Matrix factorizations and families of curves of genus 15

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ABSTRACT

In this note we explain how certain matrix factorizations on cubic threefolds lead to families of curves of genus 15 and degree 16 in $\mathbb{P}^4$. We prove that the moduli space $\widetilde{\mathcal{M}}_{15,16}^4 = \{(C, L) \mid C \in \mathcal{M}_{15}, L \in W_{16}^4(C) \subset \text{Pic}^{16}(C)\}$ is uniruled, and that $\widetilde{\mathcal{M}}_{15,16}^4$ is birational to a space of certain matrix factorizations on cubics. Our attempt to prove the unirationality of this space failed with our methods. Instead, one can interpret our findings as evidence for the conjecture that the maximal rational connected fibration of $\widetilde{\mathcal{M}}_{15,16}^4$ has a 3-dimensional base.

1. Introduction

The moduli spaces $\mathcal{M}_g$ of curves of genus g are known to be unirational for $g \leq 14$ [Sev21, Ser81, CR84, Ver05]. For $g = 22$ and $g \geq 24$ they are known to be of general type [HM82, EH87, Far00, Far09]. The cases in between are not fully understood: $\mathcal{M}_{23}$ has positive Kodaira dimension [Far00], $\mathcal{M}_{15}$ is rationally connected [CR86, BV05], and $\mathcal{M}_{16}$ [CR91, Far09] is uniruled. In this paper we are mainly concerned with $\mathcal{M}_{15}$ and an attempt to prove its unirationality.

By Brill–Noether theory, a general curve of genus 15 has a smooth model of degree 16 in $\mathbb{P}^4$. Let

$$\mathcal{H} \subset \text{Hilb}_{16t+1-15}(\mathbb{P}^4)$$

be the component of the Hilbert scheme of curves of degree $d = 16$ and genus $g = 15$ in $\mathbb{P}^4$ which dominates the moduli space $\mathcal{M}_{15}$. (This component is unique because the Brill–Noether dual models form the Severi variety of plane curves of degree $d = 12$ and geometric genus $g = 15$ with $\delta = 40$ nodes, which is known to be irreducible [Har86].) Let

$$\widetilde{\mathcal{M}}_{15,16}^4 \subset \{(C, L) \mid C \in \mathcal{M}_{15}, L \in W_{16}^4(C)\}$$

be the component which dominates $\mathcal{M}_{15}$. So $\mathcal{H} // \text{PGL}(5)$ is birational to $\widetilde{\mathcal{M}}_{15,16}^4$. Our main result connects this moduli space to a moduli space of certain matrix factorizations.

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THEOREM 1.1. *The moduli space $\widetilde{\mathcal{M}}_{15,16}^4$ of curves of genus 15 together with a g_{16}^4 is birational to a component of the moduli space of matrix factorizations of type $(\psi: \mathcal{O}^{18}(-3) \rightarrow \mathcal{O}^{15}(-1) \oplus \mathcal{O}^3(-2), \varphi: \mathcal{O}^{15}(-1) \oplus \mathcal{O}^3(-2) \rightarrow \mathcal{O}^{18})$ of cubic forms on $\mathbb{P}^4$.*

As a corollary of our proof we obtain the following dimension statement.

THEOREM 1.2. *A general cubic threefold in $\mathbb{P}^4$ contains a 32-dimensional uniruled family of smooth curves of genus 15 and degree 16.*

Since a general curve in $\mathcal{H}$ lies on a unique cubic threefold, and cubic threefolds depend on 10 parameters up to projectivities, the dimension 32 fits with $\dim \mathcal{M}_{15} = 42$.

Our approach to construct a family of curves of genus 15 builds upon the construction of a matrix factorization on a cubic as a syzygy module of an auxiliary module N . We use Boij–Söderberg theory [BS08, ES09, BS12, SE10] and the `Macaulay2` package [ESS] to get a list of candidate Betti tables. In all our cases the sheaf $\mathcal{L} = \widetilde{N}$ will be a line bundle on an auxiliary curve E . The choice of E and $\mathcal{L}$ is motivated by a dimension count and the shape of the Betti table of N . We have succeeded to construct altogether 20 families of curves in $\mathcal{H}$, and 17 of the families are unirational. However, the unirational families do not dominate $\mathcal{M}_{15}$, although the number of parameters in the construction exceeds 42. Three of these families have a non-unirational step in their construction. (We need to choose an effective divisor on the auxiliary curve E . Since E is not rational, this step in the construction is not unirational). Precisely, those three families dominate $\mathcal{M}_{15}$. We use the family from Theorem 5.6 to prove the uniruledness in Theorem 1.2 and Theorem 1.3.

THEOREM 1.3. *The moduli space $\widetilde{\mathcal{M}}_{15,16}^4$ is uniruled.*

Furthermore, we obtain the following result.

THEOREM 1.4. *There exists a probabilistic algorithm which randomly produces curves of genus $g = 15$ over a finite field $\mathbb{F}_q$ with q elements from a Zariski open subset of $\mathcal{M}_{15}$ in running time $O((\log q)^3)$.*

The existence of such an algorithm in principle is no surprise. Important is that the algorithm actually runs in reasonable time on current computer algebra systems.

The proofs of the theorems in this article rely on computer algebra. An implementation of all necessary computations can be found in the `Macaulay2` package [MF15].

Many of the images of the unirational families have dimension 39. There is one of dimension 41, one of dimension 40, and there are some of dimension less than 39. A good explanation why we failed to prove the unirationality of $\mathcal{M}_{15}$ with this method could be the following.

CONJECTURE 1.5. The maximal rationally connected fibration of $\widetilde{\mathcal{M}}_{15,16}^4$ has a 3-dimensional base.

2. Matrix factorizations

Matrix factorizations were introduced 1980 by Eisenbud in his seminal paper [Eis80]. We recall basic facts. Let R be a regular local ring and $f \in R$ not a unit. A *matrix factorization* of f is a pair (φ, ψ) of matrices satisfying $\psi \circ \varphi = fid$ and $\varphi \circ \psi = fid$. Then φ, ψ are necessarily square matrices of the same size. If (φ, ψ) is a matrix factorization, then $\text{coker } \varphi$ is a maximal

Cohen–Macaulay module (MCM) on the hypersurface ring R/f . Conversely, a finitely generated maximal Cohen–Macaulay module M over R/f has a short minimal free resolution

$$0 \longleftarrow M \longleftarrow F \longleftarrow G \longleftarrow 0$$

as an R -module, and multiplication with f on this complex is null homotopic:

$$\begin{array}{ccccccc} 0 & \longleftarrow & M & \longleftarrow & F & \xleftarrow{\varphi} & G \longleftarrow 0 \\ & & \downarrow 0 & & \downarrow f & \searrow \psi & \downarrow f \\ 0 & \longleftarrow & M & \longleftarrow & F & \xleftarrow{\varphi} & G \longleftarrow 0, \end{array}$$

which yields a matrix factorization (φ, ψ) . As an R/f -module, M has the infinite 2-periodic resolution

$$0 \longleftarrow M \longleftarrow \overline{F} \xleftarrow{\overline{\varphi}} \overline{G} \xleftarrow{\overline{\psi}} \overline{F} \xleftarrow{\overline{\varphi}} \overline{G} \xleftarrow{\overline{\psi}} \dots,$$

where $\overline{F} = F \otimes R/f$ and $\overline{G} = G \otimes R/f$. In particular, this sequence is exact, and the dual sequence corresponding to the matrix factorization (ψ^t, φ^t) is exact as well.

