Weight functions on non-Archimedean analytic spaces and the Kontsevich–Soibelman skeleton

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ABSTRACT

We associate a weight function to pairs (X, ω) consisting of a smooth and proper variety X over a complete discretely valued field and a pluricanonical form ω on X . This weight function is a real-valued function on the non-Archimedean analytification of X . It is piecewise affine on the skeleton of any regular model with strict normal crossings of X , and strictly ascending as one moves away from the skeleton. We apply these properties to the study of the Kontsevich–Soibelman skeleton of (X, ω) , and we prove that this skeleton is connected when X has geometric genus one and ω is a canonical form on X . This result can be viewed as an analog of the Shokurov–Kollár connectedness theorem in birational geometry.

1. Introduction

1.1 Skeleta of non-Archimedean spaces

An important property of Berkovich spaces over non-Archimedean fields is that, in many geometric situations, they can be contracted onto some subspace with piecewise affine structure, a so-called skeleton. These skeleta are usually constructed by choosing appropriate formal models for the space, associating a simplicial complex to the reduction of the formal model and embedding its geometric realization into the Berkovich space. For instance, if C is a smooth projective curve of genus at least two over an algebraically closed non-Archimedean field, then it has a canonical skeleton, which is homeomorphic to the dual graph of the special fiber of its stable model [Ber90, § 4]. Likewise, if (X, x) is a normal surface singularity over a perfect field k , then one can endow k with its trivial absolute value and construct a k -analytic punctured tubular neighborhood of x in X by removing x from the generic fiber of the formal k -scheme $\mathrm{Spf} \hat{\mathcal{O}}_{X,x}$. This k -analytic space contains a canonical skeleton, homeomorphic to the product of $\mathbb{R}_{>0}$ with the dual graph of the exceptional divisor of the minimal log-resolution of (X, x) [Thu07].

In higher dimensions, one can still associate skeleta to models with normal crossings (or to so-called pluristable models as in [Ber99]). However, it is no longer clear how to construct a canonical skeleton, since we usually cannot find a canonical model with normal crossings. The

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aim of this paper is to show how one can identify certain essential pieces that must appear in every skeleton by means of weight functions associated to pluricanonical forms.

1.2 The work of Kontsevich and Soibelman

One of the starting points of this work was the following construction by Kontsevich and Soibelman [KS06]. Let X be a smooth projective family of varieties over a punctured disc around the origin of the complex plane, and let ω be a relative differential form of maximal degree on X . Let t be a local coordinate on $\mathbb{C}$ at 0. Kontsevich and Soibelman defined a skeleton in the analytification of X over $\mathbb{C}((t))$ by taking the closure of the set of divisorial valuations that satisfy a certain minimality property with respect to the differential form ω . Their goal was to find a non-Archimedean interpretation of mirror symmetry. They studied in detail the maximally unipotent semi-stable degenerations of $K3$ -surfaces, in which case the skeleton is homeomorphic to a two-dimensional sphere, and described how a degeneration can be reconstructed from the skeleton, equipped with a certain affine structure. This idea was further developed by Gross and Siebert in their theory of toric degenerations, using tropical and logarithmic geometry instead of non-Archimedean geometry; see for instance the survey paper [Gro13].

1.3 The weight function

We generalize the definition of the Kontsevich–Soibelman skeleton to smooth varieties X over a complete discretely valued field K , endowed with a non-zero pluricanonical form ω . We define the weight of ω at a divisorial point of X^{an} (that is, a point corresponding to a divisorial valuation on the function field of X) and define the skeleton $\text{Sk}(X, \omega)$ of the pair (X, ω) as the closure of the set of divisorial points with minimal weight. A precise definition is given in § 4.7.1 and it is compared to the one of Kontsevich and Soibelman in § 4.7.3.

Next, we show how the weight function can be extended to the Berkovich skeleton associated to any *sncd*-model of X (a regular model whose special fiber has strict normal crossings), and even to the entire space X^{an} if K has residue characteristic zero or X is a curve. A remarkable property of this weight function is that it is piecewise affine on the Berkovich skeleton $\text{Sk}(\mathcal{X})$ associated to any proper *sncd*-model $\mathcal{X}$, and strictly descending under the retraction from X^{an} to $\text{Sk}(\mathcal{X})$ (Proposition 4.5.5). We use this property to show that $\text{Sk}(X, \omega)$ is the union of the faces of the Berkovich skeleton of $\mathcal{X}$ on which the weight function is constant and of minimal value (Theorem 4.7.5). This generalizes Theorem 3 in [KS06, § 6.6]; whereas the proof in that paper relied on the weak factorization theorem, we only use elementary computations on divisorial valuations and approximation of arbitrary points on X^{an} by divisorial points.

1.4 The connectedness theorem

Besides the construction of the weight function, the other main result of this paper is a connectedness theorem for the skeleton $\text{Sk}(X, \omega)$ of a smooth and proper K -variety X of geometric genus one (for instance, a Calabi–Yau variety) endowed with a non-zero differential form of maximal degree ω . This skeleton does not depend on ω , since $\text{Sk}(X, \omega)$ is invariant under multiplication of ω by a non-zero scalar. We show that $\text{Sk}(X, \omega)$ is always connected if K has residue characteristic zero (Theorem 5.5.3). Our proof is based on a variant of Kollár’s torsion-free theorem for schemes over power series rings (Theorem 5.3.1), which we deduce from the torsion-free theorem for complex varieties by means of Greenberg approximation.

1.5 The relation with birational geometry

All these constructions and results have natural analogs in the birational geometry of complex varieties, replacing the pair (X, ω) by a smooth complex variety Y equipped with a coherent ideal sheaf $\mathcal{I}$ and *sncd*-models by log resolutions. In particular, one can define in a very similar way a weight function on the non-Archimedean punctured tubular neighborhood of the zero locus of $\mathcal{I}$ in X ; this is explained in Section 6, together with the close relation with the constructions in [BFJ08] and [JM12]. If $h: Y' \rightarrow Y$ is a log resolution of $\mathcal{I}$ and E is an irreducible component of the zero locus $Z(\mathcal{I}\mathcal{O}_{Y'})$ of $\mathcal{I}\mathcal{O}_{Y'}$, then the value of the weight function at a divisorial point associated to E is equal to μ/N , where $\mu - 1$ is the multiplicity of E in the relative canonical divisor $K_{Y'/Y}$, and N is the multiplicity of E in $Z(\mathcal{I}\mathcal{O}_{Y'})$. This is a classical invariant in birational geometry, closely related to the log discrepancy of $(X, \mathcal{I})$ at E , and its infimum over all possible log resolutions h and divisors E is the log canonical threshold of $(X, \mathcal{I})$. The counterpart of the Kontsevich–Soibelman skeleton coincides with the dual complex of the union of irreducible components of $Z(\mathcal{I}\mathcal{O}_{Y'})$ that compute the log canonical threshold, and our connectedness theorem translates into the connectedness theorem of Shokurov and Kollár; see Section 6.4. Of course, the latter result was our main source of inspiration for the proof of the connectedness theorem for smooth and proper K -varieties of geometric genus one.

1.6 Connections with the existing literature

As we have explained in Section 1.2, the principal source of inspiration for this paper is the construction of the skeleton by Kontsevich and Soibelman in [KS06]. We generalize this construction in two ways. First, we attach skeleta to pluricanonical forms, instead of only canonical forms. This generalization is straightforward and does not require any new ideas. Second, we work over an arbitrary complete discretely valued field K instead of over the field of germs of meromorphic functions at the origin of the complex plane. For the computation of the skeleton in Section 4.7, we need to assume the existence of a proper *sncd*-model, but our main improvement is that we eliminate the use of the weak factorization theorem in the proof of Theorem 3 in [KS06, § 6.6]. To our best knowledge, weak factorization for models of K -varieties in the form needed for this proof is not even available in the literature if K has equal characteristic zero (although it is very likely that it can be deduced from [AKMW02]). In positive or mixed characteristic, it is a much heavier assumption than the existence of one proper *sncd*-model for a fixed proper and smooth K -variety X (and it might be too much to ask for if the residue field of K is not perfect). Another aim of our exposition is to supply full details (some of the proofs were only sketched in [KS06]) because we believe that the Kontsevich–Soibelman skeleton is an object of broad interest.

The results in Sections 2 and 4 owe much to the papers [BFJ08, JM12]. As we explain in Section 6, the weight function from Section 4 is closely related to the thinness function of [BFJ08] and the log discrepancy function of [JM12], which are defined on certain valuation spaces attached to regular schemes of characteristic zero, and the basic properties of the weight function that we prove in this paper are very similar to the results in [BFJ08, JM12]. Likewise, if K has equal characteristic zero, the results on the Berkovich skeleton in Section 3 can be found in [BFJ12, § 3]. More generally, if K has equal characteristic $p \geq 0$, then one can obtain most statements in Sections 2, 3 and 4 by easy adaptations of the arguments in [BFJ08, BFJ12, JM12]. This is not the case when K has mixed characteristic: many proofs in [BFJ08, BFJ12, JM12] use that the completed local rings of regular schemes over a field are rings of formal power series over a field, which is of course false in the mixed characteristic case. The essential extra ingredients we need are Proposition 2.4.4, which is used to define the notion of monomial point in mixed

characteristic, and the calculus of canonical sheaves in Section 4.1, in particular Proposition 4.2.1. To keep the paper self-contained, and accessible to a broad scope of readers, we have chosen to give careful proofs of all the statements in Sections 2, 3 and 4.

1.7 Further questions

As we have seen, the skeleton of a pair (X, ω) as above is contained in the Berkovich skeleton of every regular proper model with strict normal crossings. We define the essential skeleton of X as the union of the skeleta $\mathrm{Sk}(X, \omega)$ as ω runs through the set of non-zero pluricanonical forms on X (Section 4.9). It would be quite interesting to know whether one can define a suitable class of models of X whose skeleta coincide with the essential skeleton (assuming that the Kodaira dimension of X is non-negative). We are investigating this question in an ongoing project.

1.8 Structure of the paper

To conclude this introduction, we give a brief survey of the structure of the paper. In Section 2 we introduce divisorial and monomial points on analytic spaces and we prove some basic properties. In Section 3 we explain the construction of the Berkovich skeleton associated to a regular model with strict normal crossings and we define its piecewise affine structure. This is a fairly straightforward generalization of the construction by Berkovich for pluristable formal schemes, but since the non-semistable case is not covered by the existing literature, we include some details here. The main new result is Proposition 3.1.7, which says that every proper morphism of models gives rise to an inclusion of skeleta. The core of the paper is Section 4, where we construct the weight function and prove its fundamental properties; see in particular Proposition 4.5.5. The weight function is constructed in several steps, first defining it on divisorial and monomial points and then extending it to the entire analytic space by approximation. The applications to the Kontsevich–Soibelman skeleton are discussed in Section 4.7. In Section 5 we deduce a variant of Kollár’s torsion-free theorem for schemes over formal power series and use it to prove our connectedness theorem for skeleta of varieties of geometric genus one. Finally, in Section 6, we sketch the analogs of these results in the setting of complex birational geometry.

1.9 Notation

1.9.1 We denote by R a complete discrete valuation ring with residue field k and quotient field K . We denote by $\mathfrak{m}$ the maximal ideal in R and by v_K the valuation

$$K^\times \rightarrow \mathbb{Z},$$

normalized in such a way that v_K is surjective. We define an absolute value $|\cdot|_K$ on K by setting

$$|x|_K = \exp(-v_K(x))$$

for all x in $K^\times$. This absolute value turns K into a complete non-Archimedean field.

1.9.2 All schemes in this paper are assumed to be separated. We will consider the special fiber functor

$$(\cdot)_k: (R - \mathrm{Sch}) \rightarrow (k - \mathrm{Sch}), \quad \mathcal{X} \mapsto \mathcal{X}_k = \mathcal{X} \times_R k$$

from the category of R -schemes to the category of k -schemes, as well as the generic fiber functor

$$(\cdot)_K: (R - \mathrm{Sch}) \rightarrow (K - \mathrm{Sch}), \quad \mathcal{X} \mapsto \mathcal{X}_K = \mathcal{X} \times_R K$$

from the category of R -schemes to the category of K -schemes.

2. Monomial points on K -analytic spaces

2.1 Birational points

2.1.1 Let X be an integral K -scheme of finite type. Its analytification X^{an} is endowed with a canonical morphism of locally ringed spaces

$$i: X^{\text{an}} \rightarrow X.$$

For every point x of X^{an} , we denote by $K(x)$ the residue field of X at $i(x)$. This field carries a natural valuation, and the residue field $\mathcal{H}(x)$ of X^{an} at x is the completion of $K(x)$ with respect to this valuation. The dimension of the $\mathbb{Q}$ -vector space

$$|\mathcal{H}(x)^\times| \otimes_{\mathbb{Z}} \mathbb{Q}$$

will be called the *rational rank* of X at x .

2.1.2 We call a point of X^{an} *birational* if its image in X is the generic point of X . We will denote the set of birational points of X^{an} by X^{bir} . For every birational point x on X^{an} we can define a real valuation v_x on the function field $K(X)$ of X by

$$v_x: K(X)^\times \rightarrow \mathbb{R}, \quad f \mapsto -\ln |f(x)|.$$

The map $x \mapsto v_x$ is a bijection between X^{bir} and the set of real valuations $K(X)^\times \rightarrow \mathbb{R}$ that extend the valuation v_K on K . The induced topology on $X^{\text{bir}} \subset X^{\text{an}}$ is the weakest topology such that, for every non-zero rational function f on X , the map

$$X^{\text{bir}} \rightarrow \mathbb{R}: x \mapsto v_x(f)$$

is continuous. It is clear that every birational map of integral K -varieties $Y \dashrightarrow X$ induces a homeomorphism $Y^{\text{bir}} \rightarrow X^{\text{bir}}$.

2.2 Models

2.2.1 Let X be a normal integral K -scheme of finite type. An R -model for X is a normal flat R -scheme of finite type $\mathcal{X}$ endowed with an isomorphism of K -schemes $\mathcal{X}_K \rightarrow X$. Note that we do not impose any properness conditions on X or $\mathcal{X}$. If $\mathcal{X}$ and $\mathcal{Y}$ are R -models of X , then a morphism of R -schemes $h: \mathcal{X} \rightarrow \mathcal{Y}$ is called a *morphism of R -models* of X if $h_K: \mathcal{Y}_K \rightarrow \mathcal{X}_K$ is an isomorphism that commutes with the isomorphisms $\mathcal{X}_K \rightarrow X$ and $\mathcal{Y}_K \rightarrow X$. Thus there exists at most one morphism of R -models $\mathcal{Y} \rightarrow \mathcal{X}$. If it exists, then we say that $\mathcal{Y}$ *dominates* $\mathcal{X}$.

2.2.2 For every R -scheme $\mathcal{X}$, we denote by $\widehat{\mathcal{X}}$ its $\mathfrak{m}$ -adic formal completion. If $\mathcal{X}$ is an R -model of X , then $\widehat{\mathcal{X}}$ is a flat separated formal R -scheme of finite type, and we can consider its generic fiber $\widehat{\mathcal{X}}_\eta$ in the category of K -analytic spaces. This is a compact analytic domain in the analytification X^{an} of X . A point x of X^{an} belongs to $\widehat{\mathcal{X}}_\eta$ if and only if the morphism

$$\text{Spec} \mathcal{H}(x) \rightarrow X$$

extends to a morphism

$$\text{Spec} \mathcal{H}(x)^\circ \rightarrow \mathcal{X},$$

where $\mathcal{H}(x)^\circ$ denotes the valuation ring of the valued field $\mathcal{H}(x)$. Thus $X^{\text{an}} = \widehat{\mathcal{X}}_\eta$ if $\mathcal{X}$ is proper over R . The generic fiber $\widehat{\mathcal{X}}_\eta$ is endowed with an anti-continuous reduction map

$$\text{red}_{\mathcal{X}}: \widehat{\mathcal{X}}_\eta \rightarrow \mathcal{X}_k$$

that sends a point x to the image of the closed point of $\mathrm{Spec} \mathcal{H}(x)^o$ under the morphism

$$\mathrm{Spec} \mathcal{H}(x)^o \rightarrow \mathcal{X}.$$

In particular, a birational point x of X^{an} belongs to $\widehat{\mathcal{X}}_\eta$ if and only if the valuation v_x has a center on $\mathcal{X}$, and in that case, the reduction map $\mathrm{red}_\mathcal{X}$ sends x to the center of v_x .

2.2.3 Assume that $\mathcal{X}$ is a normal integral R -scheme of finite type. Let x be a point in $\widehat{\mathcal{X}}_\eta$ and let D be a Cartier divisor on $\mathcal{X}$ whose support does not contain $i(x)$. Then we set

$$v_x(D) = -\ln |f(x)|$$

where f is any element of $K(X)^\times$ such that, locally at $\mathrm{red}_\mathcal{X}(x)$, we have $D = \mathrm{div}(f)$. It is obvious that the function $v_x(\cdot)$ is $\mathbb{Z}$ -linear in D and that it behaves well with respect to pull-backs: if $h: \mathcal{X}' \rightarrow \mathcal{X}$ is a proper morphism of R -models of X , then $v_x(D) = v_x(h^*D)$ for every Cartier divisor D on $\mathcal{X}$.

PROPOSITION 2.2.4. *Assume that $\mathcal{X}$ is a normal integral R -scheme of finite type. Then for every Cartier divisor D on $\mathcal{X}$, the function*

$$\{x \in \widehat{\mathcal{X}}_\eta \mid i(x) \notin \mathrm{Supp}(D)\} \rightarrow \mathbb{R}, \quad x \mapsto v_x(D)$$

is continuous.

Proof. For every open subscheme $\mathcal{Y}$ of $\mathcal{X}$, the space $\widehat{\mathcal{Y}}_\eta$ is a closed analytic domain of $\widehat{\mathcal{X}}_\eta$ by anti-continuity of the reduction map

$$\mathrm{red}_\mathcal{X}: \widehat{\mathcal{X}}_\eta \rightarrow \mathcal{X}_k$$

and the fact that $\widehat{\mathcal{Y}}_\eta = \mathrm{red}_\mathcal{X}^{-1}(\mathcal{Y}_k)$ [Ber94, § 1]. Thus we may assume $D = \mathrm{div}(f)$ for some rational function f on $\mathcal{X}$. Then

$$v_x(D) = -\ln |f(x)|$$

is a continuous function in x . □

2.2.5 Let X be a connected regular K -scheme of finite type. An *sncd*-model for X is a regular R -model $\mathcal{X}$ such that $\mathcal{X}_k$ is a divisor with strict normal crossings. Again, we do not impose any properness conditions on X or $\mathcal{X}$; for instance, according to our definition, X is an *sncd*-model of itself. If X is proper and either k has characteristic zero or X is a curve, then every proper R -model of X can be dominated by a proper *sncd*-model, by Hironaka's resolution of singularities. If k has positive characteristic and X has dimension at least two, the existence of a proper *sncd*-model is not known.

2.3 The Zariski Riemann space

2.3.1 Let X be a normal integral proper K -scheme. We denote by $\mathcal{M}_X$ the category of proper R -models $\mathcal{X}$ of X , where the morphisms are morphisms of R -models. We can dominate any pair of proper R -models $\mathcal{X}, \mathcal{X}'$ by a common proper R -model of X , and we have already observed that there exists at most one morphism of R -models $\mathcal{X}' \rightarrow \mathcal{X}$. These properties imply that the category $\mathcal{M}_X$ is cofiltered. The *Zariski Riemann space* of X is defined as the limit

$$X^{\mathrm{ZR}} = \lim_{\mathcal{X} \in \mathcal{M}_X} \mathcal{X}_k$$

in the category of locally ringed spaces.

