The classification of minimal irregular surfaces of general type with $K^2 = 2p_g$

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ABSTRACT

According to a result of O. Debarre from 1982 minimal irregular surfaces of general type satisfy $K^2 \geq 2p_g$. In this paper we classify those surfaces for which the equality $K^2 = 2p_g$ holds.

1. Introduction

By a result of O. Debarre a minimal complex irregular surface S of general type satisfies $K^2 \geq 2p_g$ ([Deb82, Théorème 6.1]). In this paper we complete the classification of the case $K^2 = 2p_g$, which can be summarized as follows.

THEOREM 1.1. *Let S be a minimal complex surface of general type of irregularity $q > 0$ satisfying $K^2 = 2p_g$. Then $q \leq 4$.*

If $q = 1$, then the Albanese fibration is a genus two fibration with 2-connected fibres and S is a double cover of a $\mathbb{P}^1$ -bundle over the elliptic curve $\text{Alb}(S)$.

If $q \geq 2$, then $\chi = 1$ and:

- (i) *for $q = 2$, the surface S is the minimal desingularization of a double cover of a principally polarized abelian surface (A, Θ) branched over an effective divisor of class 2Θ with at most non-essential singularities;*
- (ii) *for $q = 3$, the surface S is the symmetric product of a curve of genus three;*
- (iii) *for $q = 4$, the surface S is the product of two curves of genus two.*

This classification is the result of the work of many authors.

Debarre, besides establishing the inequality $K^2 = 2p_g$ for minimal complex surfaces of general type of irregularity $q > 0$, has shown that if equality holds, then $q \leq 4$, and that if $q = 4$, then S is the product of two curves of genus two ([Deb82, Théorème 6.3]).

For $q = 1$, $K^2 = 2p_g$ is the same as $K^2 = 2\chi$. These surfaces had already been classified by Horikawa: their Albanese pencil has 2-connected genus two fibres, and the relative canonical map determines a 2–1 cover of a $\mathbb{P}^1$ -bundle over the elliptic curve $\text{Alb}(S)$ (see [Hor82, Theorem 5.2] for a detailed description). We also note that in [Cat81] Catanese studied the case $q = \chi = 1$.

Received 5 September 2013, accepted in final form 10 May 2014.

2010 Mathematics Subject Classification 14J29

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Surfaces with $p_g = q = 3$ and $K^2 = 6$ have been shown to be the symmetric product of a curve of genus three by Catanese and the first two authors ([CCM98, Prop. (3.22)]).

To obtain Theorem 1.1, we start by completing the classification of such irregular surfaces with $\chi = 1$ (see §3). As explained above, the only yet unknown case was $p_g = q = 2$. In Theorem 3.2, which is the main result in this paper, we show that these surfaces are exactly the so-called *Catanese surfaces* described in part (i) of Theorem 1.1. This had already been proved by Manetti in [Man03] under the extra assumption that K be ample and by the first two authors in [CML02] under the assumption that the bicanonical map not be birational, but the lack of a complete description of this case has for many years been an annoying gap in the classification of surfaces with small invariants.

Surfaces with $p_g = q = 2$ and $K^2 = 4$ are also interesting from the point of view of the Severi inequality $K^2 \geq 4\chi$ for minimal surfaces with Albanese dimension two, proven by Manetti in [Man03] for the case where K is ample and then in general by the third author in [Par05]. Indeed, in [Man03] Manetti showed that for K ample the Severi inequality is an equality exactly when $q = 2$ and the Albanese map is a double cover of an abelian surface branched over a smooth curve B . Examples with K non-ample and $K^2 = 4\chi$ can be obtained by letting the branch curve B acquire non-essential singularities and taking the minimal resolution of the double cover. It is natural to conjecture that these are the only surfaces with Albanese dimension two satisfying the equality in the Severi inequality. The present result proves the conjecture in the first case, that is, for $\chi = 1$.

The new ingredient that allows us settle the case $p_g = q = 2$, in a way that is in our view more conceptually transparent, is the recent classification in [CMP14] of curves C with the following property: a Brill–Noether locus $W_d^s(C)$, properly contained in the Jacobian $J(C)$ of C , contains a variety Z stable under translations by the elements of a positive dimensional abelian subvariety $A \subsetneq J(C)$ such that $\dim(Z) = d - \dim(A) - 2s$, that is, the maximum possible dimension for such Z (see §2.1 and [CMP14]). Besides this, the proof relies, as usual in these matters, on two main tools: the paracanonical system and the Albanese map (see §2.2). In fact we prove that the general paracanonical curve C is smooth of genus five and apply the results in [CMP14] to the Abelian surface $A = \text{Pic}^0(S)$, which is contained in the Theta divisor of $J(C)$, to deduce that the Albanese map of S has degree two. The result then follows by “classical” arguments.