If N is an arbitrary R/f -module, then the minimal free resolution becomes eventually 2-periodic: if

$$0 \longleftarrow N \longleftarrow F_0 \longleftarrow F_1 \longleftarrow \dots \longleftarrow F_c \longleftarrow 0$$

is the minimal free resolution of N as an R -module, then a (not necessarily minimal) free resolution of N as an R/f -module starts with

$$0 \longleftarrow N \longleftarrow \overline{F}_0 \longleftarrow \overline{F}_1 \longleftarrow \overline{F}_2 \oplus \overline{F}_0 \longleftarrow \overline{F}_3 \oplus \overline{F}_1 \longleftarrow \dots \longleftarrow \overline{F}_{ev} \longleftarrow \overline{F}_{odd} \longleftarrow \dots,$$

where

$$F_{ev} = \bigoplus_{i \equiv 0 \pmod{2}} F_i \quad \text{and} \quad F_{odd} = \bigoplus_{i \equiv 1 \pmod{2}} F_i.$$

The high syzygy modules over a Cohen–Macaulay ring are MCM. In case of a hypersurface $M = \text{coker}(\overline{F}_{odd} \rightarrow \overline{F}_{ev})$ is an MCM module. There is a natural surjection from $M \oplus F$ to N with kernel P ,

$$0 \longleftarrow N \longleftarrow M \oplus F \longleftarrow P \longleftarrow 0,$$

where F is a free R/f -module and P is a module of finite projective dimension. In the examples relevant later on, we will find that we can choose $F = 0$.

Thus, an arbitrary R/f -module can be built from an MCM-module and a module of finite projective dimension. In the remarkable paper [AB89] of Auslander and Buchweitz on Maximal Cohen–Macaulay approximation this phenomenon is studied in much wider generality.

In the case of interest for this paper, we replace R by the standard graded polynomial ring S , the homogeneous coordinate ring of some $\mathbb{P}^n$, and f by a homogeneous form of degree d . We have to take the grading into account. A matrix factorization is a pair $(\varphi: G \rightarrow F, \psi: F \rightarrow G(d))$, where $F = \bigoplus_{\ell=1}^r S(-a_\ell)$ and $G = \bigoplus_{\ell=1}^r S(-b_\ell)$. If N is an S/f -module with minimal free resolution $F_\bullet$ as an S -module, then the free resolution as an S/f -module has as i th term the module $\overline{F}_i \oplus \overline{F}_{i-2}(-d) \oplus \dots \oplus \overline{F}_0(-id/2)$ or $\overline{F}_i \oplus \overline{F}_{i-2}(-d) \oplus \dots \oplus \overline{F}_1(-(i-1)d/2)$ in case i is even or odd, respectively.

The associated sheaf $\mathcal{F} = \widetilde{M}$ of $M = \text{coker}(F \rightarrow G)$ is a sheaf of maximal Cohen–Macaulay modules on the scheme $X \subset \mathbb{P}^n$ defined by f . Thus, if X is smooth, then $\mathcal{F}$ is locally free, that is, a vector bundle on X . In this case $\det \varphi = \lambda f^k$ for a unit $\lambda \in K \subset S$ and $k = \text{rank } \mathcal{F}$. We

frequently use the sheafified notation

$$\left(\varphi: \bigoplus_{\ell=1}^r \mathcal{O}(-b_\ell) \longrightarrow \bigoplus_{\ell=1}^r \mathcal{O}(-a_\ell), \psi: \bigoplus_{\ell=1}^r \mathcal{O}(-a_\ell) \longrightarrow \bigoplus_{\ell=1}^r \mathcal{O}(d - b_\ell) \right)$$

for matrix factorizations. From the short exact sequence

$$0 \longrightarrow \bigoplus_{\ell=1}^r \mathcal{O}(-b_\ell) \xrightarrow{\varphi} \bigoplus_{\ell=1}^r \mathcal{O}(-a_\ell) \longrightarrow \mathcal{F} \longrightarrow 0$$

we obtain that $\mathcal{F}$ has no middle cohomology:

$$H^i(X, \mathcal{F}(j)) = 0 \quad \text{for all } i \text{ with } 1 \leq i \leq \dim X - 1 \text{ and all } j \in \mathbb{Z};$$

that is, $\mathcal{F}$ is an arithmetically Cohen–Macaulay (ACM) bundle on X . Conversely, if $\mathcal{F}$ is an ACM-bundle on a (smooth) hypersurface $X \subset \mathbb{P}^n$, then

$$M = H_*^0(\mathcal{F}) = \sum_{j \in \mathbb{Z}} H^0(\mathcal{F}(j))$$

is an MCM-module over S/f , where $\langle f \rangle$ is the homogeneous ideal of X . The investigation of ACM-bundles on hypersurfaces is a widely studied subject which fairly recently caught the attention even of physicists in mirror symmetry to model D-branes; see, for example, [AM12].

3. Syzygies of the general curve in $\mathcal{H}$

Recall that $\mathcal{H}$ denotes the component of the Hilbert scheme of curves of degree $d = 16$ and genus $g = 15$ in $\mathbb{P}^4$ which dominate $\mathcal{M}_{15}$.

PROPOSITION 3.1. *Let $C \in \mathcal{H}$ be a general point. The homogeneous coordinate ring $S_C = S/I_C$ and the section ring $\Gamma_*(\mathcal{O}_C) = \bigoplus_{n \in \mathbb{Z}} H^0(\mathcal{O}_C(n))$ have minimal free resolutions with the following Betti tables:*

$$\begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & . & . & . & . \\ 1 & . & . & . & . & . \\ 2 & . & 1 & . & . & . \\ 3 & . & 15 & 30 & 18 & 3 \end{array} \quad \text{and} \quad \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & . & . & . \\ 1 & . & . & . & . \\ 2 & 3 & 16 & 15 & . \\ 3 & . & . & . & 3 \end{array}$$

respectively. In particular, $C \subset \mathbb{P}^4$ lies on a unique smooth cubic threefold X . The minimal resolution of $\Gamma_*(\mathcal{O}_C)$ as a module over the homogeneous coordinate ring of X is eventually 2-periodic with Betti numbers

$$\begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & \dots \\ \hline 0 & 1 & . & . & . & . & . \\ 1 & . & . & . & . & . & . \\ 2 & 3 & 15 & 15 & . & . & . \\ 3 & . & . & 3 & 18 & 15 & . \\ \vdots & . & . & . & . & 3 & 18 \end{array}$$

Proof. Assuming that the maps $H^0(\mathcal{O}_{\mathbb{P}^4}(n)) \rightarrow H^0(\mathcal{O}_C(n))$ are of maximal rank for all n , in other words, that C has maximal rank, we find, using the Riemann–Roch theorem and the fact that $\mathcal{O}_C(n)$ is non-special for $n \geq 2$, that

- the Hilbert series of the homogeneous coordinate ring of C is

$$H_C(t) = 1 + 5t + 15t^2 + 34t^3 + (34 + 16)t^4 + (34 + 2 \cdot 16)t^5 + \cdots;$$

- the Hartshorne–Rao module

$$H_*^1(\mathcal{I}_C) = \sum_{n \in \mathbb{Z}} H^1(\mathcal{I}_C(n)) \cong K^3(-2)$$

is a 3-dimensional vector space concentrated in degree 2;

- the ideal sheaf $\mathcal{I}_C$ is 4-regular; and
- the homogeneous ideal $I_C = H_*^0(\mathcal{I}_C)$ has a single generator in degree 3 and 15 further generators in degree 4.

Hence, the Hilbert numerator has shape

$$(1 - t)^5 H_C(t) = 1 - t^3 - 15t^4 + 30t^5 - 18t^6 + 3t^7,$$

and smooth maximal-rank curves in $\mathcal{H}$ have a Betti table as claimed in the proposition. To establish that a general point in $C \in \mathcal{H}$ is a maximal-rank curve, it suffices to produce a single maximal-rank example. We will explicitly construct such examples in Section 5. Moreover, by inspection we find that the general C lies on a smooth cubic hypersurface X .

