(C) = C\}$$

is an index 2 closed subgroup of the real Lie group $O(L)(\mathbb{R})$. Thus $O^+(L_{\mathbb{R}})$ is a real Lie subgroup of $O(L)(\mathbb{R})$, which is not the subgroup of $\mathbb{R}$ -points of an algebraic subgroup of $O(L)$.

One defines

$$\begin{aligned} O^+(L) &:= O(L)(\mathbb{Z}) \cap O^+(L_{\mathbb{R}}), \\ SO^+(L) &:= O(L)(\mathbb{Z}) \cap O^+(L_{\mathbb{R}}) \cap SO(L)(\mathbb{R}). \end{aligned}$$

The groups $O^+(L)$ and $SO^+(L)$ are abstract subgroups of $O(L)(\mathbb{Z})$ of index at most 4.

For application to K3 surfaces, we take L to be the Néron–Severi lattice $\text{NS}(X)$ and C to be the connected component of P containing the ample cone. Thus $\text{Aut}(X)^*$, the representation of $\text{Aut}(X)$ on $\text{NS}(X)$, lies in $O^+(\text{NS}(X))$. Moreover, $\text{Aut}(X)^{*0} := \text{Aut}(X)^* \cap SO^+(\text{NS}(X))$ is a subgroup of $\text{Aut}^*(X)$ of index at most 2.

We call a polynomial $P(x) \in \mathbb{Z}[x]$ a *Salem polynomial* if it is irreducible, monic, and of even degree $2d \geq 2$, and the complex zeroes of $P(x)$ consist of two real zeroes a_1 and a_2 and $d - 1$

pairs of complex conjugate zeroes $\alpha_i, \bar{\alpha}_i$ ($1 \leq i \leq d-1$) satisfying

$$0 < a_2 = \frac{1}{a_1} < 1, \quad \alpha_i, \bar{\alpha}_i \in S^1 := \{z \in \mathbb{C} \mid |z| = 1\} \setminus \{\pm 1\}.$$

PROPOSITION 3.1. *Let $f \in O^+(L)$. Then, the characteristic polynomial of f is the product of cyclotomic polynomials and at most one Salem polynomial counted with multiplicities.*

Proof. As mentioned in [EO15, Proposition 3.1], this is well known. See [McM02, Ogu10]. $\square$

DEFINITION 3.2. (i) For f as in Proposition 3.1, we define the *entropy* $h(f)$ of f by

$$h(f) = \log(r(f)) \geq 0,$$

where $r(f)$ is the spectral radius of f , that is, the maximum of the absolute values of the complex eigenvalues of f acting on L .

(ii) For a smooth projective surface S and an automorphism f on it, we define the *entropy* $h(f)$ by

$$h(f) = \log r(f^*|NS(S)),$$

where f^* is the action on $NS(S)$ induced by f .

This definition is consistent with the topological entropy of automorphisms of smooth complex projective surfaces [ES13].

4. Two observations from group theory

In this section, we shall prove Theorem 4.1, relying on [BdlH04] and [BC13], and Theorem 4.6, relying on [Ogu08]. They are crucial for our main results, Theorems 1.2 and 1.1.

THEOREM 4.1. *Let L be a hyperbolic lattice of even rank $2d$, and let $G \subset SO^+(L)$ be a subgroup. Assume that G has no G -stable $\mathbb{R}$ -linear subspace of $L_{\mathbb{R}}$ other than $\{0\}$ and $L_{\mathbb{R}}$. Then there is an element $g \in G$, the characteristic polynomial of which is a Salem polynomial of degree $2d$.*

The proof below mimics that of [BC13, Proposition 3.1], where the authors handle the case of $L = NS(X(2))$, and deduces the statement from [BdlH04, Proposition 1]. We slightly clarify their argument to make it fit with Theorem 4.1.

Proof. Recall [BdlH04, Proposition 1], in which neither the evenness of the rank of L nor the inclusion $G \subset SO^+(L)$ are necessary assumptions.

