On a certain duality of Néron–Severi lattices of supersingular $K3$ surfaces

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Dedicated to Professor Shigeru Mukai on the occasion of his sixtieth birthday

ABSTRACT

Let X and Y be supersingular $K3$ surfaces defined over an algebraically closed field. Suppose that the sum of their Artin invariants is 11. Then there exists a certain duality between their Néron–Severi lattices. We investigate geometric consequences of this duality. As an application, we classify genus one fibrations on supersingular $K3$ surfaces with Artin invariant 10 in characteristic 2 and 3, and give a set of generators of the automorphism group of the nef cone of these supersingular $K3$ surfaces. The difference between the automorphism group of a supersingular $K3$ surface X and the automorphism group of its nef cone is determined by the period of X . We define the notion of genericity for supersingular $K3$ surfaces in terms of the period, and prove the existence of generic supersingular $K3$ surfaces in odd characteristics for each Artin invariant larger than one.

1. Introduction

A $K3$ surface X defined over an algebraically closed field k is said to be *supersingular* (in the sense of Shioda) if the rank of its Néron–Severi lattice S_X is 22. Supersingular $K3$ surfaces exist only when the base field k is of positive characteristic. Let X be a supersingular $K3$ surface in characteristic $p > 0$. Artin [Art74] proved that the discriminant group of S_X is a p -elementary abelian group of rank 2σ , where σ is a positive integer such that $\sigma \leq 10$. This integer σ is called the *Artin invariant* of X . The isomorphism class of the lattice S_X depends only on p and σ (Rudakov and Shafarevich [RS81]). Moreover, supersingular $K3$ surfaces with Artin invariant σ form a $(\sigma - 1)$ -dimensional family, and a supersingular $K3$ surface with Artin invariant one in characteristic p is unique up to isomorphisms (Ogus [Ogu79], [Ogu83], Rudakov and Shafarevich [RS81]).

Recently, many studies of supersingular $K3$ surfaces in small characteristics with Artin invariant one have appeared: for example, for $p = 2$, Dolgachev and Kondō [DK03], Katsura and Kondō [KK12a], Elkies and Schütt [ES12]; for $p = 3$, Katsura and Kondō [KK12b], Kondō and Shimada [KS12], Sengupta [Sen11]; and for $p = 5$, Shimada [Shi14], and Katsura, Kondō and

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Shimada [KKS13]. On the other hand, not much is known about the geometric properties of supersingular $K3$ surfaces with big Artin invariant (see, for example, Rudakov and Shafarevich [RS78], [RS81], Shioda [Shi87], Shimada [Shi04b], [Shi04c]).

In this paper, we present some methods to investigate supersingular $K3$ surfaces with big Artin invariant by means of the following simple observation. Let $X_{p,\sigma}$ be a supersingular $K3$ surface in characteristic p with Artin invariant σ , and let $S_{p,\sigma}$ denote its Néron–Severi lattice.

LEMMA 1.1. *Suppose that $\sigma + \sigma' = 11$. Then $S_{p,\sigma'}$ is isomorphic to $S_{p,\sigma}^\vee(p)$, where $S_{p,\sigma}^\vee(p)$ is the lattice obtained from the dual lattice $S_{p,\sigma}^\vee$ of $S_{p,\sigma}$ by multiplying the symmetric bilinear form with p .*

Lemma 1.1 is proved in Section 3. We use this duality between $S_{p,\sigma}$ and $S_{p,\sigma'}$ in the study of genus one fibrations and the automorphism groups of supersingular $K3$ surfaces.

First, we apply Lemma 1.1 to the classification of genus one fibrations. Note that the Néron–Severi lattice S_Y of a $K3$ surface Y is a hyperbolic lattice. The orthogonal group $O(S_Y)$ of S_Y contains the stabilizer subgroup $O^+(S_Y)$ of a positive cone of $S_Y \otimes \mathbb{R}$ as a subgroup of index two.

DEFINITION 1.2. Let Y be a $K3$ surface, and let $\phi : Y \rightarrow \mathbb{P}^1$ be a genus one fibration. We denote by $f_\phi \in S_Y$ the class of a fiber of ϕ . Let $\psi : Y \rightarrow \mathbb{P}^1$ be another genus one fibration on Y . We say that ϕ and ψ are *Aut-equivalent* if there exist $g \in \text{Aut}(Y)$ and $\bar{g} \in \text{Aut}(\mathbb{P}^1)$ such that $\phi \circ g = \bar{g} \circ \psi$ holds. We say that ϕ and ψ are *lattice equivalent* if there exists an element $g \in O^+(S_Y)$ such that $f_\phi^g = f_\psi$. We denote by $\mathbb{E}(Y)$ the set of lattice equivalence classes of genus one fibrations on Y , and by $[\phi] \in \mathbb{E}(Y)$ the lattice equivalence class containing ϕ .