The Betti table of the resolution $F_\bullet$ of the section ring as an S -module can be deduced with the same method, since $H_{\Gamma_*(\mathcal{O}_C)}(t) = H_C(t) + 3t^2$. The only questionable entry of the Betti table is $\beta_{2,5}^S(\Gamma_*(\mathcal{O}_C)) = 0$, for which we argue as follows. The complex $\text{Hom}(F_\bullet, S(-5))$ resolves $H_*^0(\mathcal{E}xt^3(\mathcal{O}_C, \omega_{\mathbb{P}^4})) \cong H_*^0(\omega_C)$. By Brill–Noether theory, there are no linear relations among the three generators in $H^0(\omega_C(-1))$ if and only if $(C, \mathcal{O}_C(1))$ does not correspond to a ramification point of the map $\widehat{\mathcal{M}}_{15,16}^4 \rightarrow \mathcal{M}_{15}$. So $\beta_{2,5}^S(\Gamma_*(\mathcal{O}_C)) = 0$ holds for $C \in \mathcal{H}$ outside the ramification divisor. Finally, we compute the Betti number of $\Gamma_*(\mathcal{O}_C)$ as an S_X -module. The (possibly non-minimal) resolution from Section 2 has Betti table

	0	1	2	3	4	...
0	1	.	.	.	.	.
1	.	.	1	.	.	.
2	3	16	15	.	1	.
3	.	.	3	19	15	.
$\vdots$	.	.	.	.	3	19

So this resolution is non-minimal. The minimal version has the desired Betti table. $\square$

Consider the matrix factorization

$$(\varphi: \mathcal{O}^{15}(-4) \oplus \mathcal{O}^3(-5) \longrightarrow \mathcal{O}^{18}(-3), \psi: \mathcal{O}^{18}(-3) \longrightarrow \mathcal{O}^{15}(-1) \oplus \mathcal{O}^3(-2))$$

corresponding (up to twist) to the 2-periodic part of the resolution of $\Gamma_*(\mathcal{O}_C)$ as an S_X -module. Let $\mathcal{F} = \text{coker } \varphi$. The sheaf $\mathcal{F}$ is a vector bundle of rank $\mathcal{F} = 7$ on X , since $\deg \det \varphi = 15 + 2 \cdot 3 = 3 \cdot 7$. We have a short exact sequence

$$0 \longleftarrow \text{coker } \psi \longleftarrow \mathcal{O}_X^{15}(-1) \oplus \mathcal{O}_X^3(-2) \longleftarrow \mathcal{F} \longleftarrow 0.$$

The composition $\mathcal{O}_X^3(-2) \leftarrow \mathcal{F} \leftarrow \mathcal{O}_X^{18}(-3)$ is surjective with a summand $\mathcal{O}_X^3(-3)$ in the kernel. Indeed, the composition has the sheafified presentation matrix $S^3(-2) \leftarrow S^{15}(-3)$ of the

Hartshorne–Rao module of C restricted to X as a summand, and surjectivity follows because

$$\operatorname{coker}(S^{15}(-3) \longrightarrow S^3(-2)) \cong K^3(-2)$$

is a module of finite length. Thus, we obtain a complex

$$0 \longrightarrow \mathcal{O}_X^3(-3) \xrightarrow{\beta} \mathcal{F} \xrightarrow{\alpha} \mathcal{O}_X^3(-2) \longrightarrow 0.$$

THEOREM 3.2. *Let $C \in \mathcal{H}$ be a general point. Then the complex*

$$0 \longrightarrow \mathcal{O}_X^3(-3) \xrightarrow{\beta} \mathcal{F} \xrightarrow{\alpha} \mathcal{O}_X^3(-2) \longrightarrow 0$$

obtained from the resolution of $\Gamma_(\mathcal{O}_C)$ as an S_X -module is a monad for the ideal sheaf $\mathcal{I}_{C/X}$ of C in X ; that is, α is surjective, β is injective, and $\ker \alpha / \operatorname{image} \beta \cong \mathcal{I}_{C/X}$.*

Proof. We have already proved the surjectivity of α . Thus $\mathcal{G} = \ker \alpha$ is a rank 4 subbundle of $\mathcal{F}$, and β induces a homomorphism $\beta': \mathcal{O}_X(-3)^3 \rightarrow \mathcal{G}$ between locally free sheaves on X . We expect that β' drops rank along a codimension 2 subscheme of X . This is the case if $\operatorname{coker}(\beta^*: \mathcal{G}^* \rightarrow \mathcal{O}_X^3(3))$ has support in codimension 2 on X . The composition $\mathcal{O}_X^{15}(1) \rightarrow \mathcal{F}^* \rightarrow \mathcal{G}^* \rightarrow \mathcal{O}_X^3(3)$ coincides with the restriction to X of $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^4}}(\mathcal{O}^3(-6) \rightarrow \mathcal{O}^{15}(-4), \omega_{\mathbb{P}^4})$ up to twist, where $\mathcal{O}^{15}(-4) \leftarrow \mathcal{O}^3(-6)$ denotes the sheafified last map in the resolution of $\Gamma_*(\mathcal{O}_C)$ as an S -module. Hence $\operatorname{coker}(\beta^*(-2): \mathcal{G}^*(-2) \rightarrow \mathcal{O}_X^3(1)) \cong \omega_C$ has support on C , which has codimension 2 on X . Finally, $\omega_X \cong \mathcal{O}_X(-2)$ implies $\ker(\beta^*: \mathcal{G}^* \rightarrow \mathcal{O}_X^3(3)) \cong \mathcal{O}_X$ and $\ker \alpha / \operatorname{image} \beta \cong \operatorname{coker}(\beta') \cong \mathcal{I}_{C/X} \subset \mathcal{O}_X$. $\square$

Proof of Theorem 1.1. One direction follows from Theorem 3.2. For the other direction, consider an arbitrary matrix factorization of type

$$(\psi: \mathcal{O}^{18}(-3) \longrightarrow \mathcal{O}^{15}(-1) \oplus \mathcal{O}^3(-2), \varphi: \mathcal{O}^{15}(-1) \oplus \mathcal{O}^3(-2) \longrightarrow \mathcal{O}^{18})$$

of some cubic form as in Theorem 1.1. The pair $(\varphi(-3), \psi)$ is a matrix factorization of the shape used to derive the monad of Theorem 3.2. In particular, we have again a short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_X^{15}(-1) \oplus \mathcal{O}_X^3(-2) \longrightarrow \operatorname{coker} \psi \longrightarrow 0$$

with $\mathcal{F} = \operatorname{coker} \varphi(-3)$. The composition $\mathcal{O}_X^{18}(-3) \rightarrow \mathcal{F} \rightarrow \mathcal{O}_X^3(-2)$ has a summand $\mathcal{O}_X^3(-3)$ in the kernel simply because there are only five linearly independent linear forms on $\mathbb{P}^4$. Thus we can derive a complex

$$0 \longrightarrow \mathcal{O}_X^3(-3) \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_X^3(-2) \longrightarrow 0$$

again. It is an open condition on the matrices ψ that the summand $\mathcal{O}_X^3(-3)$ in the kernel be uniquely determined, and that $\mathcal{F} \rightarrow \mathcal{O}_X^3(-2)$ be surjective. Further open conditions on matrix factorizations are the conditions that the complex above be a monad, that its homology be the ideal sheaf $\mathcal{I}_{C/X}$ of a smooth curve of degree $d = 16$ and genus $g = 15$ on a smooth cubic threefold X , that $C \subset \mathbb{P}^4$ be a maximal-rank curve, and that the pair $(C, \mathcal{O}_C(1))$ not lie in the ramification divisor of $\widehat{\mathcal{M}}_{15,16}^4 \rightarrow \mathcal{M}_{15}$. Thus the theorem follows if we establish the existence of such matrix factorizations. We will prove the existence computationally in Section 5 with Theorem 5.1. $\square$

4. Betti Tables

One approach to the desired matrix factorizations is via the study of $\mathcal{M}_X(7, c_1(\mathcal{F}), c_2(\mathcal{F}), c_3(\mathcal{F}))$, the moduli space of vector bundles on a general cubic threefold X . We choose a different, more direct approach.