PROPOSITION 2.3.2. *The map $j: X^{\text{an}} \rightarrow X^{\text{ZR}}$ induced by the reduction maps $\text{red}_{\mathcal{X}}: X^{\text{an}} \rightarrow \mathcal{X}_k$ is injective. It has a continuous retraction $r: X^{\text{ZR}} \rightarrow X^{\text{an}}$ such that the topology on X^{an} is the quotient topology with respect to r and such that, for every point x in X^{ZR} and every proper R -model $\mathcal{X}$ of X , the image of x under the natural projection $p_{\mathcal{X}}: X^{\text{ZR}} \rightarrow \mathcal{X}_k$ lies in the closure of $\{\text{red}_{\mathcal{X}} \circ r(x)\}$.*

Proof. Combining Theorems 3.4 and 3.5 in [vdPS95], we obtain a natural surjection $r: X^{\text{ZR}} \rightarrow X^{\text{an}}$ with all the required properties. To be precise, the results in [vdPS95] are formulated for affinoid rigid K -varieties, but as noted at the end of [vdPS95], they extend immediately to quasi-separated and quasi-compact rigid K -varieties. To pass to our algebraic setting, it then suffices to observe that the algebraizable formal R -models of X^{an} are cofinal in the category of admissible formal R -models: if $\mathcal{X}$ is a proper R -model of X , then by Raynaud's theorem [BL93, Theorem 4.1] the admissible blow-ups of the $\mathfrak{m}$ -adic completion $\widehat{\mathcal{X}}$ of $\mathcal{X}$ are cofinal in the category of admissible formal R -models of X . The center of such a blow-up is a closed subscheme of

$$\widehat{\mathcal{X}} \times_R (R/\mathfrak{m}^n) \cong \mathcal{X} \times_R (R/\mathfrak{m}^n)$$

for some $n > 0$. In particular, admissible blow-ups are algebraizable (simply blow up the corresponding closed subscheme of $\mathcal{X}$). $\square$

COROLLARY 2.3.3. *Let Z be a closed subscheme of X , and let x be any point of $(X \setminus Z)^{\text{an}}$. Then there exists a proper R -model $\mathcal{X}$ of X such that the closure of $\text{red}_{\mathcal{X}}(x)$ in $\mathcal{X}_k$ is disjoint from the closure of Z in $\mathcal{X}$.*

Proof. The fiber $r^{-1}(x)$ is closed in X^{ZR} and disjoint from $j(Z^{\text{an}})$. The closure of $\{j(x)\}$ in X^{ZR} is the intersection of the sets $p_{\mathcal{X}}^{-1}(C_{\mathcal{X}})$ where $\mathcal{X}$ runs through the proper R -models of X , $C_{\mathcal{X}}$ denotes the closure of $\text{red}_{\mathcal{X}}(x)$ in $\mathcal{X}_k$ and $p_{\mathcal{X}}: X^{\text{ZR}} \rightarrow \mathcal{X}_k$ is the natural projection. Thus we can find a proper R -model $\mathcal{X}$ of X such that $\text{red}_{\mathcal{X}}^{-1}(C_{\mathcal{X}})$ is disjoint from $j(Z^{\text{an}})$. If we denote by $\mathfrak{Z}$ the $\mathfrak{m}$ -adic completion of the schematic closure of Z in $\mathcal{X}$, then $\mathfrak{Z}_{\eta} = Z^{\text{an}}$ by properness of Z and the reduction map $Z^{\text{an}} \rightarrow \mathfrak{Z}_k$ is surjective because $\mathfrak{Z}$ is R -flat (its image is closed by [Ber94, p. 542] and it contains all the closed points of $\mathfrak{Z}_k$ by [Liu02, Corollary 10.1.38]). It follows that $C_{\mathcal{X}}$ must be disjoint from the closure of Z in $\mathcal{X}$. $\square$

2.4 Divisorial and monomial points

2.4.1 Let X be a normal integral K -scheme of finite type. Let $\mathcal{X}$ be an R -model for X and let E be an irreducible component of $\mathcal{X}_k$. Denote by ξ the generic point of E . The local ring $\mathcal{O}_{\mathcal{X}, \xi}$ is a discrete valuation ring with fraction field $K(X)$, the field of rational functions on X . We denote by v_E the associated discrete valuation on $K(X)$, normalized in such a way that its restriction to K coincides with v_K . This is the unique valuation on $\mathcal{X}$ that extends v_K and is centered at ξ . The index of $\mathbb{Z} = |K^{\times}|$ in the value group of v_E is equal to the multiplicity of E in the divisor $\mathcal{X}_k$. The valuation v_E defines a birational point x of X^{an} such that $v_x = v_E$. We call this point the divisorial point associated to the pair $(\mathcal{X}, E)$. It is the unique point of $\text{red}_{\mathcal{X}}^{-1}(\xi)$. We call the pair $(\mathcal{X}, E)$ a divisorial presentation for x . Removing a closed subset of $\mathcal{X}_k$ that does not contain E has no influence on the divisorial valuation v_E , so that we can always assume that $\mathcal{X}$ is regular and that $\mathcal{X}_k$ is irreducible.

2.4.2 The construction of divisorial points can be generalized in the following way. Let $\mathcal{X}$ be an R -model of X and let $(E_1, \dots, E_r)$ be a tuple of distinct irreducible components of $\mathcal{X}_k$ such

that the intersection

$$E = \cap_{i=1}^r E_i$$

is non-empty. Let ξ be a generic point of E , and assume that, locally at ξ , the R -scheme $\mathcal{X}$ is regular and $\mathcal{X}_k$ is a divisor with strict normal crossings. In this case, we call $(\mathcal{X}, (E_1, \dots, E_r), \xi)$ an *sncd-triple* for X . Then there exist a regular system of local parameters $z_1, \dots, z_r$ in $\mathcal{O}_{\mathcal{X}, \xi}$, positive integers $N_1, \dots, N_r$ and a unit u in $\mathcal{O}_{\mathcal{X}, \xi}$ such that

$$\pi := uz_1^{N_1} \cdot \dots \cdot z_r^{N_r}$$

is a uniformizer in R and such that, locally at ξ , the prime divisor E_i is defined by the equation $z_i = 0$.

We say that an element f of $\widehat{\mathcal{O}}_{\mathcal{X}, \xi}$ is *admissible* if it has an expansion of the form

$$f = \sum_{\beta \in \mathbb{Z}_{\geq 0}^r} c_\beta z^\beta, \quad (2.4.1)$$

where each coefficient c_β is either zero or a unit in $\widehat{\mathcal{O}}_{\mathcal{X}, \xi}$. Such an expansion is called an *admissible expansion* of f . Let α be an element of $\mathbb{R}_{\geq 0}^r$ such that

$$\alpha_1 N_1 + \dots + \alpha_r N_r = 1.$$

For every β in $\mathbb{R}^r$, we set

$$\alpha \cdot \beta = \sum_{i=1}^r \alpha_i \beta_i.$$

LEMMA 2.4.3. *Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}_A$ and residue field κ_A , and let $(y_1, \dots, y_m)$ be a system of generators for $\mathfrak{m}_A$. We denote by $\widehat{A}$ the $\mathfrak{m}_A$ -adic completion of A . Let B be a subring of A such that the elements $y_1, \dots, y_m$ belong to B and generate the ideal $B \cap \mathfrak{m}_A$ in B . Then, in the ring $\widehat{A}$, every element f of B can be written as*

$$f = \sum_{\beta \in \mathbb{Z}_{\geq 0}^m} c_\beta y^\beta, \quad (2.4.2)$$

where the coefficients c_β belong to $(A^\times \cap B) \cup \{0\}$.

Proof. Such an expansion for f is constructed inductively in the following way. Since A is local, f belongs either to $A^\times$ or to the maximal ideal $\mathfrak{m}_A$. In the latter case, we can write f as a B -linear combination of the elements $y_1, \dots, y_m$. Now suppose that i is a positive integer and that we can write every f in B as a sum of an element f_i of the form (2.4.2) and a B -linear combination of degree i monomials in the elements $y_1, \dots, y_m$. Applying this assumption to the coefficients of the B -linear combination, we see that we can write f as the sum of an element f_{i+1} of the form (2.4.2) and a B -linear combination of degree $i+1$ monomials in the elements $y_1, \dots, y_m$ such that f_i and f_{i+1} have the same coefficients in degree less than i . Repeating this construction, we find an expansion for f of the form (2.4.2). $\square$

PROPOSITION 2.4.4. *We keep the notation of § 2.4.2.*

- (i) *Every element of $\widehat{\mathcal{O}}_{\mathcal{X}, \xi}$ is admissible.*
- (ii) *There exists a unique real valuation*

$$v_\alpha: K(X)^\times \rightarrow \mathbb{R}$$

such that

$$v_\alpha(f) = \min \{ \alpha \cdot \beta \mid \beta \in \mathbb{Z}_{\geq 0}^r, c_\beta \neq 0 \}$$

for every non-zero element f in $\mathcal{O}_{\mathcal{X}, \xi}$ and every admissible expansion of f of the form (2.4.1). This valuation does not depend on the choice of $z_1, \dots, z_r$, and its restriction to K coincides with v_K .

Proof. (i) This follows at once from Lemma 2.4.3, taking $A = B = \widehat{\mathcal{O}}_{\mathcal{X}, \xi}$.

(ii) Uniqueness is clear from statement (i). Let us show that our definition of the value $v_\alpha(f)$ does not depend on the chosen admissible expansion of f . For notational convenience, we set $A = \widehat{\mathcal{O}}_{\mathcal{X}, \xi}$ and we denote by $\mathfrak{m}_A$ and κ_A its maximal ideal and residue field, respectively.

To every admissible expansion (2.4.1) we can associate a Newton polyhedron Γ . This is the convex hull of the set

$$\{ \beta \in \mathbb{Z}_{\geq 0}^r \mid c_\beta \neq 0 \} + \mathbb{R}_{\geq 0}^r$$

in $\mathbb{R}^r$. We denote by Γ_c the set of points in $\mathbb{Z}_{\geq 0}^r$ that lie on a compact face of Γ and we set

$$f_c = \sum_{\beta \in \Gamma_c} \bar{c}_\beta z^\beta,$$

where $\bar{c}_\beta$ denotes the residue class of c_β in κ_A . Thus f_c is an element of $\kappa_A[z_1, \dots, z_r]$. We claim that it depends only on f , and not on the chosen admissible expansion (2.4.1). To see this, let

$$f = \sum_{\beta \in \mathbb{Z}_{\geq 0}^r} c'_\beta z^\beta$$

be another admissible expansion of f , with associated set Γ'_c and polynomial f'_c . Then

$$\sum_{\beta \in \mathbb{Z}_{\geq 0}^r} (c_\beta - c'_\beta) z^\beta = 0.$$

Choosing admissible expansions for the elements $c_\beta - c'_\beta$ that do not lie in $A^\times \cup \{0\}$, we can rewrite this expression into an admissible expansion

$$0 = \sum_{\beta \in \mathbb{Z}_{\geq 0}^r} d_\beta z^\beta$$

such that $\bar{d}_\beta = \bar{c}_\beta - \bar{c}'_\beta$ for all β in $\Gamma_c \cup \Gamma'_c$. Suppose that one of the coefficients d_β is different from zero. Then we can write

$$\sum_{\beta \in \mathbb{Z}_{\geq 0}^r} d_\beta z^\beta \in g_{i_0}(z_1, \dots, z_d) + \mathfrak{m}_A^{i_0+1},$$

where $g_{i_0}(z_1, \dots, z_d) \in A[z_1, \dots, z_d]$ is the homogeneous part of minimal degree i_0 of the power series $\sum_{\beta \in \mathbb{Z}_{\geq 0}^r} d_\beta z^\beta$. By [EGA4.1, 17.1.1], the graded κ_A -algebra $\oplus_{i \geq 0} \mathfrak{m}_A^i / \mathfrak{m}_A^{i+1}$ of the regular local ring A is isomorphic to the polynomial ring over κ_A in the residue classes of $z_1, \dots, z_d$ modulo $\mathfrak{m}_A^2$. Thus the image of $g_{i_0}(z_1, \dots, z_d)$ in $\mathfrak{m}_A^{i_0} / \mathfrak{m}_A^{i_0+1}$ is non-zero, which contradicts the equality

$$0 = \sum_{\beta \in \mathbb{Z}_{\geq 0}^r} d_\beta z^\beta.$$

It follows that all d_β are zero, so that $\Gamma_c = \Gamma'_c$ and $f_c = f'_c$.

Now we set

$$m = \min \{ \alpha \cdot \beta \mid \beta \in \Gamma_c, \bar{c}_\beta \neq 0 \}$$

and we denote by Γ_c^α the set of points β in Γ_c such that $\alpha \cdot \beta = m$. We set

$$f_\alpha = \sum_{\beta \in \Gamma_c^\alpha} \bar{c}_\beta z^\beta \in \kappa_A[z_1, \dots, z_r].$$

Then m and f_α are completely determined by f_c , so that they do not depend on the chosen admissible expansion for f . Moreover,

$$v_\alpha(f) = \min\{\alpha \cdot \beta \mid \beta \in \Gamma_c, c_\beta \neq 0\} = m \quad (2.4.3)$$

because for every linear form L on $\mathbb{R}^r$ with non-negative coefficients, the minimal value of L on Γ is reached on Γ^c . It follows that $v_\alpha(f)$ depends only on f , and not on the chosen admissible expansion.

Now it is easy to check that v_α is a valuation: if f and g are non-zero elements in $\mathcal{O}_{\mathcal{X}, \xi}$, then clearly

$$v_\alpha(f + g) \geq \min\{v_\alpha(f), v_\alpha(g)\}$$

and we also have $v_\alpha(f \cdot g) = v_\alpha(f) + v_\alpha(g)$ because $(f \cdot g)_\alpha = f_\alpha \cdot g_\alpha$. It is obvious that $v_\alpha(f)$ does not depend on the choice of $z_1, \dots, z_r$ since these elements are determined up to a unit in $\mathcal{O}_{\mathcal{X}, \xi}$. To see that v_α extends v_K , note that

$$\pi = uz_1^{N_1} \cdot \dots \cdot z_r^{N_r}$$

is an admissible expansion for the uniformizer π in R so that

$$v_\alpha(\pi) = \sum_{i=1}^r \alpha_i N_i = 1. \quad \square$$

2.4.5 The valuation v_α from Proposition 2.4.4 extends the discrete valuation v_K on K , and thus defines a birational point x of $\widehat{\mathcal{X}}_\eta \subset X^{\text{an}}$ such that $v_x = v_\alpha$. We call this birational point the *monomial point* associated to the *sncd*-triple $(\mathcal{X}, (E_1, \dots, E_r), \xi)$ and the tuple α , and we will say that these data represent the point x . Replacing $\mathcal{X}$ by an open neighborhood of ξ has no influence on the associated monomial point x , so that we can always assume that $\mathcal{X}$ is regular and that $\mathcal{X}_k$ is a divisor with strict normal crossings.

2.4.6 Permuting the indices, we may assume that there exists an element r' in $\{1, \dots, r\}$ such that $\alpha_i \neq 0$ for all $i \in \{1, \dots, r'\}$ and $\alpha_i = 0$ for all $i \in \{r' + 1, \dots, r\}$. If we denote by ξ' the unique generic point of $\bigcap_{i=1}^{r'} E_i$ whose closure contains ξ , then it is clear from the constructions that the data

$$((E_1, \dots, E_r), \xi, (\alpha_1, \dots, \alpha_r)) \quad \text{and} \quad ((E_1, \dots, E_{r'}), \xi', (\alpha_1, \dots, \alpha_{r'}))$$

define the same monomial point x of X^{an} . Moreover, ξ' is the center of v_x , and $\alpha_i = v_x(E_i)$ for every $i \in \{1, \dots, r\}$. Thus $\xi' = \text{red}_{\mathcal{X}}(x)$ and $\xi', (E_1, \dots, E_{r'})$ and $(\alpha_1, \dots, \alpha_{r'})$ are completely determined by the model $\mathcal{X}$ and the monomial point x , up to permutation of the indices $\{1, \dots, r'\}$. It is clear that the rational rank of X at x is the dimension of the $\mathbb{Q}$ -vector subspace of $\mathbb{R}$ generated by the coordinates of α .

2.4.7 We say that a point of X^{an} is *divisorial* if it is the divisorial point associated to some R -model $\mathcal{X}$ of X and some irreducible component of $\mathcal{X}_k$. We denote the set of divisorial points on X^{an} by X^{div} . The notion of a *monomial point* on X^{an} is defined analogously, and the set of monomial points on X^{an} is denoted by X^{mon} . Note that every divisorial point is monomial. If x

is a monomial point on X^{an} and $\mathcal{X}$ is an R -model of X , then we say that $\mathcal{X}$ is *adapted* to x if there exist irreducible components $(E_1, \dots, E_r)$ of $\mathcal{X}_k$, a generic point ξ of $\cap_{i=1}^r E_i$ and a tuple α in $\mathbb{R}_{>0}^r$ such that $(\mathcal{X}, (E_1, \dots, E_r), \xi)$ is an *sncd*-triple and x is the monomial point associated to this *sncd*-triple and the tuple α .

PROPOSITION 2.4.8. *Let x be a monomial point of X^{an} . Then the following are equivalent:*

- (i) *The point x is divisorial.*
- (ii) *The valuation v_x is discrete.*
- (iii) *The analytic space X^{an} has rational rank one at x .*

Proof. It suffices to prove that x is divisorial if X^{an} has rational rank one at x . Let $(\mathcal{X}, (E_1, \dots, E_r), \xi)$ be an *sncd*-triple for X and let α be an element of $\mathbb{R}_{>0}^r$ such that these data represent the point x . Since X^{an} has rational rank one at x , the tuple α must belong to $\mathbb{Q}_{>0}^r$. Permuting the indices, we may assume that α_1 is minimal among the coordinates of α . We consider the blow-up $h: \mathcal{X}' \rightarrow \mathcal{X}$ at the closure of ξ . We denote by E'_i the strict transform of E_i , for all i in $\{2, \dots, r\}$, and we denote by E'_1 the exceptional divisor of the blow-up. Let ξ' be the generic point of $E'_1 \cap \dots \cap E'_r$. Then a straightforward computation shows that

$$(\mathcal{X}', (E'_1, \dots, E'_r), \xi')$$

is still an *sncd*-triple, and together with the tuple

$$\alpha' = (\alpha_1, \alpha_2 - \alpha_1, \dots, \alpha_r - \alpha_1)$$

it represents the point x . By the construction in § 2.4.6, we can eliminate the zero entries in α' . The rational rank of X^{an} at x is equal to one, so that all the α_i are integer multiples of a common rational number $q > 0$. An elementary induction argument shows that, starting from a finite tuple of positive integers and repeatedly choosing one of the minimal non-zero coordinates and subtracting it from the other coordinates, we arrive after a finite number of steps at a tuple with only one non-zero coordinate. Thus, repeating our blow-up procedure finitely many times, we arrive at a monomial presentation with $r = 1$, that is, a divisorial presentation for x . $\square$

PROPOSITION 2.4.9. *The set X^{div} of divisorial points on X^{an} is dense in X^{an} .*

Proof. If $j: X \rightarrow \bar{X}$ is a normal compactification of X , then $j^{\text{an}}: X^{\text{an}} \rightarrow \bar{X}^{\text{an}}$ is an open immersion of K -analytic spaces. It identifies the divisorial points on X^{an} and $\bar{X}^{\text{an}}$: every R -model for X can be extended to an R -model of $\bar{X}$ by gluing $\bar{X}$ to its generic fiber, which shows that the image of a divisorial point in X^{an} is divisorial in $\bar{X}^{\text{an}}$. Conversely, if x is a divisorial point on $\bar{X}^{\text{an}}$ associated to an R -model $\bar{\mathcal{X}}$ of $\bar{X}$ and an irreducible component E of $\bar{\mathcal{X}}_k$, then by removing the closure of $\bar{X} \setminus X$ we get a divisorial presentation of x as a point of X^{an} . Thus we may assume that X is proper over K .

