Maps between local Picard groups

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ABSTRACT

The paper investigates maps between local Picard groups. The main goal is to understand the kernel in two cases: pull-back to the normalization and restriction to a hyperplane section. We give sufficient (and almost necessary) conditions that guarantee that the kernel is finite or of finite type.

1. Introduction

Let X be a scheme and $x \in X$ a point. The *local Picard group* of X at x , denoted by $\mathrm{Pic}^{\mathrm{loc}}(x, X)$, is the Picard group of the *punctured neighborhood* $\mathrm{Spec}_X \mathcal{O}_{x,X} \setminus \{x\}$. Our aim is to study the pull-back map on the local Picard group in two situations.

Question 1.1 (Normalization). Let $\pi: \bar{X} \rightarrow X$ denote the normalization, and let $\bar{x}_i \in \bar{X}$ be the preimages of x . What is the kernel of the pull-back map

$$\pi^*: \mathrm{Pic}^{\mathrm{loc}}(x, X) \rightarrow \sum_i \mathrm{Pic}^{\mathrm{loc}}(\bar{x}_i, \bar{X})?$$

Question 1.2 (Restriction to a divisor). Let $x \in D \subset X$ be an effective Cartier divisor. What is the kernel of the restriction map

$$r_D^X: \mathrm{Pic}^{\mathrm{loc}}(x, X) \rightarrow \mathrm{Pic}^{\mathrm{loc}}(x, D)?$$

In both cases we are interested in conditions that guarantee that the kernels of these maps are “small”. Here “small” can mean trivial, or finite, or “naturally” a subgroup of a linear algebraic group.

Grothendieck’s local Lefschetz-type theorem [Gro68, XI.3.16] is the first major result on Question 1.2. He proves that r_D^X is injective if $\mathrm{depth}_x X \geq 4$. Conjecture 1.2 in [Kol13a] asserts that this can be relaxed to $\mathrm{depth}_x X \geq 3$ and $\dim X \geq 4$. This was proved in [Kol13a] when X is log canonical and in [BdJ14] when X is normal.

I was led to Question 1.1 while investigating stable varieties and their moduli. The method of [Kol13b] first studies the normalization $\bar{X}$ of a stable variety X and then descends the information from $\bar{X}$ to X . It is especially important to know when the canonical divisor of X is $\mathbb{Q}$ -Cartier; this was settled in [Kol11]. For semi-log canonical varieties Question 1.1 is answered in [Kol13b, Section 5.7].

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In many applications, the natural assumption is that X satisfies Serre's condition S_2 only. To understand this case, one needs to know that, as in the global situation, the local Picard group is not just a group but a group scheme. The local definition is more involved than the projective one, and the natural setting turns out to be the following.

CONDITION 1.3. We consider a scheme X that is essentially of finite type over a field k , and a 0-dimensional closed subscheme $x \subset X$. The scheme X satisfies Serre's condition S_2 and has pure dimension at least 3 (that is, every associated prime of $\mathcal{O}_X$ has the same dimension, which is at least 3).

Boutot [Bou78] defines a *local Picard functor* $\text{Pic}^{\text{loc}}(x, X)$ and proves that, under the above assumptions, it is represented by the *local Picard scheme* $\mathbf{Pic}^{\text{loc}}(x, X)$; see Definition 3.3 and [Bou78, Theorem 1.ii] for details. Frequently, $\mathbf{Pic}^{\text{loc}}(x, X)$ has infinitely many connected components. The identity component, usually denoted by $\mathbf{Pic}^{\text{loc}-\circ}(x, X)$, is a commutative, connected algebraic group of finite type, but it need not be a linear algebraic group. The union of those components of $\mathbf{Pic}^{\text{loc}}(x, X)$ that become torsion modulo $\mathbf{Pic}^{\text{loc}-\circ}(x, X)$ is denoted by $\mathbf{Pic}^{\text{loc}-\tau}(x, X)$.

The following theorem gives a unified answer to both questions.

THEOREM 1.4. *Let X and Y be schemes that satisfy Condition 1.3 and $\pi: Y \rightarrow X$ a finite morphism. Let $x \subset \pi(Y) \subset X$ be a 0-dimensional closed subscheme, $y \subset Y$ the preimage of x and $r_Y^X: \mathbf{Pic}^{\text{loc}}(x, X) \rightarrow \mathbf{Pic}^{\text{loc}}(y, Y)$ the natural pull-back map.*

Assume that $\pi(Y)$ contains the support of a complete intersection subscheme $x \subset Z \subset X$ of dimension at least 3. Then

$$(r_Y^X)^{-1} \mathbf{Pic}^{\text{loc}-\tau}(x, X) = \mathbf{Pic}^{\text{loc}-\tau}(y, Y).$$

1.5 (Local Néron–Severi groups). One can restate the above result in terms of the *local Néron–Severi groups*

$$\text{NS}^{\text{loc}}(x, X) := \mathbf{Pic}^{\text{loc}}(x, X) / \mathbf{Pic}^{\text{loc}-\circ}(x, X).$$

The local Néron–Severi group is a finitely generated abelian group if $\text{char } k = 0$, but only the prime-to- p part is known to be finitely generated if $\text{char } k = p > 0$; see Definition 3.4. Theorem 1.4 is equivalent to saying that the kernel of the natural pull-back map $\text{NS}^{\text{loc}}(x, X) \rightarrow \text{NS}^{\text{loc}}(y, Y)$ is torsion.

The global variant of this was proved in [Kle66, Remark 3, p. 305].

Theorem 1.4 implies that $\ker r_Y^X \subset \mathbf{Pic}^{\text{loc}-\tau}(x, X)$, but for both questions considered above we can say more. Our answers are optimal in characteristic 0 but in characteristic $p > 0$ I have not been able to exclude a possibly infinite, discrete, p^∞ -torsion quotient. The precise statements are the following.

THEOREM 1.6. *Assume that (x, X) satisfies Condition 1.3, in particular $\dim X \geq 3$. Let $\pi: Y \rightarrow X$ be a finite surjection, $x \subset X$ a 0-dimensional closed subscheme and $y \subset Y$ the preimage of x . Then the kernel of the pull-back map $\pi^*: \mathbf{Pic}^{\text{loc}}(x, X) \rightarrow \mathbf{Pic}^{\text{loc}}(y, Y)$ is*

- (i) *a linear algebraic group if $\text{char } k = 0$ and*
- (ii) *the extension of a discrete p^∞ -torsion group by a linear algebraic group if $\text{char } k = p > 0$.*

I do not know any example where the kernel of π^* is not a linear algebraic group.

We have to be careful with the scheme-theoretic formulation of Question 1.2 since $D \subset X$ need not be S_2 ; thus $\mathbf{Pic}^{\text{loc}}(x, D)$ need not exist. Nonetheless, for now we use the suggestive notation $r_D^X: \mathbf{Pic}^{\text{loc}}(x, X) \rightarrow \mathbf{Pic}^{\text{loc}}(x, D)$; see Definition 3.3 for its precise meaning.

THEOREM 1.7. *Assume that (x, X) satisfies Condition 1.3 and $\dim X \geq 4$. Let $x \in D \subset X$ be an effective Cartier divisor. Then the kernel of the restriction map $r_D^X: \mathbf{Pic}^{\text{loc}}(x, X) \rightarrow \mathbf{Pic}^{\text{loc}}(x, D)$ is*

- (i) *a unipotent algebraic group if $\text{char } k = 0$,*
- (ii) *a unipotent algebraic group if X is normal and*
- (iii) *an extension of a discrete p^∞ -torsion group by a unipotent algebraic group if $\text{char } k = p > 0$.*

As before, I do not know any example where the kernel of r_D^X is not unipotent.

Remark 1.8. In both cases the dimension restrictions are optimal. For example, for the nonnormal surface $S := (xyz = 0) \subset \mathbb{A}^3$ with normalization $\pi: \bar{S} \rightarrow S$ we have

$$\ker[\mathbf{Pic}^{\text{loc}}(0, S) \xrightarrow{\pi^*} \mathbf{Pic}^{\text{loc}}(\bar{0}, \bar{S})] \cong \mathbb{Z}^3.$$

For the ordinary 3-fold node $X := (x^2 + y^2 + z^2 + t^2 = 0) \subset \mathbb{A}^4$ and $D := (t = 0)$ we have

$$\ker[\mathbf{Pic}^{\text{loc}}(x, X) \xrightarrow{r_D^X} \mathbf{Pic}^{\text{loc}}(x, D)] \cong \mathbb{Z}.$$

There are also many instances where the kernels in Theorem 1.6(1) and Theorem 1.7(1) are positive dimensional; see Examples 4.3 and 4.7, [Kol13a, Example 12] and [BdJ14, Examples 1.31–35].

It is also important to assume in Theorem 1.7 that D be a Cartier divisor. For instance, let $H \subset \mathbb{P}^n$ be a hyperplane and $P \subset \mathbb{P}^n \setminus H$ a finite set of points. Let (v, X) be a cone over the blow-up $B_P \mathbb{P}^n$ with vertex v and $v \in D \subset X$ the cone over H . Then

$$\ker[\mathbf{Pic}^{\text{loc}}(v, X) \xrightarrow{r_D^X} \mathbf{Pic}^{\text{loc}}(v, D)] \cong \mathbb{Z}^{|P|}.$$

If $\text{depth}_x X \geq 3$, then $\mathbf{Pic}^{\text{loc-}\circ}(x, X)$ is trivial and we get the following results.

THEOREM 1.9. *We take the notation and assumptions of Theorems 1.6–1.7. Assume in addition that $\text{depth}_x X \geq 3$. Then*

- (i) *$r_D^X: \mathbf{Pic}^{\text{loc}}(x, X) \rightarrow \mathbf{Pic}^{\text{loc}}(x, D)$ is injective,*
- (ii) *$\ker(\pi^*)$ is finite if $\text{char } k = 0$ and*
- (iii) *the prime-to- p part of $\ker(\pi^*)$ is finite if $\text{char } k = p > 0$.*

1.10 (Numerical criteria for Cartier divisors). Let T be an irreducible, regular, 1-dimensional scheme and $f: X \rightarrow T$ a flat, projective morphism of relative dimension n . Assume for simplicity that f has normal fibers.

Let D be a divisor on X such that $D_t := D|_{X_t}$ is Cartier for every $t \in T$. In general D need not be Cartier. For example, let

$$X = (x^2 - y^2 + z^2 - t^2 = 0) \subset \mathbb{A}_{xyz}^3 \times \mathbb{A}_t \quad \text{and} \quad D = (x - y = z - t = 0).$$

The divisor D is Cartier, except at the origin, where it is not even $\mathbb{Q}$ -Cartier. However, D_0 is a line on a quadric cone, hence $2D_0 = (x - y = 0)$ is Cartier. Thus $2D$ is Cartier on every fiber, but it is not Cartier.

We discuss several criteria in Section 11. The following is an easy-to-state special case of Theorem 11.6.

THEOREM 1.11. *Using the above notation, assume in addition that D_t is ample for every $t \in T$. Then D is a Cartier divisor on X if and only if the self-intersection number (D_t^n) is independent of $t \in T$.*

1.12 (Complex analytic case). Even if $k = \mathbb{C}$, the proofs of Theorems 1.6–1.7 proceed by reduction to positive characteristic. There are nonisolated complex analytic singularities that do not lie on any algebraic variety. Our proof does not apply to them, but the conclusions are hopefully valid.

Over $\mathbb{C}$ one can use the first Chern class to realize $\mathrm{NS}^{\mathrm{loc}}(x, X)$ as a subgroup of $H^2(\mathrm{link}(x, X), \mathbb{Z})$, the second cohomology of the *link* of (x, X) ; see § 5.2. Thus it would be natural to try to prove the analytic case by showing that the kernels of the corresponding maps between these cohomology groups are torsion. This is, however, not true.

Example 5.5 shows a semi-log canonical hypersurface singularity (x, X) of dimension 3 whose normalization $\pi: (\bar{x}, \bar{X}) \rightarrow (x, X)$ is also a hypersurface singularity and for which

$$\ker[\pi^*: H^2(\mathrm{link}(x, X), \mathbb{Z}) \rightarrow H^2(\mathrm{link}(\bar{x}, \bar{X}), \mathbb{Z})] \cong \mathbb{Z}^2.$$

For Theorem 1.7, similar examples are given in [Koll13a, Section 5].

2. Outline of the proofs

The proofs of Theorems 1.4–1.9 have four major components:

- the proof of Theorem 1.7 when X is normal and $\mathrm{char} k > 0$ (This was done in [BdJ14].)
- the proof of Theorem 1.6 over finite fields (This is done in Section 6.)

(These two together imply Theorem 1.4 over finite fields.)

- a lifting argument that derives the general assertions from the finite field cases (This is done in Sections 7–9.)
- the treatment of torsion elements in $\ker r_D^X$. (The nonexistence of prime-to- p torsion follows from [Gro68, XIII.2.1]; see Paragraph 2.5. The p^∞ -torsion is excluded in § 10.4.)

The key to the results is to find a good answer to the following.

Question 2.1. Let (x, X) be a local scheme and L a line bundle on $X \setminus \{x\}$. How can one check if L is in $\mathbf{Pic}^{\mathrm{loc}-\tau}(x, X)$?

In the global case, when Y is a proper scheme over a field k , there is a simple numerical criterion: a line bundle L is in $\mathbf{Pic}^\tau(Y)$ if and only if L has degree 0 on every reduced, irreducible curve $C \subset Y$.

For the local Picard group there are no proper curves to work with and I do not know any similar numerical criterion to identify $\mathbf{Pic}^{\mathrm{loc}-\circ}(x, X)$ or $\mathbf{Pic}^{\mathrm{loc}-\tau}(x, X)$ in general. (If X is normal and it has a resolution of singularities $X' \rightarrow X$, then one can work on X' and use the exceptional curves. This was used in [Bou78] to prove that $\mathrm{NS}^{\mathrm{loc}}(x, X)$ is finitely generated.)

Over a finite field the $\mathbb{F}_q$ -points of a finite-type group scheme form a finite group, which gives the following.

CLAIM 2.2. Let (x, X) be a local scheme over a finite field $\mathbb{F}_q$ that satisfies Condition 1.3, and let L be a line bundle on $X \setminus \{x\}$. Then $L \in \mathbf{Pic}^{\mathrm{loc}-\tau}(x, X)$ if and only if L is torsion.

This leads to a somewhat roundabout way of proving that a line bundle L is in the connected component of a Picard group:

- Reduce everything to finite fields.
- Check that we get torsion line bundles.
- Lift back to the original setting.

The last step is the critical one; let us consider it in more detail for local schemes (x, X) of finite type over a field k . We may assume that k is finitely generated over its prime field, thus we can view k as the function field of an integral scheme S that is of finite type over $\mathbb{Z}$. By a suitable choice of S we may even assume that we have

- (i) a scheme flat and of finite type $X_S \rightarrow S$ with a section $\sigma: S \rightarrow X_S$ and
- (ii) a line bundle L_S on $X_S \setminus \sigma(S)$ such that over the generic fiber we recover (x, X) and L .

The key technical result that we need is the following.

CLAIM 2.3. We have $L \in \mathbf{Pic}^{\text{loc}-\tau}(x, X)$ if and only if the set of closed points

$$\{s \in S \text{ such that } L_S|_{X_s} \text{ is torsion}\}$$

is Zariski dense in S .

This is quite easy to prove if the local Picard groups of the fibers (x_s, X_s) are themselves fibers of a “reasonable” group scheme $\mathbf{Pic}_S^{\text{loc}}(\sigma, X)$. Even in the proper case, such relative Picard groups exist only under some restrictions; see [BLR90, Chapter 8] for a detailed discussion. In the local case, the existence of such a group scheme $\mathbf{Pic}_S^{\text{loc}}(\sigma, X)$ is not known. We prove that, at least after replacing S by a dense open subset of $\text{red } S$, there is a good-enough approximation of $\mathbf{Pic}_S^{\text{loc}-\circ}(\sigma, X)$ to make the rest of the proof work; see Section 9.

Constructing $\mathbf{Pic}_S^{\text{loc}-\circ}(\sigma, X)$ is relatively easy if the generic point $s_g \in S$ has characteristic 0, since then $\mathbf{Pic}^{\text{loc}-\circ}(x_g, X_g)$ is a *smooth* algebraic group. However, in positive characteristic we need to understand the obstruction theory of the Picard functor. A delicate technical point is that the obstruction theory is governed by $H_x^2(X, \mathcal{O}_X)$, which is usually infinite dimensional. A section of a coherent sheaf over S vanishes at the generic point if and only if it vanishes at a Zariski-dense set of closed points, but this is no longer true for quasi-coherent sheaves; see Example 10.1. Thus the general theory does not exclude the possibility that the dimension of $\mathbf{Pic}^{\text{loc}-\circ}(x_s, X_s)$ jumps at every closed point.

Once this issue is settled, we complete the proofs as follows.

Since S is of finite type over $\mathbb{Z}$, the residue fields at closed points are all finite.

Over $\mathbb{F}_q$ we prove Theorem 1.6 for the normalization $\pi: \bar{X} \rightarrow X$ by factoring it as

$$\pi: \bar{X} \xrightarrow{\pi_3} X^{\text{wn}} \xrightarrow{\pi_2} \text{red } X \xrightarrow{\pi_1} X,$$

where X^{wn} is the weak normalization of X . The nontrivial part is the finiteness of the kernel for π_3^* . This is established in Proposition 6.12, using the quotient theory of [Kol12] and Seifert $\mathbb{G}_m$ -bundles as in [Kol13b, 9.53]. Besides proving Theorem 1.6 over $\mathbb{F}_q$, this reduces the proof of Theorem 1.7 in positive characteristic to the case when X is normal. The latter was done in [BdJ14].