Consider an S -module N , annihilated by the equation of a cubic hypersurface X , such that the 2-periodic part of the minimal free S_X -resolution gives the desired matrix factorization or its transpose. If we require, in addition, that the S_X -resolution derived from the minimal free S -resolution as in Section 2 be minimal right away, then up to twist there are only finitely many Betti tables possible.

PROPOSITION 4.1. *The Boij–Söderberg cone of S -Betti tables contains up to twist precisely 39 different integral tables $\beta^S(N)$ of projective dimension $\text{pd } \beta^S(N) \leq 4$ and codimension $\text{codim } \beta^S(N) \geq 3$ such that the induced possibly non-minimal S_X -resolution from Section 2 is minimal and its 2-periodic part corresponds to a matrix factorization of desired shape. All of these tables satisfy $\text{codim } \beta^S(N) = 3$.*

Proof. The possible shapes of the Betti table $\beta^S(N)$ are

$$\begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & a & . & . & . & . \\ 1 & b & c & d & . & . \\ 2 & . & . & e & f & h \\ 3 & . & . & . & . & i \end{array} \quad \text{and} \quad \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & a & b & . & . & . \\ 1 & . & c & d & e & . \\ 2 & . & . & . & f & h \end{array}$$

with $(a + d + h, b + e + i, c + f) = (3, 15, 18)$ or $(15, 3, 18)$ for the first shape, and $(a + d + h, b + e, c + f) = (18, 15, 3)$ or $(18, 3, 15)$ for the second shape. Since all entries are non-negative, there are only finitely many tables to start with, and as a computation shows, 39 of the tables lie in the Boij–Söderberg cone and satisfy $\text{codim } \beta^S(N) \geq 3$. The last assertion follows by inspection of this list, which we produced with `Macaulay2` using our package [MF15]. $\square$

Remark 4.2. The Picard group of a non-singular cubic is generated by the hyperplane class. This motivates $\text{codim } \beta^S(N) \geq 3$ since otherwise we have to guarantee that the class of the codimension 2 part of the support of N is a multiple of the hyperplane class. The condition $\text{pd}_S \beta^S(N) \leq 4$ is motivated by the wish to think of N as a submodule of the global section module $\Gamma_*(\mathcal{L})$ of some auxiliary sheaf $\mathcal{L} = \tilde{N}$.

Example 4.3. The Betti tables of Proposition 4.1 with $\deg \beta^S(N) = 11$ are the following:

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 5 & 9 & . & . \\ 1 & . & 3 & 13 & 6 \end{array} \quad \text{and its dual} \quad \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 1 & 6 & 13 & 3 & . \\ 2 & . & . & 9 & 5 \end{array}$$

$$\begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & . & . & . & . \\ 1 & 3 & 10 & 1 & . & . \\ 2 & . & . & 12 & 8 & 1 \end{array} \quad \text{and} \quad \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 6 & 12 & . & . & . \\ 1 & . & . & 11 & 3 & . \\ 2 & . & . & . & 3 & 1 \end{array}$$

Example 4.4. The Betti tables of Proposition 4.1 with $\deg \beta^S(N) = 13$ are the following:

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 2 & . & . & . \\ 1 & 2 & 15 & 13 & . \\ 2 & . & . & 1 & 3 \end{array} \quad \text{and its dual} \quad \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 3 & 1 & . & . \\ 1 & . & 13 & 15 & 2 \\ 2 & . & . & . & 2 \end{array}$$

Example 4.5. The Betti tables of Proposition 4.1 with $\text{pd } \beta^S(N) = 3$, that is, ACM-tables, are the four ACM-tables above, the tables

	0	1	2	3				0	1	2	3	
0	1	.	.	.	and			0	4	3	.	.
1	4	12	2	.				1	.	12	14	.
2	.	.	11	6				2	.	.	.	3

	0	1	2	3				0	1	2	3	
0	2	.	.	.	and			0	3	.	.	.
1	2	11	1	.				1	.	10	.	.
2	.	.	13	7				2	.	.	15	8

of degree $\deg \beta^S(N) = 14, 16, 17$ and 20 , respectively, and their duals.

In Section 5 we will construct unirational families of pairs (N, X) of a module and a cubic threefold X whose equation annihilates N , for many of the Betti tables above. Naturally we seek families which depend on at least 42 parameters modulo projectivities. Our approach is the following: The module N will sheafify to a line bundle $\mathcal{L}$ on an auxiliary (smooth and irreducible) curve E of degree $d_E = \deg \beta^S(N)$. The geometric genus g_E and the degree $\deg \mathcal{L}$ of the line bundle are not determined by $\beta^S(N)$. However, $h^0(\mathcal{L})$ and $h^1(\mathcal{L})$ are determined by $\beta^S(N)$, at least if we make some plausible assumptions on the local cohomology module

$$H_{\mathfrak{m}}^1(N) \cong \Gamma_*(\mathcal{L})/N.$$

Here $\mathfrak{m} \subset S$ denotes the homogeneous maximal ideal. It is natural to assume $h^0(\mathcal{O}_E(1)) = 5$. However, the speciality $h^1(\mathcal{O}_E(1))$ is another undetermined quantity. Our choice of $g_E, h^1(\mathcal{O}_E(1))$ and $\deg \mathcal{L}$ is motivated by a dimension count. We will construct generically reduced families of dimension ≥ 42 for the following Betti tables:

$d_E = 11$					$d_E = 14$					
	0	1	2	3		0	1	2	3	4
0	5	9	.	.	0	2	.	.	.	.
1	.	3	13	6	1	1	9	.	.	.
					2	.	.	14	9	1

	0	1	2	3	4		0	1	2	3
0	6	12	.	.	.	0	6	11	.	.
1	.	.	11	3	.	1	.	2	12	4
2	.	.	.	3	1	2	.	.	.	1

	0	1	2	3	4		0	1	2	3
0	1	.	.	.	.	0	7	15	4	.
1	3	10	1	.	.	1	.	.	8	3
2	.	.	12	8	1	2	.	.	.	1

The tables in the first column are realized by modules N such that the line bundle $\mathcal{L} = \tilde{N}$ has support on a curve E residual to a line in the cubic threefold. The tables in the second column are realized by modules on curves E of degree 14. The last table is an example not covered by Proposition 4.1.

5. Constructions

Let us call a matrix factorization (ϕ, ψ) of $18 \times (15 + 3)$ and $(15 + 3) \times 18$ matrices on a cubic threefold X *good*, if the complex as in Proposition 3.2 is a monad of an ideal sheaf of a *smooth* curve C of degree $d_C = 16$ and genus $g_C = 15$ such that $\mathcal{O}_C(1) \in W_{16}^4(C)$ is a *smooth isolated point*, that is, $(C, \mathcal{O}_C(1))$ does not lie in the ramification loci of $\widetilde{\mathcal{M}}_{15,16}^4 \rightarrow \mathcal{M}_{15}$. If (ϕ, ψ) is good, then we call $(\psi, \phi(-3)), (\psi^t, \phi^t), (\phi^t, \psi^t(-3))$ and twists of these matrix factorizations *good* as well. In our constructions below, an auxiliary curve E , a line bundle $\mathcal{L}$ on E , and a submodule $N \subset \Gamma_*(\mathcal{L})$ play a role. We will always denote by

$$d_E, g_E, d_{\mathcal{L}}$$

the degree $\deg \mathcal{O}_E(1)$ of E , the genus of E , and the degree of $\mathcal{L}$, respectively.