Many combinatorial properties of a genus one fibration $\phi : Y \rightarrow \mathbb{P}^1$ depend only on the lattice equivalence class $[\phi]$ (see Proposition 4.1). Moreover, when $\sigma = 10$, the classification of genus one fibrations by Aut-equivalence seems to be too fine (see Remark 7.6). Therefore we concentrate on the study of lattice equivalence classes.

Using Lemma 1.1, we prove the following result.

THEOREM 1.3. *Suppose that $\sigma + \sigma' = 11$. Then there exists a canonical one-to-one correspondence $[\phi] \mapsto [\phi']$ between $\mathbb{E}(X_{p,\sigma})$ and $\mathbb{E}(X_{p,\sigma'})$.*

We say that a genus one fibration is *Jacobian* if it admits a section.

THEOREM 1.4. *Suppose that a genus one fibration $\phi : X_{p,\sigma} \rightarrow \mathbb{P}^1$ is a Jacobian fibration, and let $\phi' : X_{p,\sigma'} \rightarrow \mathbb{P}^1$ be a genus one fibration on $X_{p,\sigma'}$ with $\sigma' = 11 - \sigma$ such that $[\phi'] \in \mathbb{E}(X_{p,\sigma'})$ corresponds to $[\phi] \in \mathbb{E}(X_{p,\sigma})$ by Theorem 1.3. Then ϕ' does not admit a section.*

Elkies and Schütt [ES12] proved the following result.

THEOREM 1.5 [ES12]. *Any genus one fibration on $X_{p,1}$ admits a section.*

Thus we obtain another proof of [EvdG11, Proposition 12.1].

COROLLARY 1.6 [EvdG11]. *There exist no Jacobian fibrations on $X_{p,10}$.*

By an *ADE-type*, we mean a finite formal sum of the symbols A_i ($i \geq 1$), D_i ($i \geq 4$) and E_k ($k = 6, 7, 8$) with positive integer coefficients. For a genus one fibration $\phi : Y \rightarrow \mathbb{P}^1$ on a $K3$ surface Y , we have the *ADE-type* of the reducible fibers of ϕ . This *ADE-type* depends only

on the lattice equivalence class $[\phi] \in \mathbb{E}(Y)$ (see Proposition 4.1). Therefore we can use $R_{[\phi]}$ to denote the *ADE*-type of the reducible fibers of ϕ .

From the classification of lattice equivalence classes of genus one fibrations of $X_{2,1}$ by Elkies and Schütt [ES12], and that of $X_{3,1}$ by Sengupta [Sen11], we obtain the classification of lattice equivalence classes of genus one fibrations on $X_{2,10}$ and $X_{3,10}$. In particular, we obtain the list of *ADE*-types $R_{[\phi']}$ of the reducible fibers of genus one fibrations ϕ' on $X_{2,10}$ and $X_{3,10}$ (see Theorems 4.8 and 4.9).

In Elkies and Schütt [ES12] and Sengupta [Sen11], the authors also obtained explicit defining equations for the Jacobian fibrations. Note that the lattice equivalence classes of all *extremal* (quasi-)elliptic fibrations (that is, Jacobian fibrations with Mordell–Weil rank zero) on supersingular *K3* surfaces are classified in Shimada [Shi04a].

As the second application of Lemma 1.1, we investigate the automorphism group of the nef cone of a supersingular *K3* surface. For a *K3* surface Y , let $\text{Nef}(Y) \subset S_Y \otimes \mathbb{R}$ denote the nef cone. We denote by $\text{Aut}(\text{Nef}(Y)) \subset \text{O}^+(S_Y)$ the group of isometries of S_Y that preserve $\text{Nef}(Y)$. Since $\text{Aut}(X_{p,\sigma})$ acts on $S_{p,\sigma}$ faithfully (Rudakov and Shafarevich [RS81, Section 8, Proposition 3]), we have

$$\text{Aut}(X_{p,\sigma}) \subset \text{Aut}(\text{Nef}(X_{p,\sigma})) \subset \text{O}^+(S_{p,\sigma}). \quad (1.1)$$

Using the description of $\text{Aut}(X_{2,1})$ by Dolgachev and Kondō [DK03], and that of $\text{Aut}(X_{3,1})$ by Kondō and Shimada [KS12], we give a set of generators of $\text{Aut}(\text{Nef}(X_{2,10}))$ and $\text{Aut}(\text{Nef}(X_{3,10}))$ in Theorems 6.4 and 6.8, respectively.