THEOREM 4.2. *Let $G \subset O(L)(\mathbb{R})$ be an abstract subgroup. Assume that G has no G -stable $\mathbb{R}$ -linear subspace of $L_{\mathbb{R}}$ other than $\{0\}$ and $L_{\mathbb{R}}$. Then the Zariski closure of G in $O(L_{\mathbb{R}})$ contains $SO(L_{\mathbb{R}})$. In particular, if in addition $G \subset SO(L)(\mathbb{R})$, then its Zariski closure in $SO(L_{\mathbb{R}})$ is $SO(L_{\mathbb{R}})$.*

As is well known, if L is any non-degenerate quadratic lattice and $g \in SO(L)(\mathbb{Z})$, then if the rank of L is odd, 1 is an eigenvalue of g . Indeed, if Q (respectively, M) is the matrix of the quadratic form of the lattice L (respectively, g) in a chosen basis, then ${}^tMQ(M - \text{Id}) = (\text{Id} - {}^tM)Q$, thus $\det(M - \text{Id}) = -\det(M - \text{Id}) \in \mathbb{Z}$. We now use that in Theorem 4.1, the rank of L is even.

PROPOSITION 4.3. *There is an element $g \in SO(L)(\mathbb{R})$ such that no eigenvalue of g is a root of unity.*

Proof. We may choose a real basis

$$\langle v_1, v_2, v_3, \dots, v_{2d} \rangle$$

of $L_{\mathbb{R}}$ under which the bilinear form $(*, **)$ is represented by the matrix $Q := (1, -1, -1, \dots, -1)$. We identify $\mathbb{R}$ -linear maps and $2d \times 2d$ -matrices with real entries via this basis. Consider the $\mathbb{R}$ -linear map of $L_{\mathbb{R}}$ given by the matrix

$$M = P \oplus R_2 \oplus \dots \oplus R_d,$$

where

$$P = \begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix}, \quad R_i = \begin{pmatrix} \cos 2\pi\sqrt{2} & -\sin 2\pi\sqrt{2} \\ \sin 2\pi\sqrt{2} & \cos 2\pi\sqrt{2} \end{pmatrix}$$

for all $2 \leq i \leq d$. Then ${}^tMQM = Q$, $\det M = 1$, and the complex eigenvalues of M are

$$\sqrt{2} \pm 1, \quad e^{\pm 2\pi\sqrt{-1}\cdot\sqrt{2}} = \cos(2\pi\sqrt{2}) \pm \sqrt{-1}\sin(2\pi\sqrt{2}).$$

Note that $e^{\pm 2\pi\sqrt{-1}\cdot s}$ ($s \in \mathbb{R}$) is a root of unity if and only if s is rational. Since $\sqrt{2}$ is an irrational real number, it follows that no eigenvalue of M is a root of unity. Thus M satisfies all the requirements. $\square$

Now assume $G \subset \mathrm{SO}^+(L)$. The argument now closely follows the proof of [BC13, Proposition 3.1].

LEMMA 4.4. *There are finitely many cyclotomic polynomials of degree at most $2d$.*

Proof. This is because the number of complex numbers with $x^{(2d)!} = 1$ is at most $(2d)!$. $\square$

Let $P_{2d} \subset \mathbb{Z}[t]$ be the set of monic polynomials of degree $2d$. Then P_{2d} is identified with the affine variety $\mathbb{A}^{2d}$ defined over $\mathbb{Z}$. The map

$$\mathrm{char}: \mathrm{SO}(L) \rightarrow P_{2d}, \quad h \mapsto \Phi_h(t) := \det(tI_{2d} - h)$$

is a morphism of affine varieties. Let

$$u_1(t) = t - 1, u_2(t) := t + 1, \dots, u_N(t)$$

be the cyclotomic polynomials in $\mathbb{Z}[t]$ of degree at most $2d$, where N is the cardinality of the cyclotomic polynomials of degree at most $2d$ (Lemma 4.4). The subsets

$$P_i := \{p(t) \in P_{2d}(\mathbb{C}) \mid u_i(t) \mid p(t)\}$$

define proper closed algebraic subvarieties of $P_{2d} \otimes_{\mathbb{Z}} \mathbb{Q}$, so their finite union

$$Q_{2d} := \cup_{i=1}^N P_i \subset P_{2d} \otimes_{\mathbb{Z}} \mathbb{Q}$$

is also one.