For all the theorems, we use Claim 2.3 to pass to arbitrary base fields k .

2.4. Here we show that Theorem 1.6 is implied by the special case when $Y = \bar{X}$. To see this, let $\sigma: \bar{Y} \rightarrow Y$ be the normalization of Y . It is enough to show that the kernel of the composition

$$\mathbf{Pic}^{\text{loc}}(x, X) \xrightarrow{\pi^*} \mathbf{Pic}^{\text{loc}}(y, Y) \xrightarrow{\sigma^*} \mathbf{Pic}^{\text{loc}}(\bar{y}, \bar{Y})$$

is contained in $\mathbf{Pic}^{\mathrm{loc}-\tau}(x, X)$. The map $\sigma^* \circ \pi^*$ is also the composition of

$$\mathbf{Pic}^{\mathrm{loc}}(x, X) \xrightarrow{\pi^*} \mathbf{Pic}^{\mathrm{loc}}(\bar{x}, \bar{X}) \xrightarrow{\bar{\pi}^*} \mathbf{Pic}^{\mathrm{loc}}(\bar{y}, \bar{Y}).$$

We already know that the kernel of π^* is contained in $\mathbf{Pic}^{\mathrm{loc}-\tau}(x, X)$.

If $q: U \rightarrow V$ is a finite surjection between irreducible normal schemes, then for any line bundle L on V we have $L^{\deg U/V} \cong \mathrm{norm}_{U/V} q^* L$. Thus the kernel of $\bar{\pi}^*$ is torsion and so the kernel of $\sigma^* \circ \pi^* = \bar{\pi}^* \circ \pi^*$ is also contained in $\mathbf{Pic}^{\mathrm{loc}-\tau}(x, X)$.

2.5. We take the notation and assumptions of Theorem 1.7. Here we show that once we know that $\ker(r_D^X) \subset \mathbf{Pic}^{\mathrm{loc}-\tau}(x, X)$, the remaining claims about unipotence follow.

Let $L \in \mathbf{Pic}^{\mathrm{loc}}(x, X)$ be a nontrivial line bundle such that $L|_D \cong \mathcal{O}_D$ and $L^m \cong \mathcal{O}_X$ for some $m > 0$ not divisible by $\mathrm{char} k$. Choose m to be the smallest with these properties. Then L determines a degree m , irreducible, cyclic cover $\tau: \tilde{X} \rightarrow X$ that induces a trivial cover on $D \setminus \{x\}$. Thus $\tilde{D} := \tau^{-1}(D)$ is a Cartier divisor such that $\tilde{D} \setminus \{\tilde{x}\}$ has $m > 1$ connected components. By [Gro68, XIII.2.1] this is impossible.

Thus $\ker(r_D^X)$ is torsion free if $\mathrm{char} k = 0$ and contains only p^∞ -torsion if $\mathrm{char} k = p > 0$. Using Lemma 2.6, this implies that once $\ker(r_D^X)$ is known to be of finite type, it is unipotent if $\mathrm{char} k = 0$ and unipotent up to p^∞ -torsion if $\mathrm{char} k = p > 0$. (Note that a finite p^∞ -torsion group is unipotent if $\mathrm{char} k = p > 0$, so $\ker(r_D^X)$ is unipotent unless the p^∞ -torsion part is not finitely generated.)

LEMMA 2.6. *Let G be a locally of finite type, commutative algebraic group over a field k . Assume that G is torsion free if $\mathrm{char} k = 0$ and contains only p^∞ -torsion if $\mathrm{char} k = p > 0$.*

Then G° is unipotent and G/G° is torsion free if $\mathrm{char} k = 0$ and contains only p^∞ -torsion if $\mathrm{char} k = p > 0$.

Proof. A connected commutative algebraic group H has a unique connected subgroup $0 \subset H_l \subset H$ such that H/H_l is an Abelian variety and $H_l \cong H_m + H_u$ is a linear algebraic group, where H_m is multiplicative and H_u is unipotent.

The existence of H_l is called Chevalley's theorem; the first published proof is in [Bar55], see [BSU13] for a modern treatment. The decomposition $H_l \cong H_m + H_u$ is in most books on linear algebraic groups; see for instance [Bor91, Theorem 4.7].

Note that both Abelian varieties and multiplicative groups contain many torsion elements. These lift back to torsion elements in H using the following elementary observation.

Let H be a group and $K \subset H$ a central subgroup. Assume that $\bar{h} \in H/K$ is m -torsion and that K is m -divisible. Then $\bar{h}$ lifts to an m -torsion element $h \in H$. $\square$

3. Definition of local Picard groups

The literature is very inconsistent, there are at least four variants of what authors call the local Picard group.

DEFINITION 3.1 (Local Picard group). Let X be a scheme and $x \in X$ a point. Assume for simplicity that X is excellent and $\mathrm{depth}_x \mathcal{O}_X \geq 2$.

The *local Picard group* $\mathbf{Pic}^{\mathrm{loc}}(x, X)$ is a group whose elements are S_2 sheaves F on some neighborhood $x \in U \subset X$ such that F is locally free on $U \setminus \{x\}$. Two such sheaves give the same element if they are isomorphic over some neighborhood of x . The product is given by the S_2 -hull of the tensor product.

One can also realize the local Picard group as $\mathrm{Pic}(\mathrm{Spec} \mathcal{O}_{x,X} \setminus \{x\})$ or as the direct limit of $\mathrm{Pic}(U \setminus \{x\})$ as U runs through all open Zariski neighborhoods of x . Usually it is necessary to take the limit, for example if X is a smooth curve of genus at least 1.

If X is normal and $X \setminus \{x\}$ is smooth (or locally factorial), then $\mathrm{Pic}^{\mathrm{loc}}(x, X)$ is isomorphic to the divisor class group of $\mathcal{O}_{x,X}$.

In many contexts it is more natural to work with the *étale-local Picard group* $\mathrm{Pic}^{\mathrm{et}-\mathrm{loc}}(x, X) := \mathrm{Pic}(\mathrm{Spec} \mathcal{O}_{x,X}^h \setminus \{x\})$, where $\mathcal{O}_{x,X}^h$ is the Henselization of the local ring $\mathcal{O}_{x,X}$. Alternatively, $\mathrm{Pic}^{\mathrm{et}-\mathrm{loc}}(x, X)$ is the direct limit of $\mathrm{Pic}^{\mathrm{loc}}(x', X')$ as (x', X') runs through all étale neighborhoods of (x, X) . Usually $\mathrm{Pic}^{\mathrm{loc}}(x', X') \rightarrow \mathrm{Pic}^{\mathrm{et}-\mathrm{loc}}(x, X)$ is neither injective nor surjective and it is necessary to take the limit; see Example 4.4.

Even for isolated singularities over $\mathbb{C}$, it is quite hard to understand the relationship between $\mathrm{Pic}^{\mathrm{loc}}(x, X)$ and $\mathrm{Pic}^{\mathrm{et}-\mathrm{loc}}(x, X)$. By [PS94] there are many singularities such that $\mathrm{Pic}^{\mathrm{loc}}(x, X) = 0$, yet $\mathrm{Pic}^{\mathrm{et}-\mathrm{loc}}(x, X)$ is large.

Example 4.4 shows that for some rather simple singularities we always have $\mathrm{Pic}^{\mathrm{loc}}(x, X) \neq \mathrm{Pic}^{\mathrm{et}-\mathrm{loc}}(x, X)$.

DEFINITION 3.2 (Picard group of a local ring). Let (R, m) be a semilocal ring. Assume for simplicity that R is excellent and $\mathrm{depth}_m R \geq 2$. We can define its local Picard group $\mathrm{Pic}^{\mathrm{loc}}(R, m)$ purely algebraically as follows. Its elements are isomorphism classes of finite R -modules M such that $\mathrm{depth}_m M \geq 2$ and that M_r is locally free of rank 1 over R_r for every non-zero-divisor $r \in m$. The product is given by the S_2 -hull of the tensor product.

If (R, m) is the (semi)local ring of a point x (or of a 0-dimensional subscheme) on a scheme X , then $\mathrm{Pic}^{\mathrm{loc}}(R, m) = \mathrm{Pic}^{\mathrm{loc}}(x, X)$.

In particular, $\mathrm{Pic}^{\mathrm{loc}}(x, X)$ does not depend on our choice of the base scheme.

Next, assume that (x, X) is essentially of finite type over (s, S) . Then $k(x)$ is a finitely generated field extension of $k(s)$. Pick a transcendence basis $\bar{t}_1, \dots, \bar{t}_m \in k(x)/k(s)$, lift it back to $t_1, \dots, t_m \in \mathcal{O}_{x,X}$, and localize $\mathcal{O}_{s,S}[t_1, \dots, t_m]$ at the generic point of its intersection with m_x to get (m_R, R) . We can now view $\mathcal{O}_{x,X}$ as an essentially of finite type R -algebra. The advantage is that now $\mathcal{O}_{x,X}/m_x$ is a finite extension of R/m_R . Thus $\mathcal{O}_{x,X}$ is the localization of a finite type R -algebra at a closed point.

That is, in the study of local Picard groups on schemes of finite type, it is sufficient to work with closed points.

DEFINITION 3.3 (Local Picard functor and scheme, [Bou78]). Let k be a field and (x, X) a local, Noetherian k -scheme. Set $U := \mathrm{Spec}_X \mathcal{O}_{x,X} \setminus \{x\}$.

For a local k -algebra A , consider the presheaf

$$A \mapsto \mathrm{Pic}((U \times_k \mathrm{Spec} A)^h),$$

where the superscript h denotes the Henselization. Sheafifying in the étale topology gives the *local Picard functor* $\mathrm{Pic}^{\mathrm{loc}}(x, X)$.

Thus the local Picard functor works with objects $(\pi: (\tilde{x}, \tilde{X}) \rightarrow (x, X), \tilde{L})$, where π is étale and $\tilde{L}$ is a line bundle on $\tilde{X} \setminus \tilde{x}$.

By [Bou78, Theorem 1.ii], if $\mathrm{depth}_x X \geq 2$ and $H^1(U, \mathcal{O}_U) \cong H_x^2(X, \mathcal{O}_X)$ is finite dimensional, then the local Picard functor is represented by a k -group scheme $\mathbf{Pic}^{\mathrm{loc}}(x, X)$ that is locally of finite type. The tangent space of $\mathbf{Pic}^{\mathrm{loc}}(x, X)$ at the identity is naturally isomorphic to $H_x^2(X, \mathcal{O}_X)$.

If $H_x^2(X, \mathcal{O}_X) = 0$, then $\mathbf{Pic}^{\text{loc}}(x, X)$ is essentially the same as $\text{NS}^{\text{loc}}(x, X)$. The interesting case is when $0 < \dim_k H_x^2(X, \mathcal{O}_X) < \infty$. This holds if

- (1.a) $k(x)$ is finite over k ,
- (1.b) X is S_2 and
- (1.c) X is pure of dimension at least 3.

Note that $H_x^2(X, \mathcal{O}_X) = 0$ if and only if $\text{depth}_x \mathcal{O}_X \geq 3$ and that the conditions (1.a–c) are almost necessary; see [Gro68, VIII.2.3].

Algebraic equivalence is given by

- (2.a) a connected k -scheme T with two points t_1, t_2 ,
- (2.b) an étale morphism $\pi: Y \rightarrow X \times T$ such that the injection $\{x\} \times T \hookrightarrow X \times T$ lifts to $\sigma: \{x\} \times T \hookrightarrow Y$ and
- (2.c) a line bundle L_Y on $Y \setminus \sigma(\{x\} \times T)$.

Let a subscript i denote restriction to the fiber over t_i . The two line bundles

$$(\pi_1: (y_1, Y_1) \rightarrow (x, X), L_1) \quad \text{and} \quad (\pi_2: (y_2, Y_2) \rightarrow (x, X), L_2)$$

are declared *algebraically equivalent*. The following elementary property is quite useful.

- (3) Two line bundles $L_1, L_2 \in \mathbf{Pic}^{\text{loc}}(x, X)$ are algebraically equivalent if and only if they are algebraically equivalent after some field extension $K \supset k$.

All line bundles algebraically equivalent to the trivial bundle $\mathcal{O}_X$ form the identity component of $\mathbf{Pic}^{\text{loc}}(x, X)$, denoted by $\mathbf{Pic}^{\text{loc}-o}(x, X)$. As usual, $\mathbf{Pic}^{\text{loc}-\tau}(x, X) \subset \mathbf{Pic}^{\text{loc}}(x, X)$ denotes the union of those components that become torsion elements in $\mathbf{Pic}^{\text{loc}}(x, X)/\mathbf{Pic}^{\text{loc}-o}(x, X)$.

If $H_x^2(X, \mathcal{O}_X)$ is infinite dimensional, then $\mathbf{Pic}^{\text{loc}}(x, X)$ does not exist. However, as long as $\text{depth}_x X \geq 2$, the unit section of $\mathbf{Pic}^{\text{loc}}(x, X)$ is represented by a finitely presented closed immersion, by [Bou78, Theorem 1.i]. We can use this to define the kernel used in Theorem 1.7.

Thus assume that X is S_2 , that $\dim X \geq 4$ and that $x \in D \subset X$ is a Cartier divisor. Set $\tilde{D} := \text{Spec}_D j_* \mathcal{O}_{D \setminus x}$, and let $\tilde{x} \subset \tilde{D}$ be the preimage of x . Then $\text{depth}_{\tilde{x}} \tilde{D} \geq 2$; hence, by [Bou78, Theorem 1.i], there is a subgroup scheme $\mathbf{K}(X|D) \subset \mathbf{Pic}^{\text{loc}}(x, X)$ representing those line bundles that become trivial when restricted to $D \setminus \{x\} \cong \tilde{D} \setminus \{\tilde{x}\}$. This $\mathbf{K}(X|D)$ gives the precise definition of $\ker[r_D^X: \mathbf{Pic}^{\text{loc}}(x, X) \rightarrow \mathbf{Pic}^{\text{loc}}(x, D)]$ used in Theorem 1.7.

DEFINITION 3.4 (Local Néron–Severi group). Let k be a field and (x, X) a local, Noetherian k -scheme such that $\mathbf{Pic}^{\text{loc}}(x, X)$ exists. The quotient

$$\text{NS}^{\text{loc}}(x, X) := \mathbf{Pic}^{\text{loc}}(x, X) / \mathbf{Pic}^{\text{loc}-o}(x, X)$$

is called the *local Néron–Severi group*. The analytic methods show that if k has characteristic 0, then $\text{NS}^{\text{loc}}(x, X)$ is a finitely generated abelian group; see Proposition 5.3. Combining the method of [Bou78] with [dJ96], we see that, even in characteristic $p > 0$, the local Néron–Severi group is finitely generated if X is normal and finitely generated modulo p^∞ -torsion in general. Our computations show that the p^∞ -torsion part has bounded exponent, but say nothing about finite generation. I do not know any examples satisfying Condition 1.3 where $\text{NS}^{\text{loc}}(x, X)$ is not finitely generated.

The above is probably not a completely agreed-upon definition; another candidate is the usually smaller

$$\mathbf{Pic}^{\text{loc}}(x, X) / (\mathbf{Pic}^{\text{loc}}(x, X) \cap \mathbf{Pic}^{\text{loc}-o}(x, X)).$$

3.5 (Comparing Pic and $\mathbf{Pic}$). Let (x, X) be a local, Noetherian scheme over a field k . By definition there is a natural injection

$$\text{Pic}^{\text{loc}}(x, X) \hookrightarrow \mathbf{Pic}^{\text{loc}}(x, X)(k),$$

which is usually not a surjection. If k is algebraically closed, then, essentially by definition,

$$\text{Pic}^{\text{et-loc}}(x, X) \cong \mathbf{Pic}^{\text{loc}}(x, X)(k).$$

If, in addition, X is complete or henselian, then

$$\text{Pic}^{\text{loc}}(x, X) \cong \mathbf{Pic}^{\text{loc}}(x, X)(k).$$

4. Examples of local Picard groups

Here we discuss a series of examples of local Picard groups. They show that the assumptions of Theorems 1.6 and 1.7 are essentially optimal.

Example 4.1. Let $(0, X)$ be a local scheme with an isolated singularity with a resolution $p: Y \rightarrow X$ that is an isomorphism over $X \setminus \{0\}$. Let $\{E_i: i \in I\}$ be the exceptional divisors. There is a natural exact sequence

$$0 \rightarrow \sum_i \mathbb{Z}[E_i] \rightarrow \text{Pic}(Y) \rightarrow \text{Pic}^{\text{loc}}(0, X) \rightarrow 0. \quad (4.1.1)$$

(Left exactness follows from [KM98, 3.39], but there are many other ways to prove it.)

Assume in addition that there is an effective, exceptional divisor E such that $\mathcal{O}_Y(-E)$ is p -ample. (Any resolution obtained by repeatedly blowing up subschemes whose support lies over $\{0\}$ has this property.) Let L denote the line bundle $\mathcal{O}_X(-E)|_E$; it is ample by assumption.