Then we turn to the case $\chi > 1$, $q \geq 2$. Since $q \leq 4$ and the case $q = 4$ has already been described by Debarre, to complete the proof of Theorem 1.1 one needs to rule out the existence of minimal irregular surfaces with $K^2 = 2p_g$, $3 \geq q \geq 2$, $\chi > 1$. This is done in §4, once again using “classical” arguments.

2. Preliminaries

2.1. Abelian subvarieties of Theta divisors of Jacobians. In [CMP14], following [AH91, DF93], we considered the following situation:

(*) C is a smooth, projective, complex curve of genus g , Z is an irreducible r -dimensional subvariety of a Brill–Noether locus $W_d^s(C) \subsetneq J^d(C)$, and Z is stable under translations by the elements of an abelian subvariety $A \subsetneq J(C)$ of dimension $a > 0$ (if so, we will say that Z is *A-stable*).

Here $J^d(C)$ is the set of equivalence classes of divisors of degree d on C and $J(C) := J^0(C)$

is the *Jacobian variety* of C . The Brill–Noether locus $W_d^s(C)$ is the subscheme of $J^d(C)$ corresponding to classes of divisors D such that $h^0(\mathcal{O}_C(D)) \geq s + 1$ and we set $W_d(C) := W_d^0(C)$ (see [ACGH85, Ch. IV]).

It was proved in [DF93] that if $(*)$ holds, then $r + a + 2s \leq d$. In [CMP14], improving on partial results in [AH91, DF93], we give the full classification of the cases in which $(*)$ holds and $r + a + 2s = d$. We will not need here the full strength of the results in [CMP14], only the part concerning the case $(d, s) = (g - 1, 0)$ (see [CMP14, Theorem 3.1]), which is as follows.

THEOREM 2.1. *Let C be a curve of genus g . Let $A \subsetneq J(C)$ be an abelian variety of dimension $a > 0$ and let $Z \subset W_{g-1}(C)$ be an irreducible, A -stable variety of dimension $r = g - 1 - a$. Then there is a degree two morphism $\varphi: C \rightarrow C'$, with C' smooth of genus g' , such that one of the following occurs:*

- (a) *we have $g' = a$, $A = \varphi^*(J(C'))$ and $Z = W_{g-1-2a}(C) + \varphi^*(J^a(C'))$;*
- (b) *we have $g' = a + 1$, $r = a$, φ is étale, A is the Prym variety of φ , and $Z \subset W_{g-1}(C)$ is the connected component of $\varphi_*^{-1}(K_{C'})$ consisting of divisor classes D with $h^0(C, \mathcal{O}_C(D))$ odd, where $\varphi_*: J^{g-1}(C) \rightarrow J^{g-1}(C')$ is the norm map.*

In particular, $Z \cong A$ is an abelian variety if and only if either we are in case (a) and $g = 2a + 1$, or we are in case (b).

2.2. Some generalities on irregular surfaces. Let S be a smooth projective irreducible surface over $\mathbb{C}$. We will use the standard notation $q(S) = h^1(S, \mathcal{O}_S)$, $p_g(S) = h^0(S, \Omega_S^2)$, $\chi(S) = \chi(\mathcal{O}_S) = p_g(S) - q(S) + 1$ for the *irregularity*, the *geometric genus*, and the Euler characteristic of the structure sheaf, respectively. We will often use the simplified notation p_g, q, χ . Note that, since we are working over the complex numbers, one has also $q = h^0(S, \Omega_S^1)$.

Numerical and linear equivalence will be denoted by $\sim$ and $\equiv$, respectively.

We will denote by $a: S \rightarrow \text{Alb}(S)$ the *Albanese morphism* of S . The dimension of $a(S)$ is denoted by $\text{Albdim}(S)$ and is called the *Albanese dimension* of S .

A *pencil of genus b* on S is a rational map $f: S \dashrightarrow B$ with connected fibres, with B a smooth curve of genus b . The indeterminacy points of f are the *base points* of the pencil. If $b > 0$, the pencil is said to be *irrational* and has no base points.