Let $g \in G$. Its characteristic polynomial $\Phi_g(t) \in \mathbb{Z}[t]$ is monic and of degree $2d$. By Proposition 3.1, the polynomial $\Phi_g(t)$ is the product of cyclotomic polynomials and at most one Salem polynomial, counted with multiplicities. Thus, $\Phi_g(t)$ is a Salem polynomial of degree $2d$ if and only if $\Phi_g(t)$ is irreducible and is not a cyclotomic polynomial of degree $2d$, which is equivalent to saying that no $u_i(t)$ divides $\Phi_g(t)$ in $\mathbb{Z}[t]$. Since $\Phi_g(t)$ and $u_i(t)$ are monic polynomials in $\mathbb{Z}[t]$, it follows that no $u_i(t)$ divides $\Phi_g(t)$ in $\mathbb{Z}[t]$ if and only if no $u_i(t)$ divides $\Phi_g(t)$ in $\mathbb{C}[t]$. The last condition is, by definition, equivalent to $\Phi_g(t) \in P_{2d}(\mathbb{C}) \setminus Q_{2d}$. The following lemma completes the proof.

LEMMA 4.5. *There is an element $g \in G$ such that $\Phi_g(t) \in P_{2d}(\mathbb{C}) \setminus Q_{2d}$.*

Proof. By our assumption, we can apply Theorem 4.2 to G , so the Zariski closure of G in $\mathrm{SO}(L_{\mathbb{R}})$ is $\mathrm{SO}(L_{\mathbb{R}})$. On the other hand, $\mathrm{char}^{-1}(P_{2d} \otimes_{\mathbb{Z}} \mathbb{Q} \setminus Q_{2d})$ is Zariski open in $\mathrm{SO}(L) \otimes_{\mathbb{Z}} \mathbb{Q}$, and not empty by Proposition 4.3. Thus it intersects $G \subset \mathrm{SO}(L)(\mathbb{Q})$ non-trivially. This finishes the proof. $\square$

This completes the proof of Theorem 4.1. $\square$

The following theorem is deduced from [Ogu08, Proof of Lemma 3.6, Claim 3.8].

THEOREM 4.6. *Let L be a hyperbolic lattice of signature $(1, r+1)$ with $r \geq 0$, and let $e \in L$ be a primitive element such that $(e, e) = 0$. Let $G \subset \mathrm{SO}(L)(\mathbb{Z})$ such that $G \simeq \mathbb{Z}^r$ as a group and such that $g(e) = e$ for all $g \in G$. Then any G -stable $\mathbb{R}$ -linear subspace M of $L_{\mathbb{R}}$ either is in the hypersurface $e^{\perp}$ in $L_{\mathbb{R}}$ or satisfies $M = L_{\mathbb{R}}$.*

Proof. Note that $G \subset \mathrm{SO}^+(L)$. Indeed, for $g \in G$, we have $g(e) = e$. Any small enough open ball in $L_{\mathbb{R}}$ in the classical topology, centered in e , meets C in an open $\mathcal{U}$ such that for any $x \in \mathcal{U}$, the distance between $g(x)$ and x is small enough that it forces $g(x)$ to lie in C .