Suppose that p is odd. We fix a lattice N isomorphic to $S_{p,\sigma}$. Then we define a quadratic space (N_0, q_0) of dimension 2σ over $\mathbb{F}_p$ by

$$N_0 := pN^\vee/pN \quad \text{and} \quad q_0(px \bmod pN) := px^2 \bmod p \quad (x \in N^\vee). \quad (1.2)$$

We fix a *marking* $\eta : N \xrightarrow{\sim} S_{p,\sigma}$ for a supersingular *K3* surface $X := X_{p,\sigma}$ defined over k . Then $\text{Aut}(\text{Nef}(X))$ acts on (N_0, q_0) , and the *period* $K_{(X,\eta)} \subset N_0 \otimes k$ of the marked supersingular *K3* surface (X, η) is defined as the Frobenius pull-back of the kernel of the natural homomorphism

$$N \otimes k \rightarrow S_X \otimes k \rightarrow H_{\text{DR}}^2(X/k)$$

(see Section 7). In virtue of the Torelli theorem for supersingular *K3* surfaces by Ogus [Ogu79], [Ogu83], the subgroup $\text{Aut}(X)$ of $\text{Aut}(\text{Nef}(X))$ is equal to the stabilizer subgroup of the period $K_{(X,\eta)}$. In particular, the index of $\text{Aut}(X_{p,\sigma})$ in $\text{Aut}(\text{Nef}(X_{p,\sigma}))$ is finite. On the other hand, the classification of 2-reflective lattices due to Nikulin [Nik81] implies that $\text{Aut}(\text{Nef}(X_{p,\sigma}))$ is infinite. Hence, at least when p is odd, $\text{Aut}(X_{p,\sigma})$ is an infinite group. See Sections 5 and 7 for details. Moreover, Lieblich and Maulik [LM11] proved that if $p > 2$, then $\text{Aut}(X_{p,\sigma})$ is finitely generated and its action on $\text{Nef}(X_{p,\sigma})$ has a rational polyhedral fundamental domain.

We say that a supersingular *K3* surface X is *generic* if there exists a marking $\eta : N \xrightarrow{\sim} S_X$ such that the only isometries of (N_0, q_0) that preserve the period $K_{(X,\eta)} \subset N_0 \otimes k$ are scalar multiplications (see Definition 7.4). Using the surjectivity of the period mapping proved by Ogus [Ogu83], we prove the following result.

THEOREM 1.7. *Suppose that p is odd and $\sigma > 1$. Then there exist an algebraically closed field k and a supersingular *K3* surface X with Artin invariant σ defined over k that is generic.*

We observe that if $X_{3,10}$ is generic, then the index of $\text{Aut}(X_{3,10})$ in $\text{Aut}(\text{Nef}(X_{3,10}))$ is very large (see Remark 7.6). An analogous result for a generic complex Enriques surface was obtained by Barth and Peters [BP83].

Recently, Liedtke [Lie13] showed the existence of a rational map between supersingular $K3$ surfaces in characteristic $p \geq 5$, and also the existence and nonexistence of Jacobian fibrations on them. It would be interesting to realize our duality as a correspondence of varieties. The authors have not yet seen any clear relation between our duality and Liedtke's rational maps.

This paper is organized as follows. In Section 2, we fix notation and terminology about lattices and $K3$ surfaces. In Section 3, Lemma 1.1 is proved by means of the fundamental results of Rudakov and Shafarevich [RS81] on the Néron–Severi lattices of supersingular $K3$ surfaces. In Section 4, we study genus one fibrations on supersingular $K3$ surfaces, and prove Theorems 1.3 and 1.4. Moreover, the bijections $\mathbb{E}(X_{p,1}) \cong \mathbb{E}(X_{p,10})$ for $p = 2$ and 3 are given explicitly in Tables 4.1 and 4.2. In Section 5, we review the classical method to investigate the orthogonal group of a hyperbolic lattice by means of a chamber decomposition of the associated hyperbolic space, and fix some notation and terminology. We then apply this method to the nef cone of a supersingular $K3$ surface. In Section 6, we give a set of generators of $\text{Aut}(\text{Nef}(X_{2,10}))$ and of $\text{Aut}(\text{Nef}(X_{3,10}))$. In Section 7, we review the theory of the period mapping and the Torelli theorem for supersingular $K3$ surfaces in odd characteristics due to Ogus [Ogu79], [Ogu83], and describe the relation between $\text{Aut}(X_{p,\sigma})$ and $\text{Aut}(\text{Nef}(X_{p,\sigma}))$. In Section 8, we prove Theorem 1.7.