Perhaps the easiest case is the construction for a Betti table of type

	0	1	2	3
0	5	9	.	.
1	.	3	13	6

We have to choose for $(E, \mathcal{O}_E(1))$ a curve together with a very ample line bundle $\mathcal{O}_E(1)$ of degree $d_E = 11$ with $h^0(\mathcal{O}_E(1)) = 5$, a cubic form $f \in H^0(\mathcal{I}_E(3))$ up to a scalar, and the line bundle $\mathcal{L}$. By the Riemann–Roch theorem, we have

$$h^0(\mathcal{O}_E(1)) - h^1(\mathcal{O}_E(1)) = d_E + 1 - g_E,$$

so that we obtain $h^1(\mathcal{O}_E(1)) = g_E - 7$. Hence we expect that the pair $(E, \mathcal{O}_E(1))$ depends on

$$4g_E - 3 - h^0(\mathcal{O}_E(1))h^1(\mathcal{O}_E(1)) = 32 - g_E$$

parameters. Assuming that cubics cut a complete non-special linear series on E , we obtain

$$34 - (3d_E + 1 - g_E) = g_E$$

parameters for the choice of X . Finally, since N is an ACM-module, we have $N = \Gamma_*(\mathcal{L})$ and

$$h^1(\mathcal{L}) = h^0(\omega_E \otimes \mathcal{L}^{-1}) = \dim \operatorname{Ext}_S^3(N, S(-5))_0 = \beta_{3,5}^S(N) = 0.$$

Thus, the line bundle is non-special of degree $\deg \mathcal{L} = g_E - 1 + 5$ by the Riemann–Roch theorem, and depends on g_E parameters. Altogether we have

$$32 + g_E$$

parameters.

THEOREM 5.1. *There exists a 42-dimensional unirational family of tuples*

$$(E, \mathcal{O}_E(1), X, \mathcal{L}) \quad \text{with} \quad (d_E, g_E, d_{\mathcal{L}}) = (11, 10, 14)$$

consisting of a smooth curve E , a very ample line bundle $\mathcal{O}_E(1)$ of degree $d_E = 11$ with $h^0(\mathcal{O}_E(1)) = 5$, a smooth cubic hypersurface $X \subset \mathbb{P}(H^0(\mathcal{O}_E(1))) \cong \mathbb{P}^4$ containing the image of E , and a non-special line bundle $\mathcal{L}$ on E such that $N = \Gamma_(\mathcal{L})$ has an S -resolution with Betti table $\beta^S(N)$*

	0	1	2	3
0	5	9	.	.
1	.	3	13	6

such that for a general tuple the S_X -resolution of N gives a good matrix factorization of desired shape.

Proof. Since $h^1(\mathcal{O}_E(1)) = 3$, we expect that E has a plane model of degree $2g_E - 2 - d_E = 7$ and $\delta = \binom{6}{2} - g_E = 5$ double points. So we start with 5+10 general points $p_1, \dots, p_5, q_1, \dots, q_{10} \in \mathbb{P}^2$ and a curve $E' \subset \mathbb{P}^2$ of degree 7 with double points in $p_1, \dots, p_5$ and simple points in $q_1, \dots, q_{10}$. Let E be the normalization of E' , let $\mathcal{O}_E(1) = \omega_E(-H)$, where H denotes a general hyperplane section of $E' \subset \mathbb{P}^2$, and let $\mathcal{L} = \omega_E(q_1 + q_2 + q_3 - (q_4 + \dots + q_{10}))$. The complete linear system $|\mathcal{O}_E(1)|$ is cut out by plane cubics through $p_1, \dots, p_5$, so re-embeds E into $\mathbb{P}^4$ as a curve on a Del Pezzo surface Y of degree 4. Hence $H^0(Y, \mathcal{I}_{E/Y}(3)) \cong H^0(\mathbb{P}^2, \mathcal{I}_{\{p_1, \dots, p_5\}}(2))$ is 1-dimensional and $h^0(\mathbb{P}^4, \mathcal{I}_E(3)) = 2 \cdot 5 + 1 = g_E + 1$, as desired. Note that E is residual to a line in the complete intersection of the cubic with two quadrics. Counting parameters, we find $2 \cdot 15 - 8 = 22$ parameter for the choice of the points up to projectivities, $\binom{7+2}{2} - 3 \cdot 5 - 10 - 1 = 10$ parameters for the choice of E , and another 10 for the choice of X . So altogether we get the desired 42. Clearly our parameter space is unirational.

To verify for a general point in this unirational parameter space, that the curve E and the cubic X are smooth, that the module $N = \Gamma_*(\mathcal{L})$ has syzygies as expected, and that the matrix factorization leads to a smooth curve C of degree $d_C = 16$ and genus $g_C = 15$ such that $\mathcal{O}_C(1) \in W_{16}^4(C)$ is a smooth isolated point can be done by producing a single example with these properties, because these properties are open conditions. It is even enough to check this in an example defined over a finite prime field $\mathbb{F}_p$, since we may regard such an example as the reduction modulo p of an example defined over the integers. By semicontinuity the example over the generic point of $\text{Spec } \mathbb{Z}_{(p)}$ is an example defined over $\mathbb{Q}$ which has all desired properties. We pick our example over a moderate-size prime field at random, and check all assertions with the computer algebra system `Macaulay2` using the package [\[MF15\]](#). This computation completes the proof of Theorem 5.1 and also the proof of Theorem 1.1. $\square$

Remark 5.2. The reader might wonder why we did not try to construct a family of such modules N using a curve E of genus $g_E = 11$. In this case the line bundle $\mathcal{O}_E(2)$ of degree 22 is non-special, hence $E \subset \mathbb{P}^4$ would lie on at least $15 - (22 + 1 - 11) = 3$ quadrics, which by Bézout must have a surface Y in common of degree at most 3. Since $35 - (33 + 1 - 11) = 12$, we expect that $h^1(\mathcal{I}_E(3)) = 1$, that $h^0(\mathcal{I}_E(3)) = 13$, and that there are two linear syzygies among the quadrics. In this case, the surface Y would be a cubic scroll, and the equation of the cubic hypersurface X would be a linear combination of the quadrics, hence the determinant of a 3×3 linear matrix. So X would be singular. Since a general curve $C \in \mathcal{H}$ lies on a smooth cubic threefold by Theorem 5.1, the deduced family of curves of genus $g_C = 15$ cannot dominate $\mathcal{M}_{15}$.

Next we discuss the third table of Example 4.3. As before we set $\mathcal{L} = \tilde{N}$. Since $H_{\mathfrak{m}}^1(N)$ is dual to $\text{Ext}_S^4(N, S(-5))$, it is reasonable to assume $H_{\mathfrak{m}}^1(N) \cong K(-2)$, so that the Betti tables $\beta^S(N)$ and $\beta^S(\Gamma_*(\mathcal{L}))$ differ by a Koszul complex on the five linear forms:

$$\begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & . & . & . & . \\ 1 & 3 & 10 & 1 & . & . \\ 2 & . & . & 12 & 8 & 1 \end{array} \quad \text{and} \quad \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & . & . & . \\ 1 & 4 & 15 & 11 & . \\ 2 & . & . & 2 & 3 \end{array}$$

respectively. This time $\mathcal{L}$ is a line bundle with $h^0(\mathcal{L}) = 1$ and $h^1(\mathcal{L}) = 3$, and degree $\deg \mathcal{L} = g_E - 3$. So $\mathcal{L}$ is determined by the choice of $g_E - 3$ general points on E . On the other hand, the choice of $N \subset \Gamma_*(\mathcal{L})$ corresponds to choosing a 3-dimensional subspace of the 4-dimensional

space of generators of $\Gamma_*(\mathcal{L})$ in degree 1, that is, to a point in $\mathbb{P}^3$. Thus the pair $(N, \mathcal{L})$ depends, given $E \subset \mathbb{P}^4$, again on g_E parameters. Thus, if we choose $E \subset X$ as in the proof of Theorem 5.1 as a curve of genus $g_E = 10$ residual to a line, we get again a 42-dimensional family.