An effective non-zero divisor D on a smooth surface is said to be *m -connected* if for every decomposition $D = A + B$ with A, B effective non-zero divisors one has $A \cdot B \geq m$.

A *non-essential singularity* of the branch locus B of a double cover $X \rightarrow Y$ of surfaces, with Y smooth and X normal, is a double point or a triple point that has no infinitely near triple points (see [BPvdV04, Ch. III, Section 7], where these are called *simple singularities*).

As usual, we denote by $|K_S|$ (or simply $|K|$) the *canonical system* of S . If $\eta \in \text{Pic}^0(S)$, the linear system $|K + \eta|$ is called a *twisted canonical system* of S . A curve $C_\eta \in |K + \eta|$ is a *paracanonical curve* on S .

Assume that S is irregular of general type. We will denote by $\mathfrak{K}_S$ (or simply by $\mathfrak{K}$) the *paracanonical system* of S , that is, the Hilbert scheme of paracanonical curves on S . The morphism $p: \mathfrak{K} \rightarrow \text{Pic}^0(S)$ is defined by $C \mapsto \mathcal{O}_S(C - K)$. As $\chi(\mathcal{O}_S) \geq 1$, there is a unique component $\mathcal{K}_S$ (or simply $\mathcal{K}$) of $\mathfrak{K}$ dominating $\text{Pic}^0(S)$ via p . It is called the *main paracanonical system* of S .

If $\text{Albdim}(S) = 2$ and $\eta \in \text{Pic}^0(S)$ is general, one has $h^1(S, \eta) = 0$, hence $\dim(|K + \eta|) = \chi - 1$ (this is the so-called *generic vanishing theorem*; see [GL87, Theorem 1]). If $\eta \in \text{Pic}^0(S)$ and $C \in |K + \eta|$ are general, then C corresponds to the general point of $\mathcal{K}$, which has dimension $q + \dim(|K + \eta|) = p_g$.

3. Surfaces with $K^2 = 4$ and $p_g = q = 2$

Let S be minimal, of general type, with $p_g = q = 2$. One has $K^2 \geq 4$ by [Deb82, Théorème 6.1] and the equality is attained in the following example.

Example 3.1. Let (A, Θ) be a principally polarized abelian surface. Let $p: S \rightarrow A$ be the double cover branched over an effective, smooth divisor B in the class of 2Θ , so that $p_*\mathcal{O}_S = \mathcal{O}_A \oplus \mathcal{O}_A(-\theta)$, with θ in the class of Θ and S smooth and minimal. Then $K = p^*(\theta)$ and the invariants of S are $p_g = q = 2$, $K^2 = 4$. We can also allow non-essential singularities on the divisor B . In that case S will be the minimal resolution of the double cover of A branched over B . One has $\text{Alb}(S) \cong A$.

A special case is when Θ is *reducible*, that is, A is the product of elliptic curves, and $\Theta = E_1 + E_2$, with E_i elliptic curves such that $E_1 \cdot E_2 = 1$. In that case S has two elliptic pencils $\mathcal{M}_1, \mathcal{M}_2$ of curves of genus two (see [CML02, Example 7.1]; according to [Zuc03], this is the only case in which S has an irrational pencil of curves of genus two; we will not use this result, though).

Note that the curves in $\mathcal{K}_S$ are the proper transforms via p of the curves in the class of Θ . If Θ is irreducible, then the general curve in $\mathcal{K}_S$ is smooth; otherwise, the general curve in $\mathcal{K}_S$ is the sum of a curve in $\mathcal{M}_1$ and a curve in $\mathcal{M}_2$.

In this section we prove the following classification result.

THEOREM 3.2. *A minimal surface of general type with $p_g = q = 2$, $K^2 = 4$ is as in Example 3.1.*

By [CML02, Proposition 2.3], minimal surfaces S of general type with $K^2 = 4$ and $p_g = q = 2$ have $\text{Albdim}(S) = 2$; that is, $a: S \rightarrow \text{Alb}(S)$ is surjective. Surfaces of general type with $p_g = q = 2$ and with an irrational pencil have been studied in [Pen11, Zuc03], but we will not use their results here.