Let nE denote the subscheme of X defined by $\mathcal{O}_X(-nE)$. There is an exact sequence

$$0 \rightarrow L^n \xrightarrow{h \mapsto 1+h} \mathcal{O}_{(n+1)E}^* \rightarrow \mathcal{O}_{nE}^* \rightarrow 1. \quad (4.1.2)$$

Since L is ample, $H^1(E, L^n) = H^2(E, L^n)$ for $n \gg 1$. Thus the natural restriction maps $\text{Pic}((n+1)E) \rightarrow \text{Pic}(nE)$ are isomorphisms for $n \gg 1$. If $\mathcal{O}_X$ is complete, then

$$\text{Pic}(Y) \cong \text{Pic}(nE) \quad \text{for } n \gg 1. \quad (4.1.3)$$

(This probably needs Grothendieck's existence theorem [Gro60, III.5.1.4].) Thus

$$\text{Pic}(0, X) = \text{Pic}(nE) / \sum_i \mathbb{Z}[E_i] \quad \text{for } n \gg 1. \quad (4.1.4)$$

For normal surfaces over a field of characteristic 0, the equalities (4.1.4) are used in [Mum61] to define a local Picard scheme. For $\dim X \geq 3$ it is an alternate way of constructing the local Picard scheme. Note, however, that the scheme structures of the two sides can differ in positive characteristic, as we see next.

Example 4.2 (Cones). Let $W \subset \mathbb{P}_k^N$ be a smooth projective variety and $(0, X)$ the affine cone over W ; see [Kol13b, 3.8] for our conventions on cones. By blowing up the vertex we get a resolution $p: Y \rightarrow X$, where

$$X = \text{Spec}_k \sum_{r \geq 0} H^0(W, \mathcal{O}_W(r)) \quad \text{and} \quad Y = \text{Spec}_W \sum_{r \geq 0} \mathcal{O}_W(r).$$

Thus (4.1.3) suggests that

$$\dim T_0 \mathbf{Pic}(Y) \stackrel{?}{=} H^1(Y, \mathcal{O}_Y) = \sum_{r \geq 0} H^1(W, \mathcal{O}_W(r)). \quad (4.2.1)$$

On the other hand, $\dim T_0 \mathbf{Pic}^{\text{loc}}(0, X)$ equals $H^1(U, \mathcal{O}_U)$, where $U = X \setminus \{0\}$. Since $U = \text{Spec}_W \sum_{-\infty}^{\infty} \mathcal{O}_W(r)$, we see that

$$\dim T_0 \mathbf{Pic}^{\text{loc}}(0, X) = \sum_{-\infty}^{\infty} H^1(W, \mathcal{O}_W(r)). \quad (4.2.2)$$

Comparing this with (4.2.1), we see extra summands for $r < 0$. This is not a problem in characteristic 0, where these vanish by Kodaira's theorem. However, the two formulas can give different answers in positive characteristic.

Next, assume that the characteristic is 0. Then the above considerations give an exact sequence

$$0 \rightarrow \sum_{r > 0} H^1(W, \mathcal{O}_W(r)) \rightarrow \mathbf{Pic}^{\text{loc}}(0, X) \rightarrow \mathbf{Pic}(W)/\mathbb{Z}[\mathcal{O}_W(1)] \rightarrow 0. \quad (4.2.3)$$

Let $H \subset W$ a smooth hyperplane section and $D \subset X$ the cone over it. A similar computation gives an exact sequence

$$0 \rightarrow \sum_{r > 0} H^1(H, \mathcal{O}_H(r)) \rightarrow \mathbf{Pic}^{\text{loc}}(0, D) \rightarrow \mathbf{Pic}(H)/\mathbb{Z}[\mathcal{O}_H(1)] \rightarrow 0. \quad (4.2.4)$$

Thus, we see that the restriction map $\mathbf{Pic}^{\text{loc}}(0, X) \rightarrow \mathbf{Pic}^{\text{loc}}(0, D)$ has a positive-dimensional kernel if the maps $H^1(W, \mathcal{O}_W(r)) \rightarrow H^1(H, \mathcal{O}_H(r))$ are not all injective; cf. [BdJ14, Examples 1.31–35].

Example 4.3 (Weighted cones). In Example 4.2, let $(t = 0)$ be an equation of $H \subset W$. We can view t as a map $X \rightarrow \mathbb{A}^1$. After base change to $t = s^m$ we get a new singularity $(0, X_m)$ containing the same $(0, D)$ as a hyperplane section. Thinking of X as $\text{Spec}_k \sum_{r \geq 0} t^r H^0(W, \mathcal{O}_W(r))$, we now have

$$X_m = \text{Spec}_k \sum_{r \geq 0} s^r H^0(W, \mathcal{O}_W(\lfloor r/m \rfloor)).$$

Computing as in Example 4.2, we get

$$H_0^2(X_m, \mathcal{O}_{X_m}) \cong \sum_{-\infty}^{\infty} H^1(W, \mathcal{O}_W(\lfloor r/m \rfloor)) \cong \bigoplus_1^m H_0^2(X, \mathcal{O}_X).$$

Thus, if $\text{char } k = 0$, then $\dim \mathbf{Pic}^{\text{loc}}(0, X_m) = m \cdot \dim \mathbf{Pic}^{\text{loc}}(0, X)$ and so

$$\dim \ker [\mathbf{Pic}^{\text{loc}}(0, X_m) \rightarrow \mathbf{Pic}^{\text{loc}}(0, D)]$$

grows with m , save when $\mathbf{Pic}^{\text{loc}}(0, X)$ is 0 dimensional.

Example 4.4. We give examples of normal surface singularities $(0, S^h)$ such that $\mathbf{Pic}^{\text{et-loc}}(0, S^h)$ is positive dimensional but $\mathbf{Pic}^{\text{loc}}(0, S)$ is finitely generated for every algebraic model S of S^h .

Let S be a normal, projective surface with a single isolated singularity at $(0, S)$. Assume that $\mathbf{Pic}^{\text{loc-o}}(0, S)$ has no Abelian subvarieties. We claim that the image $\mathbf{Pic}(S \setminus 0) \rightarrow \mathbf{Pic}^{\text{loc}}(0, S)$ is finitely generated.

To see this, let $\pi: T \rightarrow S$ be a resolution of singularities. Then $\text{Pic}(T) \rightarrow \text{Pic}(S \setminus 0)$ is onto and we can factor $\text{Pic}(T) \rightarrow \text{Pic}^{\text{loc}}(0, S)$ as $\text{Pic}(T) \twoheadrightarrow \text{Pic}(S \setminus 0) \rightarrow \text{Pic}^{\text{loc}}(0, S)$. Since $\text{Pic}^\circ(T)$ is an Abelian variety, its image in $\text{Pic}^{\text{loc}}(0, S)$ is also an Abelian variety, thus trivial. Hence $\text{Pic}^\circ(T) \rightarrow \mathbf{Pic}^{\text{loc}-\circ}(0, S)$ is the constant map and so $\text{Pic}(T) \rightarrow \text{Pic}^{\text{loc}}(0, S)$ factors through $\text{NS}(T) \rightarrow \text{Pic}^{\text{loc}}(0, S)$.

Here are some concrete equations with the above properties.

(4.4.1) Cusps, for example $(xyz + x^4 + y^4 + z^4 = 0)$ or $(z^2 = x^2(x^2 + y^2) + y^5)$, have $\mathbf{Pic}^{\text{loc}-\circ}(0, S) \cong \mathbb{G}_m$.

(4.4.2) Let C be an irreducible curve with a single node whose normalization $E := \bar{C}$ is a smooth elliptic curve. There is an extension

$$1 \rightarrow \mathbb{G}_m \rightarrow \text{Pic}^\circ(C) \rightarrow \text{Pic}^\circ(E) \cong E \rightarrow 0.$$

The extension is nonsplit; even more, $\text{Pic}^\circ(C)$ does not contain any compact curves if and only if the difference of the two preimages of the node is nontorsion on $\text{Pic}(E)$. For example, the exceptional divisor of the minimal resolution of $(z^2 = x^2(x^4 + cy^4) + y^7)$ is such a curve; the nontorsion condition holds for very general $c \in \mathbb{C}$.

In the nonnormal case, several new phenomena occur.

Example 4.5 (Nonnormal surfaces). We compute the local Picard group for three of the simplest nonnormal surfaces. Let $S := (xy = 0) \subset \mathbb{A}^3$ be the union of two planes in $\mathbb{A}^3$, let $T := (z^2 = 0) \subset \mathbb{A}^3$ be the double plane in $\mathbb{A}^3$, and let $W := (y^2 - x^3 = 0) \subset \mathbb{A}^3$ be the product of a cuspidal cubic with a line. The schemes S , T and W are all S_2 .

Let $s \in S$ denote the origin; then $\text{depth}_s \mathcal{O}_S = 2$. The scheme S has two irreducible components, $S_x = (x = 0) \cong \mathbb{A}^2$ and $S_y = (y = 0) \cong \mathbb{A}^2$. The normalization $\bar{S}$ is the disjoint union $S_x \amalg S_y$. Thus $\text{Pic}^{\text{loc}}(\bar{s}, \bar{S}) \cong 0$. We claim that

$$\text{Pic}^{\text{loc}}(s, S) \cong \ker[\text{Pic}^{\text{loc}}(s, S) \xrightarrow{\pi^*} \text{Pic}^{\text{loc}}(\bar{s}, \bar{S})] \cong \mathbb{Z}. \quad (4.5.1)$$

To see this, let L be an invertible sheaf on $S \setminus s$. Both $L|_{S_x}$ and $L|_{S_y}$ are trivial; let σ_x and σ_y be generating sections. Then σ_x and σ_y restrict to global sections of L on the punctured z -axis. Their quotient is a regular function on the punctured z -axis; its order of pole or zero at the origin gives the isomorphism $\text{Pic}^{\text{loc}}(s, S) \cong \mathbb{Z}$.

Next, let $t \in T$ denote the origin; then $\text{depth}_t \mathcal{O}_T = 2$. The normalization of T is $\bar{T} = (z = 0) \subset \mathbb{A}^3$. There is an exact sequence

$$0 \rightarrow \mathcal{O}_{\bar{T} \setminus t} \xrightarrow{e} \mathcal{O}_{T \setminus t}^* \rightarrow \mathcal{O}_{\bar{T} \setminus t}^* \rightarrow 1,$$

where $e(g) = 1 + zg$. Taking cohomology and using that $\text{Pic}^{\text{loc}}(t, \bar{T}) = 0$ gives the isomorphism

$$\text{Pic}^{\text{loc}}(t, T) \cong H^1(\bar{T} \setminus t, \mathcal{O}_{\bar{T} \setminus t}^*). \quad (4.5.2)$$

The latter can be naturally identified with $\sum_{m \geq 1} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-m))$. Thus

$$\begin{aligned} \text{Pic}^{\text{loc}}(t, T) &\cong \ker[\text{Pic}^{\text{loc}}(t, T) \xrightarrow{\pi^*} \text{Pic}^{\text{loc}}(\bar{t}, \bar{T})] \\ &\cong (\text{infinite dimensional vector space}). \end{aligned} \quad (4.5.3)$$

Let $w \in W$ denote the origin. We can write W as $\text{Spec } k[t^2, t^3, z]$ with normalization $\bar{W} \cong \text{Spec } k[t, z]$. Let $C \subset W$ denote the z -axis; this is the singular locus of W . There is an exact sequence

$$1 \rightarrow \mathcal{O}_W^* \rightarrow \mathcal{O}_{\bar{W}}^* \xrightarrow{d} \mathcal{O}_C \rightarrow 0,$$

where $d(f_0(z) + f_1(z)t + \cdots) = f_1(z)/f_0(z)$. Then we have the exact sequence

$$H^0(\bar{W} \setminus \{w\}, \mathcal{O}_{\bar{W}}^*) \xrightarrow{d} H^0(C \setminus \{w\}, \mathcal{O}_C) \rightarrow \text{Pic}(W \setminus \{w\}) \rightarrow \text{Pic}(\bar{W} \setminus \{w\}) = 1.$$

Note that d factors as

$$H^0(\bar{W} \setminus \{w\}, \mathcal{O}_{\bar{W}}^*) = H^0(\bar{W}, \mathcal{O}_{\bar{W}}^*) \xrightarrow{d} H^0(C, \mathcal{O}_C) \rightarrow H^0(C \setminus \{w\}, \mathcal{O}_C).$$

This shows that

$$\begin{aligned} \text{Pic}^{\text{loc}}(w \in W) &\cong H_w^1(\mathcal{O}_C) \cong k[z, z^{-1}]/k[z] \\ &\cong (\text{infinite dimensional vector space}). \end{aligned} \tag{4.5.4}$$

Example 4.6. As a slight variation of Example 4.5, let $X \subset \mathbb{A}^{2n+1}$ be the union of the linear spaces $X_1 = (x_1 = \cdots = x_n = 0)$ and $X_2 = (x_{n+2} = \cdots = x_{2n+1} = 0)$. Here $X_1 \cap X_2 \cong \mathbb{A}^1$ and the normalization is $\bar{X} = X_1 \amalg X_2$. Here X is not S_2 if $n \geq 2$, but $\text{depth}_x \mathcal{O}_X = 2$, where $x \in X$ denotes the origin.

As in the previous example, we see that

$$\text{Pic}^{\text{loc}}(x, X) \cong \ker[\text{Pic}^{\text{loc}}(x, X) \xrightarrow{\pi^*} \text{Pic}^{\text{loc}}(\bar{x}, \bar{X})] \cong \mathbb{Z}. \tag{4.6.1}$$

Example 4.7 (Deminormal varieties, [Kol13b, Section 5.1]). Fix a ground field k , and let X_i be cones over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^n$ with vertices v_i . Note that the X_i have rational singularities at the origin and $\text{Pic}^{\text{loc}}(v_i, X_i) \cong \mathbb{Z}$.

Pick distinct points $p_j \in \mathbb{P}^1$, and let $D_{ij} \subset X_i$ be the cone over $\{p_j\} \times \mathbb{P}^n$. Note that $D_{ij} \cong \mathbb{A}^{n+1}$.

The first example, Y_2 , is obtained by gluing X_1 to X_2 using the natural identifications $D_{1j} \cong D_{2j}$ for $j = 1, 2$. Then $\bar{Y}_2 = X_1 \amalg X_2$ and so $\text{Pic}^{\text{loc}}(\bar{v}, \bar{Y}_2) \cong \mathbb{Z}^2$. We claim that

$$\begin{aligned} \text{Pic}^{\text{loc}}(v, Y_2) &\cong k^* + \mathbb{Z}^2 \quad \text{and} \\ \ker[\text{Pic}^{\text{loc}}(v, Y_2) \xrightarrow{\pi^*} \text{Pic}^{\text{loc}}(\bar{v}, \bar{Y}_2)] &\cong k^*. \end{aligned} \tag{4.7.1}$$

The extra k^* is obtained as follows. Set $D_i = D_{i1} + D_{i2} \subset X_i$. Take the trivial bundles $\mathcal{O}_{X_i}$ and an isomorphism

$$\phi^0: \mathcal{O}_{D_1 \setminus v} \cong \mathcal{O}_{D_2 \setminus v}$$

to get an element of $\text{Pic}^{\text{loc}}(v, Y_2)$. Two such isomorphisms give the same line bundle if and only if they differ by multiplication by sections of $H^0(X_i \setminus v, \mathcal{O}_{X_i}^*)$. Since $\dim D_{ij} \geq 2$, the isomorphism ϕ^0 extends to a pair of isomorphisms

$$\phi_j: \mathcal{O}_{D_{1j}} \cong \mathcal{O}_{D_{2j}} \quad \text{for } j = 1, 2.$$

These give isomorphisms

$$\phi_j(v): k \cong k(v_1) \cong k(v_2) \cong k.$$

The quotient $\phi_1(v)/\phi_2(v)$ gives a well-defined element of k^* .

The second example, Y_3 , is obtained by gluing X_1 to X_2 using the natural identifications $D_{1j} \cong D_{2j}$ for $j = 1, 2, 3$. We claim that here the kernel is a direct product:

$$\begin{aligned} \text{Pic}^{\text{loc}}(v, Y_3) &\cong k^{n+1} \times (k^*)^2 + \mathbb{Z}^2 \quad \text{and} \\ \ker[\text{Pic}^{\text{loc}}(v, Y_3) \xrightarrow{\pi^*} \text{Pic}^{\text{loc}}(\bar{v}, \bar{Y}_3)] &\cong k^{n+1} \times (k^*)^2. \end{aligned} \tag{4.7.2}$$

To see this, set $D_i = D_{i1} + D_{i2} + D_{i3} \subset X_i$. The trivial bundles $\mathcal{O}_{X_i}$ can be glued using a section of $H^0(D_i \setminus v, \mathcal{O}_{D_i \setminus v}^*)$. Two such sections give the same line bundle if and only if they differ by multiplication by a section of $H^0(X_i \setminus v, \mathcal{O}_{X_i \setminus v}^*)$. The previous considerations explain the $(k^*)^2$, but now we also get new conditions from sections of the form $1 + (\text{linear functions})$. On D_i , we have $n+1$ independent linear functions on each D_{ij} , giving $3n+3$ independent linear functions all together. On $X_i \subset \mathbb{A}^{2n+2}$ we only have $2n+2$ independent linear functions. This accounts for the k^{n+1} part.

In principle, we could get more conditions by considering $1 + (\text{quadratic functions})$, but it is easy to see that this does not happen for three or four copies of D_{ij} . For more copies, we obtain conditions coming from higher-degree polynomials.