LEMMA 4.7. *There are an integral basis*

$$\langle e, w_1, \dots, w_r \rangle$$

of $e^{\perp} \subset L$, so necessarily $(w_j, w_j) < 0$ for all $1 \leq j \leq r$, a $\mathbb{Q}$ -basis

$$\langle e, w_1, \dots, w_r, u \rangle$$

of $L_{\mathbb{Q}}$ with $(e, u) = 1$, and a finite-index subgroup

$$H := \langle g_1, \dots, g_r \rangle \simeq \mathbb{Z}^r$$

of G such that

$$g_i = \begin{pmatrix} 1 & \mathbf{a}_i^t & c_i \\ \mathbf{0} & I_r & q_i \mathbf{e}_i \\ 0 & \mathbf{0}^t & 1 \end{pmatrix}$$

under the $\mathbb{Q}$ -basis above. Here $1, 0 \in \mathbb{Q}$ are the unit and the zero, respectively, c_i and $q_i \neq 0$ are in $\mathbb{Q}$, and $\mathbf{e}_i$ is the i th unit vector of $\mathbb{Q}^r$. Furthermore, I_r is the $r \times r$ identity matrix, $\mathbf{0} \in \mathbb{Q}^r$ is the zero vector, $\mathbf{a}_i^t$ is the transpose of a column vector $\mathbf{a}_i \in \mathbb{Q}^r$, and similarly for $\mathbf{0}^t$.

Proof. This is observed in [Ogu08, Proof of Lemma 3.6, Claim 3.8]. The essential part is that $e^{\perp}/\mathbb{Z}e$ is a negative-definite lattice with respect to the $\mathbb{Z}$ -valued bilinear form induced by $(*, **)$ and G acts on this negative-definite lattice $e^{\perp}/\mathbb{Z}e$. Note that

$$H := \mathrm{Ker}(G \rightarrow \mathrm{O}(e^{\perp}/\mathbb{Z}e)(\mathbb{Z}))$$

is a finite-index subgroup of G , as $\mathrm{O}(e^{\perp}/\mathbb{Z}e)(\mathbb{Z})$ is a finite group. Since $G \simeq \mathbb{Z}^r$, we then have $H \simeq \mathbb{Z}^r$ by the fundamental theorem of finitely generated abelian groups. Next, choose a $\mathbb{Z}$ -basis

$$\langle e, w_1, \dots, w_r \rangle$$

of $e^{\perp}$. As $e^{\perp}$ is of signature $(0, 1, r-1)$ (degenerate lattice), it follows that $(w_i, w_i) < 0$ and $(e, w_i) = 0$, in addition to $(e^2) = 0$. Choose $u \in L_{\mathbb{Q}}$ such that $(e, u) = 1$. Such a vector u exists as L is hyperbolic. Then,

$$\langle e, w_1, \dots, w_r, u \rangle$$

is a $\mathbb{Q}$ -basis of $L_{\mathbb{Q}}$ and the matrix representation of H with respect to this $\mathbb{Q}$ -basis is of the form

$$g = \begin{pmatrix} 1 & \mathbf{a}^t(g) & c(g) \\ \mathbf{0} & I_r & \mathbf{b}(g) \\ 0 & \mathbf{0}^t & 1 \end{pmatrix}$$

for all $g \in H \subset G \subset \mathrm{SO}(L)(\mathbb{Z})$. It is proved in [Ogu08, Claim 3.8] that the map

$$H \rightarrow \mathbb{Q}^r, \quad g \mapsto \mathbf{b}(g)$$

is an injective group homomorphism. This is easily checked by an explicit computation of matrices. Since $H \simeq \mathbb{Z}^r$, it follows that

$$\mathbb{Z}^r \simeq H \simeq \langle \mathbf{b}(g) \mid g \in H \rangle \subset \mathbb{Q}^r.$$

Thus, again by the fundamental theorem of finitely generated abelian groups, we obtain a $\mathbb{Z}$ -basis of H with the required property. $\square$

LEMMA 4.8. *There are integers i, j such that $1 \leq i, j \leq r$ and $\mathbf{a}_j^t \cdot \mathbf{e}_i \neq 0$. Here the left-hand side is a product of matrices and we naturally identify 1×1 matrices with their entry.*