LEMMA 3.3. *Let S be a minimal surface of general type with $p_g = q = 2$, $K^2 = 4$. Then*

- (i) *either the general curve in the main paracanonical system $\mathcal{K}$ is smooth*
- (ii) *or S is as in Example 3.1, with Θ reducible.*

Proof. Let $C \in \mathcal{K}$ be general. Write $C = F + M$, where F is the fixed divisor of $\mathcal{K}$. Since $\dim(\mathcal{K}) = 2$, the curve M moves in a two-dimensional system $\mathcal{M}$ of curves. First suppose that M is reducible. Then M must consist of two distinct irreducible components M_i , each moving in a one-dimensional family of curves $\mathcal{M}_i$, with $1 \leq i \leq 2$. The index theorem (see [BPvdV04, Ch. IV, Corollary 2.16]) yields $K \cdot M_i \geq 2$, for $1 \leq i \leq 2$. On the other hand,

$$4 = K^2 = K \cdot F + K \cdot M_1 + K \cdot M_2,$$

hence $K \cdot F = 0, K \cdot M_1 = K \cdot M_2 = 2$. Then $M_1^2 = M_2^2 = 0$ and $\mathcal{M}_i$ is a pencil of curves of genus two, for $1 \leq i \leq 2$. Such a pencil is not rational by [Bea82, Lemme on p. 345], hence it is of genus one, because $q(S) = 2$ and $\text{Albdim}(S) = 2$. This implies $M_1 \cdot F = M_2 \cdot F = 0$, hence $F = 0$ by the 2-connectedness of paracanonical divisors (see, for example, [BPvdV04, Ch. VII, Proposition 6.2]) and $M_1 \cdot M_2 = 2$. Let $f_i: S \rightarrow E_i$ be the elliptic pencils $\mathcal{M}_i$, for $1 \leq i \leq 2$. The morphism $f = f_1 \times f_2: S \rightarrow E_1 \times E_2$ is a double cover and we are in case (ii).

Next we may assume that M is irreducible, and we prove that C is irreducible.

We claim that $M^2 \geq 3$. In fact, by [CCM98, Lemma 0.14], $M^2 = 0$ would mean that $\mathcal{M}$ is a pencil and this is impossible because it has dimension two. Now, by [CCM98, Proposition 0.16],

a curve M with $M^2 = 1$ moving in a two-dimensional system of curves varies in a linear system of rational curves and this is impossible because S is of general type. Finally, $M^2 = 2$ is also excluded because, by [CCM98, Theorem 0.20], S would be birational to a product of curves, which is not the case.

Suppose $F \neq 0$. Then by the 2-connectedness of paracanonical divisors one has

$$K^2 \geq K \cdot M = F \cdot M + M^2 \geq 5,$$

a contradiction.

Finally, we prove that C is smooth. Assume, to the contrary, that C is singular. Let $c \in \mathcal{K}$ be the point corresponding to C . The two-dimensional tangent space $T_{\mathcal{K},c}$ is contained in $H^0(C, N_{C|S}) \cong H^0(C, \mathcal{O}_C(C))$, which is also two-dimensional, so that equality holds. It therefore corresponds to a g_4^1 on C . Every section of $H^0(C, \mathcal{O}_C(C))$ in $T_{\mathcal{K},c}$ vanishes at each singular point of C . This implies that C has a unique singular point x_C . Consider the rational map $\text{Pic}^0(S) \dashrightarrow S$ which associates with C its singular point x_C . Since this map cannot be dominant, it follows that for C general in $\mathcal{K}$ there is a one-dimensional system $\mathcal{K}_C$ of curves in $\mathcal{K}$ having, like C , a singular point at x_C . Since the curves in $\mathcal{K}_C$ have no variable intersection off x_C , the system $\mathcal{K}_C$ is a pencil, and it is rational because it has the base point x_C . Thus the surface P parametrizing $\mathcal{K}$ would be ruled, a contradiction since P is birational to $\text{Pic}^0(S)$. $\square$

By Lemma 3.3 we may assume from now on that the general curve $C \in \mathcal{K}$ is smooth of genus five. Consider the *restriction morphism* $\mathfrak{r}: \text{Pic}^0(S) \rightarrow J^4(C)$ sending η to $\mathcal{O}_C(C + \eta)$. This map is injective by [CFM97, Proposition 1.6]. By generic vanishing, if $\eta \in \text{Pic}^0(S)$ is general, one has

$$h^0(C, \mathcal{O}_C(C + \eta)) = h^0(S, \mathcal{O}_S(C + \eta)) = 1.$$

Hence the image A of $\mathfrak{r}$ is an abelian surface contained in $W_4(C)$.