THEOREM 5.3. *There exists a 42-dimensional unirational family of tuples*

$$(E, \mathcal{O}_E(1), X, \mathcal{L}, N) \quad \text{with} \quad (d_E, g_E, d_{\mathcal{L}}) = (11, 10, 7)$$

consisting of a smooth curve of genus $g_E = 10$, a very ample line bundle $\mathcal{O}_E(1)$ of degree $d_E = 11$ with $h^0(\mathcal{O}_E(1)) = 5$, a smooth cubic hypersurface $X \subset \mathbb{P}(H^0(\mathcal{O}_E(1))) \cong \mathbb{P}^4$ containing the image of E , an effective line bundle $\mathcal{L}$ on E , and a submodule $N \subset \Gamma_*(\mathcal{L})$ such that N has an S -resolution with Betti table

	0	1	2	3	4
0	1	.	.	.	.
1	3	10	1	.	.
2	.	.	12	8	1

such that for general tuples the S_X -resolution of N gives a good matrix factorization of desired type.

Proof. This time we take E' as a septic with five nodes, passing through seven additional simple points, which we use to define the line bundle $\mathcal{L}$. The dimension count reads

$2(5 + 7) - 8$	$=$	16	for the points in the plane up to projectivities
$36 - 3 \cdot 5 - 7 - 1$	$=$	13	for E'
		3	for $N \subset \Gamma_*(\mathcal{L})$
		10	for the cubic $X \supset E$
		42	parameters altogether

That general choices lead to a tuple with all the desired properties follows again by a computation of a random example over a finite prime field. See [MF15] for details. $\square$

For the last table of Example 4.3, it is reasonable to assume $\text{Ext}_S^4(N, S(-6)) \cong S/(I_L + \mathfrak{m}^2)$, where I_L denotes the homogeneous ideal of a line L . Then $H_{\mathfrak{m}}^1(N)$ has Betti table

	0	1	2	3	4	5
0	2	9	15	11	3	.
1	.	.	1	3	3	1

Since $\beta^S(N)$ and $\beta^S(\Gamma_*(\mathcal{L}))$ differ by this table, we get

	0	1	2	3	4
0	6	12	.	.	.
1	.	.	11	3	.
2	.	.	.	3	1

and

	0	1	2	3
0	8	21	15	.
1	.	.	1	3

respectively, for these tables. Note that we can recover the line L from $\mathcal{L}$: its equations are given by the linear entries in the last syzygy matrix of $\Gamma_*(\mathcal{L})$. So with $N' = \text{Hom}_K(S/(I_L + \mathfrak{m}^2), K(-1))$ and $\phi \in \text{Hom}_S(\Gamma_*(\mathcal{L}), N')_0$ a surjective morphism, we can take $N = \ker \phi$. In all cases computed, we found $\dim \text{Hom}_S(\Gamma_*(\mathcal{L}), N')_0 = 1$, so that in these cases N is determined by $\mathcal{L}$. If this is true in general, then we can obtain another 42-dimensional unirational family as follows: Start again

with $5 + 10$ points $p_1, \dots, p_5, q_1, \dots, q_{10} \in \mathbb{P}^2$ and a general septic E' with nodes in $p_1, \dots, p_5$ and simple points in $q_1, \dots, q_{10}$, hence geometric genus $g_E = 10$. As the non-special line bundle $\mathcal{L}$ on E of degree $\deg \mathcal{L} = 17$ we can take $\mathcal{L} = \mathcal{O}_E(1) \otimes \mathcal{O}_E(q_1 + \dots + q_8 - q_9 - q_{10})$. Then $\Gamma_*(\mathcal{L})$ determines a line L and hence a module N' as above.

PROPOSITION 5.4. *If for general choices, $\text{Hom}_S(\Gamma_*(\mathcal{L}), N')_0 \neq 0$ holds for the construction above, then this gives a 42-dimensional unirational family of pairs (N, X) with invariants*

$$(d_E, g_E, d_{\mathcal{L}}) = (11, 10, 17)$$

such that for general tuples the S_X -syzygies give a good matrix factorization of desired shape.

Proof. This is another computer algebra verification documented in [MF15]. $\square$

Remark 5.5. Note that from our examples of Proposition 5.4, we can conclude by semi-continuity that the dimension $\dim \text{Hom}_S(\Gamma_*(\mathcal{L}), N')_0$ is at most 1 for general $\Gamma_*(\mathcal{L})$. It is very unlikely that $\text{Hom}_S(\Gamma_*(\mathcal{L}), N')_0 = 1$ does not hold for general choices, because this would mean that our randomly chosen examples, by accident all lie in a proper subfamily. Having tested several examples over a field of approximate size 10^4 , this seems nearly impossible. This is no rigorous proof, which might be actually be easy. We did not seriously try to prove this, in view of Proposition 6.1.

Next we discuss a family which rises from modules with Betti table $\beta^S(N)$

$$\begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 2 & . & . & . & . \\ 1 & 1 & 9 & . & . & . \\ 2 & . & . & 14 & 9 & 1 \end{array} \quad \text{and} \quad \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 2 & . & . & . \\ 1 & 2 & 14 & 10 & . \\ 2 & . & . & 4 & 4 \end{array}$$

for $\beta^S(\Gamma_*(\mathcal{L}))$ which differ by a Koszul complex. In this case E has degree $d_E = 14$ and assuming $h^0(\mathcal{O}_E(1)) = 5$, we obtain $h^1(\mathcal{O}_E(1)) = g_E - 10$; hence we expect $4g_E - 3 - h^0(\mathcal{O}_C(1)) \cdot h^1(\mathcal{O}_C(1)) = 47 - g_E$ parameters for the pair $(E, \mathcal{O}_E(1))$. The number of cubic hypersurfaces containing the image of E in $\mathbb{P}H^0(\mathcal{O}_C(1)) \cong \mathbb{P}^4$ is expected to be $g_E - 9$. The line bundle $\mathcal{L} = \tilde{N}$ has degree $\deg \mathcal{L} = g_E - 3$ and depends on $g_E - 2 \cdot 4$ parameters. Finally, choosing $N \subset \Gamma_*(\mathcal{L})$ corresponds to the choice of a point in $\mathbb{P}^1$, which gives one more parameter. Altogether we have $g_E + 31$ parameters. Choosing $g_E = 11$ we can hope for a dominant family. The model of $E \subset \mathbb{P}^3$ embedded by $\omega_E \otimes \mathcal{L}^{-1}$ is the space model of degree 12 used by Chang and Ran to prove the unirationality of $\mathcal{M}_{11}$. The line bundle $\mathcal{O}_E(1) \cong \omega_E(-(p_1 + \dots + p_6))$ is the Brill–Noether dual to an effective divisor. We do not know how to construct E together with six points in a unirational way. But over a finite field one can easily find points in E with a probabilistic method. Thus we are able to produce random elements in this family.

THEOREM 5.6. *There exists a 42-dimensional family of tuples*

$$(E, \mathcal{O}_E(1), X, \mathcal{L}, N) \quad \text{with} \quad (d_E, g_E, d_{\mathcal{L}}) = (14, 11, 8)$$

consisting of a smooth curve E , a very ample line bundle $\mathcal{O}_E(1)$ of degree $d_E = 14$ with $h^0(\mathcal{O}_E(1)) = 5$, a smooth cubic hypersurface $X \subset \mathbb{P}(H^0(\mathcal{O}_E(1))) \cong \mathbb{P}^4$ containing the image of E , a line bundle $\mathcal{L}$ on E such that $h^0(\mathcal{L}) = 2$, and a submodule $N \subset \Gamma_*(\mathcal{L})$ such that N has an S -resolution with Betti table $\beta^S(N)$

$$\begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 2 & . & . & . & . \\ 1 & 1 & 9 & . & . & . \\ 2 & . & . & 14 & 9 & 1 \end{array}$$

such that the S_X -resolution gives a good matrix factorization of desired type.