We denote by X^{ZR} the Zariski Riemann space of X as defined in Section 2.3. Let $\mathcal{X}$ be a proper R -model of X and denote by $p: X^{\text{ZR}} \rightarrow \mathcal{X}_k$ the projection morphism. Then for every generic point ξ of $\mathcal{X}_k$, the fiber $p^{-1}(\xi)$ consists of a unique point, that we will denote by ξ^{ZR} . To see this, note that every morphism of proper R -models $\mathcal{X}' \rightarrow \mathcal{X}$ is an isomorphism over an open neighborhood of ξ by Zariski's main theorem, because $\mathcal{X}$ is normal. Since X^{ZR} carries the limit topology, the points of the form ξ^{ZR} form a dense subset of X^{ZR} as $\mathcal{X}$ varies over the class of proper R -models of X . To conclude the proof, it suffices to observe that the image of ξ^{ZR} under the retraction $r: X^{\text{ZR}} \rightarrow X^{\text{an}}$ is the divisorial point x on X^{an} associated to $\mathcal{X}$ and the closure of ξ . Indeed, $p(\xi^{\text{ZR}}) = \xi$ must lie in the closure of $(\text{red}_{\mathcal{X}} \circ r)(\xi^{\text{ZR}})$ by Proposition 2.3.2. Thus $\text{red}_{\mathcal{X}}(r(\xi^{\text{ZR}})) = \xi$, but x is the only element in $\text{red}_{\mathcal{X}}^{-1}(\xi)$. $\square$

3. The Berkovich skeleton of an *sncd*-model

3.1 Definition of the Berkovich skeleton

3.1.1 In [Ber99], Berkovich associates a skeleton to a certain class of formal schemes $\mathfrak{X}$ over R , the so-called pluristable formal R -schemes. This skeleton is a closed subspace of the generic fiber $\mathfrak{X}_\eta$ of $\mathfrak{X}$, and Berkovich has shown that it is a strong deformation retract of $\mathfrak{X}_\eta$. This construction was a crucial ingredient of his proof of the local contractibility of smooth analytic spaces over a non-Archimedean field. If $\mathcal{X}$ is a regular R -scheme of finite type such that $\mathcal{X}_k$ is a divisor with strict normal crossings, then its $\mathfrak{m}$ -adic completion $\widehat{\mathcal{X}}$ is not pluristable if $\mathcal{X}_k$ is not reduced. Nevertheless, the definition of the skeleton can be generalized to this setting in a fairly straightforward way. In this section, we explain the construction of the skeleton and we prove some basic properties that we will need further on. If R has equal characteristic, most of the results can be found already in [BFJ12, §3].

3.1.2 Let X be a connected regular K -scheme of finite type and let $\mathcal{X}$ be an *sncd*-model for X . Then the reduced special fiber $(\mathcal{X}_k)_{\text{red}}$ is a strictly semi-stable k -variety, to which one can associate an unoriented simplicial set $\Delta(\mathcal{X}_k)$ in a standard way; see for instance [Nic11, §3.1]. If $\mathcal{X}_k$ is a curve, then $\Delta(\mathcal{X}_k)$ is simply the unweighted dual graph of $\mathcal{X}_k$. The geometric realization $|\Delta(\mathcal{X}_k)|$ is a compact simplicial space whose points can be uniquely represented by couples (ξ, w) where ξ is a generic point of an intersection of r distinct irreducible components of $\mathcal{X}_k$, for some $r > 0$, and w is an element of the open simplex

$$\Delta_\xi^o = \left\{ x \in \mathbb{R}_{>0}^{\Psi(\xi)} \mid \sum_{i \in \Psi(\xi)} x_i = 1 \right\}.$$

Here $\Psi(\xi)$ denotes the set of irreducible components of $\mathcal{X}_k$ that pass through ξ . In [Nic11, §3.2], we called (ξ, w) the barycentric representation of the point, and w the tuple of barycentric coordinates.

3.1.3 We define a map

$$\text{Sk}_{\mathcal{X}}: |\Delta(\mathcal{X}_k)| \rightarrow X^{\text{an}}$$

by sending a point P with barycentric representation (ξ, w) as above to the monomial point on X^{an} associated to the *sncd*-triple $(\mathcal{X}, (E_1, \dots, E_r), \xi)$ and the tuple

$$\alpha = \left(\frac{w_{E_1}}{N_1}, \dots, \frac{w_{E_r}}{N_r} \right) \in \mathbb{R}_{>0}^r,$$

where $E_1, \dots, E_r$ are the irreducible components of $\mathcal{X}_k$ passing through ξ and $N_1, \dots, N_r$ are their multiplicities in $\mathcal{X}_k$. We call the image of $\text{Sk}_{\mathcal{X}}$ the *Berkovich skeleton* of $\mathcal{X}$, and denote it by $\text{Sk}(\mathcal{X})$. We endow it with the induced topology from X^{an} . Beware that it depends not only on X , but also on the chosen *sncd*-model $\mathcal{X}$. Note that the Berkovich skeleton of $\mathcal{X}$ is contained in $\widehat{\mathcal{X}}_\eta$, and that a monomial point on X^{an} lies on $\text{Sk}(\mathcal{X})$ if and only if the *sncd*-model $\mathcal{X}$ is adapted to x , in the sense of §2.4.7. It is clear from the definition that, for every open subscheme $\mathcal{Y}$ of $\mathcal{X}$, the skeleton $\text{Sk}(\mathcal{Y}) \subset \mathcal{Y}_K^{\text{an}}$ is equal to the closed subset $\text{red}_{\mathcal{X}}^{-1}(\mathcal{Y}_k)$ of $\text{Sk}(\mathcal{X})$.

PROPOSITION 3.1.4. *The map $\text{Sk}_{\mathcal{X}}: |\Delta(\mathcal{X}_k)| \rightarrow \text{Sk}(\mathcal{X})$ is a homeomorphism.*

Proof. We have seen in §2.4.6 that ξ and w are completely determined by $\mathcal{X}$ and $\text{Sk}_{\mathcal{X}}(P)$. Thus $\text{Sk}_{\mathcal{X}}$ is injective. Since $|\Delta(\mathcal{X}_k)|$ is compact and X^{an} is Hausdorff, it suffices to show that $\text{Sk}_{\mathcal{X}}$

is continuous.

Let $\mathcal{U}$ be an open subscheme of $\mathcal{X}$, and denote by $\Delta(\mathcal{U}_k)$ the simplicial set associated to the semi-stable k -variety $(\mathcal{U}_k)_{\text{red}}$. Then $\widehat{\mathcal{U}}_\eta$ is a closed analytic domain in $\widehat{\mathcal{X}}_\eta$. The open immersion $\mathcal{U}_k \rightarrow \mathcal{X}_k$ induces a closed embedding $|\Delta(\mathcal{U}_k)| \rightarrow |\Delta(\mathcal{X}_k)|$, and $\text{Sk}_{\mathcal{X}}$ coincides with $\text{Sk}_{\mathcal{U}}$ on $|\Delta(\mathcal{U}_k)|$. If we cover $\mathcal{X}$ by finitely many affine open subschemes $\mathcal{U}$, then the spaces $|\Delta(\mathcal{U}_k)|$ will form a closed cover of $|\Delta(\mathcal{X}_k)|$. Thus we may assume that $\mathcal{X}$ is affine.

By definition of the Berkovich topology on X^{an} , it is enough to prove that for every regular function f on $X = \mathcal{X}_K$, the map

$$|\Delta(\mathcal{X}_k)| \rightarrow \mathbb{R}_{\geq 0}, \quad x \mapsto |f(\text{Sk}_{\mathcal{X}}(x))|$$

is continuous. This follows at once from the fact that the quasi-monomial valuation v_α in Proposition 2.4.4 is continuous in the parameters α . $\square$

3.1.5 We can construct a continuous retraction

$$\rho_{\mathcal{X}}: \widehat{\mathcal{X}}_\eta \rightarrow \text{Sk}(\mathcal{X})$$

for the inclusion $\text{Sk}(\mathcal{X}) \rightarrow \widehat{\mathcal{X}}_\eta$ as follows. Let x be a point of $\widehat{\mathcal{X}}_\eta$, and denote by ζ its image under the reduction map

$$\text{red}_{\mathcal{X}}: \widehat{\mathcal{X}}_\eta \rightarrow \mathcal{X}_k.$$

Let $E_1, \dots, E_r$ be the irreducible components of $\mathcal{X}_k$ that pass through ζ . Let ξ be the generic point of the connected component of $E_1 \cap \dots \cap E_r$ that contains ζ , and set

$$w_{E_i} = v_x(E_i)$$

for all i . Then $\rho_{\mathcal{X}}(x)$ is the point with barycentric representation (ξ, w) , where we used the map $\text{Sk}_{\mathcal{X}}$ to identify $|\Delta(\mathcal{X}_k)|$ and $\text{Sk}(\mathcal{X})$. In other words, $\rho_{\mathcal{X}}(x)$ is the monomial point on X^{an} represented by $(\mathcal{X}, (E_1, \dots, E_r), \xi)$ and $w = (w_{E_1}, \dots, w_{E_r})$. This is the unique monomial point y on X^{an} such that $\mathcal{X}$ is adapted to x , $v_y(E) = v_x(E)$ for every prime component E of $\mathcal{X}_k$ and $\text{red}_{\mathcal{X}}(x)$ belongs to the closure of $\text{red}_{\mathcal{X}}(y)$. It is straightforward to check that $\rho_{\mathcal{X}}$ is continuous and right inverse to the inclusion $\text{Sk}(\mathcal{X}) \rightarrow \widehat{\mathcal{X}}_\eta$.

PROPOSITION 3.1.6. *Let $\mathcal{X}$ be an sncd-model for X and let x be a point of $\widehat{\mathcal{X}}_\eta$. Then*

$$|f(x)| \leq |f(\rho_{\mathcal{X}}(x))|$$

for every f in $\mathcal{O}_{\mathcal{X}, \text{red}_{\mathcal{X}}(x)}$.

Proof. Let $E_1, \dots, E_r$ be the irreducible components of $\mathcal{X}_k$ passing through $\text{red}_{\mathcal{X}}(x)$, and set $\alpha_i = v_x(E_i)$ for all i . Denote by ξ the generic point of the connected component of $\cap_{i=1}^r E_i$ containing $\text{red}_{\mathcal{X}}(x)$. Then $\rho_{\mathcal{X}}(x)$ is the monomial point of X^{an} represented by $(\mathcal{X}, (E_1, \dots, E_r), \xi)$ and α , and $\xi = \text{red}_{\mathcal{X}}(\rho_{\mathcal{X}}(x))$.

Let $z_i = 0$ be a local equation for E_i on $\mathcal{X}$ at $\text{red}_{\mathcal{X}}(x)$, for all i . Then $(z_1, \dots, z_r)$ is a regular system of local parameters in $\mathcal{O}_{\mathcal{X}, \xi}$. By Lemma 2.4.3, one can always find an admissible expansion

$$f = \sum_{\beta \in \mathbb{Z}_{\geq 0}^r} c_\beta z^\beta$$

for f in $\widehat{\mathcal{O}}_{\mathcal{X}, \xi}$ such that all the coefficients c_β belong to $\mathcal{O}_{\mathcal{X}, \text{red}_{\mathcal{X}}(x)}$. We have $|h(x)| \leq 1$ and $|h(\rho_{\mathcal{X}}(x))| \leq 1$ for every function h in $\mathcal{O}_{\mathcal{X}, \text{red}_{\mathcal{X}}(x)} \subset \mathcal{O}_{\mathcal{X}, \xi}$. Since, moreover, $|z_i(x)| =$

$|z_i(\rho_{\mathcal{X}}(x))| < 1$ for all i in $\{1, \dots, r\}$, we can rewrite the expression for f as

$$f = \sum_{\beta \in S} c_{\beta} z^{\beta} + g,$$

where S is a finite subset of $\mathbb{Z}_{\geq 0}^r$ and g is an element of $\mathcal{O}_{\mathcal{X}, \text{red}_{\mathcal{X}}(x)}$ such that $|g(x)| < |f(x)|$ and $|g(\rho_{\mathcal{X}}(x))| < |f(\rho_{\mathcal{X}}(x))|$.

Then we have

$$\begin{aligned} |f(x)| &\leq \max\{|c_{\beta}(x)| \cdot |z^{\beta}(x)| \mid \beta \in S\} \leq \max\{|z^{\beta}(x)| \mid \beta \in S\} \\ &= |(f - g)(\rho_{\mathcal{X}}(x))| = |f(\rho_{\mathcal{X}}(x))|. \end{aligned}$$

□

PROPOSITION 3.1.7. *If $h: \mathcal{X}' \rightarrow \mathcal{X}$ is an R -morphism of sned-models of X , then $\widehat{\mathcal{X}'}_{\eta} \subset \widehat{\mathcal{X}}_{\eta}$ and*

$$(\rho_{\mathcal{X}} \circ \rho_{\mathcal{X}'})(x) = \rho_{\mathcal{X}}(x)$$

for every point x in $\widehat{\mathcal{X}'}_{\eta}$. If h is proper, then $\widehat{\mathcal{X}'}_{\eta} = \widehat{\mathcal{X}}_{\eta}$ and $\text{Sk}(\mathcal{X}) \subset \text{Sk}(\mathcal{X}')$.

Proof. It is obvious that $\widehat{\mathcal{X}'}_{\eta} \subset \widehat{\mathcal{X}}_{\eta}$. Let x be a point of $\widehat{\mathcal{X}'}_{\eta}$ and set $y = \rho_{\mathcal{X}'}(x)$. Then $\text{red}_{\mathcal{X}'}(x)$ lies in the closure of $\{\text{red}_{\mathcal{X}'}(y)\}$, which implies the analogous statement for $\text{red}_{\mathcal{X}}(x) = h(\text{red}_{\mathcal{X}'}(x))$ and $\text{red}_{\mathcal{X}}(y) = h(\text{red}_{\mathcal{X}'}(y))$. Thus by definition of the map $\rho_{\mathcal{X}}$, it is enough to show that $v_x(E) = v_y(E)$ for each prime component E of $\mathcal{X}_k$. Writing

$$h^*E = \sum_{i=1}^r \alpha_i E'_i,$$

where $E'_1, \dots, E'_r$ are prime components of $\mathcal{X}'_k$, we compute

$$v_x(E) = \sum_{i=1}^r \alpha_i v_x(E'_i) = \sum_{i=1}^r \alpha_i v_y(E'_i) = v_y(E),$$

where the second equality follows from the definition of the map $\rho_{\mathcal{X}'}$.

Now assume that h is proper. Then $\widehat{\mathcal{X}'}_{\eta} = \widehat{\mathcal{X}}_{\eta}$ by the valuative criterion for properness. Let x be a point of $\text{Sk}(\mathcal{X})$ and set $\xi = \text{red}_{\mathcal{X}}(x)$, $x' = \rho_{\mathcal{X}'}(x)$ and $\xi' = \text{red}_{\mathcal{X}'}(x')$. To prove that $\text{Sk}(\mathcal{X}) \subset \text{Sk}(\mathcal{X}')$, we must show that $x = x'$. It suffices to show that $v_x(f) = v_{x'}(f)$ for every element f in $\mathcal{O}_{\mathcal{X}, \xi}$. We choose a system of coordinates $(z_1, \dots, z_r)$ and an admissible expansion

$$f = \sum_{\beta \in \mathbb{Z}_{\geq 0}^r} c_{\beta} z^{\beta}$$

for f as in § 2.4.2. Let $(z'_1, \dots, z'_s)$ be a regular system of local parameters in $\mathcal{O}_{\mathcal{X}', \xi'}$ such that there exist a unit u' in $\mathcal{O}_{\mathcal{X}', \xi'}$ and positive integers $N'_1, \dots, N'_s$ such that

$$u' \prod_{j=1}^s (z'_j)^{N'_j}$$

is a uniformizer in R . For each i in $\{1, \dots, r\}$, we can write

$$z_i = v_i \prod_{j=1}^s (z'_j)^{\gamma_{ij}}$$

in $\mathcal{O}_{\mathcal{X}', \xi'}$, with v_i a unit and γ_{ij} non-negative integers. We view $\gamma_i = (\gamma_{i1}, \dots, \gamma_{is})$ as a vector

in $\mathbb{Z}_{\geq 0}^s$, for each i . Let β and β' be elements of $\mathbb{Z}_{\geq 0}^r$ such that

$$\sum_{i=1}^r \beta_i \gamma_i = \sum_{i=1}^r \beta'_i \gamma_i.$$

This means that the monomials z^β and $z^{\beta'}$ have the same multiplicity along each irreducible component of $\mathcal{X}'_k$, and thus that the rational function $z^{\beta-\beta'}$ on X is a unit in

$$\mathcal{O}_{\mathcal{X}, \xi} \cong \mathcal{O}(\mathcal{X}' \times_{\mathcal{X}} \operatorname{Spec} \mathcal{O}_{\mathcal{X}, \xi})$$

(the isomorphism follows from the fact that h is proper, $\mathcal{X}$ and $\mathcal{X}'$ are R -flat, h_K is an isomorphism and $\mathcal{X}$ is normal). This can happen only if $\beta = \beta'$. It follows that

$$f = \sum_{\beta \in \mathbb{Z}_{\geq 0}^r} c_\beta v^\beta(z')^{\sum_{i=1}^r \beta_i \gamma_i}$$

is an admissible expansion for f in $\widehat{\mathcal{O}}_{\mathcal{X}', \xi'}$. Thus

$$\begin{aligned} v_{x'}(f) &= \min \left\{ \sum_{i,j} v_{x'}(z'_j) \beta_i \gamma_{ij} \mid \beta \in \mathbb{Z}_{\geq 0}^r, c_\beta \neq 0 \right\} \\ &= \min \left\{ \sum_i \left(\sum_j v_x(z'_j) \gamma_{ij} \right) \beta_i \mid \beta \in \mathbb{Z}_{\geq 0}^r, c_\beta \neq 0 \right\} \\ &= \min \left\{ \sum_i v_x(z_i) \beta_i \mid \beta \in \mathbb{Z}_{\geq 0}^r, c_\beta \neq 0 \right\} = v_x(f). \end{aligned} \quad \square$$

3.1.8 If $h: \mathcal{X}' \rightarrow \mathcal{X}$ is a proper morphism of *sncd*-models of X , then in general, the skeleton $\operatorname{Sk}(\mathcal{X}')$ will be strictly larger than the skeleton $\operatorname{Sk}(\mathcal{X})$. The following proposition shows, however, that the skeleton does not change if we blow up a connected component of an intersection of irreducible components of $\mathcal{X}_k$.

PROPOSITION 3.1.9. *Let $\mathcal{X}$ be an *sncd*-model for X , let $E_1, \dots, E_r$ be irreducible components of $\mathcal{X}_k$, and let ξ be a generic point of $\cap_{i=1}^r E_i$. Denote by $h: \mathcal{X}' \rightarrow \mathcal{X}$ the blow-up of $\mathcal{X}$ at the closure of $\{\xi\}$. Then $\operatorname{Sk}(\mathcal{X}') = \operatorname{Sk}(\mathcal{X})$. More precisely, if we denote by σ the face of $\operatorname{Sk}(\mathcal{X})$ corresponding to ξ , then the simplicial structure on $\operatorname{Sk}(\mathcal{X}')$ is obtained by adding a vertex at the barycenter of σ , corresponding to the exceptional divisor of h , and joining it to all the faces of σ .*

Proof. This follows from a straightforward computation. $\square$

3.2 Piecewise affine structure on the Berkovich skeleton

3.2.1 Let X be a connected regular K -scheme of finite type, and let $\mathcal{X}$ be an *sncd*-model for X . We will use the simplicial structure of $\Delta(\mathcal{X}_k)$ to define an integral piecewise affine structure on the Berkovich skeleton $\operatorname{Sk}(\mathcal{X})$. We use the homeomorphism

$$\operatorname{Sk}_{\mathcal{X}}: |\Delta(\mathcal{X}_k)| \rightarrow \operatorname{Sk}(\mathcal{X})$$

to identify $\operatorname{Sk}(\mathcal{X})$ with the geometric realization $|\Delta(\mathcal{X}_k)|$ of $\Delta(\mathcal{X}_k)$. Let U be a subset of $\operatorname{Sk}(\mathcal{X})$ and consider a function

$$f: U \rightarrow \mathbb{R}.$$

We say that f is *affine* (with respect to the model $\mathcal{X}$) if the following two conditions are fulfilled:

- (i) The function f is continuous.
- (ii) For every closed face σ of $\text{Sk}(\mathcal{X})$, we can cover $\sigma \cap U$ by open subsets V of $\sigma \cap U$ such that the restriction of f to V is an affine function with coefficients in $\mathbb{Z}$ in the variables

$$\left(\frac{w_{E_1}}{N_1}, \dots, \frac{w_{E_r}}{N_r} \right),$$

where $E_1, \dots, E_r$ are the irreducible components of $\mathcal{X}_k$ corresponding to the vertices of σ , $N_1, \dots, N_r$ are their multiplicities in $\mathcal{X}_k$ and $(w_{E_1}, \dots, w_{E_r})$ are the barycentric coordinates on σ .