LEMMA 3.4. *In the above setup there exist a smooth genus two curve C' and a degree two morphism $\varphi: C \rightarrow C'$ such that $A = \varphi^*(J^2(C'))$.*

Proof. We can apply Theorem 2.1 with $g = 5$ and $r = a = 2$. Hence to prove the assertion it suffices to show that case (b) of Theorem 2.1 does not occur. Suppose that we are in case (b). Then, by Theorem 2.1 (cf. [BH04, § 12.6]), for all $\eta \in \text{Pic}^0(S)$, one has $h^0(C, L_\eta) \equiv 1 \pmod{2}$. On the other hand, $L_0 = \mathcal{O}_C(C)$ and, by generic vanishing, $h^0(C, \mathcal{O}_C(C)) = 2$, a contradiction. $\square$

End of the proof of Theorem 3.2. By Lemma 3.3, we may assume $C \in \mathcal{K}$ general to be smooth. By Lemma 3.4, there is a degree two morphism $\varphi: C \rightarrow C'$, with C' of genus two, such that $A = \varphi^*(J^2(C'))$. Since φ is ramified at four points, $A \cong J(C')$ is a principally polarized abelian surface. So $\text{Pic}^0(S)$ is also principally polarized and we may identify both $\text{Pic}^0(S)$ and $\text{Alb}(S)$ with A .

Next, we prove that there is an involution $\mathfrak{i}$ of S such that $\mathfrak{i}(C) = C$ and $\mathfrak{i}|_C$ coincides with the involution ι determined by the double cover $\varphi: C \rightarrow C'$ (then $\mathfrak{i}$ will be biregular, since S is minimal of general type). To define $\mathfrak{i}$ we argue as follows.

The curve C is smooth by Lemma 3.3; let $x \in C$ be general and let $\mathcal{K}_x$ be the family of curves in $\mathcal{K}$ containing x . We claim that the set $\{x, \iota(x)\}$ coincides with the base locus B_x of $\mathcal{K}_x$, and therefore is independent of C . Indeed, by the generality of x , there exists a $D \in \mathcal{K}_x$ such that $h^0(C, \mathcal{O}_C(D)) = 1$ and the divisor $D|_C$ is reduced and equal to $x + \iota(x) + y + \iota(y)$ for some $y \in C$. Since B_x is invariant under ι by construction, we have two possibilities:

$$(a) \quad B_x = \{x, \iota(x), y, \iota(y)\} \quad \text{or} \quad (b) \quad B_x = \{x, \iota(x)\}.$$

In case (a) all the curves of $\mathcal{K}_x$ would cut the same divisor on C . Since the map $\text{Pic}^0(C) \rightarrow \text{Pic}^0(S)$ is injective, this would mean that all the curves of $\mathcal{K}_x$ are linearly equivalent, which goes against the assumption that C be general.

Consider the minimal desingularization X of the quotient S/i . It contains a two-dimensional family of (generically smooth and irreducible) curves Γ of genus two, with $\Gamma^2 = 2$, that is, the genus two quotients of the curves in $\mathcal{K}_S$ under the involution i . By [CCM98, Theorem 0.20], all curves Γ are isomorphic and X is birational to the symmetric product $\Gamma(2)$, which is birational to $J(\Gamma)$. So we have a degree two map $S \dashrightarrow J(\Gamma)$, which is in fact a morphism since $J(\Gamma)$ is an abelian variety and S is smooth. The universal property of the Albanese map immediately implies that $A = J(\Gamma)$ and that $S \rightarrow J(\Gamma)$ is the Albanese morphism $a: S \rightarrow A$. Note that via $X \dashrightarrow A$ the curves Γ are mapped to Θ -divisors, hence the paracanonical curves are mapped 2-to-1 to Θ -divisors by a .