Example 4.8 (Pinch point). Let k be any field and $S := (x^2 = y^2z) \subset \mathbb{A}^3$ the pinch point. Its normalization is the map $\pi: \bar{S} := \mathbb{A}_{uv}^2 \rightarrow S$ given by $(u, v) \mapsto (uv, u, v^2)$. The line $(x = z = 0)$ generates $\text{Pic}^{\text{loc}}(0, S) \cong \mathbb{Z}/2$.

Note that π is a homeomorphism if $\text{char } k = 2$. This suggests that in positive characteristic there may not be a perfect analog of the first Chern class as a mapping to topological cohomology.

A more general version is the following.

Example 4.9. Start with $(\bar{X}, \bar{D}) \cong (\mathbb{A}^n, \mathbb{A}^{n-1})$, and let $\bar{x} \in \mathbb{A}^{n-1}$ be the origin. Let $\tau: \mathbb{A}^{n-1} \rightarrow \mathbb{A}^{n-1}$ be coordinate-wise multiplication by a primitive r th root of unity. Assume that the characteristic does not divide r .

Construct $\pi: \bar{X} \rightarrow X$ by identifying the points in the τ -orbits with one another. Thus X is an affine variety, even Cohen–Macaulay. For $r = 2$ it has only double normal crossing singularities along $D := \pi(\bar{D})$. We claim that

$$\text{Pic}^{\text{loc}}(x, X) = \ker[\text{Pic}^{\text{loc}}(x, X) \xrightarrow{\pi^*} \text{Pic}^{\text{loc}}(\bar{x}, \bar{X})] \cong \mathbb{Z}/r.$$

For any r th root of unity ϵ , the τ -action on $\bar{D}$ can be lifted to $\mathcal{O}_{\bar{D}}$ as $g(x) \mapsto \epsilon \cdot g(\tau^{-1}(x))$. Taking the quotient, we obtain line bundles $L(\epsilon)$ over X corresponding to the r th roots of unity.

Choose coordinates such that $\bar{D} = (z_n = 0)$. A local trivialization of $L(\epsilon)$ would correspond to an invertible function $\phi(z_1, \dots, z_n)$ such that $\phi(\epsilon z_1, \dots, \epsilon z_{n-1}, 0) = \epsilon \phi(z_1, \dots, z_{n-1}, 0)$. This would imply $\phi(0, \dots, 0) = \epsilon \phi(0, \dots, 0)$; thus $\phi(0, \dots, 0) = 0$ if $\epsilon \neq 1$. Therefore ϕ is not invertible near the origin if $n \geq 2$.

5. Analytic local Picard groups

DEFINITION 5.1 (Analytic local Picard groups). Let X be a complex analytic space and $x \in X$ a point. Assume for simplicity that $\text{depth}_x \mathcal{O}_X \geq 2$.

Let $W \subset X$ be the intersection of X with a small (open) ball around x . The *analytic local Picard group* $\text{Pic}^{\text{an-loc}}(x, X)$ can be defined as in Definition 3.1 using (analytic) S_2 sheaves on W . By [Art69], if X is an algebraic variety over $\mathbb{C}$, then there is a natural isomorphism

$$\text{Pic}^{\text{et-loc}}(x, X) \cong \text{Pic}^{\text{an-loc}}(x, X^{\text{an}}). \quad (5.1.1)$$

By [Siu69], if X is S_2 and has pure dimension at least 3, then

$$\text{Pic}^{\text{an}}(W \setminus \{x\}) \cong \text{Pic}^{\text{an-loc}}(x, X). \quad (5.1.2)$$

Note that if $\dim X = 2$, then $\text{Pic}^{\text{an}}(W \setminus \{x\})$ is infinite dimensional but $\text{Pic}^{\text{an-loc}}(x, W)$ is finite dimensional if X is normal.

5.2 (Exponential sequence). Let U be a complex space. Then $\mathrm{Pic}(U) \cong H^1(U, \mathcal{O}_U^*)$ and the exponential sequence

$$0 \rightarrow \mathbb{Z}_U \xrightarrow{2\pi i} \mathcal{O}_U \xrightarrow{\exp} \mathcal{O}_U^* \rightarrow 1$$

gives an exact sequence

$$H^1(U, \mathcal{O}_U) \rightarrow \mathrm{Pic}(U) \xrightarrow{c_1} H^2(U, \mathbb{Z}).$$

Let X be a complex space and $x \in X$ a point, and set $U := X \setminus \{x\}$. A piece of the local cohomology exact sequence is

$$H^1(X, \mathcal{O}_X) \rightarrow H^1(U, \mathcal{O}_U) \rightarrow H_x^2(X, \mathcal{O}_X) \rightarrow H^2(X, \mathcal{O}_X).$$

Thus if X is Stein, then we have an isomorphism

$$H^1(U, \mathcal{O}_U) \cong H_x^2(X, \mathcal{O}_X).$$

For local questions we should replace X by a contractible open neighborhood $x \in W \subset X$. Then $W \setminus \{x\}$ is homotopy equivalent to $\mathrm{link}(x, X)$, which is the intersection of X with a small sphere centered at x . Thus $H^2(W \setminus \{x\}, \mathbb{Z}) = H^2(\mathrm{link}(x, X), \mathbb{Z})$.

Combining the above with [Siu69] and [Art69], we obtain the following well-known result.

PROPOSITION 5.3. *Let (x, X) be a $\mathbb{C}$ -scheme of finite type that is S_2 and has pure dimension at least 3. Then taking the first Chern class gives an exact sequence*

$$0 \rightarrow \mathbf{Pic}^{\mathrm{loc}-\circ}(x, X) \rightarrow \mathbf{Pic}^{\mathrm{loc}}(x, X) \xrightarrow{c_1} H^2(\mathrm{link}(x, X), \mathbb{Z}).$$

This suggests a topological way to proving Theorem 1.6 in characteristic 0: we should prove that the kernel of the pull-back map

$$H^2(\mathrm{link}(x, X), \mathbb{Z}) \rightarrow H^2(\mathrm{link}(\bar{x}, \bar{X}), \mathbb{Z})$$

is torsion, where $\bar{X} \rightarrow X$ denotes the normalization. However, Examples 5.4 and 5.5 show that these maps can have nontorsion kernel, even for hypersurface singularities. We start with a projective example and then we take a cone over it to get a local example.

Example 5.4 (Singular K3 surface). Let $g_4(x_0, x_1, x_2)$ be a general quartic form. Then

$$S := (u^2 = x_0^2 g_4(x_0, x_1, x_2)) \subset \mathbb{P}^3(1, 1, 1, 3)$$

is a K3 surface with a double line $L = (x_0 = u = 0)$. Its normalization $\pi: \bar{S} \rightarrow S$ is the smooth Del Pezzo surface of degree 2

$$\bar{S} := (v^2 = g_4(x_0, x_1, x_2)) \subset \mathbb{P}^3(1, 1, 1, 2).$$

Thus $H^2(\bar{S}, \mathbb{Z}) = \mathrm{Pic}(\bar{S}) \cong \mathbb{Z}^8$.

The preimage of the line L is a smooth elliptic curve $E = (x_0 = 0)$. We claim that

$$\ker[H^2(S, \mathbb{Z}) \xrightarrow{\pi^*} H^2(\bar{S}, \mathbb{Z})] \cong H^1(E, \mathbb{Z}) \cong \mathbb{Z}^2. \quad (5.4.1)$$

To see this, we start with the short exact sequence defining Q ,

$$0 \rightarrow \mathbb{Z}_L \rightarrow \pi_* \mathbb{Z}_E \rightarrow Q \rightarrow 0,$$

which shows that

$$H^1(L, Q) \cong H^1(E, \mathbb{Z}) \quad \text{and} \quad H^2(L, Q) = 0. \quad (5.4.2)$$

The sheaf Q also sits in the short exact sequence

$$0 \rightarrow \mathbb{Z}_S \rightarrow \pi_* \mathbb{Z}_{\bar{S}} \rightarrow Q \rightarrow 0,$$

and from this we get an exact sequence

$$0 \rightarrow H^1(L, Q) \rightarrow H^2(S, \mathbb{Z}) \rightarrow H^2(\bar{S}, \mathbb{Z}) \rightarrow H^2(L, Q) = 0. \quad (5.4.3)$$

Putting (5.4.2) and (5.4.3) together gives (5.4.1).

Example 5.5. Using the previous notation, set

$$X := (u^2 = x_0^2 g_4(x_0, x_1, x_2)) \subset \mathbb{A}^4.$$

Its normalization $\pi: \bar{X} \rightarrow X$ is given as

$$\bar{X} := (v^2 = g_4(x_0, x_1, x_2)) \subset \mathbb{A}^4.$$

Note that X and $\bar{X}$ are weighted homogeneous cones over S and $\bar{S}$, respectively. Thus, we have isomorphisms modulo torsion:

$$\mathrm{Pic}^{\mathrm{loc}}(0, X) \cong_{\mathbb{Q}} \mathrm{Pic}(S)/[\mathcal{O}_S(1)] \quad \text{and} \quad \mathrm{Pic}^{\mathrm{loc}}(0, \bar{X}) \cong_{\mathbb{Q}} \mathrm{Pic}(\bar{S})/[\mathcal{O}_{\bar{S}}(1)].$$

More precisely, set $U := X \setminus \{0\}$ and $\bar{U} := \bar{X} \setminus \{\bar{0}\}$. Note that U (respectively, $\bar{U}$) is a Seifert $\mathbb{C}^*$ -bundle over S (respectively, $\bar{S}$). (See [Kol13b, 9.49–51] for a quick introduction to Seifert bundles in algebraic geometry.) This implies that

$$H^2(\bar{U}, \mathbb{Z}) \cong \mathbb{Z}^7 \quad \text{and} \quad H^2(U, \mathbb{Z}) \cong \mathbb{Z}^9.$$

(The Seifert $\mathbb{C}^*$ -bundle structure gives these modulo torsion. Since both X and $\bar{X}$ are hypersurface singularities, U and $\bar{U}$ are simply connected; thus the above H^2 are torsion free.) Thus π^* is not injective on H^2 and

$$\mathbb{Z}^2 \cong H^1(E, \mathbb{Z}) \cong \ker[H^2(U, \mathbb{Z}) \xrightarrow{\pi^*} H^2(\bar{U}, \mathbb{Z})]. \quad (5.5.1)$$

By contrast, we claim that $\mathrm{Pic}^{\mathrm{loc}}(0, X) = 0$ for very general g_4 . To see this, first note that $\mathbb{Q}$ -Cartier divisors D on S correspond to Cartier divisors $\bar{D}$ on $\bar{S}$ for which $\bar{D}|_E$ is invariant under the Galois involution of $E \rightarrow L$.

Since $\bar{S}$ is a Del Pezzo surface, the pair $(E, \bar{S})$ is obtained from a pair $(E, \mathbb{P}^2)$ by blowing up seven points $P_1, \dots, P_7 \in E$. Thus the image of $\mathrm{Pic}(\bar{S}) \rightarrow \mathrm{Pic}(E)$ is generated by $\mathcal{O}_{\mathbb{P}^2}(1)|_E$ and the $\mathcal{O}_E(P_i)$. For a very general choice of the P_i , these are independent in $\mathrm{Pic}(E)$ and the only Galois-invariant divisor classes are given by the multiples of $\mathcal{O}_{\mathbb{P}^2}(3)|_E \otimes \mathcal{O}_E(-P_1 - \dots - P_7)$. This is also the pull-back of the hyperplane class under the projection $S \rightarrow \mathbb{P}_{\mathbf{x}}^2$. Therefore $\mathrm{Pic}^{\mathrm{loc}}(0, X) = 0$.

6. Schemes over finite fields

We start with the following strengthening of Theorem 1.6 over finite fields.

THEOREM 6.1. *Let k be a finite field and X an excellent k -scheme of pure dimension at least 3 that is S_2 (or at least topologically S_2 ; see Definition 6.8). Let $x \in X$ be a point with finite residue field $k(x)$. Let $\pi: \bar{X} \rightarrow X$ denote the normalization and $\bar{x}$ the preimage of x . Then the kernel of the pull-back map*

$$\pi^*: \mathrm{Pic}^{\mathrm{loc}}(x, X) \rightarrow \mathrm{Pic}^{\mathrm{loc}}(\bar{x}, \bar{X})$$

is torsion.

Note that, by our Examples 4.5–4.7, the assumptions that $\dim X \geq 3$, X be topologically S_2 and $k(x)$ be finite are all necessary.

Proof. We factor the normalization $\pi: \bar{X} \rightarrow X$ as

$$\pi: \bar{X} \xrightarrow{\pi_3} X^{\text{wn}} \xrightarrow{\pi_2} \text{red } X \xrightarrow{\pi_1} X$$

and show that for each step, the kernel of the pull-back map on the local Picard groups is torsion.

For $\pi_1: \text{red } X \rightarrow X$, this is done in Lemma 6.2, and for $\pi_2: X^{\text{wn}} \rightarrow \text{red } X$, this is done in Corollary 6.7. Both of these results are well known and hold in much greater generality. The most delicate part is the finiteness of the kernel for

$$\pi_3^*: \text{Pic}^{\text{loc}}(x^{\text{wn}}, X^{\text{wn}}) \rightarrow \text{Pic}^{\text{loc}}(\bar{x}, \bar{X}).$$

This is established in Proposition 6.12, using the quotient theory of [Kol12] and Seifert $\mathbb{G}_m$ -bundles as in [Kol13b, 9.53]. $\square$

LEMMA 6.2. *Let k be a field of positive characteristic and X a Noetherian k -scheme. Then the kernel of the restriction map $\text{Pic}(X) \rightarrow \text{Pic}(\text{red } X)$ is killed by some power of $\text{char } k$.*

Proof. Let $J \subset \mathcal{O}_X$ be an ideal sheaf such that $J^2 = 0$. Set $X_J := \text{Spec } \mathcal{O}_X/J$. Note that $j \mapsto 1 + j$ identifies J with the kernel of $\mathcal{O}_X^* \rightarrow \mathcal{O}_{X_J}^*$. This gives an exact sequence

$$H^1(X, J) \rightarrow H^1(X, \mathcal{O}_X^*) \rightarrow H^1(X_J, \mathcal{O}_{X_J}^*).$$

Since $H^1(X, J)$ is a k -vector space, it is killed by multiplication by $\text{char } k$.

Using this for the powers of the ideal sheaf defining $\text{red } X$, we see that the kernel of the restriction map $\text{Pic}(X) \rightarrow \text{Pic}(\text{red } X)$ is killed by multiplication by a power of $\text{char } k$. $\square$

The above proof even shows the following stronger result.

COROLLARY 6.3. *Let k be a field of positive characteristic and (x, X) a local Noetherian k -scheme that is S_2 and has pure dimension at least 3. Then the kernel of the restriction map $\text{Pic}^{\text{loc}}(x, X) \rightarrow \text{Pic}^{\text{loc}}(x, \text{red } X)$ is a connected unipotent group scheme.*

DEFINITION 6.4. Let X be a scheme. A *partial weak normalization* is a finite morphism $\pi: X' \rightarrow X$ such that

- (i) X' is reduced,
- (ii) p is a finite, universal homeomorphism and
- (iii) the induced morphism $X' \rightarrow \text{red } X$ is an isomorphism at all generic points.

The scheme X is called *weakly normal* if X is reduced and its sole partial weak normalization is the identity $X \cong X$.

If X is excellent, more generally, if the normalization $\bar{X}$ is finite over X , then there is a unique maximal partial weak normalization $X^{\text{wn}} \rightarrow X$, called the *weak normalization* of X . Note that X^{wn} is weakly normal.

If X has residue characteristic 0, then the weak normalization agrees with the seminormalization. See [Kol96, Section I.7.2] for details.

A weakly normal scheme is S_2 if and only if it is connected in codimension 1 as in [Har62]. Therefore, if X is S_2 , then so is its weak normalization.

For the proof of the next lemma, see [Kol97, Section 6] or [Kol12, Proposition 35].

LEMMA 6.5. *Let k be a field of positive characteristic and $g: Y \rightarrow X$ a finite morphism of Noetherian k -schemes. The following are equivalent:*

- (i) The morphism g is a universal homeomorphism.
- (ii) The scheme Y is dominated by a Frobenius twist $F_q: X_q \rightarrow Y \xrightarrow{g} X$ for some power q of $\text{char } k$.

COROLLARY 6.6. *Let k be a field of positive characteristic and $g: Y \rightarrow X$ a finite, universal homeomorphism of Noetherian k -schemes. Then the kernel of the pull-back map $g^*: \text{Pic}(X) \rightarrow \text{Pic}(Y)$ is killed by some power of $\text{char } k$.*

Proof. By Lemma 6.5, there is a $q = (\text{char } k)^m$ such that F_q factors as

$$F_q: X_q \rightarrow Y \xrightarrow{g} X.$$

Thus it is enough to prove that the kernel of the pull-back map $F_q^*: \text{Pic}(X) \rightarrow \text{Pic}(X_q)$ is killed by q .

If a line bundle L is given by an open cover $\{U_i\}$ and transition functions $\{\phi_{ij}\}$, then F_q^*L can be given by the open cover $\{F_q^{-1}U_i\}$ and transition functions $\{F_q^*\phi_{ij}\}$.