We say that f is *piecewise affine* (with respect to the model $\mathcal{X}$) if f is continuous and if we can cover each face σ of $\text{Sk}(\mathcal{X})$ by finitely many polytopes P such that the vertices of P have rational barycentric coordinates and the restriction of f to $U \cap P$ is affine.

PROPOSITION 3.2.2. *Let $\mathcal{X}$ be an sncd-model of X , and let h be a non-zero rational function on X . Then the function*

$$f_h: \text{Sk}(\mathcal{X}) \rightarrow \mathbb{R}, \quad x \mapsto v_x(h)$$

is piecewise affine. If σ is a closed face of $\text{Sk}(\mathcal{X})$ corresponding to a generic point ξ of an intersection of irreducible components of $\mathcal{X}_k$, then the function

$$f_h|_{\sigma}: \sigma \rightarrow \mathbb{R}, \quad x \mapsto v_x(h)$$

- (i) *is concave if ξ is not contained in the closure of the locus of poles of h on X ;*
- (ii) *is convex if ξ is not contained in the closure of the locus of zeroes of h on X ;*
- (iii) *is affine if ξ is not contained in the closure of the locus of zeroes and poles of h on X .*

Proof. Continuity of f_h follows immediately from the definition of the Berkovich topology. The remainder of the statement can be checked on the restrictions of f to the faces of the skeleton. Let $E_1, \dots, E_r$ be irreducible components of $\mathcal{X}_k$, let ξ be a generic point of their intersection and let σ be the closed face of $\text{Sk}(\mathcal{X})$ corresponding to ξ . We may assume that ξ does not lie in the closure of the locus of poles of h on X , because $f_{1/h} = -f_h$ so that statement (ii) follows from statement (i). Multiplying h by a suitable element in R , we can further reduce to the case where h belongs to $\mathcal{O}_{\mathcal{X}, \xi}$. Taking an admissible expansion for h in $\widehat{\mathcal{O}}_{\mathcal{X}, \xi}$, we deduce from (2.4.3) that $f_h|_{\sigma}$ is a minimum of finitely many affine functions on σ and therefore a concave piecewise affine function. Statement (iii) now follows from statements (i) and (ii), because a piecewise affine concave and convex function is affine. $\square$

PROPOSITION 3.2.3. *Let $\mathcal{X}$ and $\mathcal{Y}$ be sncd-models of X . Then the map*

$$\rho_{\mathcal{X}}|_{\widehat{\mathcal{X}}_{\eta} \cap \text{Sk}(\mathcal{Y})}: \widehat{\mathcal{X}}_{\eta} \cap \text{Sk}(\mathcal{Y}) \rightarrow \text{Sk}(\mathcal{X}), \quad y \mapsto \rho_{\mathcal{X}}(y)$$

is piecewise affine, in the following sense: if U is a subset of $\text{Sk}(\mathcal{X})$ and $f: U \rightarrow \mathbb{R}$ is a piecewise affine function on U , then $f \circ \rho_{\mathcal{X}}$ is a piecewise affine function on $\rho_{\mathcal{X}}^{-1}(U) \cap \text{Sk}(\mathcal{Y})$.

Proof. This is an easy consequence of Proposition 3.2.2 and the definition of the map $\rho_{\mathcal{X}}$. $\square$

As a corollary, we see that the notion of piecewise affine function is intrinsic on X and independent of the choice of an sncd-model.

COROLLARY 3.2.4. *If $\mathcal{X}$ and $\mathcal{Y}$ are sncd-models of X and U is a subset of $\mathrm{Sk}(\mathcal{X}) \cap \mathrm{Sk}(\mathcal{Y})$, then a function $f: U \rightarrow \mathbb{R}$ is piecewise affine with respect to $\mathcal{X}$ if and only if it is piecewise affine with respect to $\mathcal{Y}$.*

Proof. This follows immediately from Proposition 3.2.3. $\square$

Thus we get a canonical piecewise affine structure on the set of monomial points in X^{an} . Beware, however, that the notion of affine function depends on the chosen model.

4. The weight function and the Kontsevich–Soibelman skeleton

4.1 Some reminders on canonical sheaves

4.1.1 Let $f: Y \rightarrow X$ be a morphism of finite type of locally Noetherian schemes, and assume that f is a local complete intersection. Then one can define the canonical sheaf $\omega_{Y/X}$ of the morphism f ; see §6.4.2 and Exercise 6.4.6 in [Liu02]. It is a line bundle on Y , whose restriction to the smooth locus of f is canonically isomorphic to the sheaf of relative differential forms of maximal degree of f . We will mainly be interested in the situation where Y and X are regular; then every morphism of finite type $Y \rightarrow X$ is a local complete intersection, by [Liu02, Example 6.3.18]. If $g: Z \rightarrow Y$ is another local complete intersection morphism between locally Noetherian schemes, then the canonical sheaves of f , g and $f \circ g$ are related by the adjunction formula: there is a canonical isomorphism

$$\omega_{Z/X} \cong \omega_{Z/Y} \otimes_{\mathcal{O}_Z} g^* \omega_{Y/X}, \quad (4.1.1)$$

by Theorem 6.4.9 and Exercise 6.4.6 in [Liu02].

4.1.2 If X and Y are integral and f is dominant and smooth at the generic point of Y , then the function field $F(Y)$ of Y is a separable extension of the function field $F(X)$ of X , of finite transcendence degree d . If ξ is the generic point of Y , then we can view $\omega_{Y/X}$ as a sub- $\mathcal{O}_Y$ -module of the constant sheaf on Y associated to the rank one $F(Y)$ -vector space

$$(\omega_{Y/X})_\xi \cong \Omega_{F(Y)/F(X)}^d.$$

4.1.3 Let us recall how, in suitable situations, the canonical sheaf $\omega_{Y/X}$ can be explicitly computed. Assume that X and Y are regular integral schemes, and let $f: Y \rightarrow X$ be a morphism of finite type such that f is dominant and smooth at the generic point of Y . Denote by d the transcendence degree of $F(Y)$ over $F(X)$, and let y be a point of Y . Locally at y , the morphism f is of the form

$$\mathrm{Spec} A[T_1, \dots, T_n] / (F_1, \dots, F_r) \rightarrow \mathrm{Spec} A,$$

where the sequence $(F_1, \dots, F_r)$ is regular at y and $r \leq n$. Then $n = r + d$. Renumbering the polynomials F_i , we may assume that

$$\Delta = \det \left(\left[\frac{\partial F_i}{\partial T_j} \right]_{1 \leq i, j \leq r} \right)$$

is different from zero in $\mathcal{O}_{Y,y}$. Then by [Liu02, Corollary 6.4.14], the stalk $(\omega_{Y/X})_y$ of the canonical sheaf $\omega_{Y/X}$ at the point y is the sub- $\mathcal{O}_{Y,y}$ -module of $\Omega_{F(Y)/F(X)}^d$ generated by

$$\Delta^{-1} (dT_{r+1} \wedge \dots \wedge dT_n).$$

4.1.4 Now let $f: Y \rightarrow X$ be a dominant morphism of finite type of regular integral schemes, and assume that f is étale over the generic point of X . This means that the function field $F(Y)$ is a finite separable extension of $F(X)$. Then we can view $\omega_{Y/X}$ as a sub- $\mathcal{O}_Y$ -module of the constant sheaf

$$\Omega_{F(Y)/F(X)}^0 \cong F(Y)$$

on Y . This embedding defines a canonical element in the linear equivalence of Cartier divisors on Y associated to the line bundle $\omega_{Y/X}$, namely, the divisor of the element $1 \in F(Y)$ viewed as a rational section of $\omega_{Y/X}$. We call this divisor the relative canonical divisor of f and denote it by $K_{Y/X}$. It follows from § 4.1.3 that $K_{Y/X}$ is effective and that its defining ideal sheaf $\mathcal{O}(-K_{Y/X})$ is precisely $\text{Fitt}^0 \Omega_{Y/X}^1$, the zeroth Fitting ideal of $\Omega_{Y/X}^1$. If $g: Z \rightarrow Y$ is a dominant morphism of finite type of regular integral schemes such that $F(Z)$ is separable over $F(Y)$ of transcendence degree d , then the isomorphism in the adjunction formula (4.1.1) is an equality of sub- $\mathcal{O}_Z$ -modules of the constant sheaf

$$\Omega_{F(Z)/F(Y)}^d \cong \Omega_{F(Z)/F(X)}^d$$

on Z .

Example 4.2. Let X be a regular integral scheme, and let Z be a regular integral closed subscheme of codimension r . If $f: Y \rightarrow X$ is the blow-up of X at Z , then one can use § 4.1.3 to compute that the relative canonical divisor of f is equal to $(r-1)E$, with $E = f^{-1}(Z)$ the exceptional divisor of f .

PROPOSITION 4.2.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be regular flat R -schemes of finite type, and assume that $\mathcal{Y}_k$ and $\mathcal{X}_k$ are irreducible and $(\mathcal{X}_k)_{\text{red}}$ is regular. Let $h: \mathcal{Y} \rightarrow \mathcal{X}$ be a dominant R -morphism such that h is étale over the generic point of $\mathcal{X}$. We denote by M and N the multiplicities of $\mathcal{X}_k$ and $\mathcal{Y}_k$, respectively, and by $\nu - 1$ the multiplicity of $(\mathcal{Y}_k)_{\text{red}}$ in the relative canonical divisor $K_{\mathcal{Y}/\mathcal{X}}$. Then M divides N and $\nu \geq N/M$.*

Proof. First, we prove that M divides N . Let y be the generic point of $\mathcal{Y}_k$ and set $x = h(y)$. Then we can find a prime element f and a unit u in $\mathcal{O}_{\mathcal{X},x}$ such that $\pi = f^M u$ is a uniformizer in R . Thus the rational function $h^* f^M$ on $\mathcal{Y}$ has order N along $\mathcal{Y}_k$, which implies that M divides N .

Now we prove that $\nu \geq N/M$. Shrinking $\mathcal{X}$ and $\mathcal{Y}$, we may assume that the following properties hold:

- the schemes $\mathcal{X}$ and $\mathcal{Y}$ are affine;
- the element $f \in \mathcal{O}_{\mathcal{X},x}$ is defined at every point of $\mathcal{X}$;
- there exist an integer $m > 0$, a polynomial P in $\mathcal{O}(\mathcal{X})[T_1, \dots, T_m]$ and a surjective morphism of $\mathcal{O}(\mathcal{X})$ -algebras

$$\varphi: A := \mathcal{O}(\mathcal{X})[T_1, \dots, T_m] / (PT_1^{N/M} - f) \rightarrow \mathcal{O}(\mathcal{Y}).$$

Clearly, $\text{Spec} A$ is regular at every point lying above x , so that the morphism $\mathcal{Y} \rightarrow \text{Spec} A$ is a regular immersion at the point y . We set $F_1 = PT_1^{N/M} - f$. Further shrinking $\mathcal{Y}$ around y , we may assume that there exist an integer $n \geq m$ and polynomials $F_2, \dots, F_n$ in $\mathcal{O}(\mathcal{X})[T_1, \dots, T_n]$ such that the sequence $(F_1, \dots, F_n)$ is a regular sequence in $\mathcal{O}(\mathcal{X})[T_1, \dots, T_n]$ and such that the morphism φ factors through an isomorphism of $\mathcal{O}(\mathcal{X})$ -algebras

$$\mathcal{O}(\mathcal{X})[T_1, \dots, T_n] / (PT_1^{N/M} - f, F_2, \dots, F_n) \rightarrow \mathcal{O}(\mathcal{Y}).$$

Applying § 4.1.3, we now compute that $\nu \geq N/M$. □

4.3 The weight of a differential form at a divisorial valuation

4.3.1 Let X be a connected smooth K -scheme of dimension n and let ω be a non-zero rational m -pluricanonical form on X , for some $m > 0$. Thus ω is a non-zero rational section of the m -pluricanonical line bundle $\omega_{X/K}^{\otimes m}$ on X . In the following subsections, we will explain how one can use ω to define a weight function on the analytic space X^{an} . As we will see, if X is proper and ω is regular at every point of X , then the weight function has the interesting property that it is strictly increasing if one moves away from the Berkovich skeleton associated to any proper *sncd*-model of X . We will use this property to compute the so-called Kontsevich–Soibelman skeleton of the pair (X, ω) in Theorem 4.7.5.

4.3.2 For each regular R -model $\mathcal{X}$ of X , the pluricanonical form ω defines a rational section of the relative m -pluricanonical line bundle $\omega_{\mathcal{X}/R}^{\otimes m}$ and thus a divisor $\text{div}_{\mathcal{X}}(\omega)$ on $\mathcal{X}$. If $h: \mathcal{X}' \rightarrow \mathcal{X}$ is a morphism of regular R -models of X , then it follows from § 4.1.4 that

$$\text{div}_{\mathcal{X}'}(\omega) = h^* \text{div}_{\mathcal{X}}(\omega) + mK_{\mathcal{X}'/\mathcal{X}}.$$

4.3.3 We first define the weight function on divisorial points. Let x be a divisorial point of X^{an} , associated to a regular R -model $\mathcal{X}$ of X and an irreducible component E of $\mathcal{X}_k$. We denote by μ the unique integer such that $\mu - m$ equals the multiplicity of E in $\text{div}_{\mathcal{X}}(\omega)$, and by N the multiplicity of E in $\mathcal{X}_k$.

PROPOSITION 4.3.4. *The rational number μ/N only depends on X , ω and x , and not on the choice of the model $\mathcal{X}$.*

Proof. Let $\mathcal{Y}$ be another regular R -model of X , and let F be an irreducible component of $\mathcal{Y}_k$ such that x is the divisorial point associated to $(\mathcal{Y}, F)$. Then the isomorphism between the generic fibers of $\mathcal{X}$ and $\mathcal{Y}$ will extend to an isomorphism between some open neighborhoods of the generic points of E and F , respectively, because the local rings at these points are the same when viewed as subrings of the field of rational functions on X : they both coincide with the valuation ring of v_x . Thus the value μ/N does not depend on the choice of the R -model $\mathcal{X}$. $\square$

4.3.5 We call μ/N the weight of ω at the point x , and denote it by $\text{wt}_{\omega}(x)$. In this way, we obtain a function

$$\text{wt}_{\omega}: X^{\text{div}} \rightarrow \mathbb{Q}, \quad x \mapsto \text{wt}_{\omega}(x)$$

on the set of divisorial points of X^{an} , which we call the weight function associated to ω . Note that

$$\text{wt}_{\omega^{\otimes d}}(x) = d \cdot \text{wt}_{\omega}(x)$$

for every integer $d > 0$ and

$$\text{wt}_{f\omega}(x) = \text{wt}_{\omega}(x) + v_x(f)$$

for every non-zero rational function f on X .

4.3.6 In the following subsections, it will be convenient to have a formula for the weight of ω at a divisorial point in terms of a monomial presentation. Let x be a monomial point on X^{an} , represented by an *sncd*-triple $(\mathcal{X}, (E_1, \dots, E_r), \xi)$ and a tuple α in $\mathbb{R}_{>0}^r$. Replacing $\mathcal{X}$ by its open subscheme of regular points does not influence the associated monomial valuation, so that we may assume that $\mathcal{X}$ is regular. As before, the rational m -pluricanonical form ω defines a

rational section of the relative m -pluricanonical sheaf $\omega_{\mathcal{X}/R}^{\otimes m}$, and thus a divisor $\operatorname{div}_{\mathcal{X}}(\omega)$ on $\mathcal{X}$. For each i in $\{1, \dots, r\}$, we denote by μ_i the unique integer such that $\mu_i - m$ equals the multiplicity of $\operatorname{div}_{\mathcal{X}}(\omega)$ along E_i . Recall that we associated a value $v_x(E)$ to every divisor E on $\mathcal{X}$ in § 2.2.1.

LEMMA 4.3.7. *If x is divisorial, then*

$$\operatorname{wt}_{\omega}(x) = v_x(\operatorname{div}_{\mathcal{X}}(\omega) + m(\mathcal{X}_k)_{\operatorname{red}}).$$

In particular, if ξ is not contained in the closure of the locus of zeroes and poles of ω on X , then

$$\operatorname{wt}_{\omega}(x) = \sum_{i=1}^r \alpha_i \mu_i.$$

Proof. The second assertion follows immediately from the first, since in this case, locally around ξ , we have

$$\operatorname{div}_{\mathcal{X}}(\omega) = \sum_{i=1}^r (\mu_i - m) E_i, \quad \text{so that} \quad v_x(\operatorname{div}_{\mathcal{X}}(\omega)) = \sum_{i=1}^r (\mu_i - m) \alpha_i.$$

Thus it suffices to prove the first assertion. If $r = 1$, this is simply the definition of the weight of ω at x , and we will reduce to this situation. We construct an *sncd*-triple

$$(\mathcal{X}', (E'_1, \dots, E'_r), \xi')$$

and a tuple α' in $\mathbb{Q}_{>0}^r$ as in the proof of Proposition 2.4.8, by blowing up $\mathcal{X}$ at the unique connected component of $\cap_{i=1}^r E_i$ that contains ξ . As we have explained in the proof of Proposition 2.4.8, repeating this operation a finite number of times, we can reduce to the case where $r = 1$. Thus it is enough to show that the value

$$v_x(\operatorname{div}_{\mathcal{X}}(\omega) + m(\mathcal{X}_k)_{\operatorname{red}})$$

does not change under such blow-ups, that is,

$$v_x(\operatorname{div}_{\mathcal{X}}(\omega)) + m \sum_{i=1}^r \alpha_i = v_x(\operatorname{div}_{\mathcal{X}'}(\omega)) + m \sum_{i=1}^r \alpha'_i.$$

The relative canonical divisor of the blow-up morphism $\mathcal{X}' \rightarrow \mathcal{X}$ is equal to $(r-1)E'_1$, so that

$$h^* \operatorname{div}_{\mathcal{X}}(\omega) = \operatorname{div}_{\mathcal{X}'}(\omega) - m(r-1)E'_1.$$

Since $\alpha'_1 = \alpha_1$ and $\alpha'_i = \alpha_i - \alpha_1$ for $2 \leq i \leq r$, we obtain

$$\begin{aligned} v_x(\operatorname{div}_{\mathcal{X}'}(\omega)) + m \sum_{i=1}^r \alpha'_i &= v_x(h^* \operatorname{div}_{\mathcal{X}}(\omega)) + m(r-1)\alpha_1 + m \sum_{i=1}^r \alpha'_i \\ &= v_x(\operatorname{div}_{\mathcal{X}}(\omega)) + m \sum_{i=1}^r \alpha_i. \end{aligned} \quad \square$$

LEMMA 4.3.8. *Let $\mathcal{Y}$ be an *sncd*-model for X and let y be a divisorial point on $\widehat{\mathcal{Y}}_{\eta}$. Then*

$$\operatorname{wt}_{\omega}(y) \geq v_y(\operatorname{div}_{\mathcal{Y}}(\omega) + m(\mathcal{Y}_k)_{\operatorname{red}})$$

and equality holds if and only if y lies on the Berkovich skeleton $\operatorname{Sk}(\mathcal{Y})$.

Proof. Denote by $E_1, \dots, E_r$ the irreducible components of $\mathcal{Y}_k$ that contain $\operatorname{red}_{\mathcal{Y}}(y)$, and let ξ be the generic point of the connected component of $\cap_{i=1}^r E_i$ containing $\operatorname{red}_{\mathcal{Y}}(y)$. We set $y' = \rho_{\mathcal{Y}}(y)$.