Let B be the branch divisor of $a: S \rightarrow A$. Since the curves in $\mathcal{K}$ have genus five, we have $B \cdot \Theta = 4$. By the index theorem we have $B^2 \leq 8$. On the other hand, B is divisible by 2 in the Neron–Severi group $NS(A)$, hence $B^2 \geq 8$. In conclusion, $B^2 = 8$ and the index theorem implies that the class of B is 2Θ and the assertion follows. $\square$

Remark 3.5. Let S be as in Example 3.1 with B smooth. Then the ramification curve R is a canonical curve isomorphic to B , hence smooth. So the general curve $C \in |K|$ is smooth and, as above, we can consider the restriction map $\text{Pic}^0(S) \rightarrow W_4(C)$, whose image is an abelian surface $A \subset W_4(C)$. We can use Theorem 2.1, but now the argument of Lemma 3.4 does not apply because C is not a paracanonical curve. In this situation the Prym case (b), and not case (a), of Theorem 2.1 occurs. Indeed, the bicanonical morphism is not birational for S : it is in fact composed with an involution σ such that $\Sigma = S/\sigma$ is a surface with 20 nodes (over which the double cover $\phi: S \rightarrow \Sigma$ is ramified). If X is the minimal desingularization of Σ , one has $p_g(X) = 2, q(X) = 0, K_X^2 = 2$ (see [CML02, Proposition 6.5]). So the general curve $C \in |K_S|$ is the étale double cover of the general curve $C' \in |K_X|$. One has the following commutative diagram:

$$\begin{array}{ccc} S & \xrightarrow{\phi} & \Sigma \\ p \downarrow & & \downarrow q \\ A & \xrightarrow{\kappa} & Y, \end{array}$$

where $Y \subset \mathbb{P}^3$ is the 16-nodal Kummer surface of A (κ is the obvious double cover), and $q: \Sigma \rightarrow Y$ is a double cover branched along a smooth plane section H of Y (which pulls back via κ to the branch divisor of p), plus six nodes lying on a conic $\Gamma \subset Y$.

By [Bea88, Proposition 4], there is a unique curve C_0 in the intersection of $|K_S|$ with $\mathcal{K}$, that is, the proper transform of Γ on S . It is interesting to notice that for C_0 both cases (a) and (b) of Theorem 2.1 occur at the same time (see [CMP14, Remark 3.2]).

We finally notice that the idea of the proof of Theorem 3.2 can also be applied to recover the classification of minimal surfaces S with $p_g = q = 3$ (see [CCM98, Proposition (3.22)]). In this case q is odd, the analogue of Lemma 3.4 does not hold, and the Prym case (b) of Theorem 2.1 occurs. We do not dwell on this here.

4. Proof of Theorem 1.1

As explained in the introduction, once we have Theorem 3.2, to prove Theorem 1.1, it suffices to show the following result.

PROPOSITION 4.1. *There is no minimal complex surface of general type S with $2 \leq q \leq 3$, $\chi \geq 2$, and $K^2 = 2p_g$.*

Proof. Suppose by contradiction that such a surface S exists.

CLAIM 4.2. One has $\text{Albdim}(S) = 2$, $q = 3$, $p_g = 4$.

Proof of Claim 4.2. Suppose, by contradiction, that $\text{Albdim}(S) = 1$, and let $a: S \rightarrow B$ be the Albanese pencil, with B of genus q . We denote by g the genus of the fibres of a ; one has $g \geq 2$ because S is of general type,

By [Hor82, Theorem 3.1], if $K^2 < 3\chi$, the general fibre of a is hyperelliptic of genus $g = 2$ or 3. Since B has genus q , applying the slope inequality for hyperelliptic fibrations ([Hor82, Theorem 2.1]), one has

$$K^2 \geq 4 \frac{g-1}{g} (\chi + (g+1)(q-1)).$$

From $K^2 = 2p_g$, one concludes that $q = 1$, a contradiction.

So $K^2 \geq 3\chi$. Then the inequality $2p_g = K^2 \geq 3p_g - 3q + 3$ yields $p_g \leq 6$ for $q = 3$ and $p_g \leq 3$ for $q = 2$. Since $K^2 = 2p_g$, one has $K^2 \leq 12$ if $q = 3$ and $K^2 \leq 6$ if $q = 2$. But, by Arakelov's theorem (see [Bea82, Corollaire on p. 344]), we have

$$K^2 \geq 8(g-1)(q-1) \geq 8(q-1),$$

a contradiction.

So $\text{Albdim}(S) = 2$. Then the Severi inequality proved in [Par05] gives $2p_g = K^2 \geq 4\chi$, that is, $p_g \leq 2q - 2$. Since $\chi \geq 2$ and $q \leq 3$, the only possibility is $p_g = 4$, $q = 3$. $\square$

CLAIM 4.3. The surface S has no irrational pencil $f: S \rightarrow B$ with B of genus $b > 1$. Therefore

- (i) the map $\Phi: \bigwedge^2 H^0(S, \Omega_S^1) \rightarrow H^0(S, \mathcal{O}_S(K_S))$ is injective;
- (ii) the Albanese image Ξ of S is not covered by elliptic curves.