Proof. This follows from another computation over a finite field documented in [MF15]. $\square$

Our next construction is a family of modules N with Betti table

	0	1	2	3
0	6	11	.	.
1	.	2	12	4
2	.	.	.	1

The support E has degree $d_E = 14$. We will construct E as a curve residual to a rational normal curve R of degree $d_R = 4$ in a complete intersection $(2, 3, 3)$ of degree 18. Such curves have Betti table

	0	1	2	3
0	1	.	.	.
1	.	1	.	.
2	.	2	.	.
3	.	3	10	5

and genus $g_E = 15$. The line bundle $\mathcal{L} = \tilde{N}$ has to have degree $\deg \mathcal{L} = 19$ since $h^0(\mathcal{L}) - h^1(\mathcal{L}) = 6 - 1 = \deg \mathcal{L} + 1 - g_E$. So it has the form $\mathcal{L} = \omega_E(-D)$, where D is an effective divisor of degree 9 on E . A unirational construction of N runs as follows: Start with a rational normal curve $R \subset \mathbb{P}^4$ and choose one point p_0 . Choose a quadric Q containing R and p_0 . Choose eight lines ℓ_i through p_0 and take p_i as the second intersection point of $\ell_i \cap Q$. Then choose two general cubic hypersurfaces X, X' through $R \cup \{p_0, \dots, p_9\}$. The residual E of R in $Q \cap X \cap X'$ is the desired curve, and $D = p_0 + \dots + p_8$ is the desired effective divisor of degree 9 on E .

THEOREM 5.7. *Up to projectivities, there is a 47-dimensional unirational family of tuples*

$$(X, E, \mathcal{L}) \quad \text{with} \quad (d_E, g_E, d_{\mathcal{L}}) = (14, 15, 19)$$

consisting of a cubic hypersurface X , a curve E residual to a rational normal curve R of degree $d_R = 4$ in a complete intersection $Q \cap X \cap X'$, where Q is a quadric and X' is a further cubic hypersurface, and $\mathcal{L} = \omega_E(-D)$ for D an effective divisor of degree 9 on E such that $N = H_^0(\mathcal{L})$ has Betti table $\beta^S(N)$*

	0	1	2	3
0	6	11	.	.
1	.	2	12	4
2	.	.	.	1

For general choices, the S_X -resolution gives a good matrix factorization of desired type.

Proof. Most of the result follows from a computation in Macaulay2 documented in [MF15]. For the dimension count we note that the stabilizer of R in $\mathrm{PGL}(5)$ has dimension 3. Thus $R \cup \{p_0\}$ depends up to projectivities on one parameter (the cross ratio if we think of $\mathbb{P}^4 = \mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}(4)))$ as the linear system of quartic polynomials). Choosing Q gives four parameters since $h^0(\mathbb{P}^4, \mathcal{I}_{R \cup \{p_0\}}(2)) = 5$. The lines give $24 = 8 * 3$ parameters, and X gives another $12 = 6 * 5 - 8 - 9 - 1 = h^0(\mathcal{I}_{R \cup \{p_0, \dots, p_7\}}(3)) - 1$ parameters. Finally, the choice of X' gives $12 - 5 - 1 = 6$ further parameters, since the construction depends on the equations of X' only modulo the

equation of Q and X . Altogether this gives $1 + 4 + 24 + 12 + 6 = 47$ parameters. A tangent space computation at a general point shows that this space is generically smooth with its natural scheme structure. $\square$

THEOREM 5.8. *Up to projectivities, there is a 46-dimensional family of tuples*

$$(X, E, \mathcal{L}) \quad \text{with} \quad (d_E, g_E, d_{\mathcal{L}}) = (14, 14, 18)$$

consisting of a cubic hypersurface X , a curve E , and a line bundle $\mathcal{L} = \omega_E(-D)$ for D an effective divisor of degree 8 on E such that $N = H_^0(\mathcal{L})$ has Betti table $\beta^S(N)$*

	0	1	2	3
0	6	11	.	.
1	.	2	12	4
2	.	.	.	1

For general choices, the S_X -resolution gives a good matrix factorization of desired type.

Proof. A maximal-rank curve E of degree 14 and genus 14 lies on no quadric. However, since $h^1(E, \mathcal{O}_E(1)) = 4$, these curves have a model in $\mathbb{P}^3$ and the corresponding component of the Hilbert scheme is unirational. The space of pairs (E, X) up to projectivities is unirational of dimension 38. The choice of eight points on E gives further eight parameters. Checking an example by a computation in Macaulay2 documented in [MF15] implies the result. $\square$

Our last two examples concern the construction of matrix factorizations from modules not covered by Proposition 4.1. We will construct a unirational family of modules with Betti table

	0	1	2	3
0	7	15	4	.
1	.	.	8	3
2	.	.	.	1

whose support is as in the family of Theorem 5.7 a curve E which is residual to a rational normal curve R of degree 4 in a complete intersection $(2, 3, 3)$. This time the line bundle $\mathcal{L} = \tilde{N}$ must have degree $\deg \mathcal{L} = 20$ by the Riemann–Roch theorem. Thus $\mathcal{L} = \omega_E(-D)$, where D is an effective divisor of degree 8. By the same construction as for the family in Theorem 5.7, we get the following result.

THEOREM 5.9. *Up to projectivities, there is a 46-dimensional unirational family of tuples*

$$(X, E, \mathcal{L}) \quad \text{with} \quad (d_E, g_E, d_{\mathcal{L}}) = (14, 15, 20)$$

consisting of a cubic hypersurface X , a curve E residual to a rational normal curve R of degree $d_R = 4$ in a complete intersection $Q \cap X \cap X'$, where Q is a quadric and X' is a further cubic hypersurface, and $\mathcal{L} = \omega_E(-D)$ for D an effective divisor of degree 8 on E such that $N = H_^0(\mathcal{L})$ has Betti table $\beta^S(N)$*

	0	1	2	3
0	7	15	4	.
1	.	.	8	3
2	.	.	.	1

For general choices, the S_X -resolution gives a good matrix factorization of desired type.

Proof. Most of the results follow from a computation in `Macaulay2` documented in [MF15]. The dimension count gives $1+4+3\cdot 7+13+7 = 46$ parameters this time. A tangent space computation at a general point of the parameter space shows that this space is generically smooth with its natural scheme structure. $\square$

THEOREM 5.10. *Up to projectivities, there is 45-dimensional family of tuples*

$$(X, E, \mathcal{L}) \quad \text{with} \quad (d_E, g_E, d_{\mathcal{L}}) = (14, 14, 19)$$

consisting of a cubic hypersurface X , a curve E , and $\mathcal{L} = \omega_E(-D)$ for D an effective divisor of degree 7 on E such that $N = H_^0(\mathcal{L})$ has Betti table $\beta^S(N)$*

	0	1	2	3
0	7	15	4	.
1	.	.	8	3
2	.	.	.	1

For general choices, the S_X -resolution gives a good matrix factorization of desired type.