The point y' is divisorial because the value group of $v_{y'}$ is contained in the value group of v_y . Multiplying ω with a suitable rational function on X , we may assume that $\text{red}_{\mathcal{Y}}(y)$ is not contained in the closure of the locus of zeroes and poles of ω on X . Then

$$v_y(\text{div}_{\mathcal{Y}}(\omega) + m(\mathcal{Y}_k)_{\text{red}}) = v_{y'}(\text{div}_{\mathcal{Y}}(\omega) + m(\mathcal{Y}_k)_{\text{red}}) = \text{wt}_{\omega}(y')$$

by Lemma 4.3.7, so that it is enough to show that

$$\text{wt}_{\omega}(y) \geq \text{wt}_{\omega}(y'),$$

with equality if and only if $y = y'$.

By means of the blowing-up procedure explained in the proof of Proposition 2.4.8, we construct an *sncd*-model $\mathcal{X}$ of X and an irreducible component E of $\mathcal{X}_k$ such that y' is the divisorial point associated to $(\mathcal{X}, E)$. Then Propositions 3.1.9 and 3.1.7 imply $y' = \rho_{\mathcal{X}}(y)$. Thus we can assume without loss of generality that $r = 1$. We set $E = E_1$.

Since every element of $\mathcal{O}_{\mathcal{Y}, \text{red}_{\mathcal{Y}}(y)}$ belongs to the valuation ring of v_y , we can find a regular R -model $\mathcal{Z}$ of X such that $\mathcal{Z}_k$ is irreducible and y is the divisorial point associated to $(\mathcal{Z}, (\mathcal{Z}_k)_{\text{red}})$, together with a morphism $h: \mathcal{Z} \rightarrow \mathcal{Y}$ of R -models of X . This morphism sends the generic point of $\mathcal{Z}_k$ to ξ . We set $F = (\mathcal{Z}_k)_{\text{red}}$.

We denote by $\mu - m$ the multiplicity of E in $\text{div}_{\mathcal{Y}}(\omega)$, by M the multiplicity of E in $\mathcal{Y}_k$, by $\nu - 1$ the multiplicity of F in $K_{\mathcal{Z}/\mathcal{Y}}$ and by N the multiplicity of F in $\mathcal{Z}_k$. If $f = 0$ is a local equation for E in $\mathcal{Y}$ at ξ , then

$$v_y(f) = v_{y'}(f) = \frac{1}{M}$$

because, up to an invertible factor, f^M is a uniformizer in R . This implies that N is a multiple of M and that the multiplicity of F in $\text{div}_{\mathcal{Z}}(\omega)$ is equal to

$$(\mu - m)\frac{N}{M} + m\nu - m.$$

Now we compute

$$\text{wt}_{\omega}(y) = \frac{1}{N} \left((\mu - m)\frac{N}{M} + m\nu - m + m \right) = \frac{\mu}{M} + \frac{m(\nu - N/M)}{N}.$$

On the other hand,

$$\text{wt}_{\omega}(y') = \frac{\mu}{M}.$$

It follows from Proposition 4.2.1 that $\nu \geq N/M$, so that

$$\text{wt}_{\omega}(y) \geq \text{wt}_{\omega}(y').$$

Now assume that $y \neq y'$. We denote by Y the closure of $\{\text{red}_{\mathcal{Y}}(y)\}$ in $\mathcal{Y}$, endowed with its reduced induced structure. Shrinking $\mathcal{Y}$, we may assume that Y is regular. Note that Y is a strict subvariety of $(\mathcal{Y}_k)_{\text{red}}$, since otherwise, y would be equal to y' . We denote by $g: \mathcal{Y}' \rightarrow \mathcal{Y}$ the blow-up of $\mathcal{Y}$ at Y . Then

$$\begin{aligned} v_y(\text{div}_{\mathcal{Y}}(\omega) + m(\mathcal{Y}_k)_{\text{red}}) &= v_y(g^* \text{div}_{\mathcal{Y}'}(\omega) + mg^*(\mathcal{Y}_k)_{\text{red}}) \\ &= v_y(\text{div}_{\mathcal{Y}'}(\omega) - mK_{\mathcal{Y}'/\mathcal{Y}} + m(\mathcal{Y}'_k)_{\text{red}}) \\ &< v_y(\text{div}_{\mathcal{Y}'}(\omega) + m(\mathcal{Y}'_k)_{\text{red}}). \end{aligned}$$

Applying the first part of the proof to the model $\mathcal{Y}'$, we get

$$v_y(\text{div}_{\mathcal{Y}'}(\omega) + m(\mathcal{Y}'_k)_{\text{red}}) \leq \text{wt}_{\omega}(y),$$

and therefore $\text{wt}_\omega(y') < \text{wt}_\omega(y)$. $\square$

4.4 The weight function on a Berkovich skeleton

4.4.1 Let X be a connected smooth proper K -scheme of dimension n and let ω be a non-zero rational m -pluricanonical form on X . We suppose that X has a proper *sncd*-model $\mathcal{X}$. Note that, at this point, we are not assuming resolution of singularities, nor that any two proper *sncd*-models can be dominated by a common one. We denote by E_i , for $i \in I$, the irreducible components of $\mathcal{X}_k$ and by $m\mu_i - m$ the multiplicity of E_i in $\text{div}_{\mathcal{X}}(\omega)$, for each $i \in I$.

4.4.2 Let x be a point of the Berkovich skeleton $\text{Sk}(\mathcal{X})$. Then we set

$$\text{wt}_{\mathcal{X},\omega}(x) = v_x(\text{div}(\omega) + m(\mathcal{X}_k)_{\text{red}}).$$

In this way, we obtain a function

$$\text{wt}_{\mathcal{X},\omega}: \text{Sk}(\mathcal{X}) \rightarrow \mathbb{R}$$

that we call the *weight function* associated to $\mathcal{X}$ and ω . It is clear from the definition that, for each point x of $\text{Sk}(\mathcal{X})$, we have

$$\text{wt}_{\mathcal{X},\omega^{\otimes d}}(x) = d \cdot \text{wt}_{\mathcal{X},\omega}(x)$$

for all $d > 0$ and

$$\text{wt}_{\mathcal{X},f\omega}(x) = \text{wt}_{\mathcal{X},\omega}(x) + v_x(f)$$

for every non-zero rational function f on X .

PROPOSITION 4.4.3. *The function*

$$\text{wt}_{\mathcal{X},\omega}: \text{Sk}(\mathcal{X}) \rightarrow \mathbb{R}$$

is piecewise affine. If J is a non-empty subset of I , ξ is a generic point of $\cap_{j \in J} E_j$ and σ is the closed face of $\text{Sk}(\mathcal{X})$ corresponding to ξ , then the restriction of $\text{wt}_{\mathcal{X},\omega}(x)$ to σ is

- *concave if ξ is not contained in the closure of the locus of poles of ω on X ;*
- *convex if ξ is not contained in the closure of the locus of zeroes of ω on X ;*
- *affine if ξ is not contained in the closure of the locus of zeroes and poles of ω on X .*

Proof. The function $\text{wt}_{\mathcal{X},\omega}$ is continuous, by Proposition 2.2.4. The remaining properties can be checked on the restriction of $\text{wt}_{\mathcal{X},\omega}$ to a closed face σ as in the statement. We can write ω as $h \cdot \omega_0$, where h is a non-zero rational function on X and ω_0 is a generator of the stalk of $\omega_{\mathcal{X}/R}^{\otimes m}$ at ξ . If we choose, for each j in J , a local equation $x_j = 0$ for the prime divisor E_j on $\mathcal{X}$ at ξ , then we have

$$\text{wt}_{\mathcal{X},\omega}(x) = v_x \left(h \prod_{j \in J} x_j^m \right)$$

for every x in σ , so that the result follows from Proposition 3.2.2. $\square$

PROPOSITION 4.4.4.

- (i) *If x is a divisorial point on $\text{Sk}(\mathcal{X})$, then*

$$\text{wt}_{\mathcal{X},\omega}(x) = \text{wt}_\omega(x).$$

- (ii) If x is a divisorial point on X^{an} such that $\text{red}_{\mathcal{X}}(x)$ is not contained in the closure of the locus of poles of ω on X , then

$$\text{wt}_{\omega}(x) \geq \text{wt}_{\mathcal{X}, \omega}(\rho_{\mathcal{X}}(x)),$$

with equality if and only if $x \in \text{Sk}(\mathcal{X})$.

- (iii) If $\mathcal{Y}$ is a proper *sncd*-model of X and y is a point of $\text{Sk}(\mathcal{Y})$ such that $\text{red}_{\mathcal{X}}(y)$ is not contained in the closure of the locus of poles of ω on X , then

$$\text{wt}_{\mathcal{Y}, \omega}(y) \geq \text{wt}_{\mathcal{X}, \omega}(\rho_{\mathcal{X}}(y))$$

and equality can occur only if $y \in \text{Sk}(\mathcal{X})$.

- (iv) If $\mathcal{Y}$ is a proper *sncd*-model of X and if x is a point in $\text{Sk}(\mathcal{X}) \cap \text{Sk}(\mathcal{Y})$, then

$$\text{wt}_{\mathcal{X}, \omega}(x) = \text{wt}_{\mathcal{Y}, \omega}(x).$$

Proof. (i) This follows from Lemma 4.3.7.

(ii) We may assume that $\text{red}_{\mathcal{X}}(x)$ is not contained in the closure of the zero locus of ω on X , by dividing ω by a suitable element of $\mathcal{O}_{\mathcal{X}, \zeta}$ and invoking Proposition 3.1.6. Then it follows from Lemma 4.3.8 that

$$\text{wt}_{\omega}(x) \geq v_x \left(\sum_{i \in I} (\mu_i - m) E_i + m(\mathcal{X}_k)_{\text{red}} \right) = \text{wt}_{\mathcal{X}, \omega}(\rho_{\mathcal{X}}(x))$$

with equality if and only if x belongs to $\text{Sk}(\mathcal{X})$.

(iii) Denote by P the set of points of $\mathcal{X}_k$ that belong to the closure of the locus of poles of ω on X . This is a closed subset of $\mathcal{X}_k$. Let Z be the set of points y' on $\text{Sk}(\mathcal{Y})$ such that $\text{red}_{\mathcal{X}}(y')$ does not lie in P . This is a closed subset of $\text{Sk}(\mathcal{Y})$ containing y , by anti-continuity of the reduction map $\text{red}_{\mathcal{X}}$. We claim that the divisorial points in Z are dense in Z . If we accept this claim for now, it suffices to prove the inequality in the statement in the case where y is divisorial, by continuity of $\text{wt}_{\mathcal{X}, \omega}$, $\text{wt}_{\mathcal{Y}, \omega}$ and $\rho_{\mathcal{X}}$. But when y is divisorial, the inequality follows from statements (i) and (ii). Likewise, if we denote by Z' the locus of points y' in Z such that

$$\text{wt}_{\mathcal{Y}, \omega}(y') = \text{wt}_{\mathcal{X}, \omega}(\rho_{\mathcal{X}}(y')),$$

then the divisorial points in Z' form a dense subset, because the functions $\text{wt}_{\mathcal{Y}, \omega}$ and $\text{wt}_{\mathcal{X}, \omega} \circ \rho_{\mathcal{X}}$ are piecewise affine on $\text{Sk}(\mathcal{Y})$, by Propositions 3.2.3 and 4.4.3. Statement (ii) implies that $X^{\text{div}} \cap Z'$ is contained in $\text{Sk}(\mathcal{X})$, so that Z' is contained in $\text{Sk}(\mathcal{X})$.

It remains to prove our claim that the set of divisorial points in Z is dense in Z . Since y is an arbitrary point of Z , it is enough to show that y belongs to the closure of the set of divisorial points in Z . Denote by ξ the center of v_y on $\mathcal{X}$ and by C the closure of $\{\xi\}$ in $\mathcal{X}$, endowed with its reduced induced structure. We choose finitely many affine open neighborhoods $\mathcal{U}_1, \dots, \mathcal{U}_r$ of ξ in $\mathcal{X}$ such that C is contained in their union. For each i in $\{1, \dots, r\}$, we choose a regular function f_i in $\mathcal{O}(\mathcal{U}_i)$ such that the support of the closed subscheme of $\mathcal{U}_i$ defined by $f_i = 0$ is equal to $\mathcal{U}_i \cap P$. Then the set

$$S = \{z \in \text{red}_{\mathcal{X}}^{-1}(C) \mid v_z(f_i) = 0 \text{ for } i = 1, \dots, r\}$$

is a locally closed subset of Z containing y . The divisorial points in S form a dense subset of S , because $\text{red}_{\mathcal{X}}^{-1}(C)$ is open in $\text{Sk}(\mathcal{Y})$ and the functions

$$\text{Sk}(\mathcal{Y}) \rightarrow \mathbb{R}, \quad z \rightarrow v_z(f_i)$$

are piecewise affine. It follows that y belongs to the closure in Z of the set of divisorial points in Z .

(iv) Since the statement is symmetric in $\mathcal{X}$ and $\mathcal{Y}$, it is enough to prove

$$\mathrm{wt}_{\mathcal{Y}, \omega}(x) \geq \mathrm{wt}_{\mathcal{X}, \omega}(x).$$

Multiplying ω by a suitable non-zero rational function on X , we can reduce to the case where the center of v_x on $\mathcal{X}$ is not contained in the closure of the locus of poles of ω on X . In that case, the result follows from statement (iii). $\square$

4.5 The weight function on X^{an}

4.5.1 Let X be a connected smooth K -scheme of dimension n . We will show how weight functions can be defined on the whole space X^{an} . Throughout this section, we make the following assumption: there exists a smooth compactification $\bar{X}$ such that, for every proper R -model $\bar{\mathcal{X}}$ of $\bar{X}$, there exists a morphism of R -models $h: \bar{\mathcal{X}}' \rightarrow \bar{\mathcal{X}}$ such that $\bar{\mathcal{X}}'$ is a proper *sncd*-model and h is an isomorphism over the open subscheme of $\bar{\mathcal{X}}$ consisting of the points where $\bar{\mathcal{X}}$ is regular and $\bar{\mathcal{X}}'_k$ is a divisor with strict normal crossings. This assumption is satisfied when k has characteristic zero or X is a curve.

PROPOSITION 4.5.2. *Let x be a point of X^{an} . Every proper R -model $\bar{\mathcal{Y}}$ of $\bar{X}$ can be dominated by a proper *sncd*-model $\bar{\mathcal{X}}$ that has an open subscheme $\mathcal{X}$ such that $\mathcal{X}$ is an *sncd*-model for X and $\widehat{\mathcal{X}}_\eta$ is a neighborhood of x in X^{an} . If x is monomial, we can moreover arrange that $x \in \mathrm{Sk}(\mathcal{X})$.*

Proof. If x is monomial, we can first choose an *sncd*-model $\mathcal{Y}'$ of X such that x lies in $\mathrm{Sk}(\mathcal{Y}')$ and compactify it to an R -model $\bar{\mathcal{Y}}'$ of $\bar{X}$. Replacing $\bar{\mathcal{Y}}$ by a proper R -model of $\bar{X}$ that dominates both $\bar{\mathcal{Y}}$ and $\bar{\mathcal{Y}}'$, we can assume that x lies in $\mathrm{Sk}(\bar{\mathcal{Y}})$, by Proposition 3.1.7.

We set $Z = \bar{X} \setminus X$ and we endow this closed subset of $\bar{X}$ with its reduced induced structure. By Corollary 2.3.3, we can dominate $\bar{\mathcal{Y}}$ by a proper *sncd*-model $\bar{\mathcal{X}}$ of $\bar{X}$ such that the closure C of $\mathrm{red}_{\bar{\mathcal{X}}}(x)$ in $\bar{\mathcal{X}}_k$ is disjoint from the closure of Z in $\bar{\mathcal{X}}$. The set $\mathrm{red}_{\bar{\mathcal{X}}}^{-1}(C)$ is open in X^{an} by anti-continuity of the reduction map. Removing from $\bar{\mathcal{X}}$ the closure of Z , we get an open subscheme $\mathcal{X}$ which is an *sncd*-model of X and whose generic fiber $\widehat{\mathcal{X}}_\eta$ is a neighborhood of x in X^{an} . If x lies in $\mathrm{Sk}(\bar{\mathcal{Y}})$, then it will also lie in $\mathrm{Sk}(\mathcal{X})$, by Proposition 3.1.7. $\square$

PROPOSITION 4.5.3. *Let ω be a non-zero rational m -pluricanonical form on X . There exists a unique function*

$$\mathrm{wt}_\omega: X^{\mathrm{mon}} \rightarrow \mathbb{Q}$$

on the set of monomial points of X^{an} such that

$$\mathrm{wt}_\omega(x) = v_x(\mathrm{div}_{\mathcal{X}}(\omega) + m(\mathcal{X}_k)_{\mathrm{red}})$$

*for every monomial point x of X and every *sncd*-model $\mathcal{X}$ of X for which $x \in \mathrm{Sk}(\mathcal{X})$.*

Proof. This follows immediately from Proposition 4.4.4, since we can compactify $\mathcal{X}$ to a proper *sncd*-model of $\bar{X}$. $\square$

4.5.4 Let ω be a non-zero regular m -pluricanonical form on X , for some $m > 0$, and let x be a point of X^{an} . We set

$$\mathrm{wt}_\omega(x) = \sup_{\mathcal{X}} \{ \mathrm{wt}_\omega(\rho_{\mathcal{X}}(x)) \} \in \mathbb{R} \cup \{+\infty\},$$

where the supremum is taken over all *sncd*-models $\mathcal{X}$ of X such that x lies in $\widehat{\mathcal{X}}_\eta$. In this way, we obtain a function

$$\mathrm{wt}_\omega: X^{\mathrm{an}} \rightarrow \mathbb{R} \cup \{+\infty\}.$$

PROPOSITION 4.5.5.

(i) *The function*

$$\mathrm{wt}_\omega: X^{\mathrm{an}} \rightarrow \mathbb{R} \cup \{+\infty\}$$

is lower semi-continuous.

(ii) *We have*

$$\mathrm{wt}_\omega(x) = v_x(\mathrm{div}_{\mathcal{X}}(\omega) + m(\mathcal{X}_k)_{\mathrm{red}})$$

*for every monomial point x on X^{an} and every *sncd*-model $\mathcal{X}$ of X such that $x \in \mathrm{Sk}(\mathcal{X})$. In particular, the definition in § 4.5.4 agrees with the one in § 4.3.5 for divisorial points and with the one in Proposition 4.5.3 for points on the skeleton of an *sncd*-model of X .*

(iii) *For every point x of X^{an} and every *sncd*-model $\mathcal{X}$ of X such that $x \in \widehat{\mathcal{X}}_\eta$, we have*

$$\mathrm{wt}_\omega(x) \geq \mathrm{wt}_\omega(\rho_{\mathcal{X}}(x))$$

and equality holds if and only if x lies in $\mathrm{Sk}(\mathcal{X})$.

(iv) *If $h: Y \rightarrow X$ is an open immersion, then*

$$\mathrm{wt}_{h^*\omega}(y) = \mathrm{wt}_\omega(h(y))$$

for every point y on Y^{an} .

(v) *For each point x in X^{an} , we have*

$$\mathrm{wt}_{\omega \otimes d}(x) = d \cdot \mathrm{wt}_\omega(x)$$

for all integers $d > 0$ and

$$\mathrm{wt}_{f\omega}(x) = \mathrm{wt}_\omega(x) + v_x(f)$$

for all regular functions $f \neq 0$ on X .

Proof. (i) Let x be a point of X^{an} . By Proposition 4.5.2, we can find an *sncd*-model $\mathcal{X}$ of X such that $\widehat{\mathcal{X}}_\eta$ is a neighborhood of x . For every point y of $\widehat{\mathcal{X}}_\eta$, we have

$$\mathrm{wt}_\omega(x) = \sup_{\mathcal{X}'} \{ \mathrm{wt}_\omega(\rho_{\mathcal{X}'}(x)) \} \in \mathbb{R} \cup \{+\infty\},$$

where the supremum is taken over all proper R -morphisms $\mathcal{X}' \rightarrow \mathcal{X}$ of *sncd*-models of X . Thus wt_ω is lower semi-continuous on $\widehat{\mathcal{X}}_\eta$, since it is the supremum of the continuous functions $\mathrm{wt}_\omega \circ \rho_{\mathcal{X}'}$.