Proof. Assume to the contrary that $\mathbf{a}_j^t \cdot \mathbf{e}_i = 0$ for all i, j . Then $\mathbf{a}_j^t = 0$ for all j . Thus

$$g_j(u) = c_j e + q_j w_j + u$$

by the explicit matrix form. Then by induction one has

$$g_j^k(u) = g_j((k-1)c_j e + (k-1)w_j + u) = k(c_j e + q_j w_j) + u$$

for all positive integer k . Since g_j^k preserves the intersection form, it follows that

$$(u, u) = (g^k(u), g^k(u)) = (u, u) + 2k(q_j(w_j, u) + c_j) + k^2 q_j^2(w_j, w_j),$$

whence

$$(w_j, w_j) q_j^2 k + 2(c_j + q_j(w_j, u)) = 0$$

for all positive integers k , which contradicts the inequalities $(w_j, w_j) < 0$ and $q_j \neq 0$ in Lemma 4.7. $\square$

To conclude the proof of Theorem 4.6, it suffices to confirm the following.

LEMMA 4.9. *Let M be a G -stable $\mathbb{R}$ -linear subspace of $L_{\mathbb{R}}$. Assume that there is a $v \in M_{\mathbb{R}}$ such that $v \notin e^{\perp}$ in $L_{\mathbb{R}}$. Then, $M = L_{\mathbb{R}}$.*

Proof. By replacing v by a multiple by $\mathbb{R}^{\times}$, we may assume without loss of generality that

$$v = x e + \sum_{s=1}^r y_s w_s + u,$$

where x and y_s are real numbers. Then, by Lemma 4.7,

$$g_i(v) = v + (\mathbf{a}_i^t \cdot \mathbf{y} + c_i) e + q_i w_i,$$

where $\mathbf{y} \in \mathbb{R}^r$ is the vector whose s th entry is y_s . But $g(v) \in M$ by assumption, and by the fact that M is $\mathbb{R}$ -linear it follows that

$$v_i := (\mathbf{a}_i^t \cdot \mathbf{y} + c_i) e + q_i w_i \in M.$$

By Lemma 4.7, it follows that

$$g_j(v_i) = v_i + q_i({}^t\mathbf{a}_j \cdot \mathbf{e}_i)e.$$

Recall from Lemma 4.7 that $q_i \neq 0$. Then, for the same reason as above, it follows that

$$({}^t\mathbf{a}_j \cdot \mathbf{e}_i)e \in M$$

for all $1 \leq i, j \leq r$. Thus by Lemma 4.8, we have $e \in M$. Combining this with $v_i \in M$, it follows that $q_i w_i \in M$ for all i . Since $q_i \neq 0$, it follows that $w_i \in M$ for all i . Combining this with $v \in M$, it follows that $u \in M$. Since e, w_i, u form a $\mathbb{Q}$ -basis of $L_{\mathbb{Q}}$, thus an $\mathbb{R}$ -basis of $L_{\mathbb{R}}$, it follows that $L_{\mathbb{R}} \subset M$. This finishes the proof. $\square$

This completes the proof of Theorem 4.6. $\square$

5. Proof of Theorems 1.2 and 1.1

First we prove Theorem 1.2. Let X be as in Theorem 1.2. We recall that $\text{NS}(X)$ is a hyperbolic lattice of signature $(1, \rho(X) - 1)$.

Recall that an elliptic fibration $\varphi: X \rightarrow \mathbb{P}^1$ on the minimal surface X is a projective, surjective morphism over a field k , which has a section $O: \mathbb{P}^1 \rightarrow X$, and is such that the generic fiber is a smooth curve of genus 1 over $k(\mathbb{P}^1)$. We denote by $\text{MW}(\varphi)$ its Mordell–Weil group, viewed as a subgroup of the group of the birational automorphisms $\text{Bir}(X)$.

The following lemma is well known.

LEMMA 5.1. *Let $\varphi: X \rightarrow \mathbb{P}^1$ be an elliptic fibration. Then*

- *the Mordell–Weil group $\text{MW}(\varphi)$ of φ is a finitely generated abelian subgroup of $\text{Aut}(X) \subset \text{Bir}(X)$;*
- *the action of $\text{MW}(\varphi)$ on $\text{NS}(X)$ is faithful.*

We denote by $\text{MW}(\varphi)^* \subset \text{O}^+(\text{NS}(X))$ the image of the induced representation on the Néron–Severi group $\text{NS}(X)$ of X . Thus $\text{MW}(\varphi) \xrightarrow{\cong} \text{MW}(\varphi)^*$.