Proof. This follows by the same strategy as for the family in Theorem 5.8. $\square$

6. Tangent space computations

Let (N, X) be a pair consisting of an auxiliary module N in one of the examples of Section 5 and a cubic hypersurface X whose equation annihilates N . We would like to estimate the dimension of the family of MCM approximations M of (N, X) , and hence the dimension of the family of curves of genus 15 obtained from our family of pairs (N, X) . For fixed X , the group $\text{Ext}_{S_X}^1(M, M)_0$ is the space of infinitesimal homogeneous deformations of M (or the matrix factorization). Since cubic threefolds depend on 10 parameters up to projectivities, we expect that $\dim \text{Ext}_{S_X}^1(M, M)_0 = 32$ for general choices of (N, X) . However, the association

$$(N, X) \longmapsto (M, X) \tag{1}$$

might not be surjective. Let $P = \ker(M \rightarrow N)$ be the kernel of the MCM approximation, which turns out to need no free summand in all cases. Then P sits in a short exact sequence

$$0 \longrightarrow P \longrightarrow M \longrightarrow N \longrightarrow 0$$

and has finite projective dimension as an S_X -module. We have a diagram

$$\begin{array}{ccccccc} \text{Ext}_{S_X}^1(M, P) & \longrightarrow & \text{Ext}_{S_X}^1(M, M) & \longrightarrow & \text{Ext}_{S_X}^1(M, N) & \longrightarrow & \text{Ext}_{S_X}^2(M, P) \\ & & & & \uparrow & & \\ & & & & \text{Ext}_{S_X}^1(N, N) & & \\ & & & & \uparrow & & \\ & & & & \text{Hom}_{S_X}(P, N) & & \end{array}$$

By the periodicity of the S_X -resolution of M , we have

$$\text{Ext}_{S_X}^i(M, P) \cong \text{Ext}_{S_X}^{i+2}(M(3), P) \quad \text{for } i \geq 1.$$

Since $\text{Ext}_{S_X}^i(M, S_X) = 0$ for $i \geq 1$, we obtain $\text{Ext}_{S_X}^i(M, P) = 0$ for $i \geq 1$, because P has finite projective dimension. Thus

$$\text{Ext}_{S_X}^1(M, M) \cong \text{Ext}_{S_X}^1(M, N).$$

For our families constructed in Section 5 in Theorems 5.1, 5.3, and 5.6 and Proposition 5.4, we expect $\dim \text{Ext}_{S_X}^1(N, N)_0 = 32$ as the total family of (N, X) depends on 42 parameters, and cubics on 10. As it turns out, $\dim \text{Ext}_{S_X}^1(N, N)_0 = 32$ holds for examples from these four families. So the map (1) is surjective on the level of tangent spaces if and only if the map

$$\text{Hom}_{S_X}(P, N)_0 \longrightarrow \text{Ext}_{S_X}^1(N, N)_0$$

is 0. For the examples from the families in Theorems 5.7, 5.8, 5.9, and 5.10, this map cannot be trivial, since $\dim \text{Ext}_{S_X}^1(M, N)_0 = 32$, while $\text{Ext}_{S_X}^1(N, N)_0$ has dimension 37, 36, 36, and 35, respectively. In all eight families of Section 5 we find that for general choices the map

$$\text{Hom}_{S_X}(N, N)_0 \longrightarrow \text{Hom}_{S_X}(M, N)_0$$

is an isomorphism between 1-dimensional spaces. Hence there is only one map $M \rightarrow N$, and $\dim \text{Hom}_{S_X}(P, N)_0$ is the dimension of the kernel

$$\text{Ext}_{S_X}^1(N, N)_0 \longrightarrow \text{Ext}_{S_X}^1(M, N)_0.$$

PROPOSITION 6.1. *The dimension $\dim \text{Hom}_{S_X}(M, N)_0$ is 1 in a randomly chosen example over a moderate-sized finite field in all eight families from Section 5. Moreover,*

$$\dim \text{Hom}_{S_X}(P, N)_0 = \begin{cases} 0 \\ 3 \\ 4 \\ 6 \\ 7 \end{cases} \quad \text{for some examples from} \quad \begin{cases} 5.6 \\ 5.1, 5.4, 5.10 \\ 5.8 \\ 5.3, 5.7 \\ 5.9 \end{cases}$$

and

$$\dim \text{Ext}_{S_X}^1(N, N)_0 = \begin{cases} 32 \\ 35 \\ 36 \\ 37 \end{cases} \quad \text{in an open set of the families in} \quad \begin{cases} 5.1, 5.4, 5.6 \\ 5.10 \\ 5.9, 5.8 \\ 5.7 \end{cases}$$

Proof. The proposition follows by computations documented in [MF15]. For the second set of dimensions note that the tangent dimension $\dim \text{Ext}_{S_X}^1(N, N)_0$ cannot be smaller by the dimension count in our families. $\square$

COROLLARY 6.2. *The family from*

$$\text{Theorem} \begin{cases} 5.1, 5.9 \\ 5.7 \\ 5.6, 5.8, 5.10 \end{cases} \quad \text{maps onto an (at least)} \quad \begin{cases} 39 \\ 41 \\ 42 \end{cases} \text{-dimensional}$$

subvariety of $\mathcal{M}_{15}$, respectively. The non-dominant families are unirational; the dominant family from Theorem 5.6 is uniruled.

Proof. In principle, it could be that the map from our families to $\mathcal{M}_{15}$ is ramified at the given randomly chosen point. So the tangent space computation of Proposition 6.1 gives only a lower

bound for the dimension of the family in $\mathcal{M}_{15}$. In reality it is very unlikely that equality does not hold. The last statement follows from Theorem 1.3. $\square$

Proof of Theorem 1.3 and Theorem 1.2. The last choice in the construction of the dominant family from Theorem 5.6 is the choice of a point in $\mathbb{P}^1$. Thus $\widetilde{\mathcal{M}}_{15,16}^4$ is ruled by lines. The same holds for a component of the Hilbert scheme of genus 15 and degree 16 curves on a general cubic threefold. $\square$

Proof of Theorem 1.4. We can use the dominant family from Theorem 5.6, Theorem 5.8, or Theorem 5.10. For the family from Theorem 5.6, all computations but the choice of an effective divisor of degree 6 on E are computations with modules of fixed degree and regularity in a fixed number of variables. So these computations run in a number of field operations in $\mathbb{F}_q$ independent of the size of q . Hence they contribute with $O((\log q)^2)$ to the running time. The computation of an effective divisor uses factorization of a univariate polynomial of fixed degree, hence Berlekamp's algorithm, which runs in $O((\log q)^3)$ field operations in $\mathbb{F}_q$; see, for example, [GG03, Theorem 14.14]. $\square$

Remark 6.3. The points in $\mathcal{M}_{15}(\mathbb{F}_q)$ which we can find with the probabilistic algorithm for elements from Theorem 5.6 form only a fraction. If B denotes the parameter space of the family, then the maps $B \rightarrow \mathcal{M}_{15,16}^4 \rightarrow \mathcal{M}_{15}$ are both generically finite. Thus, it is reasonable to expect about $0.39 \approx 0.63^2$ of the points in $\mathcal{M}_{15}(\mathbb{F}_q)$ to be in the image of $B(\mathbb{F}_q)$ for reasonably large q . See [EHS15, Section 2] for a discussion.

Remark 6.4. The construction of the families from Theorems 5.7 and 5.9 has some easy variants. Instead of starting with a rational normal curve R of degree 4, we might start with some other curves R of degree 4 and take as degree 14 curves E the residual in a complete intersection $(2, 3, 3)$. We have inspected the cases where R is

- (i) an elliptic curve R_1^4 of degree 4;
- (ii) a rational normal curve R_0^4 (the case of Theorem 5.7 and 5.9);
- (iii) the disjoint union $R_3^1 \cup L$ of a plane elliptic curve and a line;
- (iv) the disjoint union $R_0^3 \cup L$ of a twisted cubic and a line;
- (v) the disjoint union $C_1 \cup C_2$ of two conics;
- (vi) the disjoint union $C \cup L_1 \cup L_2$ of a conic and two lines;
- (vii) the disjoint union $L_1 \cup \dots \cup L_4$ of four lines.

None of these 12 further families dominates $\mathcal{M}_{15}$. Indeed, except for the family from Theorem 5.7, which is 41-dimensional, and its variant (iv), which turned out to be 40-dimensional, all of these unirational families are of dimension at most 39, with equality in five further cases; see [MF15]. One natural explanation why only the not obviously unirational families are dominant, and nearly all other constructions lead to 39-dimensional families, could be Conjecture 1.5. Of course the evidence for this conjecture is rather weak.

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