(ii) This follows from Proposition 4.4.4.

(iii) The inequality follows immediately from the definition. Proposition 4.4.4 implies that every monomial point y of $\widehat{\mathcal{X}}_\eta$ that does not lie on $\mathrm{Sk}(\mathcal{X})$ satisfies

$$\mathrm{wt}_\omega(y) > \mathrm{wt}_\omega(\rho_{\mathcal{X}}(y)).$$

Thus it suffices to prove the following property: if x does not lie on $\mathrm{Sk}(\mathcal{X})$, then we can find a proper morphism $\mathcal{X}' \rightarrow \mathcal{X}$ of *sncd*-models of X such that $x' := \rho_{\mathcal{X}'}(x)$ does not lie on $\mathrm{Sk}(\mathcal{X})$. Note that $\rho_{\mathcal{X}}(x') = \rho_{\mathcal{X}}(x)$ by Proposition 3.1.7.

We choose a proper morphism of *sncd*-models $\mathcal{Y} \rightarrow \mathcal{X}$ such that

$$\mathrm{red}_{\mathcal{Y}}(x) \neq \mathrm{red}_{\mathcal{Y}}(\rho_{\mathcal{X}}(x)).$$

The existence of such a model follows from Proposition 2.3.2. We set $y = \rho_{\mathcal{Y}}(x)$. If y does not belong to $\text{Sk}(\mathcal{X})$, then we can take $\mathcal{X}' = \mathcal{Y}$ and $x' = y$. Thus we need to consider only the case where $y \in \text{Sk}(\mathcal{X})$. Then we have $y = \rho_{\mathcal{X}}(x)$, so that

$$\text{red}_{\mathcal{Y}}(x) \neq \text{red}_{\mathcal{Y}}(y)$$

and $\text{red}_{\mathcal{Y}}(x)$ belongs to the closure of $\{\text{red}_{\mathcal{Y}}(y)\}$. It follows that $\text{red}_{\mathcal{Y}}(y)$ cannot belong to the closure of $\{\text{red}_{\mathcal{Y}}(x)\}$. Blowing up $\mathcal{Y}$ at the closure of $\{\text{red}_{\mathcal{Y}}(x)\}$ and resolving the singularities, we obtain a proper morphism of *sncd*-models $\mathcal{X}' \rightarrow \mathcal{Y}$. We set $x' = \rho_{\mathcal{X}'}(x)$. Note that $\rho_{\mathcal{Y}}(x') = y$ by Proposition 3.1.7. The inverse image in $\mathcal{X}'_k$ of the closure of $\{\text{red}_{\mathcal{Y}}(x)\}$ is a union of irreducible components of $\mathcal{X}'_k$. This implies that $\text{red}_{\mathcal{Y}}(x')$ lies in the closure of $\{\text{red}_{\mathcal{Y}}(x)\}$. In particular, $\text{red}_{\mathcal{Y}}(x')$ is different from $\text{red}_{\mathcal{Y}}(y)$, which means that x' cannot lie in $\text{Sk}(\mathcal{Y})$. Therefore, it does not lie in $\text{Sk}(\mathcal{X})$ either, since $\text{Sk}(\mathcal{X})$ is contained in $\text{Sk}(\mathcal{Y})$ by Proposition 3.1.7.

(iv) Let $\mathcal{Y}$ be an *sncd*-model of Y such that $y \in \widehat{\mathcal{Y}}_{\eta}$. Gluing $\mathcal{Y}_K$ to X by means of the open immersion h , we get an *sncd*-model $\mathcal{X}$ of X such that $h(y) \in \widehat{\mathcal{X}}_{\eta}$. Then it is clear that

$$\text{wt}_{h^*\omega}(\rho_{\mathcal{Y}}(y)) = \text{wt}_{\omega}(\rho_{\mathcal{X}}(h(y))) \leq \text{wt}_{\omega}(h(y)).$$

Since this holds for all *sncd*-models $\mathcal{Y}$ of Y with $y \in \widehat{\mathcal{Y}}_{\eta}$, we find

$$\text{wt}_{h^*\omega}(y) \leq \text{wt}_{\omega}(x).$$

Conversely, let $\mathcal{X}$ be an *sncd*-model of X such that $h(y)$ lies in $\widehat{\mathcal{X}}_{\eta}$. Then by applying Proposition 4.5.2 to Y and a compactification of $\mathcal{X}$ to a proper R -model of $\bar{X}$, we can find a proper morphism of *sncd*-models $\mathcal{X}' \rightarrow \mathcal{X}$ and an open subscheme $\mathcal{Y}$ of $\mathcal{X}'$ such that $\mathcal{Y}$ is an *sncd*-model of Y and $y \in \widehat{\mathcal{Y}}_{\eta}$. We have

$$\text{wt}_{\omega}(\rho_{\mathcal{X}}(h(y))) \leq \text{wt}_{\omega}(\rho_{\mathcal{X}'}(h(y))) = \text{wt}_{h^*\omega}(\rho_{\mathcal{Y}}(y)) \leq \text{wt}_{h^*\omega}(y)$$

and thus

$$\text{wt}_{h^*\omega}(y) = \text{wt}_{\omega}(h(y)).$$

(v) This is obvious. □

Remark 4.6. The weight function

$$\text{wt}_{\omega}: X^{\text{an}} \rightarrow \mathbb{R} \cup \{+\infty\}$$

is not continuous. Assume, for instance, that X is a curve and that $\mathcal{X}$ is a regular R -model of X . Let E be an irreducible component of $\mathcal{X}_k$ of multiplicity N , and let $(y_n)_{n \geq 0}$ be any sequence of distinct closed points on E that are not contained in any other irreducible component of $\mathcal{X}_k$. If we denote by x_n the divisorial point of X^{an} associated to the exceptional divisor of the blow-up of $\mathcal{X}$ at y_n , then the sequence $(x_n)_{n \geq 0}$ converges to the divisorial point $x \in X^{\text{an}}$ associated to $(\mathcal{X}, E)$. However, a direct computation shows that

$$\text{wt}_{\omega}(x_n) \geq \text{wt}_{\omega}(x) + 1/N$$

for every $n \geq 0$.

4.7 The Kontsevich–Soibelman skeleton

4.7.1 Let X be a connected smooth K -scheme of dimension n and let ω be a non-zero rational m -pluricanonical form on X , for some $m > 0$. We define the weight of X with respect to ω to be the value

$$\text{wt}_{\omega}(X) = \inf \{ \text{wt}_{\omega}(x) \mid x \in X^{\text{div}} \} \in \mathbb{R} \cup \{-\infty\}.$$

We say that a divisorial point x of X^{an} is ω -essential if

$$\text{wt}_\omega(x) = \text{wt}_\omega(X).$$

The closure in X^{bir} (with the induced topology from X^{an}) of the set of ω -essential points is called the *Kontsevich–Soibelman skeleton* of (X, ω) and denoted by $\text{Sk}(X, \omega)$. We endow it with the induced topology from X^{an} .

PROPOSITION 4.7.2. *The skeleton (X, ω) is a birational invariant: if $h: Y \rightarrow X$ is a birational morphism of connected smooth K -schemes, then the morphism $h^{\text{an}}: Y^{\text{an}} \rightarrow X^{\text{an}}$ induces a homeomorphism between $\text{Sk}(Y, h^*\omega)$ and $\text{Sk}(X, \omega)$.*

Proof. It is enough to consider the case where h is an open immersion. This case is trivial, since Y^{an} and X^{an} have the same divisorial and birational points and we can extend every model $\mathcal{Y}$ for Y to a model for X by gluing X to the generic fiber $\mathcal{Y}_K$. $\square$

4.7.3 The skeleton $\text{Sk}(X, \omega)$ was introduced by Kontsevich and Soibelman in [KS06, § 6.6] in the case where $R = \mathbb{C}[[t]]$ and X and ω are defined over the ring $\mathbb{C}\{t\}$ of germs of analytic functions on $\mathbb{C}$ at 0. Instead of our definition of the weight, they used the following invariant: with the notation of § 4.3.3, they considered the value

$$\frac{1}{N} \cdot \text{ord}_E(\omega \wedge dt/t),$$

where $\omega \wedge dt$ is viewed as a meromorphic differential form of maximal degree on $\mathcal{X}$. However, this invariant does not lead to the correct definition of the skeleton, and Theorem 3 in [KS06, § 6.6] is incorrect as stated. Instead, one should use the invariant

$$\frac{1}{N} \cdot (\text{ord}_E(\omega \wedge dt/t) + 1),$$

which coincides with our definition of weight up to shift by -1 because the wedge product with dt defines an isomorphism between $\omega_{\mathcal{X}/R}$ and $\Omega_{\mathcal{X}/\mathbb{C}}^{n+1}$. We will now prove a generalization of Theorem 3 in [KS06, § 6.6]. Instead of invoking the weak factorization theorem as in [KS06, § 6.6], we use only the elementary properties of the weight function that we have proven in the preceding subsections.

4.7.4 Assume that X is proper over K and that ω is regular at every point of X . We suppose furthermore that X has a proper *sncd*-model $\mathcal{X}$. We will give an explicit description of $\text{Sk}(X, \omega)$ in terms of $\mathcal{X}$. We write

$$\mathcal{X}_k = \sum_{i \in I} N_i E_i$$

and we denote by $\mu_i - m$ the multiplicity of E_i in $\text{div}_{\mathcal{X}}(\omega)$, for each i in I . Let J be a non-empty subset of I and let ξ be a generic point of $\cap_{j \in J} E_j$. We say that ξ is ω -essential if

$$\frac{\mu_j}{N_j} = \min \left\{ \frac{\mu_i}{N_i} \mid i \in I \right\}$$

for every j in J and ξ is not contained in the closure of the zero locus of ω on X .

THEOREM 4.7.5. *With the assumptions and notation of § 4.7.4, we have*

$$\text{wt}_\omega(X) = \min \{ \text{wt}_\omega(x) \mid x \in X^{\text{div}} \} = \min \{ \text{wt}_\omega(x) \mid x \in X^{\text{an}} \} = \min \left\{ \frac{\mu_i}{N_i} \mid i \in I \right\}.$$

The Kontsevich–Soibelman skeleton $\mathrm{Sk}(X, \omega)$ is equal to the set of points $x \in X^{\mathrm{an}}$ where the weight function wt_ω reaches its minimal value $\mathrm{wt}_\omega(X)$. It is the union of the open faces in the Berkovich skeleton $\mathrm{Sk}(\mathcal{X})$ corresponding to the ω -essential points of $\mathcal{X}_k$. This is a non-empty compact subspace of $\mathrm{Sk}(\mathcal{X})$.

In particular, this union of faces in the Berkovich skeleton $\mathrm{Sk}(\mathcal{X})$ depends only on X and ω , and not on the choice of the proper *sncd*-model $\mathcal{X}$.

Proof. This follows immediately from Proposition 4.4.4 and the fact that

$$\mathrm{wt}_{\mathcal{X}, \omega}(x) \geq \sum_{i \in I} \mu_i v_x(E_i)$$

for every x in $\mathrm{Sk}(\mathcal{X})$, with equality if and only if $\mathrm{red}_{\mathcal{X}}(x)$ is not contained in the closure of the zero locus of ω on X . $\square$

Example 4.8. Let X be a smooth proper K -variety with trivial canonical sheaf, and assume that X has a proper *sncd*-model $\mathcal{X}$ such that $\omega_{\mathcal{X}/R}$ is trivial. Then $\mathrm{Sk}(X)$ is the union of the closed faces of $\mathrm{Sk}(\mathcal{X})$ corresponding to the irreducible components of $\mathcal{X}_k$ of maximal multiplicity. In particular, if $\mathcal{X}_k$ is reduced, then $\mathrm{Sk}(X) = \mathrm{Sk}(\mathcal{X})$. Such models play an important role in the study of degenerations of complex $K3$ -surfaces; see [Kul77] and [PP81].

4.8.1 Note that $\mathrm{Sk}(X, \omega)$ may be empty if ω has poles on X . For instance, it is easy to see that the weight of $\mathbb{P}_K^1 = \mathrm{Proj} K[x, y]$ with respect to the one-form $d(x/y)$ is equal to $-\infty$.

4.9 The essential skeleton

4.9.1 Let X be a connected smooth and proper K -variety. If $\mathcal{X}$ is an *sncd*-model of X , then the Berkovich skeleton $\mathrm{Sk}(\mathcal{X})$ is of course highly dependent on the choice of $\mathcal{X}$. Nevertheless, we have seen in Theorem 4.7.5 that for every non-zero m -pluricanonical form ω on X , the Kontsevich–Soibelman skeleton $\mathrm{Sk}(X, \omega)$ identifies certain open faces of $\mathrm{Sk}(\mathcal{X})$ that must be contained in the Berkovich skeleton of *every* *sncd*-model of X . This naturally leads to the following definition.

DEFINITION 4.10. Let X be a connected smooth and proper K -variety. We define the *essential skeleton* of X by

$$\mathrm{Sk}(X) = \bigcup_{\omega} \mathrm{Sk}(X, \omega) \subset X^{\mathrm{an}},$$

where ω runs through the set of non-zero regular pluricanonical forms on X .

PROPOSITION 4.10.1. *The essential skeleton $\mathrm{Sk}(X)$ is a birational invariant of X .*

Proof. This follows at once from the birational invariance of $\mathrm{Sk}(X, \omega)$ and of the spaces of pluricanonical forms on X . $\square$

4.10.2 If X has Kodaira dimension $-\infty$, then $\mathrm{Sk}(X)$ is empty. If X has non-negative Kodaira dimension and there exists a proper *sncd*-model $\mathcal{X}$ of X , then $\mathrm{Sk}(X)$ is a non-empty union of closed faces of $\mathrm{Sk}(\mathcal{X})$, by Theorem 4.7.5 (it is a union of open faces and it is compact). In particular, $\mathrm{Sk}(X)$ carries a canonical piecewise affine structure, by Corollary 3.2.4. If the canonical sheaf $\omega_{X/K}$ of X is trivial, then $\mathrm{Sk}(X) = \mathrm{Sk}(X, \omega)$, where ω is any non-zero differential form of maximal degree on X .

5. The skeleton is connected

In this section we show that the Kontsevich–Soibelman skeleton of a smooth and proper K -variety of geometric genus one is connected, assuming that k has characteristic zero. The proof is based on a generalization of Kollár’s torsion-free theorem to R -schemes. We will first deduce this generalization by means of an approximation argument.

5.1 Algebraic approximation of R -schemes

5.1.1 If X is a proper scheme of pure dimension d over a field F , then a line bundle $\mathcal{L}$ on X is called *big* if there exists a constant $C > 0$ such that

$$\dim_F H^0(X, \mathcal{L}^n) \geq Cn^d$$

for all sufficiently divisible positive integers n .

PROPOSITION 5.1.2. *Assume that R has equal characteristic. Let $\mathcal{X}$ be a flat R -scheme and let $\mathcal{L}$ be a line bundle on $\mathcal{X}$. Then for every integer $n > 0$, we can find the following data:*

- (i) *a smooth connected algebraic k -curve C , a k -rational point O on C and a ring isomorphism*

$$R/\mathfrak{m}^n \rightarrow \mathcal{O}_{C,O}/\mathfrak{m}_{C,O}^n,$$

where $\mathfrak{m}_{C,O}$ denotes the maximal ideal of $\mathcal{O}_{C,O}$;

- (ii) *a flat C -scheme $\mathcal{X}'$, a line bundle $\mathcal{L}'$ on $\mathcal{X}'$, an isomorphism of $R/\mathfrak{m}^n$ -schemes*

$$f: \mathcal{X}' \times_C \operatorname{Spec}(\mathcal{O}_{C,O}/\mathfrak{m}_{C,O}^n) \rightarrow \mathcal{X} \times_R (R/\mathfrak{m}^n)$$

and an isomorphism of line bundles

$$\mathcal{L}'/\mathfrak{m}_{C,O}^n \rightarrow f^*(\mathcal{L}/\mathfrak{m}^n).$$

Here we wrote $\mathcal{L}'/\mathfrak{m}_{C,O}^n$ for the pullback of $\mathcal{L}'$ to $\mathcal{X}' \times_C \operatorname{Spec}(\mathcal{O}_{C,O}/\mathfrak{m}_{C,O}^n)$, and $\mathcal{L}/\mathfrak{m}^n$ for the pullback of $\mathcal{L}$ to $\mathcal{X} \times_R (R/\mathfrak{m}^n)$.

These data satisfy the following properties:

- (a) *If $\mathcal{X}$ is proper over R , then we can arrange that $\mathcal{X}'$ be proper over C .*
- (b) *If $\mathcal{X}$ is projective over R and $\mathcal{L}$ is ample, then $\mathcal{L}'$ is ample over some neighborhood of O in C .*
- (c) *If the restriction of $\mathcal{L}$ to $\mathcal{X}_K$ is ample, then we can arrange that $\mathcal{L}'$ be ample over the generic point of C . If $\mathcal{X}_K$ is proper over K and the restriction of $\mathcal{L}$ to $\mathcal{X}_K$ is big, then we can arrange that the generic fiber of $\mathcal{X}' \rightarrow C$ be proper and that $\mathcal{L}'$ is big over the generic point of C .*
- (d) *If $\mathcal{X}$ is regular and $n \geq 2$, then $\mathcal{X}'$ is regular at every point of the fiber over O . If, in addition, E is a divisor on $\mathcal{X}$ whose support is contained in $\mathcal{X}_k$ and has strict normal crossings, then E viewed as a divisor on $\mathcal{X}'$ via the isomorphism of k -schemes $\mathcal{X}_k \cong \mathcal{X}' \times_C O$ also has strict normal crossings. If $\mathcal{X}$ is regular and proper over R , then we can arrange that $\mathcal{X}'$ be regular and proper over C .*

Proof. We fix a k -algebra structure on R by choosing a section of the projection morphism $R \rightarrow k$. By a standard spreading out argument [EGA4.3, § 8], we can find the following data:

- a sub k -algebra A of R such that A is integrally closed and of finite type over k ;

- an A -scheme $\mathcal{X}_A$ of finite type, a line bundle $\mathcal{L}_A$ on $\mathcal{X}_A$, an isomorphism of R -schemes

$$g: \mathcal{X} \rightarrow \mathcal{X}_A \times_A R$$

and an isomorphism of line bundles

$$g^*(\mathcal{L}_A \otimes_A R) \rightarrow \mathcal{L},$$

where we wrote $\mathcal{L}_A \otimes_A R$ for the pullback of $\mathcal{L}_A$ to $\mathcal{X}_A \times_A R$.

We denote by a the image of the closed point of $\mathrm{Spec} R$ under the morphism

$$\mathrm{Spec} R \rightarrow \mathrm{Spec} A.$$

It follows from [EGA4.3, 11.6.1] that $\mathcal{X}_A$ is flat over some open neighborhood of a in $\mathrm{Spec} A$. Replacing A by a suitable localization, we may suppose that $\mathcal{X}_A$ is flat over A .

We denote by R' the henselization of the local ring of $\mathbb{A}_k^1$ at the origin, and by $\mathfrak{m}'$ its maximal ideal. The ring R' is an excellent henselian discrete valuation ring whose completion is isomorphic to R ; we fix such an isomorphism. By Greenberg's approximation theorem [Gre66, Theorem 1], we can find a ring morphism $A \rightarrow R'$ such that the composition with the projection $R' \rightarrow R'/(\mathfrak{m}')^n \cong R/\mathfrak{m}^n$ coincides with the morphism $A \rightarrow R/\mathfrak{m}^n$ induced by the inclusion $A \rightarrow R$. We set

$$\mathcal{X}_{R'} = \mathcal{X}_A \times_A R'$$

and we denote by $\mathcal{L}_{R'}$ the pullback of $\mathcal{L}_A$ to $\mathcal{X}_{R'}$.