As an abstract scheme, X_q is naturally isomorphic to X and, under this isomorphism, $F_q^*\phi_{ij} = \phi_{ij}^q$. Therefore the Frobenius pull-back F_q^*L is isomorphic to $L^{\otimes q}$ under this isomorphism. Thus, the kernel of $F_q^*: \text{Pic}(X) \rightarrow \text{Pic}(X_q)$ is precisely the set of q -torsion elements. $\square$

Since the weak normalization of an excellent scheme is a finite, universal homeomorphism, as a special case we get the following.

COROLLARY 6.7. *Let k be a field of positive characteristic and X an excellent k -scheme with weak normalization $\tau: X^{\text{wn}} \rightarrow X$. Then the kernel of the pull-back map $\text{Pic}(X) \rightarrow \text{Pic}(X^{\text{wn}})$ is killed by some power of $\text{char } k$.*

In contrast with Corollary 6.3, the kernel of τ^* need not be connected, as shown by Example 4.8.

The last step of the proof uses the concept of topologically S_2 schemes and the theory of finite, set-theoretic equivalence relations developed in [Kol12].

DEFINITION 6.8 (S_2 and topologically S_2). Recall that a scheme X is S_2 if a finite morphism $g: Y \rightarrow X$ is an isomorphism provided that

- (i) there is a closed subset $Z \subset X$ of codimension at least 2 such that g is an isomorphism over $X \setminus Z$ and
- (ii) Y has no associated primes supported in $g^{-1}(Z)$.

Similarly, a scheme X is *topologically S_2* if a finite morphism $g: Y \rightarrow X$ is a finite, universal homeomorphism provided that

- (1') there is a closed subset $Z \subset X$ of codimension at least 2 such that g is a finite, universal homeomorphism over $X \setminus Z$ and
- (2') Y has no irreducible components supported in $g^{-1}(Z)$.

It is not hard to see that a pure-dimensional scheme X is topologically S_2 if and only if the following holds:

- (3') Let $\pi: U \rightarrow X$ be an étale morphism from a connected scheme U and $Z \subset U$ a closed subscheme of codimension at least 2. Then $U \setminus Z$ is connected.

These imply that an S_2 scheme is topologically S_2 , a weakly normal scheme is S_2 if and only if it is topologically S_2 , and the weak normalization of an S_2 scheme is topologically S_2 . We also use that by [Gro68, XIII.2.1] a Cartier divisor in a topologically S_2 scheme is also topologically S_2 .

6.9 (Set-theoretic equivalence relations). (For more details, see [Kol12].)

Let X be an excellent scheme, $\pi: \bar{X} \rightarrow X$ its normalization and R the normalization of $\bar{X} \times_X \bar{X}$. Together with the coordinate projections $\sigma_1, \sigma_2: R \rightrightarrows \bar{X}$ we have a *finite, set-theoretic equivalence relation*.

Let $Q \subset R$ be a closed subscheme such that $\sigma_1|_Q, \sigma_2|_Q: Q \rightrightarrows \bar{X}$ is also a set-theoretic equivalence relation. By [Kol12, Lemma 1.7], the geometric quotient $\bar{X}/Q$ exists and the geometric fibers of $\bar{X} \rightarrow \bar{X}/Q$ are precisely the Q -equivalence classes.

In particular, if $Q = R$, then $\bar{X}/R \rightarrow X$ is a finite, universal homeomorphism. If X is weakly normal, then $\bar{X}/R = X$.

Finally, we study what happens as we go from a weakly normal scheme to its normalization.

PROPOSITION 6.10. *Let k be a field of characteristic $p > 0$ and X an excellent k -scheme of pure dimension at least 3 that is weakly normal and S_2 . Let $\pi: \bar{X} \rightarrow X$ denote the normalization. Let $x \in X$ be a closed point with residue field $k(x)$. Then there are a k -torus $\mathbb{T}$ and a linear representation*

$$I_x: \ker[\mathbf{Pic}^{\text{loc}}(x, X) \xrightarrow{\pi^*} \mathbf{Pic}^{\text{loc}}(\bar{x}, \bar{X})] \rightarrow \mathbb{T}$$

whose kernel is p^∞ -torsion.

Proof. Let L be a line bundle on $X \setminus \{x\}$ such that π^*L is trivial on $\bar{X} \setminus \{\bar{x}\}$.

We can view L as a $\mathbb{G}_m$ -bundle over $X \setminus \{x\}$; thus π^*L is a trivial $\mathbb{G}_m$ -bundle over $\bar{X} \setminus \{\bar{x}\}$. It extends to a trivial $\mathbb{G}_m$ -bundle $\bar{L}$ over $\bar{X}$. Since X is excellent, $\bar{X}$ is finite over X ; hence $\bar{L}$ is of finite type over X .

Let R be the normalization of $\bar{X} \times_X \bar{X}$. Then $\sigma_1, \sigma_2: R \rightrightarrows \bar{X}$ is a set-theoretic finite equivalence relation. Since X is weakly normal, the geometric quotient $\bar{X}/R$ equals X ; see §6.9.

Let $r \subset R$ denote the union of the reduced preimages of $\bar{x}$. Since π^*L is pulled back from X , we have isomorphisms

$$\phi_{R \setminus r}: \sigma_1^*(\pi^*L) \cong \sigma_2^*(\pi^*L) \tag{6.10.1}$$

over $R \setminus r$. By assumption π^*L is trivial, thus $\phi_{R \setminus r}$ can be viewed as an isomorphism of two trivial $\mathbb{G}_m$ -bundles on $R \setminus r$.

Let $R^{[1]} \subset R$ be the union of the irreducible components of dimension at least $\dim X - 1$ and $R^{(1)} \subset R$ the equivalence relation generated by $R^{[1]}$. We will check at the end that $\bar{X}/R = \bar{X}/R^{(1)}$.

Since $R^{[1]}$ is normal and every irreducible component has dimension at least 2, the isomorphism $\phi_{R \setminus r}$ extends to an isomorphism

$$\phi_R^{[1]}: \sigma_1^*(\bar{L}) \cong \sigma_2^*(\bar{L}) \tag{6.10.2}$$

over $R^{[1]}$. The isomorphisms (6.10.2) define a $\mathbb{G}_m$ -equivariant extension of the relation $R^{(1)}$ to a finite relation on $\bar{L}$. Let $\bar{R}_L$ denote the equivalence relation generated by this extension. In general, such an extension is a profinite equivalence relation.

In our case the only nonfiniteness can occur over $x \in X$. For $\bar{x}_i \in \bar{x}$, let $\mathbb{G}_m(\bar{x}_i)$ denote the multiplicative group scheme of the residue field $k(\bar{x}_i)$. For a point $r_\ell \in R^{(1)}$ lying over x , set

$\bar{x}_i := \sigma_1(r_\ell)$ and $\bar{x}_j := \sigma_2(r_\ell)$. (We allow $i = j$.) Then $\phi_R^{[1]}$ gives an isomorphism

$$\phi_{ij\ell}: \mathbb{G}_m(\bar{x}_i) \cong \mathbb{G}_m(\bar{x}_j) \quad (6.10.3)$$

that is defined over $k(r_\ell)$.

It is enough to prove the theorem after a finite field extension. We can thus replace k by the composition of the above $k(\bar{x}_i)$ and $k(r_\ell)$.

Let Γ_x denote the graph whose vertices are the points $\bar{x}_i$. To each r_ℓ we add an edge connecting $\bar{x}_i$ and $\bar{x}_j$. After we fix a base point $\bar{x}_0$, compositions of the above $\phi_{ij\ell}$ define a homomorphism

$$I_x(L): H_1(\Gamma_x, \mathbb{Z}) \rightarrow \mathbb{G}_m(k). \quad (6.10.4)$$

The construction is compatible with field extensions and Henselization, thus we get a homomorphism

$$I_x: \ker[\mathbf{Pic}^{\text{loc}}(x, X) \xrightarrow{\pi^*} \mathbf{Pic}^{\text{loc}}(\bar{x}, \bar{X})] \rightarrow \text{Hom}(H_1(\Gamma_x, \mathbb{Z}), \mathbb{G}_m). \quad (6.10.5)$$

Next, assume that L is in the kernel of I_x . Then $\bar{R}_L$ is a finite equivalence relation. More precisely, each $\bar{R}_L$ -equivalence class over x contains exactly one point of $\mathbb{G}_m(\bar{x}_0)$.

Since $\bar{L}$ is of finite type over X , by [Koll12, Theorem 6] the geometric quotient $\bar{L}/\bar{R}_L$ exists. It is a Seifert $\mathbb{G}_m$ -bundle over $\bar{X}/R^{(1)}$ by [Koll13b, 9.48]. (The statement there assumes that two other conditions (HN) and (HSN) are also satisfied. These are, however, used only to ensure that the geometric quotient $\bar{L}/\bar{R}_L$ exists. In our case existence is assured by [Koll12, Theorem 6], the rest of the proof then works.) By [Koll13b, 9.53], a power $\bar{L}^{\otimes q}$ of $\bar{L}$ descends to a $\mathbb{G}_m$ -bundle over $\bar{X}/R^{(1)}$ for some $q = p^r$.

Finally, let $R^{(2)} \subset R$ denote the union of all irreducible components of dimension at most $\dim X - 2$. Let $Z \subset X$ be the image of $R^{(2)}$. Then Z has codimension at least 2 in X and R agrees with $R^{(1)}$ over $X \setminus Z$. Thus $\bar{X}/R^{(1)} \rightarrow X$ is a finite, universal homeomorphism over $X \setminus Z$. Since X is S_2 , this implies that $\bar{X}/R^{(1)} \rightarrow X$ is a finite, universal homeomorphism over X . By construction, it is an isomorphism at all generic points. Since X is weakly normal, this implies that $\bar{X}/R^{(1)} = X$. Thus $L^{\otimes m}$ extends to a $\mathbb{G}_m$ -bundle over X , hence it is trivial in $\text{Pic}^{\text{loc}}(x, X)$. $\square$

6.11. Note that we have proved that I_x is a homomorphism of algebraic groups, but the proof did not establish that its image is a closed algebraic subgroup of $\mathbb{T}$. (For instance we did not exclude the possibility that $\ker \pi^* \cong \mathbb{Z}$ and I_x is an injection.)

However, once we know that $\ker \pi^*$ in Theorem 1.6 is of finite type, Proposition 6.10 implies that $\ker \pi^*$ is in fact linear.

COROLLARY 6.12. *Let k be a finite field and X an excellent k -scheme of pure dimension at least 3 that is weakly normal and S_2 . Let $x \in X$ be a closed point. Then the kernel of the pull-back map*

$$\pi^*: \text{Pic}^{\text{loc}}(x, X) \rightarrow \text{Pic}^{\text{loc}}(\bar{x}, \bar{X})$$

is torsion.

In contrast with the previous steps, the order of the torsion kernel need not be a power of $\text{char } k$.

Proof. Let L be a line bundle on $X \setminus \{x\}$ such that π^*L is trivial on $\bar{X} \setminus \{\bar{x}\}$. In (6.10.4) we constructed a homomorphism

$$I_x(L): H_1(\Gamma_x, \mathbb{Z}) \rightarrow \mathbb{G}_m(k).$$

If k is finite, then $\mathbb{G}_m(k)$ is a torsion group. Thus $I_x(L^m)$ is the trivial homomorphism for some $m > 0$. By Proposition 6.10 this implies that L^m is p^∞ -torsion; hence L is a torsion element of $\text{Pic}^{\text{loc}}(x, X)$. $\square$

We can now prove the following form of Theorem 1.4 over finite fields.

THEOREM 6.13. *Let k be a finite field, X and Y excellent k -schemes of pure dimension at least 3 that are topologically S_2 and $\pi: Y \rightarrow X$ a finite morphism. Let $x \subset \pi(Y) \subset X$ be a finite subscheme and $y \subset Y$ the preimage of x .*

Assume that $\pi(Y)$ contains the support of a complete intersection subscheme $x \subset Z \subset X$ of dimension at least 3.

Then the kernel of the natural map $\text{Pic}^{\text{loc}}(x, X) \rightarrow \text{Pic}^{\text{loc}}(y, Y)$ consists of torsion elements.

Proof. We use induction on the dimension of X . If $\dim X \leq 3$, then in fact $X = Z$ and so π is a finite surjection. This case is treated in Theorem 6.1 and § 2.4.

Thus, assume $\dim X \geq 4$. If $\tau: Y' \rightarrow Y$ is a finite morphism and $Z \subset (\pi \circ \tau)(Y')$, then the kernel of $\text{Pic}^{\text{loc}}(x, X) \rightarrow \text{Pic}^{\text{loc}}(y', Y')$ contains the kernel of $\text{Pic}^{\text{loc}}(x, X) \rightarrow \text{Pic}^{\text{loc}}(y, Y)$. Thus we may assume that $Y \rightarrow X$ factors through the normalization $\bar{X} \rightarrow X$, that $\pi(Y) = \text{Supp } Z$ and that Y is normal. Using Theorem 6.1 we may thus also assume that X is normal.

Since $Z \subset X$ is a complete intersection, it is also a complete intersection inside some Cartier divisor $D \subset X$. As we noted in Definition 6.8, the divisor D is also topologically S_2 by [Gro68, XIII.2.1]. Using [BdJ14], we can thus replace X by D and finish by induction. $\square$

7. Using the relative Picard group

Theorems 1.4, 1.6 and 1.7 assert that certain line bundles are contained in $\mathbf{Pic}^{\text{loc}-\tau}(x, X)$. We plan to prove such results by first establishing the claim over finite fields and then going back to arbitrary fields. As we already noted in Question 2.1, we need a method to decide when a line bundle on $X \setminus \{x\}$ is contained in $\mathbf{Pic}^{\text{loc}-\tau}(x, X)$.

7.1 (General set-up). Theorems 1.4, 1.6 and 1.7 can be formulated as follows.

Step 0: Starting point. Let $p: (y, Y) \rightarrow (x, X)$ be a finite morphism of local k -schemes and $L \in \text{Pic}^{\text{loc}}(x, X)$ such that π^*L is in $\mathbf{Pic}^{\text{loc}-\tau}(y, Y)$. By passing to a power of L , we may assume that π^*L is in $\mathbf{Pic}^{\text{loc}-\circ}(y, Y)$, that is, π^*L is algebraically equivalent to the trivial bundle.

We would like to prove that, under suitable conditions, $L \in \mathbf{Pic}^{\text{loc}-\tau}(x, X)$.

Step 1: Spreading out. There is an integral $\mathbb{Z}$ -scheme of finite type S such that we have the following.

- (i) There are S -schemes of finite type $Y_S \rightarrow S$ and $X_S \rightarrow S$ with sections $\sigma_Y: S \rightarrow Y_S$ and $\sigma_X: S \rightarrow X_S$.
- (ii) There is a finite morphism $p_S: Y_S \rightarrow X_S$ such that $\sigma_X = p_S \circ \sigma_Y$ and $\text{red } p_S^{-1}(\sigma_X(S)) = \sigma_Y(S)$.
- (iii) There is a line bundle L_S on $X_S \setminus \sigma_X(S)$ such that for every $s \in S$, the restriction $p_S^*L_S|_{Y_s}$ is algebraically equivalent to the trivial bundle on Y_s .
- (iv) There is a map to the generic point $\text{Spec } k \rightarrow S$ such that

$$(p: Y \rightarrow X, L) \cong \text{Spec } k \times_S (p_S: Y_S \rightarrow X_S, L_S).$$

There are several known results that say that certain good properties of the generic fiber are inherited by all fibers, at least over a dense open subset; see [Gro60, IV.12] for long lists. For example, if the generic fiber is S_2 and has pure dimension d , then, possibly after shrinking S , we may assume that every fiber is S_2 and has pure dimension d .

Step 2: Over finite fields. For every closed point $s \in S$ we have a finite morphism $p_s: (y_s, Y_s) \rightarrow (x_s, X_s)$ and an $L_s \in \text{Pic}^{\text{loc}}(x_s, X_s)$ such that $p_s^* L_s$ is algebraically equivalent to the trivial bundle.

This is an instance of the original problem over the residue field $k(s)$. Since S is a $\mathbb{Z}$ -scheme of finite type, these residue fields are all finite. Thus Theorem 6.13 implies that, under our assumptions, $L_s^{m_s}$ is trivial for some $m_s > 0$.

If $m_s = m$ is independent of s , then it is reasonable to expect that $L^m \cong \mathcal{O}_X$. However, this is usually not the case. If $\mathbf{Pic}^{\text{loc}-\circ}(x, X)$ has positive dimension and $\text{char } k = 0$, then there are nontorsion line bundles $L \in \mathbf{Pic}^{\text{loc}-\circ}(x, X)$. For these, the m_s are not even bounded.

Step 3: Lifting: easy case. If $\text{depth}_{x_s} X_s \geq 3$, then [Gro68, XI.3.16] implies that $L^{m_s} \cong \mathcal{O}_X$ and we are done. This happens precisely when $\mathbf{Pic}^{\text{loc}-\circ}(x, X) \cong \text{Spec } k$; thus $\mathbf{Pic}^{\text{loc}-\tau}(x, X)$ is identified with the torsion subgroup of $\text{NS}^{\text{loc}}(x, X)$.