Proof of Claim 4.3. Let g be the genus of the general fibre of f . Again by Arakelov's theorem (see [Bea82, Corollaire on p. 344]) we have

$$8 = K^2 \geq 8(g-1)(b-1),$$

hence $g = b = 2$. By [Hor77, Theorem 3], one has

$$K^2 \geq 2\chi - 6 + 6b = 10,$$

a contradiction. This proves the first assertion and statement (i) follows by Castelnuovo-de Franchis' theorem ([dFr05], see [Bea78, Proposition X.9] for a modern reference) and the fact that, since $q = 3$, all two-vectors are decomposable.

To prove statement (ii), suppose that Ξ is covered by elliptic curves. Then there is an elliptic curve $E \subset \text{Alb}(S)$ such that the image of S via the composition of the Albanese morphism and the morphism $\text{Alb}(S) \rightarrow \text{Alb}(S)/E$ is a curve B . Since B has to span $\text{Alb}(S)/E$, which has dimension two, the curve B has genus $b \geq 2$, contradicting the first assertion. $\square$

CLAIM 4.4. Let F be the general fibre of a pencil $f: S \dashrightarrow B$ with B of genus b (possibly $b = 0$ and the pencil is linear, with base points), and let F be irreducible of geometric genus g . Then

- (i) we have $K \cdot F \geq 4$ and $g \geq 3$;
- (ii) if $K \cdot F = 4$, then either $F^2 = 0, g = 3, b = 1$ or $F^2 = 2, b = 0, g = 4$ and $K \sim 2F$.

In particular, S has no pencil of curves of genus two and its bicanonical map is a birational morphism.

Proof of Claim 4.4. By Claim 4.3, we have $b \leq 1$. By blowing up if necessary, we may assume that the pencil has no base points and that F is smooth of genus g . Then, by [Bea82, Lemme on p. 345], one has $3 = q \leq b + g$ with equality only if S is birational to a product of curves of genus b and g . Since S is of general type and $b \leq 1$, equality cannot hold and thus either $b = 1$ and $g \geq 3$ or $b = 0$ and $g \geq 4$.

Now consider the original surface S again.

If $K \cdot F < 4$, then by the index theorem and the fact that, by the adjunction formula, $K \cdot F$ and $F^2 = 0$ have the same parity, one has either $F^2 = 0, K \cdot F = 2$ or $F^2 = 1, K \cdot F = 3$. The former case implies $g = 2$, a contradiction. The latter case gives $g = 3$ and $b = 0$ because the pencil has a base point, again a contradiction. Hence $K \cdot F \geq 4$. If equality holds, again by the index theorem and parity, either $F^2 = 0$ or $F^2 = 2$ and $K_S \sim 2F$. In the former case, $g = 3$ and thus $b = 1$.

The bicanonical map is a morphism, as it always is for a minimal surface of general type with $p_g \geq 1$ (see [BPvdV04, Ch. VII, Theorem 7.4]). Since we have just proved that S has no genus two pencil, by [CCM98, Theorem A] the bicanonical morphism of S is birational. $\square$

Finally, we prove Proposition 4.1 by showing that surfaces S as in Claim 4.2 do not exist. Suppose otherwise. Consider the map $\phi: S \dashrightarrow \mathbb{P}^2$ associated with the linear system $|\text{Im}(\Phi)|$ (see Claim 4.3(i)). A standard local computation shows that this is the composition of the Albanese map $a: S \rightarrow \Xi$ with the Gauss map $\gamma: \Xi \dashrightarrow \mathbb{P}^2$ which maps a general point $x \in \Xi$ to the point of $\mathbb{P}(T_{\text{Alb}(S),0})^* \cong \mathbb{P}^2$ corresponding to the line $\mathbb{P}(T_{\Xi,x})$ translated to the origin of $\text{Alb}(S)$. The map γ is dominant. Otherwise the line bundle determined by Ξ would not be positive definite (see [BH04, Proposition 4.4.2]) and therefore Ξ would be covered by elliptic curves, which is not the case. Hence the image of the canonical map $\varphi: S \dashrightarrow \mathbb{P}^3$ is a surface Σ .