The henselization R' is a direct limit of local rings $\mathcal{O}_{C,O}$ where C is an étale $\mathbb{A}_k^1$ -scheme and O is a k -rational point of C lying over the origin of $\mathbb{A}_k^1$. Since A is of finite type over k , we can find such a curve C such that the morphism $\mathrm{Spec} R' \rightarrow \mathrm{Spec} A$ factors through a k -morphism $C \rightarrow \mathrm{Spec} A$. If we set $\mathcal{X}' = \mathcal{X}_A \times_A C$ and we denote by $\mathcal{L}'$ the pullback of $\mathcal{L}_A$ to $\mathcal{X}'$, then the triple $(C, \mathcal{X}', \mathcal{L}')$ satisfies all the properties in (i) and (ii).

Now we prove the list of properties in the second part of the statement.

(a) If $\mathcal{X}$ is proper over R , then we can choose A and $\mathcal{X}_A$ in such a way that $\mathcal{X}_A$ is proper over A by [EGA4.3, 8.10.5]; then $\mathcal{X}'$ is proper over C .

(b) If $\mathcal{X}$ is projective over R and $\mathcal{L}$ is ample, then $\mathcal{L}/\mathfrak{m}$ is ample on $\mathcal{X}_k$, and it follows from [EGA3.1, 4.7.1] that $\mathcal{L}'$ is ample over some open neighborhood of O in C .

(c) Assume that the restriction of $\mathcal{L}$ to $\mathcal{X}_K$ is ample. Then by flat descent of ampleness [EGA4.2, 2.7.2], $\mathcal{L}_A$ is ample over the generic point of $\mathrm{Spec} A$. Thus there exists a dense open subscheme U of $\mathrm{Spec} A$ over which $\mathcal{L}_A$ is ample, by [EGA4.3, 8.10.5.2]. We denote its complement by Z . To prove our statement, we can always replace n by a larger integer n' , since the result will then also be valid for n . For n' sufficiently large, the morphism $\mathrm{Spec} R/\mathfrak{m}^{n'} \rightarrow \mathrm{Spec} A$ does not factor through Z , since $\mathrm{Spec} R \rightarrow \mathrm{Spec} A$ is dominant. Thus if n' is large enough given our choice of A , $\mathcal{X}_A$ and $\mathcal{L}_A$, then the morphism $C \rightarrow \mathrm{Spec} A$ maps the generic point of C to a point of U , which implies that $\mathcal{L}'$ is ample over the generic point of C . The statement for big line bundles can be proved in a similar way: the generic fiber of $\mathcal{X}_A \rightarrow \mathrm{Spec} A$ is proper by flat descent of properness, and the line bundle $\mathcal{L}_A$ is big over the generic point of $\mathrm{Spec} A$ by flat base change for coherent cohomology. Then $\mathcal{X}_A$ is proper over a dense open subset U of $\mathrm{Spec} A$, and $\mathcal{L}_A$ is big over every point in U by upper-semicontinuity of the function

$$U \rightarrow \mathbb{Z}_{\geq 0}, \quad x \mapsto h^0(\mathcal{X}_A \times_A x, \mathcal{L}_A^d \otimes_A \kappa(x))$$

for all $d > 0$ [EGA3.2, 7.7.5].

(d) If $n \geq 2$, then the reduction of $\mathcal{X}$ modulo $\mathfrak{m}^2$ is isomorphic to the reduction of $\mathcal{X}'$ modulo $\mathfrak{m}_{C,O}^2$. Thus the Zariski tangent space of $\mathcal{X}$ at any point of $\mathcal{X}_k$ has the same dimension as the Zariski tangent space of $\mathcal{X}'$ at the corresponding point of $\mathcal{X}' \times_C O \cong \mathcal{X}_k$. It follows that $\mathcal{X}'$ is regular at every point over O if $\mathcal{X}$ is regular. The statement about E is obvious. If $\mathcal{X}$ is regular and proper over R , then we can take $\mathcal{X}'$ to be proper over C and regular at every point in the fiber over O . Then the image in C of the singular locus of $\mathcal{X}'$ is a closed subset of C that does not contain O . Thus by shrinking C , we can arrange that $\mathcal{X}'$ be regular. $\square$

5.2 Relative vanishing theorems over a ring of formal power series

5.2.1 Assume that k has characteristic zero. We will extend some relative vanishing theorems for morphisms of complex varieties to the category of schemes of finite type over R . The standard proofs of these vanishing results use transcendental methods and cannot be adapted to R -schemes in a direct way. So we will use an approximation argument to reduce to the case where our R -scheme is defined over an algebraic curve. In the proof of the connectedness theorem, we will use only the torsion-free theorem (Theorem 5.3.1), but we decided to include the Kawamata–Viehweg vanishing theorem (Theorem 5.2.3) as well, because it is of independent interest and the method of proof is similar. Theorem 5.2.3 and Corollary 5.2.4 were also proven in Appendix B of [BFJ12], but under more restrictive conditions (there, it was assumed that $\mathcal{X}_k$ is a strict normal crossings divisor) and with more complicated arguments.

5.2.2 For notational convenience, we will denote $R/\mathfrak{m}^n$ by R_n , and $\mathcal{X} \times_R (R/\mathfrak{m}^n)$ by $\mathcal{X}_n$, for every integer $n > 0$ and every R -scheme $\mathcal{X}$. In particular, $R_0 = k$ and $\mathcal{X}_0 = \mathcal{X}_k$. If $\mathcal{X}$ is normal and E is a Cartier divisor on $\mathcal{X}$, then we denote by $\mathcal{O}_{\mathcal{X}_n}(E)$ the pullback of the line bundle $\mathcal{O}_{\mathcal{X}}(E)$ to $\mathcal{X}_n$.

THEOREM 5.2.3 [Kawamata–Viehweg vanishing]. *Assume that k has characteristic zero. Let $\mathcal{X}$ be a regular projective flat R -scheme, and denote by $\omega_{\mathcal{X}/R}$ its relative canonical sheaf. Let Δ be a $\mathbb{Q}$ -divisor on $\mathcal{X}$, supported on the special fiber $\mathcal{X}_k$, such that Δ has strict normal crossings and all multiplicities of its prime divisors lie in $[0, 1]$. Let E be a divisor on $\mathcal{X}$ such that the $\mathbb{Q}$ -divisor $E - \Delta$ is relatively nef and such that the restriction of E to $\mathcal{X}_K$ is ample. Then*

$$H^i(\mathcal{X}, \omega_{\mathcal{X}/R} \otimes \mathcal{O}_{\mathcal{X}}(E)) = 0$$

for all $i > 0$.

Proof. Set $\mathcal{L} = \mathcal{O}_{\mathcal{X}}(E)$. We set $n = 1$ and we choose C , $\mathcal{X}'$ and $\mathcal{L}'$ as in Proposition 5.1.2, such that $\mathcal{X}'$ is regular and proper over C and $\mathcal{L}'$ is ample on the generic fiber of $\mathcal{X}' \rightarrow C$. We choose a divisor E' on $\mathcal{X}'$ in the linear equivalence class defined by the line bundle $\mathcal{L}'$. Shrinking C , we may assume that C is affine and $\mathcal{L}'$ is relatively ample over $C \setminus \{O\}$, by [EGA4.3, 8.10.5.2]. Then $E' - \Delta$ is relatively nef, since the restriction of $\mathcal{L}'$ to $\mathcal{X}' \times_C O \cong \mathcal{X}_k$ is isomorphic to the restriction of $\mathcal{L}$ to $\mathcal{X}_k$.

It follows from the relative version of the Kawamata–Viehweg vanishing theorem [Kol97, Theorem 2.17(3)] that

$$H^i(\mathcal{X}', \omega_{\mathcal{X}'/C} \otimes \mathcal{L}') = 0$$

for all $i > 0$. Let t be a uniformizer in $\mathcal{O}_{C,O}$. Shrinking C , we may assume that t is regular at every point of C . Looking at the long exact cohomology sequence associated to the short exact sequence of $\mathcal{O}_{\mathcal{X}'}$ -modules

$$0 \rightarrow \omega_{\mathcal{X}'/C} \otimes \mathcal{L}' \xrightarrow{t} \omega_{\mathcal{X}'/C} \otimes \mathcal{L}' \rightarrow \omega_{\mathcal{X}_k/k} \otimes \mathcal{O}_{\mathcal{X}_k}(E) \rightarrow 0, \quad (5.2.1)$$

we find

$$H^i(\mathcal{X}_k, \omega_{\mathcal{X}_k/k} \otimes \mathcal{O}_{\mathcal{X}_k}(E)) = 0$$

for all $i > 0$. Now it follows from the semicontinuity theorem for coherent cohomology [EGA3.2, 7.7.5], combined with the implication $e) \Rightarrow d)$ in [EGA3.2, 7.8.4], that

$$H^i(\mathcal{X}, \omega_{\mathcal{X}/R} \otimes \mathcal{O}_{\mathcal{X}}(E)) = 0$$

for all $i > 0$. □

COROLLARY 5.2.4 [Kodaira vanishing]. *Assume that k has characteristic zero. Let $\mathcal{X}$ be a regular flat projective R -scheme, and denote by $\omega_{\mathcal{X}/R}$ its relative canonical sheaf. Then for every ample line bundle $\mathcal{L}$ on $\mathcal{X}$ and every integer $i > 0$, we have*

$$H^i(\mathcal{X}, \omega_{\mathcal{X}/R} \otimes \mathcal{L}) = 0.$$

Proof. This is a special case of Kawamata–Viehweg vanishing, with $\Delta = 0$ and E the Cartier divisor associated to $\mathcal{L}$. □

Remark 5.3. One can generalize Theorem 5.2.3 by requiring that E be big on $\mathcal{X}_K$ instead of ample, as in the Kawamata–Viehweg vanishing theorem formulated in [Kol97, 2.17.3]. By Proposition 5.1.2, we can find $\mathcal{X}'$, C and $\mathcal{L}'$ as in the proof of Theorem 5.2.3 such that $\mathcal{L}'$ is big on the generic fiber of $\mathcal{X}' \rightarrow C'$. But we do not know whether one can moreover arrange that $\mathcal{L}'$ be relatively nef. The problem is that nefness behaves poorly in families: even if $\mathcal{L}'$ is nef over O and over the generic point of C , this does not guarantee that $\mathcal{L}'$ is nef over some open neighborhood of O in C . However, one can circumvent this problem by means of the following variant of the Kawamata–Viehweg vanishing theorem. Let k be a field of characteristic zero and let $f: X \rightarrow S$ be a surjective projective morphism of schemes of finite type over k , with X smooth and connected. Let s be a closed point on S . Let Δ be a $\mathbb{Q}$ -divisor on X such that Δ has strict normal crossings and all multiplicities of its prime divisors lie in $[0, 1[$. Let E be a divisor on X such that the restriction of the $\mathbb{Q}$ -divisor $E - \Delta$ to $X_s = f^{-1}(s)$ is nef and such that the restriction of $E - \Delta$ to the generic fiber of f is big. Then we have $R^i f_* \mathcal{O}_X(K_X + E)_s = 0$ for every $i \geq 1$. This variant of the Kawamata–Viehweg vanishing theorem can be proven with the same arguments as in [KMM87, Theorem 1-2-3].

THEOREM 5.3.1 [torsion-free theorem]. *Assume that k has characteristic zero. Let $\mathcal{X}$ be a regular proper flat R -scheme, and denote by $\omega_{\mathcal{X}/R}$ its relative canonical sheaf. Let Δ be a $\mathbb{Q}$ -divisor on $\mathcal{X}$, supported on the special fiber $\mathcal{X}_k$, such that Δ has strict normal crossings and all multiplicities of its prime divisors lie in $[0, 1[$. Let E be a divisor on $\mathcal{X}$ such that the $\mathbb{Q}$ -divisor $E - \Delta$ is a rational multiple of $\mathcal{X}_k$. Then*

$$H^i(\mathcal{X}, \omega_{\mathcal{X}/R} \otimes \mathcal{O}_{\mathcal{X}}(E))$$

is a free R -module for all $i \geq 0$.

Proof. We fix an integer $i \geq 0$ and set

$$M_n = H^i(\mathcal{X}_n, \omega_{\mathcal{X}_n/R_n} \otimes \mathcal{O}_{\mathcal{X}_n}(E))$$

for all $n > 0$. These R -modules form a projective system, whose limit is precisely the cohomology module

$$M = H^i(\mathcal{X}, \omega_{\mathcal{X}/R} \otimes \mathcal{O}_{\mathcal{X}}(E))$$

by Grothendieck’s comparison theorem [EGA3.1, 4.1.5]. Fix an integer $n > 1$, and choose C and $\mathcal{X}'$ as in Proposition 5.1.2, with $\mathcal{X}'$ regular and proper over C . Then we can view E as a divisor

on $\mathcal{X}'$, supported on the fiber over O , which is isomorphic to $\mathcal{X}_k$. We denote by $f: \mathcal{X}' \rightarrow C$ the structural morphism. Kollár's torsion-free theorem [Kol97, Theorem 2.17(4)] implies that

$$R^j f_*(\omega_{\mathcal{X}'/C} \otimes \mathcal{O}_{\mathcal{X}'}(E))$$

is locally free for all $j \geq 0$. Thus the short exact sequence (5.2.1) induces an exact sequence

$$R^i f_*(\omega_{\mathcal{X}'/C} \otimes \mathcal{O}_{\mathcal{X}'}(E)) \rightarrow R^i f_*(\omega_{\mathcal{X}'/C} \otimes \mathcal{O}_{\mathcal{X}'}(E)) \rightarrow M_n \rightarrow 0$$

(the coboundary map must be zero because M_n is a torsion module). Hence, we get a canonical isomorphism

$$M_n \cong R^i f_*(\omega_{\mathcal{X}'/C} \otimes \mathcal{O}_{\mathcal{X}'}(E)) \otimes_{\mathcal{O}_C} (\mathcal{O}_{C,O}/\mathfrak{m}_{C,O}^n),$$

and M_n is a free R_n -module for all $n > 0$. This implies that M has no t -torsion, because every element in M_{n+1} killed by t is mapped to 0 in M_n . $\square$

Remark 5.4. With a little extra work, Theorem 5.3.1 can be generalized: one can drop the assumption that Δ and E are supported on the special fiber $\mathcal{X}_k$, and require only that $E - \Delta$ be $\mathbb{Q}$ -linearly equivalent to a rational multiple of $\mathcal{X}_k$ (instead of equal). Indeed, by choosing A sufficiently large in Proposition 5.1.2 we can assume that E , Δ and the $\mathbb{Q}$ -linear equivalence are defined over A , and by shrinking C we can assume that the pullback of Δ to $\mathcal{X}'$ has strict normal crossings, so that we can still apply Kollár's torsion-free theorem.

5.5 The skeleton is connected

5.5.1 We will now show that the skeleton of a proper smooth geometrically connected K -variety of geometric genus one is always connected. We will explain in §6.4.2 how this result can be viewed as a global version of the connectedness theorem of Kollár and Shokurov. The proof is somewhat different: we cannot use Kawamata–Viehweg vanishing because the divisor D that appears in the proof is not big on the generic fiber $\mathcal{X}_K$. Instead, we will apply the torsion-free theorem.

5.5.2 We will use the following terminology for divisors on R -schemes. Let $\mathcal{X}$ be a flat R -scheme of finite type, and let D be a Weil divisor on $\mathcal{X}$. Then we can write

$$D = D^+ - D^-$$

such that D^+ and D^- are effective and have no common prime divisors. We call D^+ and D^- the *positive* and *negative* parts of D , respectively.

THEOREM 5.5.3. *Assume that k has characteristic zero. Let X be a proper smooth geometrically connected K -variety of geometric genus one, and let ω be a non-zero differential form of maximal degree on X . Then for every sncd-model $\mathcal{X}$ of X , the union of the ω -essential components of $\mathcal{X}_k$ is connected.*

Proof. We write $\mathcal{X}_k = \sum_{i \in I} N_i E_i$ and we denote by $\mu_i - 1$ the multiplicity of E_i in $\text{div}_{\mathcal{X}}(\omega)$, for each $i \in I$. We set

$$\alpha = \min\{\mu_i/N_i \mid i \in I\}.$$

We denote by Δ the fractional part of the $\mathbb{Q}$ -divisor $\alpha \mathcal{X}_k$. Then, by definition of α , none of the prime components of Δ is ω -essential. Moreover, the negative part D^- of the divisor

$$D = \text{div}_{\mathcal{X}}(\omega) - \alpha \mathcal{X}_k + \Delta$$

is reduced, and its support is precisely the union of the ω -essential components of $\mathcal{X}_k$. Thus we must show that D^- is connected.

From the short exact sequence of $\mathcal{O}_{\mathcal{X}}$ -modules

$$0 \rightarrow \mathcal{O}_{\mathcal{X}}(D) \rightarrow \mathcal{O}_{\mathcal{X}}(D^+) \rightarrow \mathcal{O}_{D^-}(D^+) \rightarrow 0$$

we obtain an exact sequence of cohomology R -modules

$$H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(D^+)) \xrightarrow{f} H^0(D^-, \mathcal{O}_{D^-}(D^+)) \rightarrow H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(D)).$$

By Theorem 5.3.1, we know that the R -module

$$H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(D))$$

is free. On the other hand, the cokernel of

$$f: H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(D^+)) \rightarrow H^0(D^-, \mathcal{O}_{D^-}(D^+))$$

is a torsion R -module. As it injects into a free R -module, it must be zero, and f is surjective.

The restriction of D^+ to $X = \mathcal{X}_K$ is a canonical divisor. Since the geometric genus of X is one, we have $H^0(X, \mathcal{O}_{\mathcal{X}}(D^+)|_X) = K$ and thus

$$H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(D^+)) \cong R.$$

Now the surjectivity of f implies that the k -vector space

$$H^0(D^-, \mathcal{O}_{D^-}(D^+))$$

has rank one. But all locally constant k -valued functions on D^- belong to this space, so that every locally constant function on D^- is constant and D^- is connected. $\square$

COROLLARY 5.5.4 [connectedness theorem]. *Assume that k has characteristic zero. Let X be a proper smooth geometrically connected K -variety of geometric genus one, and let ω be a non-zero differential form of maximal degree on X . Then the skeleton $\text{Sk}(X, \omega)$ of X is connected. In particular, if $\omega_{X/K}$ is trivial, then the essential skeleton $\text{Sk}(X)$ of X is connected.*

Proof. If $\omega_{X/K}$ is trivial, then this is a direct consequence of Theorems 4.7.5 and 5.5.3. In the general case, we need to be more careful: if ω is a non-zero differential form of maximal degree on X , $\mathcal{X}$ is an *sncd*-model of X and Z is the schematic closure in $\mathcal{X}$ of the zero locus of ω in X , then an intersection of two ω -essential irreducible components of $\mathcal{X}_k$ does not contribute to the skeleton if it is contained in Z . However, the Kontsevich–Soibelman skeleton is a birational invariant by Proposition 4.7.2, and we can always find a proper birational R -morphism $h: \mathcal{X}' \rightarrow \mathcal{X}$ such that $\mathcal{X}'$ is regular, $\mathcal{X}'_K$ is smooth and proper over K , h is an isomorphism over $\mathcal{X} \setminus Z$ and $\mathcal{X}'_k + h^*Z$ is a divisor with strict normal crossings on $\mathcal{X}'$. Note that $\mathcal{X}'_K$ has geometric genus one, because the geometric genus is a birational invariant. Moreover, the closure Z' in $\mathcal{X}'$ of the locus of zeroes of ω on $\mathcal{X}'_K$ is contained in h^*Z . Thus the normal crossings property implies that Z' cannot contain a connected component of the intersection of two irreducible components of $\mathcal{X}'_k$. Hence, it follows from Theorems 4.7.5 and 5.5.3 that $\text{Sk}(X, \omega) = \text{Sk}(\mathcal{X}'_K, h_K^* \omega)$ is connected. $\square$

6. Relation with the birational geometry of varieties over a field of characteristic zero

Our definition of the weight function has a natural counterpart in birational geometry, that we will explain in this section. It is closely related to the thinness function constructed in [BFJ08]

and the log discrepancy function in [JM12].