Proof. The group $\text{MW}(\varphi)$ is finitely generated by [Shi90], as φ , for topological reasons, has at least one singular fiber. As X is a smooth projective minimal surface, $\text{Bir}(X) = \text{Aut}(X)$. This implies part (i). One has $|O| = \{O\}$, where $|O|$ is the complete linear system containing O . This is because $O \simeq \mathbb{P}^1$ and $(O, O) = -2 < 0$, as X is a K3 surface.

Let $f \in \text{MW}(\varphi)$. If $f^*|_{\text{NS}(X)} = \text{Id}$, then, in particular, the class of O in $\text{NS}(X)$ is invariant under f^* . Thus $|O|$ is invariant under f , and therefore the section O is invariant under f . As $f \in \text{MW}(\varphi)$, this implies $f = \text{Id}$ on X . This proves part (ii). $\square$

From here on, the notation is as in Section 3. In addition, we define

$$\text{MW}(\varphi)^{*0} := \text{MW}(\varphi)^* \cap \text{SO}^+(\text{NS}(X)_{\mathbb{R}}).$$

LEMMA 5.2. (i) *The group $\text{Aut}(X)^{*0}$ is infinite; thus $\text{Aut}(X)$ is infinite as well.*

(ii) *The abelian groups $\text{MW}(\varphi)$ and $\text{MW}(\varphi)^{*0}$ have the same rank.*

Proof. As $\text{Aut}(X)^{*0} \supset \text{MW}(\varphi)^{*0}$, statement (i) follows from statement (ii). We prove statement (ii). By Lemma 5.1, we have $\text{MW}(\varphi) \xrightarrow{\cong} \text{MW}(\varphi)^*$. On the other hand, $\text{SO}^+(\text{NS}(X)_{\mathbb{R}})$ is a finite-index subgroup of $\text{O}(\text{NS}(X))(\mathbb{R})$. Thus $\text{MW}(\varphi)^{*0}$ is a finite-index subgroup of $\text{MW}(\varphi)^*$. $\square$

Recall that X is as in Theorem 1.2. We denote by $e_1 \in \text{NS}(X)$ the class of a fiber of φ_1 .

LEMMA 5.3. *The surface X admits a $g \in \text{Aut}(X)$ such that $g^*(e_1) \neq e_1$ in $\text{NS}(X)$.*

Proof. By the assumption, X admits a different elliptic fibration $\varphi_2: X \rightarrow \mathbb{P}^1$ of positive Mordell–Weil rank. Thus, by Lemma 5.2, there is a $g \in \text{MW}(\varphi_2)$ of infinite order.

Let $f \in \text{NS}(X)$ be the fiber class of φ_2 ; thus, in particular, $g^*(f) = f$. By the Hodge index theorem, $(f + e_1, f + e_1) > 0$. Therefore, the lattice $(e_1 + f)^\perp$ in $\text{NS}(X)$ is negative definite.

Assume $g^*(e_1) = e_1$ in $\text{NS}(X)$. Then $g^*(e_1 + f) = e_1 + f$. Thus, by Lemma 5.1, the morphism g acts faithfully on $(e_1 + f)^\perp$. As $O((e_1 + f)^\perp)(\mathbb{Z})$ is a finite group, g is of finite order, which gives a contradiction. This proves the lemma. $\square$

DEFINITION 5.4. Consider all the elliptic fibrations $\Phi_i: X \rightarrow \mathbb{P}^1$ ($i \in I$) on X with maximum Mordell–Weil rank $r = \rho(X) - 2$. Let $e_i \in \text{NS}(X)$ be the class of fibers of Φ_i . Set

$$\mathcal{S} := \{e_i \in \text{NS}(X) \mid i \in I\},$$

and denote by $\mathbb{R}\langle \mathcal{S} \rangle \subset \text{NS}(X)_\mathbb{R}$ the real linear subspace spanned by $\mathcal{S}$. Note that $(e_i, e_i) = 0$, e_i is numerically effective, and e_i is primitive in $\text{NS}(X)$ for all $e_i \in \mathcal{S}$.