Step 4: Lifting: hard case. At this point we can forget about Y and $\pi: Y \rightarrow X$ in the original set-up. Thus from now on we have only the following data:

- (i) a local k -scheme (x, X) and an $L \in \text{Pic}^{\text{loc}}(x, X)$
- (ii) a finitely generated $\mathbb{Z}$ -subalgebra $A \subset k$ and its spectrum S
- (iii) an S -scheme of finite type $X_S \rightarrow S$ with a section $\sigma_X: S \rightarrow X_S$ (we set $x_s := \sigma_X(s)$ for $s \in S$)
- (iv) a line bundle L_S on $X_S \setminus \sigma_X(S)$ such that $L_s \in \text{Pic}^{\text{loc}}(x_s, X_s)$ is torsion for every closed point $s \in S$
- (v) an isomorphism $(x, X, L) \cong \text{Spec } k \times_S (\sigma_X(S), X_S, L_S)$

The proofs of Theorems 1.4, 1.6 and 1.7 will be completed by the next result.

THEOREM 7.2. *Let S be an integral scheme whose closed points are dense in S . Let $f: X \rightarrow S$ be a flat morphism whose fibers are S_2 and have pure dimension at least 3. Let $\sigma: S \rightarrow X$ be a section and L a line bundle on $X \setminus \sigma(S)$.*

Assume that for a dense set of closed point $s \in S$ there is a natural number $m_s \in \mathbb{N}$ such that $L_s^{m_s} \cong \mathcal{O}_{U_s}$, where $U_s := X_s \setminus \{x_s\}$.

Then $L_{k(S)} \in \mathbf{Pic}^{\text{loc}-\tau}(x_{k(S)}, X_{k(S)})$.

Remark 7.3. It is possible that the assumption of closed points being dense is not necessary. By Proposition 8.5 this holds if the universal deformation in Proposition 8.2 admits an algebraization. I am able to prove only a weaker version of this: an algebraization of $\mathbf{Pic}^{\text{loc}-\circ}$ over the generic fiber gives an algebraization of the universal deformation over an open subset of S . This is why we need to assume that closed points are dense in S , an assumption that always holds in our applications.

8. Formal deformation theory of Pic^{loc}

8.1. Let (s, S) be a local scheme, (y, Y) a local Henselian scheme and $g: Y \rightarrow S$ a flat, affine morphism.

Fix a line bundle L on $Y_s \setminus \{y\}$. Let (A, m_A) be a local Artin $\mathcal{O}_S$ -algebra with residue field $k = A/m_A$. Set $Y_A := Y \times_S \text{Spec } A$ and $U_A := Y_A \setminus \{y\}$.

Let $\text{Def}_S(L, A)$ denote the set of isomorphism classes of line bundles on U_A whose restriction to U_k is isomorphic to (the pull-back of) L .

We check the conditions of [Sch68] for the prorepresentability of the functor $A \mapsto \text{Def}_S(L, A)$. Consider an extension

$$0 \rightarrow M \rightarrow B \rightarrow A \rightarrow 0 \quad (8.1.1)$$

such that $M^2 = 0$. Correspondingly, there is an exact sequence

$$0 \rightarrow M \otimes \mathcal{O}_{U_s} \xrightarrow{m \mapsto 1+m} \mathcal{O}_{U_B}^* \rightarrow \mathcal{O}_{U_A}^* \rightarrow 1. \quad (8.1.2)$$

Pick any $\gamma_A \in H^0(U_A, \mathcal{O}_{U_A}^*)$. If the fibers of f are S_2 , then γ_A extends to a section $\gamma'_A \in H^0(Y_A, \mathcal{O}_{Y_A})$. Since f is affine, we can lift γ'_A to $\gamma'_B \in H^0(Y_B, \mathcal{O}_{Y_B})$ and then restrict to $\gamma_B \in H^0(U_B, \mathcal{O}_{U_B}^*)$. Thus

$$H^0(U_B, \mathcal{O}_{U_B}^*) \rightarrow H^0(U_A, \mathcal{O}_{U_A}^*) \quad (8.1.3)$$

is surjective. Thus the following is a piece of the long exact cohomology sequence of (8.1.2):

$$0 \rightarrow M \otimes H^1(U_s, \mathcal{O}_{U_s}) \rightarrow H^1(U_B, \mathcal{O}_{U_B}^*) \rightarrow H^1(U_A, \mathcal{O}_{U_A}^*) \xrightarrow{\text{obs}} M \otimes H^2(U_s, \mathcal{O}_{U_s}),$$

where obs is called the *obstruction map*. If (8.1.1) splits, then the liftings form a principal homogeneous space under $M \otimes H^1(U_s, \mathcal{O}_{U_s})$; the latter is independent of A . Furthermore, if $\dim Y_s \geq 3$, then $H^1(U_s, \mathcal{O}_{U_s})$ is finite dimensional by [Gro68, VIII.2.3].

It remains to understand liftings to $D := B \times_A C$, where $B \twoheadrightarrow A$ and $C \rightarrow A$ are maps of Artin $\mathcal{O}_S$ -algebras. A line bundle L_D on U_D is determined by a line bundle L_B on U_B , a line bundle L_C on U_C plus an isomorphism $\phi: L_C|_{U_A} \cong L_B|_{U_A}$. Any different isomorphism is given by $\gamma_A \cdot \phi$, where $\gamma_A \in H^0(U_A, \mathcal{O}_{U_A}^*)$. By (8.1.3) one can lift γ_A to $\gamma_B \in H^0(U_B, \mathcal{O}_{U_B}^*)$. Thus instead of changing ϕ by γ_A , we can change L_B by the isomorphism $\gamma_B: L_B \rightarrow L_B$ to conclude that L_D does not depend on ϕ . That is,

$$\text{Def}_S(L, B \times_A C) \cong \text{Def}_S(L, B) \times_{\text{Def}_S(L, A)} \text{Def}_S(L, C).$$

We have thus proved the following.

PROPOSITION 8.2. *Let (s, S) be a local scheme, (y, Y) a local Henselian scheme and $g: Y \rightarrow S$ a flat morphism with S_2 fibers of dimension at least 3. Let L be a line bundle on $Y_s \setminus \{y\}$. Then $A \mapsto \text{Def}_S(L, A)$ is prorepresentable by a complete, local, Noetherian $\mathcal{O}_S$ -algebra $\mathbf{Def}_S(L)$.*

DEFINITION 8.3. Let S be a (nonlocal) scheme and $f: X \rightarrow S$ a flat morphism with S_2 fibers of pure dimension at least 3. Let $\sigma: S \rightarrow X$ be a section. For each $s \in S$, we are especially interested in two deformation functors.

(8.3.1) $\mathbf{Def}_{(s, S)}(\mathcal{O}_{X_s})$ is obtained by applying Proposition 8.2 to the localization (s, S) of S at s and the Henselization (y, Y) of X at $\sigma(s)$. This is an $\mathcal{O}_{s, S}$ -algebra parametrizing deformations (over Artin algebras over $\mathcal{O}_{s, S}$) of the trivial line bundle $\mathcal{O}_{X_s}$ (pulled back to $Y_s \setminus \{y\}$).

(8.3.2) $\mathbf{Def}_{k(s)}(\mathcal{O}_{X_s})$ is obtained by applying Proposition 8.2 to $S := \{s\}$ and the Henselization (y, Y) of the fiber X_s at $\sigma(s)$. This is a $k(s)$ -algebra parametrizing deformations (over Artin algebras over the residue field $k(s)$) of the trivial line bundle $\mathcal{O}_{X_s}$ (pulled back to $Y_s \setminus \{y\}$).

Note that

$$\mathbf{Def}_{k(s)}(\mathcal{O}_{X_s}) \cong \mathbf{Def}_{(s, S)}(\mathcal{O}_{X_s}) \otimes_S k(s). \quad (8.3.3)$$

DEFINITION 8.4 (Universal families). Let S be a scheme, $f: X \rightarrow S$ a flat morphism with S_2 fibers and $\sigma: S \rightarrow X$ a section. A *family* in $\mathbf{Pic}^{\text{loc}}$ over P is given by

- (i) a morphism $p: P \rightarrow S$,
- (ii) an étale morphism $g: Y \rightarrow X \times_S P$,
- (iii) a lifting of the closed subscheme $(\sigma, 1_P): P = S \times_S P \hookrightarrow X \times_S P$ to $\sigma_P: P \rightarrow Y$ and
- (iv) a line bundle L_Y on $Y \setminus \sigma_P(P)$.

For a point $z \in P$, the corresponding line bundle on $Y_{p(z)} \setminus \sigma_P(z)$ is denoted by L_z .

These data give a *finite-type family* in $\mathbf{Pic}^{\text{loc}}$ if P is of finite type over S .

The above family is a *deformation of the trivial bundle* if, in addition,

- (v) there is a section $\rho: S \rightarrow P$ such that the restriction of L_Y to $Y_\rho \setminus \sigma_P(P)$ is isomorphic to the structure sheaf, where $Y_\rho := g^{-1}(X \times_S \rho(S))$.

For every $s \in S$ we can localize at s and get a deformation of $\mathcal{O}_{X_s}$ as in (8.1).

We say that the above family is *universal* at a line bundle L_s on $X_s \setminus \sigma(s)$ corresponding to a point $z \in P_s$ if

- (vi) the induced map

$$\mathbf{Def}_{(s,S)}(L_s) \rightarrow \hat{\mathcal{O}}_{z,P} \quad (8.4.6)$$

is an isomorphism.

It is probably not necessary, but for technical reasons it is easier to insist that a universal family satisfy the following:

- (vii) The morphism $\text{red } p: \text{red } P \rightarrow \text{red } S$ is smooth.

This always holds in characteristic 0. In positive characteristic it can be achieved after a purely inseparable, dominant (but not necessarily finite) base change $S' \rightarrow S$ and by passing to an open subset $P' \subset P$.

PROPOSITION 8.5. *Let (s, S) be an integral, local scheme with function field K . Let $f: X \rightarrow S$ be a flat morphism with S_2 fibers, $\sigma: S \rightarrow X$ a section and L a line bundle on $X \setminus \sigma(S)$. Assume that*

- (i) *there is a finite type family as in Definition 8.4(1)–(7) that is universal at $\mathcal{O}_{X_s}$ and*
- (ii) *$L_s \cong \mathcal{O}_{X_s}$.*

Then $L_K \in \mathbf{Pic}^{\text{loc-}\circ}(x_K, X_K)$.

Proof. For every Artin $\mathcal{O}_{s,S}$ -algebra A , the line bundle L defines a deformation of $\mathcal{O}_{X_s}$ over A since $L_s \cong \mathcal{O}_{X_s}$. This gives a formal deformation of $L_s \cong \mathcal{O}_{X_s}$ over $\hat{S}$, the completion of S at s .

By assumption $\mathbf{Def}_{(s,S)}(\mathcal{O}_{X_s}) \cong \hat{\mathcal{O}}_{\rho(s),P}$, thus there is a section $u: \hat{S} \rightarrow P$ such that $(u^* L_Y)|_{X_A}$ is isomorphic to $L|_{X_A}$ for every Artin $\mathcal{O}_{s,S}$ -algebra A . Proposition 8.6 then implies that $u^* L_Y \cong L|_{\hat{S}}$. Since S is integral, the section u factors through $\text{red } P$.

Now we use that $\text{red}: \text{red } P \rightarrow \text{red } S$ is smooth (Definition 8.4(7)). Thus $\hat{\mathcal{O}}_{\rho(s), \text{red } P}$ is a formal power series ring over $\hat{\mathcal{O}}_{s,S}$ and so each formal section of $\text{red } p$ can be given by an ideal $(t_1 - g_1, \dots, t_m - g_m)$, where $g_i \in \hat{\mathcal{O}}_{s,S}$. Thus any two sections are algebraically equivalent. This implies that L_K and $\mathcal{O}_{X_K}$ are algebraically equivalent over $k(\hat{S})$, hence also over K by (3.3.3). $\square$

PROPOSITION 8.6. *Let (s, S) be a local scheme with maximal ideal m . Let $f: X \rightarrow S$ be a scheme, flat over S with S_2 -fibers. Let $X_n := \operatorname{Spec}_X \mathcal{O}_X/m^{n+1}\mathcal{O}_X$ be the n th infinitesimal neighborhood of $X_0 := X_s$. Let $Z \subset X$ be a subscheme that is finite over S and $j: X \setminus Z \hookrightarrow X$ and $j_n: X_n \setminus Z_n \hookrightarrow X_n$ the natural injections. Let L be an invertible sheaf on $X \setminus Z$ and $L_n := L|_{X_n \setminus Z_0}$. Assume that one of the following holds:*

- (i) *The line bundle $(j_n)_*(L_n)$ is locally free for every $n \geq 0$.*
- (ii) *The line bundle $(j_0)_*(L_0)$ is locally free and $R^1(j_0)_*(L_0) = 0$.*

Then j_*L is invertible in a neighborhood of Z_0 .

Proof. We may assume that $\mathcal{O}_S$ is m -adically complete and, possibly after passing to a smaller neighborhood of Z_0 , we may assume that f is affine and $(j_0)_*(L_0) \cong \mathcal{O}_{X_0}$. For every n we have an exact sequence

$$0 \rightarrow (m_0^n/m_0^{n+1}) \otimes L_0 \rightarrow L_n \rightarrow L_{n-1} \rightarrow 0.$$

Pushing it forward, we get an exact sequence

$$\begin{aligned} 0 \rightarrow (m_0^n/m_0^{n+1}) \otimes (j_0)_*(L_0) &\rightarrow (j_n)_*(L_n) \xrightarrow{r_n} (j_{n-1})_*(L_{n-1}) \rightarrow \\ &\rightarrow (m_0^n/m_0^{n+1}) \otimes R^1(j_0)_*(L_0). \end{aligned}$$

If $(j_n)_*(L_n)$ is locally free, then so is its restriction to X_{n-1} , and r_n gives a map of locally free sheaves

$$\bar{r}_n: (j_n)_*(L_n)|_{X_{n-1}} \rightarrow (j_{n-1})_*(L_{n-1})$$

that is an isomorphism on $X_{n-1} \setminus Z_{n-1}$. Since $\operatorname{depth}_{Z_{n-1}} X_{n-1} \geq 2$, this implies that $\bar{r}_n$ is an isomorphism and so r_n is surjective. The vanishing of $R^1(j_0)_*(L_0)$ also implies that r_n is surjective. Thus each $(j_n)_*(L_n)$ is locally free along X_n , and the constant 1 section of $(j_0)_*(L_0) \cong \mathcal{O}_{X_0}$ lifts back to a nowhere zero global section of $\varprojlim (j_n)_*(L_n)$. Hence $\varprojlim (j_n)_*(L_n) \cong \mathcal{O}_X$.

Furthermore, we have a natural map $j_*L \rightarrow \varprojlim (j_n)_*(L_n) \cong \mathcal{O}_X$ that is an isomorphism on $X \setminus Z$. Since $\operatorname{depth}_Z j_*L \geq 2$, this implies that $j_*L \cong \mathcal{O}_X$. $\square$

The examples below show that going from formal triviality of deformations to actual triviality is not automatic.

Example 8.7. Let $(e, E) \cong (e, E')$ be an elliptic curve. Set $X := (E \setminus \{e\}) \times E'$, and let $p: X \rightarrow E'$ be the second projection. Let $\Delta \subset X$ be the diagonal and $L = \mathcal{O}_X(\Delta)$.

For $p \in E' \setminus \{e\}$, the line bundle $L|_{X_p}$ is a nontrivial element of

$$\operatorname{Pic}(X_p \setminus \{e\}) \cong \operatorname{Pic}(E \setminus \{e\}) \cong \mathbf{Pic}^\circ(E),$$

but $L|_{X_e}$ is trivial.

For $m \in \mathbb{N}$, let $X_m \subset X$ denote the m th infinitesimal thickening of the fiber $X_1 := X_e$. We have exact sequences

$$H^1(X_1, \mathcal{O}_{X_1}) \rightarrow H^1(X_{m+1}, \mathcal{O}_{X_{m+1}}^*) \rightarrow H^1(X_m, \mathcal{O}_{X_m}^*) \rightarrow H^2(X_1, \mathcal{O}_{X_1}).$$

Since $X_1 \cong E \setminus \{e\}$ is affine, this shows that

$$\operatorname{Pic}(X_m \setminus \{e\}) \cong \operatorname{Pic}(E \setminus \{e\}) \cong \mathbf{Pic}^\circ(E).$$

Thus $L|_{X_m}$ is trivial for every m .

Example 8.8. Consider the family of smooth, affine surfaces

$$S := (x^2 + y^2 = 1 + t^2 z^2) \subset \mathbb{A}_{xyz}^3 \times \mathbb{A}_t^1.$$

Set $D := (x - 1 = y - tz = 0)$ and $L := \mathcal{O}_S(D)$.

The surface S_t is a hyperboloid for $t \neq 0$; thus $\text{Pic}(S_t) \cong \mathbb{Z}$ is generated by L_t . For $t = 0$ we get a cylinder and $\text{Pic}(S_t) \cong \mathbb{Z}/2$ is generated by L_0 . As in the previous example, we see that L^2 is trivial on all infinitesimal neighborhoods of S_0 .

9. Existence of universal families

9.1. Assume that we have a field K that is finitely generated over its prime field and a local scheme (x, X) of finite type over K . Write $P_K := \mathbf{Pic}^{\text{loc-}\circ}(x, X)$.