6.1 The weight function of a coherent ideal sheaf

6.1.1 Let F be a field of characteristic zero. Let X be a connected smooth F -variety and let $\mathcal{I}$ be a non-zero coherent ideal sheaf on X . Let v be a divisorial valuation on the function field of X such that v has a center on X and this center lies in the zero locus $Z(\mathcal{I})$ of $\mathcal{I}$. Then one can find a log resolution $h: Y \rightarrow X$ of $\mathcal{I}$ such that the closure of the center of v on Y is a divisor E on Y . Recall that this means that h is a proper birational morphism such that $Z(\mathcal{I}\mathcal{O}_Y) + K_{Y/X}$ is a divisor with strict normal crossings on Y . One can moreover arrange that h be an isomorphism over $X \setminus Z(\mathcal{I})$, and we will always assume that a log resolution satisfies this property. The valuation v is equal to $r \cdot \text{ord}_E$, where ord_E is the valuation associated to the divisor E and r is a positive real number. If we denote by N the multiplicity of E in $Z(\mathcal{I}\mathcal{O}_Y)$ and by $\mu - 1$ the multiplicity of E in $K_{Y/X}$, then the quotient μ/N is an interesting geometric invariant, which we call the *weight* of v with respect to the variety X and which we denote by $\text{wt}_{\mathcal{I}}(v)$. Note that it does not depend on r . The set of the weights of all divisorial valuations v with center in $Z(\mathcal{I})$ has a minimum, called the *log canonical threshold* of the pair $(X, \mathcal{I})$ and denoted by $\text{lct}(X, \mathcal{I})$. This is a fundamental invariant in birational geometry; see [Kol97, §8]. It is well known that the log canonical threshold can be computed on a single log resolution: if $Y \rightarrow X$ is any log resolution of $\mathcal{I}$, then writing $Z(\mathcal{I}\mathcal{O}_Y) = \sum_{i \in I} N_i E_i$ and $K_{Y/X} = \sum_{i \in I} (\mu_i - 1) E_i$, one has

$$\text{lct}(X, \mathcal{I}) = \min \left\{ \frac{\mu_i}{N_i} \mid i \in I \right\}.$$

This can be viewed as a partial analog of Theorem 4.7.5.

6.1.2 The weight function can be extended to quasi-monomial valuations on X with center in $Z(\mathcal{I})$ (see [JM12, §3.1] for the definition of a quasi-monomial valuation). If v is such a quasi-monomial valuation, then we set

$$\text{wt}_{\mathcal{I}}(v) = \frac{v(K_{Y/X} + Z(\mathcal{I}\mathcal{O}_Y)_{\text{red}})}{v(Z(\mathcal{I}\mathcal{O}_Y))},$$

where $h: Y \rightarrow X$ is a log resolution of $\mathcal{I}$ such that $(Y, Z(\mathcal{I}\mathcal{O}_Y)_{\text{red}})$ is adapted to v in the sense of [JM12, §3.5] and where, for every effective divisor D on Y , we write

$$v(D) = \min\{v(f) \mid f \in \mathcal{O}(-D)_{\xi}\}$$

with ξ the center of v on Y . One can show that the definition of $\text{wt}_{\mathcal{I}}(v)$ does not depend on the choice of h ; see Section 6.2.

6.1.3 For every X -scheme of finite type Y , we denote by $\widehat{Y}$ the formal $\mathcal{I}\mathcal{O}_Y$ -adic completion of Y . In particular, $\widehat{X}$ is the formal $\mathcal{I}$ -adic completion of X . We consider the generic fiber $\widehat{X}_{\eta}$ in the sense of [Thu07, Définition et Proposition 1.7]. This is an analytic space over the field F endowed with its trivial absolute value. It is obtained by removing from the usual generic fiber of $\widehat{X}$ all the points that lie on the analytification of the closed subscheme $Z(\mathcal{I})$ of $\widehat{X}$. It carries a natural reduction map

$$\text{red}_{\widehat{X}}: \widehat{X}_{\eta} \rightarrow Z(\mathcal{I}).$$

The set of quasi-monomial valuations v on X with center in $Z(\mathcal{I})$ can be embedded in $\widehat{X}_{\eta}$ by sending v to the absolute value $f \mapsto \exp(-v(f))$ on the function field $F(X)$ of X . We will call

the points in the image *quasi-monomial points*.

6.1.4 We can extend $\text{wt}_{\mathcal{I}}$ to a function

$$\text{wt}_{\mathcal{I}}: \widehat{X}_{\eta} \rightarrow \mathbb{R} \cup \{+\infty\}$$

in a similar way as in § 4.5.4. Each log resolution $Y \rightarrow X$ of $\mathcal{I}$ gives rise to a skeleton $\text{Sk}(\widehat{Y}) \subset \widehat{X}_{\eta}$, consisting of the quasi-monomial points x in $\widehat{X}_{\eta}$ such that Y is adapted to the corresponding valuation v_x . The skeleton $\text{Sk}(\widehat{Y})$ is homeomorphic to the product of $\mathbb{R}_{>0}$ with the simplicial space associated to the strict normal crossings divisor $Z(\mathcal{I}\mathcal{O}_Y)$, and there exists a natural contraction $\rho_Y: \widehat{X}_{\eta} \rightarrow \text{Sk}(\widehat{Y})$ that can be extended to a strong deformation retraction of $\widehat{X}_{\eta}$ onto $\text{Sk}(\widehat{Y})$ [Thu07, 3.27]. One can prove that

$$\text{wt}_{\mathcal{I}}(x) \geq \text{wt}_{\mathcal{I}}(\rho_Y(x)) \quad (6.1.1)$$

for every quasi-monomial point x on $\widehat{X}_{\eta}$, with equality if and only if x lies in $\text{Sk}(\widehat{Y})$; see Section 6.2. We define the weight function on $\widehat{X}_{\eta}$ as

$$\text{wt}_{\mathcal{I}}: \widehat{X}_{\eta} \rightarrow \mathbb{R} \cup \{+\infty\}, \quad x \mapsto \text{wt}_{\mathcal{I}}(x) = \sup_{h: Y \rightarrow X} \text{wt}_{\mathcal{I}}(\rho_Y(x)),$$

where the supremum is taken over all log resolutions $h: Y \rightarrow X$ of $\mathcal{I}$. This weight function satisfies properties that are quite similar to the ones in Proposition 4.5.5.

6.2 Comparison with the log discrepancy function

6.2.1 In [JM12], Jonsson and the first-named author studied the properties of the so-called log discrepancy function A_X on a certain space of valuations. This function had previously been introduced as the *thinness function* in a slightly different setting [BFJ08]. We will now explain the relation with the weight function defined above. As in [JM12], we denote by Val_X the space of real valuations on the function field $F(X)$ of X with a center on X . We can view Val_X as a subspace of the analytification X^{an} of X with respect to the trivial absolute value on F by associating the absolute value $f \mapsto \exp(-v(f))$ on $F(X)$ to each valuation v in Val_X .

6.2.2 For every log resolution $h: Y \rightarrow X$ of $\mathcal{I}$, the skeleton $\text{Sk}(\widehat{Y})$ is contained in Val_X . In [JM12], the union of $\text{Sk}(\widehat{Y})$ and the trivial valuation on $F(X)$ was denoted by $\text{QM}(Y, Z(\mathcal{I}\mathcal{O}_Y))$. With our notation, the restriction of the log discrepancy function to $\text{Sk}(\widehat{Y})$ is given by

$$A_X: \text{Sk}(\widehat{Y}) \rightarrow \mathbb{R}, \quad x \mapsto v_x(K_{Y/X} + Z(\mathcal{I}\mathcal{O}_Y)_{\text{red}}),$$

where v_x is the quasi-monomial valuation corresponding to x . It is proven in [JM12, § 5.1] that $A_X(x)$ depends only on the point x , and not on the choice of the log resolution $Y \rightarrow X$ such that $\text{Sk}(\widehat{Y})$ contains x , so that A_X is well defined on the set of quasi-monomial points in $\widehat{X}_{\eta}$. This implies the analogous claim for $\text{wt}_{\mathcal{I}}$ in § 6.1.2, because $v_x(Z(\mathcal{I}\mathcal{O}_Y))$ does not depend on the chosen log resolution. Likewise, the inequality (6.1.1) follows from [JM12, § 5.3].

6.2.3 In [JM12, § 5.2], the log discrepancy function is extended to Val_X by means of a supremum construction similar to the one we used in § 6.1.3. It follows immediately from the definitions that we have

$$\text{wt}_{\mathcal{I}}(x) = \frac{A_X(x)}{N_x(\mathcal{I})}$$

for every x in $\text{Val}_X \cap \widehat{X}_\eta$, where $N_x(\mathcal{I})$ denotes the value $v_x(Z(\mathcal{I}\mathcal{O}_Y))$ for any log resolution $Y \rightarrow X$ of $\mathcal{I}$.

6.3 Comparison with the weight function on K -varieties

6.3.1 The weight function $\text{wt}_{\mathcal{I}}$ can also be compared to the one for K -varieties and differential forms in § 4.5.4, as follows. We set $R = F[[t]]$ and $K = F((t))$. Let X be a connected smooth F -variety and let $\mathcal{I}$ be a coherent ideal sheaf on X . We denote by $n+1$ the dimension of X . Let $h: Y \rightarrow X$ be a log resolution of $\mathcal{I}$. We write $Z(\mathcal{I}\mathcal{O}_Y) = \sum_{i \in I} N_i E_i$ and $K_{Y/X} = \sum_{i \in I} (\mu_i - 1) E_i$. For every point ξ of $Z(\mathcal{I}\mathcal{O}_Y)$, we can find an open neighborhood U of ξ in Y and a regular function f on U that generates the ideal sheaf $\mathcal{I}\mathcal{O}_U$. We view U as an $F[t]$ -scheme by means of the morphism

$$f: U \rightarrow \mathbb{A}_k^1 = \text{Spec} F[t]$$

and we denote by $\mathcal{U}$ the R -scheme obtained by extension of scalars. Note that $\mathcal{U}$ is an *sncd*-model for its generic fiber $\mathcal{U}_K$. We choose a volume form ϕ on some open neighborhood V of $h(\xi)$ in X , that is, a nowhere vanishing differential form of degree $n+1$. Shrinking U , we may assume $U \subset h^{-1}(V)$ and that $U \setminus Z(f)$ is smooth over V .

6.3.2 The differential form $h^*\phi$ on U induces a relative volume form ω in $\Omega_{U/V}^n(U \setminus Z(f))$, uniquely characterized by the property that $\omega \wedge df = h^*\phi$ in $\Omega_{U/V}^{n+1}(U \setminus Z(f))$. The form ω is called the Gelfand–Leray form associated to f and $h^*\phi$; see for instance [NS07, Definition 9.5]. It induces a volume form on $\mathcal{U}_K$ that we denote again by ω . By the adjunction formula (4.1.1), there exists a unique isomorphism

$$\varphi: \omega_{U/F[t]} \rightarrow \omega_{U/F} = \Omega_{U/F}^{n+1}$$

of line bundles on U such that $\varphi(\theta) = \theta \wedge df$ for every open subset W of $U \setminus Z(f)$ and every $\theta \in \omega_{U/V}(W) = \Omega_{U/V}^n(W)$. Since canonical sheaves are compatible with the flat base change $F[t] \rightarrow R$, it follows that

$$\text{div}_{\mathcal{U}}(\omega) = \sum_{i \in I} (\mu_i - 1)(E_i \cap U). \quad (6.3.1)$$

6.3.3 The K -analytic space $\widehat{\mathcal{U}}_\eta$ can be identified with the subspace of

$$\widehat{U}_\eta := \text{red}_{\widehat{Y}}^{-1}(U \cap Z(\mathcal{I}\mathcal{O}_Y)) \subset \widehat{X}_\eta$$

consisting of the points x where $|f(x)| = |t|_K = 1/e$, by [Nic11, 4.2]. This embedding has a retraction

$$r_{\mathcal{U}}: \widehat{U}_\eta \rightarrow \widehat{\mathcal{U}}_\eta, \quad x \mapsto r_{\mathcal{U}}(x),$$

where $r_{\mathcal{U}}(x)$ is the unique point of $\widehat{\mathcal{U}}_\eta$ such that $\text{red}_{\widehat{\mathcal{U}}}(r_{\mathcal{U}}(x)) = \text{red}_{\widehat{Y}}(x)$ and

$$|g(r_{\mathcal{U}}(x))| = |g(x)|^{-1/\ln|f(x)|}$$

for all g in $\mathcal{O}_{Y, \text{red}_{\widehat{Y}}(x)}$. One can easily deduce from (6.3.1) that the restriction of $\text{wt}_{\mathcal{I}}$ to $\widehat{U}_\eta$ coincides with $\text{wt}_\omega \circ r_{\mathcal{U}}$, where wt_ω is the weight function associated to $(\mathcal{U}_K, \omega)$.

6.4 An analog of the Kontsevich–Soibelman skeleton

6.4.1 In analogy with the definition of ω -essential divisorial points in § 4.7.1, we say that a divisorial valuation v as in § 6.1.1 is $\mathcal{I}$ -essential if

$$\mathrm{wt}_{\mathcal{I}}(v) = \mathrm{lct}(X, \mathcal{I}),$$

that is, if the divisor E computes the log canonical threshold. In that case, we also say that the divisor E is $\mathcal{I}$ -essential. We define the skeleton $\mathrm{Sk}(X, \mathcal{I})$ of the pair $(X, \mathcal{I})$ as the closure in $\widehat{X}_{\eta}$ of the set of $\mathcal{I}$ -essential divisorial valuations. It is well known that when $h: Y \rightarrow X$ is a log resolution of $\mathcal{I}$, the skeleton $\mathrm{Sk}(X, \mathcal{I})$ is the union of the open faces of $\mathrm{Sk}(\widehat{Y})$ corresponding to connected components of intersections of $\mathcal{I}$ -essential prime components of $\mathcal{I}\mathcal{O}_Y$. This is the analog of Theorem 4.7.5. In particular, the subspace $\mathrm{Sk}(X, \mathcal{I})$ of $\mathrm{Sk}(\widehat{Y}) \subset \widehat{X}_{\eta}$ is independent of the log resolution h . One can endow it with a natural piecewise affine structure as in Section 3.2.

6.4.2 The Shokurov–Kollár connectedness theorem [Kol97, 17.4] implies that for every point x in $Z(\mathcal{I})$ and every sufficiently small open neighborhood U of x in $Z(\mathcal{I})$, the inverse image of U under the reduction map

$$\mathrm{red}_{\widehat{X}}: \mathrm{Sk}(X, \mathcal{I}) \rightarrow Z(\mathcal{I})$$

is connected. Thus our connectedness theorem for skeleta of K -varieties of geometric genus one (Corollary 5.5.4) can be viewed as an analog of the Shokurov–Kollár connectedness theorem.

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REFERENCES

- AKMW02 D. Abramovich, K. Karu, K. Matsuki and J. Włodarczyk, *Torification and factorization of birational maps*, J. Amer. Math. Soc. **15** (2002), no. 3, 531–572. <http://dx.doi.org/10.1090/S0894-0347-02-00396-X>
- Ber90 V.G. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields*, Math. Surveys Monogr., vol. 33, Amer. Math. Soc., Providence, RI, 1990.
- Ber94 ———, *Vanishing cycles for formal schemes*, Invent. Math. **115** (1994), no. 3, 539–571. <http://dx.doi.org/10.1007/BF01231772>
- Ber99 ———, *Smooth p -adic analytic spaces are locally contractible*, Invent. Math. **137** (1999), no. 1, 1–84. <http://dx.doi.org/10.1007/s002220050323>
- BFJ08 S. Boucksom, C. Favre and M. Jonsson, *Valuations and plurisubharmonic singularities*, Publ. Res. Inst. Math. Sci. **44** (2008), no. 2, 449–494. <http://dx.doi.org/10.2977/prims/1210167334>
- BFJ12 ———, *Singular semipositive metrics in non-archimedean geometry*, J. Algebraic Geom., to appear, [arXiv:1201.0187](https://arxiv.org/abs/1201.0187).
- BL93 S. Bosch and W. Lütkebohmert, *Formal and rigid geometry. I. Rigid spaces*, Math. Ann. **295** (1993), no. 2, 291–317. <http://dx.doi.org/10.1007/BF01444889>

- EGA3.1 A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. Première partie*, Publ. Math. Inst. Hautes Études Sci. **11** (1961), 5–167. http://www.numdam.org/item?id=PMIHES_1961__11__5_0
- EGA3.2 ———, *Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. Seconde partie*, Publ. Math. Inst. Hautes Études Sci. **17** (1963), 5–91. http://www.numdam.org/item?id=PMIHES_1963__17__5_0
- EGA4.1 ———, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. Première partie*, Publ. Math. Inst. Hautes Études Sci. (1964), no. 20, 5–259. http://www.numdam.org/item?id=PMIHES_1964__20__5_0
- EGA4.2 ———, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. Seconde partie*, (1965), no. 24, 5–231. http://www.numdam.org/item?id=PMIHES_1965__24__5_0
- EGA4.3 ———, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. troisième partie*, Publ. Math. Inst. Hautes Études Sci. (1966), no. 28, 5–255. http://www.numdam.org/item?id=PMIHES_1966__28__5_0
- Gre66 M.J. Greenberg, *Rational points in Henselian discrete valuation rings*, Publ. Math. Inst. Hautes Études Sci. **31** (1966), 59–64. <http://dx.doi.org/10.1007/BF02684802>
- Gro13 M. Gross, *Mirror symmetry and the Strominger–Yau–Zaslow conjecture*, Current Developments in Mathematics 2012, Int. Press, Somerville, MA, 2013, 133–191.
- JM12 M. Jonsson and M. Mustață, *Valuations and asymptotic invariants for sequences of ideals*, Ann. Inst. Fourier (Grenoble) **62** (2012), no. 6, 2145–2209. <http://dx.doi.org/10.5802/aif.2746>
- KMM87 Y. Kawamata, K. Matsuda and K. Matsuki, *Introduction to the minimal model problem*, Algebraic Geometry (Sendai, 1985), Adv. Stud. Pure Math., vol. 10, North-Holland, Amsterdam, 1987, 283–360.
- Kol97 J. Kollár, *Singularities of pairs*, Algebraic Geometry (Santa Cruz, 1995), Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997, 221–287.
- KS06 M. Kontsevich and Y. Soibelman, *Affine structures and non-Archimedean analytic spaces*, The Unity of Mathematics, In Honor of the Ninetieth Birthday of I.M. Gelfand. (P. Etingof, V. Retakh and I.M. Singer, eds.), Progr. Math., vol. 244, Birkhäuser Boston, Boston, MA, 2006, 321–385. http://dx.doi.org/10.1007/0-8176-4467-9_9
- Kul77 V.S. Kulikov, *Degenerations of K3 surfaces and Enriques surfaces*, Math. USSR Izv. **11** (1977), no. 5, 957–989. <http://dx.doi.org/10.1070/IM1977v01n05ABEH001753>
- Liu02 Q. Liu, *Algebraic geometry and arithmetic curves*, Oxf. Grad. Texts Math., vol. 6, Oxford Univ. Press, Oxford, 2002.
- Nic11 J. Nicaise, *Singular cohomology of the analytic Milnor fiber, and mixed Hodge structure on the nearby cohomology*, J. Algebraic Geom. **20** (2011), no. 2, 199–237. <http://dx.doi.org/10.1090/S1056-3911-10-00526-6>
- NS07 J. Nicaise and J. Sebag, *Motivic Serre invariants, ramification, and the analytic Milnor fiber*, Invent. Math. **168** (2007), no. 1, 133–173. <http://dx.doi.org/10.1007/s00222-006-0029-7>
- PP81 U. Persson and H. Pinkham, *Degeneration of surfaces with trivial canonical bundle*, Ann. of Math. **113** (1981), no. 1, 45–66. <http://dx.doi.org/10.2307/1971133>
- vdPS95 M. van der Put and P. Schneider, *Points and topologies in rigid geometry*, Math. Ann. **302** (1995), no. 1, 81–103. <http://dx.doi.org/10.1007/BF01444488>
- Thu07 A. Thuillier, *Géométrie toroïdale et géométrie analytique non archimédienne. Application au type d'homotopie de certains schémas formels*, Manuscripta Math. **123** (2007), no. 4, 381–451. <http://dx.doi.org/10.1007/s00229-007-0094-2>

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