LEMMA 5.5. *One has $\mathbb{R}\langle \mathcal{S} \rangle \xrightarrow{\cong} \text{NS}(X)_\mathbb{R}$.*

Proof. Recall that $\mathcal{S} \neq \emptyset$ by our assumption. Let $e_1 \in \mathcal{S}$ be the class of Φ_1 . By definition of $\mathcal{S}$, $\mathbb{R}\langle \mathcal{S} \rangle$ is $\text{Aut}(X)$ -stable. Let $g \in \text{Aut}(X)$ with $e_2 := g(e_1) \neq e_1$ (Lemma 5.3). As $\mathcal{S}$ is stable under $\text{Aut}(X)$, it follows that $e_2 \in \mathcal{S}$ as well.

Assume to the contrary that $\mathbb{R}\langle \mathcal{S} \rangle \neq \text{NS}(X)_\mathbb{R}$. Let $G_1 \subset \text{MW}(\Phi_1)^{*0}$ be a free abelian group of rank $r = \rho(X) - 2$. By Theorem 4.6 applied to G_1 , one has $\mathbb{R}\langle \mathcal{S} \rangle \subset e_1^\perp$ in $\text{NS}(X)_\mathbb{R}$. By the Hodge index theorem and the fact that the e_i are primitive, one has $(e_1, e_2) > 0$, which gives a contradiction. $\square$

LEMMA 5.6. *There is no $\text{Aut}(X)^{*0}$ -stable $\mathbb{R}$ -linear subspace of $\text{NS}(X)_\mathbb{R}$ other than $\{0\}$ and $\text{NS}(X)_\mathbb{R}$.*

Proof. Let $M \neq \text{NS}(X)_\mathbb{R}$ be an $\text{Aut}(X)^{*0}$ -stable $\mathbb{R}$ -linear subspace of $\text{NS}(X)_\mathbb{R}$. For Φ_i as in Definition 5.4, let $G_i \subset \text{MW}(\Phi_i)^{*0}$ be a free abelian subgroup of maximal rank $r = \rho(X) - 2$ (Lemma 5.1). By Theorem 4.6 applied to G_i , it follows that

$$M \subset \bigcap_{i \in I} e_i^\perp \subset \text{NS}(X)_\mathbb{R}.$$

As the vectors e_i , for $i \in I$, generate $\text{NS}(X)_\mathbb{R}$ and the intersection form is non-degenerate on $\text{NS}(X)$, it follows that

$$\bigcap_{i \in I} e_i^\perp = \{0\}.$$

This proves the lemma. $\square$

Proof of Theorem 1.2. By Lemmas 5.2 and 5.6, we can apply Theorem 4.1. $\square$

Proof of Theorem 1.1. As mentioned in the introduction, Shioda [Shi13] proved that any $X(p)$ with $p = 11$ or $p > 13$ admits an elliptic fibration $\varphi_1: X \rightarrow \mathbb{P}^1$ with Mordell–Weil rank $20 = 22 - 2$. On the other hand, over $\mathbb{F}_p$, one has $X(p) \simeq \text{Km}(E \times_{\mathbb{F}_p} E)$ for a supersingular elliptic curve E over $\mathbb{F}_p$, and the fibration $\varphi_2: X \rightarrow \mathbb{P}^1$ induced by the first projection $E \times_{\mathbb{F}_p} E \rightarrow E$ is an elliptic fibration with Mordell–Weil rank 4 (thus positive) over $\mathbb{F}_p$, by the formula for the Mordell–Weil rank [Shi90]. In particular, these two fibrations are non-isomorphic. So, we apply Theorem 1.2 to deduce Theorem 1.1 for $p = 11$ or $p > 13$. The cases $p = 2$ and $p = 3$ are proved by [BC13] and [EO15]. This completes the proof of Theorem 1.1. $\square$

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