Consider the multiplication map

$$\mu: \text{Sym}^2(H^0(S, \mathcal{O}_S(K))) \rightarrow H^0(S, \mathcal{O}_S(2K)).$$

Since both spaces have dimension 10, we have two possibilities:

- (i) The map μ is an isomorphism; since the bicanonical map of S is a birational morphism (see Claim 4.4), the map φ is then also a birational morphism.
- (ii) We have $\dim(\ker(\mu)) = 1$; in this case Σ is a quadric.

Since q is odd and $\text{Albdim}(S) = 2$, the canonical system is contained in the main paracanonical system $\mathcal{K}$ (see [MPP13, Theorem 1.3]). Hence if the general canonical curve is irreducible or smooth, then the general curve in $\mathcal{K}$ is also irreducible or smooth, respectively.

Assume that we are in case (i). Since φ is a morphism, the general canonical curve is smooth. Let $C \in \mathcal{K}$ be general. Then by the above remark C is also smooth. For any paracanonical curve $D \in |2K - C|$ there exists a quadric Q_D of $\mathbb{P}^3$ such that $D + C$ is the divisor of $\varphi^*(Q_D)$. Since $h^0(S, \mathcal{O}_S(D)) = \chi = 2$, there exist at least two distinct quadrics Q_1 and Q_2 of $\mathbb{P}^3$ containing $\varphi(C)$.

Since $\varphi(C)$ is irreducible and non-degenerate of degree eight, this is a contradiction. So case (i) does not occur.

Assume now that we are in case (ii) and suppose first that Σ is non-singular. The two line rulings $|L_1|$, $|L_2|$ of Σ determine two pencils on S with general fibres F_1 , F_2 , respectively. For $1 \leq i \leq 2$, the strict transform of a general element of $|L_i|$ is numerically equivalent to $r_i F_i$, with $r_i \geq 1$, and $K - (r_1 F_1 + r_2 F_2)$ is numerically equivalent to an effective divisor. Hence we have

$$8 = K^2 \geq r_1 K \cdot F_1 + r_2 K \cdot F_2 \geq 4(r_1 + r_2)$$

(the last inequality follows by Claim 4.4). So $r_1 = r_2 = 1$, $K \cdot F_1 = K \cdot F_2 = 4$ and, by Claim 4.4, $|F_1|$ and $|F_2|$ are distinct linear pencils with $F_1^2 = F_2^2 = F_1 \cdot F_2 = 2$ (in particular, $F_1 \sim F_2$ and $K \sim 2F_1 \sim 2F_2$). The pencil $|F_i|$ has a base scheme β_i of length two, hence F_i is smooth of genus four, for $1 \leq i \leq 2$.

Set $F = F_i$ (with $i = 1$ or 2). Since $h^1(S, \mathcal{O}_S(K_S + F)) = 0$ by Ramanujam's vanishing theorem (see [BPvdV04, Ch. IV, Theorem 12.5]), the restriction sequence

$$0 \rightarrow \mathcal{O}_S(K_S) \rightarrow \mathcal{O}_S(K_S + F) \rightarrow \mathcal{O}_F(K_F) \rightarrow 0$$

yields the long exact cohomology sequence

$$0 \rightarrow H^0(S, \mathcal{O}_S(K_S)) \rightarrow H^0(S, \mathcal{O}_S(K_S + F)) \xrightarrow{r} H^0(F, \mathcal{O}_F(K_F)) \rightarrow H^1(S, \mathcal{O}_S(K_S)) \rightarrow 0.$$

This implies $\dim(\text{Im}(r)) = 1$; that is, the rational map determined by $|K_S + F|$ contracts the general curve in $|F|$ to a point. This is a contradiction, because the canonical image of S is a surface.

Finally, assume that Σ is a quadric cone. The same arguments as before show that the line ruling of Σ determines a pencil $|F|$ on S with $K \cdot F = 4$, $F^2 = 2$. Then, as above, the rational map determined by $|K_S + F|$ contracts the general curve of $|F|$ to a point, again a contradiction, which shows that case (ii) cannot occur either. This finishes the proof of the proposition. $\square$

Remark 4.5. In order to exclude the existence of surfaces S as in Claim 4.2 we could have argued, in principle, as in the proof of Theorem 3.2; that is, we could have used the results in [CMP14]. To do so, one should first prove that the general curve in $\mathcal{K}_S$ is smooth. This is not shorter, actually it is a bit more involved, than the proof presented here.

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