As in Definition 8.4(1–4), there is a universal family

$$(Y_K \xrightarrow{g_K} X_K \times_{P_K \rightarrow P_K, L_{Y_K}}). \quad (9.1.1)$$

Everything in (9.1.1) can be defined over a finitely generated subring of $A \subset K$; thus there is an integral scheme S of finite type over $\text{Spec } \mathbb{Z}$ such that (9.1.1) is the generic fiber of a family

$$(Y_S \xrightarrow{g_S} X_S \times_{P_S \rightarrow P_S, L_{Y_S}}). \quad (9.1.2)$$

The statement and the proof of the next result closely follow [Art74]. I go through the details for two reasons. I always found [Art74] rather concise and, more importantly, not all the assumptions of [Art74] are satisfied in our case. There are two main differences. The automorphism groups of our objects are all infinite dimensional, but, as was already observed in [Bou78], this does not seem to cause any problems. A more difficult point is that the obstruction spaces are also infinite dimensional.

For the knowledgeable reader, Theorem 10.3 is the only part not contained in [Art74, Bou78].

THEOREM 9.2 (Openness of universality). *Using the above notation, there is a dense open subset $T \subset S$ such that*

$$(Y_T \xrightarrow{g_T} X_T \times_{P_T \rightarrow P_T, L_{Y_T}})$$

is everywhere universal (as in Definition 8.4).

Remark 9.3. One can imagine that

$$P_T \quad \text{“=”} \quad \mathbf{Pic}^{\text{loc-}\circ}(\sigma_T, X_T),$$

but we do not claim this. The main reason is that there are families of line bundles L over $X_S \setminus \sigma(S)$ such that $L_s \in \mathbf{Pic}^{\text{loc-}\circ}(x_s, X_s)$ at the generic point but not at some special points. Consider for example the family

$$X_s := (xy = uv(u + v + s)) \subset \mathbb{C}^4 \quad \text{and} \quad D_s := (x = u = 0) + (x = v = 0).$$

For $s \neq 0$, we see that $D_s \sim (x = 0)$ is trivial in $\text{Pic}^{\text{loc}}(0, X_s)$. For $s = 0$ we can use [Kol91, 2.2.7] to see that $\text{Pic}^{\text{loc}}(0, X_0) \cong \mathbb{Z}^2$ with $(x = u = 0)$ and $(x = v = 0)$ as generators. Thus D_0 gives a nontorsion element in $\text{Pic}^{\text{loc}}(0, X_0) = \text{NS}^{\text{loc}}(0, X_0)$. I do not know whether such points $s \in S$ can be Zariski dense or not.

Thus P_T should be viewed as an open neighborhood of the zero section in the (possibly nonexistent) $\mathbf{Pic}^{\text{loc-}\circ}(\sigma_T, X_T)$.

Proof. By generic flatness we may assume that $P \rightarrow S$ is flat.

First, we prove that $\mathbf{Def}_{(s,S)}(\mathcal{O}_{Y_s}) \rightarrow \hat{\mathcal{O}}_{\rho(s),P}$ is an isomorphism if and only if the map between the fibers over the points $\mathbf{Def}_{k(s)}(\mathcal{O}_{Y_s}) \rightarrow \hat{\mathcal{O}}_{\rho(s),P_s}$ is an isomorphism. This is completely general and follows from Lemma 9.6.

Consider the tangent map

$$t_{P_K/K}: T_{\rho(K),P_K} \rightarrow R^1 j_K \mathcal{O}_{U_K}$$

defined in (9.4.5). By assumption $t_{P_K/K}$ is an isomorphism. We prove in Proposition 9.5 that the tangent maps

$$t_{P_s/S}: T_{\rho(s),P_s} \rightarrow R^1 j_s \mathcal{O}_{U_s}$$

are isomorphisms for all s in a Zariski open subset of S .

If $P \rightarrow S$ is smooth, then a simple algebra result (Lemma 9.7) shows that $\mathbf{Def}_{k(s)}(\mathcal{O}_{Y_s}) \rightarrow \hat{\mathcal{O}}_{\rho(s),P_s}$ is an isomorphism and we are done. If $\mathbf{Pic}^{\text{loc-}\circ}(x_K, X_K)$ is a smooth group scheme, then, possibly after shrinking S , we may assume that $P \rightarrow S$ is smooth. A group scheme over a field of characteristic 0 is always smooth, thus the proof of Theorem 9.2, and hence also the proofs of Theorems 1.4, 1.6 and 1.7, are complete if $\text{char } k = 0$.

Otherwise we need a more detailed study of obstruction theory; this is started in §9.8.

9.4 (Tangent map). We continue with the notation of §9.1. Let $I_{S,P} \subset \mathcal{O}_P$ be the ideal sheaf of $\rho(S) \subset P$. We identify S with $\rho(S)$ and set $R := \text{Spec}_P \mathcal{O}_P / I_{S,P}^2$. The ideal sheaf of $S \subset R$ is denoted by I_S . These data are encoded in a diagram

$$\begin{array}{ccc} Y_S & \hookrightarrow & Y_R \\ \downarrow & & g \downarrow \uparrow \sigma \\ S & \hookrightarrow & R \\ \parallel & & \downarrow \\ S & = & S \end{array} \quad \begin{array}{l} \text{where } g \text{ is flat,} \\ \sigma \text{ is a section and} \\ I_S^2 = 0. \end{array} \quad (9.4.1)$$

Set $U_R := Y_R \setminus \sigma(R)$ with natural injection $j: U_R \hookrightarrow Y_R$ and $U_S := Y_S \setminus \sigma(S)$. Let L_R be a line bundle on U_R and $L_S := L_R|_{U_S}$. There is an exact sequence

$$0 \rightarrow I_S \otimes_S L_S \rightarrow L_R \rightarrow L_S \rightarrow 0. \quad (9.4.2)$$

Pushing it forward by j , we get

$$0 \rightarrow I_S \otimes_S j_* L_S \rightarrow j_* L_R \rightarrow j_* L_S \xrightarrow{\partial} I_S \otimes_S R^1 j_* L_S. \quad (9.4.3)$$

Now, assume that $L_S \cong \mathcal{O}_{U_S}$ and that $Y_S \rightarrow S$ has S_2 fibers. Then $j_* L_S \cong \mathcal{O}_{Y_S}$ and the exact sequence becomes

$$0 \rightarrow I_S \otimes_S \mathcal{O}_{Y_S} \rightarrow j_* L_R \rightarrow \mathcal{O}_{Y_S} \xrightarrow{\partial} I_S \otimes_S R^1 j_* \mathcal{O}_{Y_S}. \quad (9.4.4)$$

Here ∂ factors through $\mathcal{O}_{Y_S} \rightarrow \mathcal{O}_Z \cong \mathcal{O}_S$; thus, if I_S is locally free over $\mathcal{O}_S$, then ∂ is equivalent to the *tangent map*

$$t_{R/S}: \mathcal{H}om_S(I_S, \mathcal{O}_S) \rightarrow R^1 j_* \mathcal{O}_{Y_S}. \quad (9.4.5)$$

(Note that $\mathcal{H}om_S(I_S, \mathcal{O}_S)$ is isomorphic to the relative tangent sheaf of R/S restricted to $S \cong \sigma(S)$.)

If we start with a set-up as in §9.1, then $R^1 j_* \mathcal{O}_{Y_S} \cong R^1 j_* \mathcal{O}_{X_S}$ and we get the following.

PROPOSITION 9.5. *Let $(Y_S \xrightarrow{g_S} X_S \times P_S \rightarrow P_S, S \xrightarrow{\rho_S} P_S, L_{Y_S})$ be a deformation as in (9.1.2). Assume that $I_{S,P}/I_{S,P}^2$ is free over S and that $R^1 j_* \mathcal{O}_{X_S}$ is free and commutes with base change. Then the tangent map*

$$t_{P/S}: \operatorname{Hom}_S(I_{S,P}, \mathcal{O}_S) \rightarrow R^1 j_* \mathcal{O}_{X_S}$$

has constant rank over a dense open subset of S .

We have used the following commutative algebra lemmas.

LEMMA 9.6. *Let (m, S) be a local ring and $\phi: (m_1, R_1) \rightarrow (m_2, R_2)$ a map of local S -algebras.*

- (i) *If R_1 is complete and $\bar{\phi}: R_1/mR_1 \rightarrow R_2/mR_2$ is surjective, then ϕ is surjective.*
- (ii) *If ϕ is surjective, $\bar{\phi}$ is an isomorphism and R_2 is flat over S , then ϕ is an isomorphism.*

Proof. For every $r \geq 1$, we have a commutative diagram

$$\begin{array}{ccc} (m^r/m^{r+1}) \otimes R_1/mR_1 & \twoheadrightarrow & (m^r/m^{r+1}) \otimes R_2/mR_2 \\ \downarrow & & \downarrow \\ m^r R_1/m^{r+1} R_1 & \xrightarrow{a_r} & m^r R_2/m^{r+1} R_2. \end{array}$$

The vertical arrows are surjective; hence so is a_r . By induction on r we obtain that the maps

$$\bar{\phi}_r: R_1/m^{r+1} R_1 \rightarrow R_2/m^{r+1} R_2$$

are surjective. Since R_1 is complete, we can pass to the inverse limit to conclude that the maps

$$R_1 = \varprojlim R_1/m^{r+1} R_1 \rightarrow \varprojlim R_2/m^{r+1} R_2$$

are surjective. This factors through the injection $R_2 \hookrightarrow \varprojlim R_2/m^{r+1} R_2$; thus $\phi: R_1 \rightarrow R_2$ is surjective.

For part (2), let J be the kernel of ϕ . We have an exact sequence

$$0 \rightarrow J \rightarrow R_1 \xrightarrow{\phi} R_2 \rightarrow 0.$$

Since R_2 is flat over S , tensoring with $S/m_S S$ is also exact, thus we get

$$0 \rightarrow J/m_S J \rightarrow R_1/m_S R_1 \xrightarrow{\bar{\phi}} R_2/m_S R_2 \rightarrow 0.$$

This implies $J/m_S J = 0$; thus $J = 0$ by the Nakayama lemma. $\square$

LEMMA 9.7. *Let (R, m) be a complete local ring and (S, n) a regular, local (R, m) -algebra. Then $R = S$ if and only if the natural maps $R/m \rightarrow S/n$ and $m/m^2 \rightarrow n/n^2$ are isomorphisms.*

Proof. If $R = S$, then clearly $R/m \rightarrow S/n$ and $m/m^2 \rightarrow n/n^2$ are isomorphisms.

Conversely, assume that $R/m \rightarrow S/n$ and $m/m^2 \rightarrow n/n^2$ are isomorphisms. By induction on r we see that the natural maps $R/m^r \rightarrow S/n^r$ are surjective. Among local rings (A, m_A) with fixed embedding dimension $\dim_{A/m} m_A/m_A^2$, the length of S/n^r is the largest possible since (S, n) is regular. Thus each $R/m^r \rightarrow S/n^r$ is an isomorphism. Since (R, m) is complete, this implies that $R = S$. $\square$

9.8 (Obstruction map). Let S be a base scheme and $R \rightarrow S$ a flat scheme with a section $\rho: S \rightarrow R$ with ideal sheaf $I_S \subset \mathcal{O}_R$. Let $T \subset R$ be a subscheme with ideal sheaf I_T . Assume $\rho(S) \subset T$ and $I_S I_T = 0$.

Let $g: Y_R \rightarrow R$ be a flat morphism with a section $\sigma: R \rightarrow Y_R$. By restriction, we get $Y_S \rightarrow S$ and $Y_T \rightarrow T$. These data are summarized in the following diagram:

$$\begin{array}{ccccc} Y_S & \hookrightarrow & Y_T & \hookrightarrow & Y_R \\ \downarrow & & \downarrow & & g \downarrow \uparrow \sigma \\ S & \hookrightarrow & T & \hookrightarrow & R \\ \parallel & & \downarrow & & \downarrow \\ S & = & S & = & S \end{array} \quad \begin{array}{l} \text{where } g \text{ is flat,} \\ \sigma \text{ is a section and} \\ I_S I_T = 0. \end{array} \quad (9.8.1)$$

Finally, write $U_R := Y_R \setminus \sigma(R)$, $U_T := Y_T \setminus \sigma(T)$ and $U_S := Y_S \setminus \sigma(S)$, and let L_T be a line bundle on U_T . We would like to understand when L_T extends to a line bundle L_R on U_R .

We have an exact sequence

$$0 \rightarrow I_T \otimes_S \mathcal{O}_{U_S} \xrightarrow{m \mapsto 1+m} \mathcal{O}_{U_R}^* \rightarrow \mathcal{O}_{U_T}^* \rightarrow 1. \quad (9.8.2)$$

This gives

$$R^1 j_* \mathcal{O}_{U_R}^* \rightarrow R^1 j_* \mathcal{O}_{U_T}^* \xrightarrow{\partial} I_T \otimes_S R^2 j_* \mathcal{O}_{U_S}. \quad (9.8.3)$$

The line bundle L_T corresponds to a section $\mathcal{O}_S \rightarrow R^1 j_* \mathcal{O}_{U_T}^*$; composing with ∂ gives

$$[L_T]: \mathcal{O}_S \rightarrow I_T \otimes_S R^2 j_* \mathcal{O}_{U_S}. \quad (9.8.4)$$

Thus L_T extends to a line bundle L_R if and only if $[L_T] = 0$.

If $I_T/I_S I_T$ is free over S , then $[L_T]$ is equivalent to a map, called the *obstruction*,

$$\text{obs}(L_T, R): \text{Hom}_S(I_T, \mathcal{O}_S) \rightarrow R^2 j_* \mathcal{O}_{Y_S}, \quad (9.8.5)$$

and L_T extends to a line bundle L_R if and only if $\text{obs}(L_T, R) = 0$.

For us the following two consequences are especially important.

Claim 9.8.6. Assume that S is integral, that $I_T/I_S I_T$ is free over S and that $R^2 j_* \mathcal{O}_{X_S}$ is free and commutes with base change over S .

- (a) The obstruction map $\text{obs}(L_T, R)$ has constant rank and commutes with base change over a dense open subset of S .
- (b) If $\text{obs}(L_T, R)$ is injective, then L_T cannot be extended over any subscheme $T \subsetneq T' \subset R$.

9.9 (End of the proof of Theorem 9.2). We return to the setting of (9.1.1) and, in addition, we choose a scheme $W_K \supset P_K$ such that W_K is smooth at the identity and has the same tangent space as P_K . (Thus $W_K = P_K$ if P_K is smooth.)

As before, everything is defined over a finitely generated subring $A \subset K$; thus there is an integral scheme S of finite type over $\text{Spec } \mathbb{Z}$ such that (9.1.1) is the generic fiber of a family

$$(Y_S \xrightarrow{g_S} X_S \times_{P_S \rightarrow P_S}, W_S \supset P_S, L_{Y_S}). \quad (9.9.1)$$

In order to apply (9.8.6.a), denote by $\widehat{\mathcal{O}}_{S,W}$ the completion of the structure sheaf of W_S along the section $\rho(S)$, by $\widehat{I}_S \subset \widehat{\mathcal{O}}_{S,W}$ the ideal sheaf of $\rho(S)$ and by $\widehat{I}_P \subset \widehat{\mathcal{O}}_{S,W}$ the ideal sheaf of P_S . Finally, set $T := \text{Spec}_S \widehat{\mathcal{O}}_{S,W}/\widehat{I}_P$ and $R := \text{Spec}_S \widehat{\mathcal{O}}_{S,W}/\widehat{I}_S \widehat{I}_P$.

Shrinking S if necessary, the following conditions can be satisfied:

- (2.a) The schemes T, R are flat over S and $I_T/I_S I_T$ is free over S .
- (2.b) The tangent map $t_{T/S}: \text{Hom}_S(I_S, \mathcal{O}_S) \rightarrow R^1 j_* \mathcal{O}_{X_S}$ is an isomorphism and commutes with base change. (This follows from Proposition 9.5.)

- (2.c) The sheaf $R^2j_*\mathcal{O}_{Y_S}$ is free and commutes with base change. (This follows by applying Theorem 10.3 to $Y_S \rightarrow P_S$ and the sheaf $F = \mathcal{O}_{Y_S}$.)

Thus, by Claim 9.8.6, the line bundle $L_S|_T$ does not extended over any subscheme $T \subsetneq T' \subset R$. Therefore $P_S \rightarrow S$ is universal along the zero section $\rho: S \rightarrow P_S$. Since $P_S \rightarrow S$ is a group scheme, this implies that it is everywhere universal. $\square$

10. Auxiliary results

We discuss two theorems, Theorem 10.3 and Theorem 10.5, that are needed to complete our arguments. The preprint version of this paper contained proofs of the special cases that were used in our applications, but in both cases we now have much more general versions available.

The first topic is the generic freeness of higher direct images. A coherent sheaf over a reduced scheme is free over a dense open set, but there are quasi-coherent sheaves that are not free over any dense open subset. Even worse, a nonzero section may vanish at every closed point. The following two examples show what can happen.

Example 10.1. Let $M \subset \mathbb{Q}$ be the $\mathbb{Z}$ -submodule consisting of all m/n such that n has no multiple prime factors. Let $\tilde{M}$ be the corresponding quasi-coherent sheaf over $\text{Spec } \mathbb{Z}$. The localization of $\tilde{M}$ at any prime ideal is free of rank 1, but $\tilde{M}$ is not free. In fact, every global section of it vanishes at all but finitely many points of $\text{Spec } \mathbb{Z}$.

Example 10.2. Let E be an elliptic curve and L the Poincaré bundle on $E \times E$. Set $X := \text{Spec}_{E \times E} \sum_{i \geq 0} L^i$, and let $g: X \rightarrow E \times E \rightarrow E$ be a projection. Note that g is smooth and has fiber dimension 2, but it is neither affine nor proper.

If $e \in E$ is not a torsion point, then $H^0(X_e, \mathcal{O}_{X_e}) = H^1(X_e, \mathcal{O}_{X_e}) = 0$. If $e \in E$ is a torsion point of order exactly m , then there are natural identifications

$$H^0(X_e, \mathcal{O}_{X_e}) \cong H^1(X_e, \mathcal{O}_{X_e}) \cong \sum_{i \in m\mathbb{N}} k(e).$$

Thus we see that $g_*\mathcal{O}_X = 0$ is free, but it does not commute with base change over any open set. By contrast, $R^1g_*\mathcal{O}_X$ is a sum of skyscraper sheaves of infinite rank supported at the torsion points. Thus it is not generically free, but it does commute with base change.

THEOREM 10.3 [Smi16]. *Let $f: X \rightarrow S$ be a finite-type morphism of Noetherian schemes with S integral. Let $Z \subset X$ be a subscheme such that $f|_Z$ is finite. Set $U := X \setminus Z$, and let $j: U \hookrightarrow X$ be the natural injection. Let F be a coherent sheaf on U that is flat over S . Then there is a dense, open subscheme $S^0 \subset S$ such that, for every $r \geq 1$,*

- (i) $f_* \circ R^r j_* F$ is free over S^0 and
- (ii) the formation of $f_* \circ R^r j_* F$ commutes with base changes $W \rightarrow S^0 \hookrightarrow S$.

10.4 (Proof of Theorem 1.9). We assume $\text{depth}_x X \geq 3$; hence, $\mathbf{Pic}^{\text{loc}}(x, X)$ is 0 dimensional. A 0-dimensional linear algebraic group is finite. This shows (1.9.2–3). Thus the only new claim is that $r_D^X: \mathbf{Pic}^{\text{loc}}(x, X) \rightarrow \mathbf{Pic}^{\text{loc}}(x, D)$ is injective (1.9.1).

We already noted in §2.5 that every torsion element in the kernel of r_D^X has p -power order, where p is the residue characteristic. The proof relied on the observation that for $p \nmid m$, a torsion element of order m in $\mathbf{Pic}^{\text{loc}}(x, X)$ corresponds to a degree m étale cover of $X \setminus \{x\}$ and then used [Gro68, XIII.2.1].

The following result completes the proof of Theorem 1.9.

THEOREM 10.5 [dJ15, Proposition 3]. *Let (x, X) be a local, Noetherian scheme and $X \in D \subset X$ a Cartier divisor. Let $x \in Z \subset D$ be a closed subscheme such that $\text{depth}_Z D \geq 2$. Then*

$$\ker[r_D^X: \text{Pic}(X \setminus Z) \rightarrow \text{Pic}(D \setminus Z)] \quad \text{is torsion free.}$$

11. Numerical criteria for relative Cartier divisors

DEFINITION 11.1. Let T be a regular 1-dimensional scheme and $f: X \rightarrow T$ a flat morphism with S_2 fibers. A *generically flat family of divisors* D on X is given by

- (i) an open set $U \subset X$ such that $\text{codim}_{X_t}(X_t \setminus U) \geq 2$ for every $t \in T$ and
- (ii) a relative Cartier divisor D_U on U , that is, a Cartier divisor whose restriction to every fiber is a Cartier divisor.

For each $t \in T$, set $U_t := X_t \cap U$. Then $D_U|_{U_t}$ extends uniquely to a divisor on X_t ; we denote it by D_t and call it the *restriction* of D to X_t .

If $|H|$ is a base-point-free linear system on X , then the restriction of D to a general $H \in |H|$ is again a generically flat family of divisors.

We say that D is a *fiberwise Cartier* family of divisors if each D_t is a Cartier divisor. It is easy to see that

$$r_t: \mathcal{O}_X(D)|_{X_t} = (j_* \mathcal{O}_U(D_U))|_{X_t} \rightarrow (j_t)_*(\mathcal{O}_{U_t}(D_{U_t})) = \mathcal{O}_{X_t}(D_t)$$

is an isomorphism for general $t \in T$; thus D is Cartier except possibly along finitely many closed fibers.

Our main interest is to find conditions that guarantee that a fiberwise Cartier family of divisors is everywhere Cartier.

We say that D is a *fiberwise ample* family of divisors if each D_t is an ample Cartier divisor.

We concentrate on the cases when T is local with closed point 0 and generic point g . The following is a direct consequence of the Nakayama lemma.

LEMMA 11.2. *We take the notation and assumptions of Definition 11.1. Then D is Cartier if and only if D_0 is Cartier and $r_0: \mathcal{O}_X(D)|_{X_0} \rightarrow \mathcal{O}_{X_0}(D_0)$ is an isomorphism.*

11.3 (Numerical inequalities). We take the notation and assumptions of Definition 11.1. Assume in addition that f is proper and has relative dimension n . By semicontinuity

$$H^0(X_0, \mathcal{O}_{X_0}(D_0)) \geq H^0(X_0, \mathcal{O}_X(D)|_{X_0}) \geq H^0(X_g, \mathcal{O}_{X_g}(D_g)). \quad (11.3.1)$$

In particular, if D_g is ample on X_g and D_0 is Cartier and ample on X_0 , then, applying (11.3.1) to multiples of D , we conclude that

$$(D_0^n) \geq (D_g^n). \quad (11.3.2)$$

One can be more precise if $\dim Z_0 = 0$. Then $\text{coker } r_0$ is artinian, thus

$$\chi(X_0, \mathcal{O}_{X_0}(D_0)) = \chi(X_0, \mathcal{O}_X(D)|_{X_0}) + \text{length}(\text{coker } r_0). \quad (11.3.3)$$

Since $\mathcal{O}_X(D)$ is flat over T ,

$$\chi(X_0, \mathcal{O}_X(D)|_{X_0}) = \chi(X_g, \mathcal{O}_X(D)|_{X_g}) = \chi(X_g, \mathcal{O}_{X_g}(D_g)).$$

Combining these, we see that

$$\chi(X_0, \mathcal{O}_{X_0}(D_0)) = \chi(X_g, \mathcal{O}_{X_g}(D_g)) + \text{length}(\text{coker } r_0). \quad (11.3.4)$$

Together with Lemma 11.2, this proves the following.

COROLLARY 11.4. *We take the notation and assumptions of Definition 11.1. Assume in addition that f is proper and $\dim Z_0 = 0$. Then D is Cartier if and only if D_0 is Cartier and*

$$\chi(X_0, \mathcal{O}_{X_0}(D_0)) = \chi(X_g, \mathcal{O}_{X_g}(D_g)).$$

Combining Theorems 1.9 and 10.5, we have the following.

THEOREM 11.5. *Let T be an irreducible, regular, 1-dimensional scheme and $f: X \rightarrow T$ a flat morphism that is essentially of finite type and has S_2 fibers.*

Let D be a generically flat family of fiberwise Cartier divisors on X . Assume that there is a closed subscheme $W \subset X$ such that $\text{codim}_{X_t}(X_t \cap W) \geq 3$ for every $t \in T$ and that D is Cartier on $U := X \setminus W$. Then D is a Cartier divisor on X .

It remains to understand what happens when the putative non-Cartier locus has codimension 2. For projective morphisms we have the following numerical criteria.

THEOREM 11.6. *Let T be an irreducible, regular, 1-dimensional scheme and $f: X \rightarrow T$ a flat, projective morphism of relative dimension n with S_2 fibers.*

Let D be a generically flat family of fiberwise Cartier divisors on X and H an f -ample Cartier divisor on X . The following are equivalent:

- (i) *The family D is a Cartier divisor on X .*
- (ii) *The intersection number $(D_t^2 \cdot H_t^{n-2})$ is independent of $t \in T$.*

If D is fiberwise ample, then these are further equivalent to the following:

- (iii) *The self-intersection number (D_t^n) is independent of $t \in T$.*

Proof. If D is Cartier, then all the intersection numbers $(D_t^i \cdot H_t^{n-i})$ are independent of $t \in T$. Thus conditions (1) implies conditions (2) and (3).

To see the converse, we may assume that T is local with closed point $0 \in T$ and generic point $g \in T$. Let $Z_0 \subset X$ be the smallest closed subset such that D is Cartier on $X \setminus Z_0$. Note that $Z_0 \subset X_0$ since D is Cartier on X_g . We can choose $U := X \setminus Z_0$ as the open set in Definition 11.1.

First, assume that $n = 2$, thus conditions (2) and (3) coincide. For each $t \in T$, the Euler characteristic is a quadratic polynomial

$$\chi(X_t, \mathcal{O}_{X_t}(mD_t)) = a_t m^2 + b_t m + c_t,$$

and we know from the Riemann–Roch theorem that $a_t = \frac{1}{2}(D_t^2)$ and $c_t = \chi(X_t, \mathcal{O}_{X_t})$. Furthermore, (11.3.4) implies that

$$a_0 m^2 + b_0 m + c_0 \geq a_g m^2 + b_g m + c_g \quad \text{for every } m \in \mathbb{Z}. \quad (11.6.4)$$

For $m \gg 1$, the quadratic terms dominate, which gives

$$(D_0^2) = 2a_0 \geq 2a_g = (D_g^2). \quad (11.6.5)$$

Now, assume $(D_0^2) = (D_g^2)$. Then $a_0 = a_g$, thus (11.6.5) implies that

$$b_0 m + c_0 \geq b_g m + c_g \quad \text{for every } m \in \mathbb{Z}. \quad (11.6.6)$$

For $m \gg 1$, this implies that $b_0 \geq b_g$, and for $m \ll -1$ it implies that $-b_0 \geq -b_g$. Thus $b_0 = b_g$ and $c_0 = \chi(X_0, \mathcal{O}_{X_0}) = \chi(X_g, \mathcal{O}_{X_g}) = c_g$ also holds since f is flat. Therefore, we have equality

in (11.6.4). Using Corollary 11.4, we see that

$$r_0: j_*\mathcal{O}_U(D_U)|_{X_0} \rightarrow (j_0)_*\mathcal{O}_{U_0}(D_0)$$

is an isomorphism and hence $\mathcal{O}_X(D) = j_*\mathcal{O}_U(D_U)$ is locally free.

In order to prove that condition (2) implies condition (1) for $n \geq 3$, we use induction on n . A suitable multiple $|mH|$ provides an embedding $X \subset \mathbb{P}_T^N$; let $X' \subset X$ be a general hyperplane section. Then $f' := f|_{X'}: X' \rightarrow T$ is a flat, projective morphism of relative dimension $n - 1$ and $Z' := Z \cap X'$ is a closed subscheme such that $\text{codim}_{X'_t}(X'_t \cap Z'_t) \geq 2$ for every $t \in T$.

Furthermore, $D' := D|_{X'}$ is a divisor, Cartier over $U' := X' \setminus Z'$, whose support does not contain any irreducible component of a fiber of f' , and $D'|_{X'_t}$ is Cartier for every t . Finally, $((D'_t)^2 \cdot H_t^{n-3}) = m(D_t^2 \cdot H_t^{n-2})$ is independent of $t \in T$.

Since f has S_2 fibers, a hyperplane section of it usually has only S_1 fibers. However, by [Gro60, IV.12.1.6], a general hyperplane section again has S_2 fibers. Thus, by induction, $D' = D|_{X'}$ is a Cartier divisor. As we noted in Definition 11.1, since X' is general, this implies that D is Cartier along X' . Hence, there is a closed subscheme $W \subset X$ such that $W \cap X_t$ is 0 dimensional for every $t \in T$ and D is Cartier on $X \setminus W$. Thus D is Cartier by Theorem 11.5.

Finally, assume that D is fiberwise ample. Choose $m > 0$ such that $mD_0 - H_0$ and $mD_t - H_t$ are ample. Thus $B := mD - H$ is also fiberwise ample and

$$m^n(D_t^n) = \sum_{i=0}^n (B_t^i \cdot H_t^{n-i}).$$

We can apply (11.3.2) to the restriction of B to an intersection of $n - i$ general members of H to obtain that $(B_0^i \cdot H_0^{n-i}) \geq (B_g^i \cdot H_g^{n-i})$ for every i . However, $(D_0^n) = (D_g^n)$ by condition (3), thus $(B_0^i \cdot H_0^{n-i}) = (B_g^i \cdot H_g^{n-i})$ for every i . We can use this for $i = 2$ and the already established fact that condition (2) implies condition (1) to conclude that B is Cartier. Thus $mD = B + H$ is also Cartier. By the same argument $(m + 1)D$ is also Cartier; this implies that D is Cartier. $\square$

12. Open problems

Finite generation of $\text{NS}^{\text{loc}}(x, X)$

The main unsolved problem is the finite generation of $\text{NS}^{\text{loc}}(x, X)$. This is known if X is normal or if the characteristic is 0 and $\dim X \geq 3$. In positive characteristic, our methods do not seem to distinguish a unipotent subgroup of $\mathbf{Pic}^{\text{loc}-\circ}$ from a discrete p -group.

For surfaces over $\mathbb{C}$, even stronger results should hold; see Conjecture 12.4.

Local Picard group of excellent schemes

While our theorems settle only the geometric cases, I see no reason why they should not hold in general.

CONJECTURE 12.1. Let X be an excellent scheme that is S_2 and has pure dimension at least 3. Let $x \in X$ be a closed point and $\bar{X} \rightarrow X$ the normalization. Then $\ker[\pi^*: \mathbf{Pic}^{\text{loc}}(x, X) \rightarrow \mathbf{Pic}^{\text{loc}}(\bar{x}, \bar{X})]$ is a linear algebraic group.

CONJECTURE 12.2. Let X be an excellent scheme that is S_2 and has pure dimension at least 4. Let $x \in X$ be a closed point and $x \in D \subset X$ an effective Cartier divisor. Then $\ker[r_D^X: \mathbf{Pic}^{\text{loc}}(x, X) \rightarrow \mathbf{Pic}^{\text{loc}}(x, D)]$ is a unipotent algebraic group.

Note that not even the existence of $\mathbf{Pic}^{\mathrm{loc}}(x, X)$ is known in the above generality, but algebraic equivalence can be defined as in Definition 3.3.

Local Picard group of surfaces

The construction of [Bou78] does not apply to surfaces; in fact, the functorial approach always gives the “wrong” answer. For normal surface singularities over $\mathbb{C}$, Mumford [Mum61] constructed a finite-dimensional local Picard group $\mathbf{Pic}^{\mathrm{loc}}(s, S)$. Example 4.5 (in particular, (4.5.3) and (4.5.4)) shows that in many nonnormal cases there is no finite-dimensional local Picard group.

CONJECTURE 12.3. Let (s, S) be an S_2 surface. Then there is a finite-dimensional local Picard group $\mathbf{Pic}^{\mathrm{loc}}(s, S)$ if and only if S is seminormal.

CONJECTURE 12.4. Let (x, X) be a local $\mathbb{C}$ -scheme of finite type that is S_2 and has pure dimension 2. Then taking the first Chern class gives an exact sequence

$$0 \rightarrow \mathbf{Pic}^{\mathrm{loc}-o}(x, X) \rightarrow \mathbf{Pic}^{\mathrm{loc}}(x, X) \xrightarrow{c_1} H^2(\mathrm{link}(x, X), \mathbb{Z}).$$

The normal case is discussed in [Mum61]. It would be nice to have something similar in positive characteristic, but I do not know what should replace $H^2(\mathrm{link}(x, X), \mathbb{Z})$. See also Example 4.8.

In connection with Example 4.4, one can ask the following.

Question 12.5. Let $(0, S^{\mathrm{an}})$ be a normal, analytic surface singularity over $\mathbb{C}$ such that $\mathbf{Pic}^{\mathrm{loc}-o}(0, S)$ is proper. Is there an algebraic model $(0, S)$ such that $\mathbf{Pic}^{\mathrm{loc}}(0, S) = \mathbf{Pic}^{\mathrm{loc}}(0, S^{\mathrm{an}})$?

Second cohomology of links

One can refine the topological approach of Section 5 using the mixed Hodge structures on the cohomology groups of the links $H^2(\mathrm{link}(x, X), \mathbb{C})$ and $H^2(\mathrm{link}(\bar{x}, \bar{X}), \mathbb{C})$ [PS08, Section 6.2]. Since the Chern class of a line bundle has pure Hodge type $(1, 1)$, the following would imply most of Conjecture 12.1 for analytic spaces.

CONJECTURE 12.6. Let X be a complex analytic space that is S_2 and has pure dimension at least 3. Let $x \in X$ be a point. Then the pull-back map

$$\pi^*: H^2(\mathrm{link}(x, X), \mathbb{C}) \rightarrow H^2(\mathrm{link}(\bar{x}, \bar{X}), \mathbb{C})$$

is injective on the weight 2 graded piece of the mixed Hodge structure.

Obstruction theory for $\mathbf{Pic}^{\mathrm{loc}}$

The usual obstruction space for $\mathbf{Pic}^{\mathrm{loc}}(x, X)$ is $H_x^3(X, \mathcal{O}_X)$. This space is infinite dimensional if $\dim X = 3$ or if $\dim X \geq 4$ and $X \setminus \{x\}$ is not S_3 .

Already, Artin [Art74] observed that an obstruction theory is an external construct imposed on a functor and that there could be different “natural” obstruction theories that work for the same functor. Theorem 10.3 suggests that one should be able to develop a finite-dimensional obstruction theory for $\mathbf{Pic}^{\mathrm{loc}}$. I do not know how to formulate such a theory.

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