Convention

We use Aut to denote automorphism groups of lattice-theoretic objects, and Aut to denote automorphism groups of geometric objects on $K3$ surfaces.

2. Preliminaries

2.1 Lattices

A $\mathbb{Q}$ -lattice is a free $\mathbb{Z}$ -module L of finite rank equipped with a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle_L : L \times L \rightarrow \mathbb{Q}$. We omit the subscript L in $\langle \cdot, \cdot \rangle_L$ if no confusions will occur. If $\langle \cdot, \cdot \rangle_L$ takes on values in $\mathbb{Z}$, we say that L is a *lattice*. For $x \in L \otimes \mathbb{R}$, we call $x^2 := \langle x, x \rangle$ the *norm* of x . A vector in $L \otimes \mathbb{R}$ of norm n is sometimes called an n -vector. A lattice L is said to be *even* if $x^2 \in 2\mathbb{Z}$ holds for any $x \in L$.

Let L be a free $\mathbb{Z}$ -module of finite rank. A submodule M of L is *primitive* if L/M is torsion free. A nonzero vector $v \in L$ is *primitive* if the submodule of L generated by v is primitive.

Let L be a $\mathbb{Q}$ -lattice of rank r . For a nonzero rational number m , we denote by $L(m)$ the free $\mathbb{Z}$ -module L with the symmetric bilinear form $\langle x, y \rangle_{L(m)} := m\langle x, y \rangle_L$. The signature of L is the signature of the real quadratic space $L \otimes \mathbb{R}$. We say that L is *negative definite* if the signature of L is $(0, r)$, and that L is *hyperbolic* if the signature is $(1, r - 1)$. A *Gram matrix* of L is an $r \times r$ matrix with entries $\langle e_i, e_j \rangle$, where $\{e_1, \dots, e_r\}$ is a basis of L . The determinant of a Gram matrix of L is called the *discriminant* of L .

For an even lattice L , the set of (-2) -vectors is denoted by $\mathcal{R}(L)$. A *negative definite* even lattice L is called a *root lattice* if L is generated by $\mathcal{R}(L)$. Let R be an *ADE*-type. The root lattice of type R is the root lattice whose Gram matrix is the Cartan matrix of type R . Suppose that L is negative definite. By the *ADE-type* of $\mathcal{R}(L)$, we mean the *ADE-type* of the root sublattice $\langle \mathcal{R}(L) \rangle$ of L generated by $\mathcal{R}(L)$. See, for example, Bourbaki [Bou68] for the classification of root lattices.

Let L be an even lattice, and let $L^\vee := \text{Hom}(L, \mathbb{Z})$ be identified with a submodule of $L \otimes \mathbb{Q}$ with the extended symmetric bilinear form. We call this $\mathbb{Q}$ -lattice $L^\vee$ the *dual lattice* of L . The

discriminant group of L is defined to be the quotient $L^\vee/L$, and is denoted by A_L . We define the *discriminant quadratic form* of L

$$q_L : A_L \rightarrow \mathbb{Q}/2\mathbb{Z}$$

by $q_L(x \bmod L) := x^2 \bmod 2\mathbb{Z}$. The order of A_L is equal to the discriminant of L up to sign. We say that L is *unimodular* if A_L is trivial, while L is *p-elementary* if A_L is p -elementary. An even 2-elementary lattice L is said to be of *type I* if $q_L(x \bmod L) \in \mathbb{Z}/2\mathbb{Z}$ holds for any $x \in L^\vee$. Note that L is p -elementary if and only if pG_L^{-1} is an integer matrix, where G_L is a Gram matrix of L .

Let $O(L)$ denote the orthogonal group of a lattice L , that is, the group of automorphisms of L preserving $\langle \cdot, \cdot \rangle_L$. We assume that $O(L)$ acts on L from the *right*, and the action of $g \in O(L)$ on $v \in L \otimes \mathbb{R}$ is denoted by $v \mapsto v^g$. Similarly, $O(q_L)$ denotes the group of automorphisms of A_L preserving q_L . There is a natural homomorphism $O(L) \rightarrow O(q_L)$.

Let L be a hyperbolic lattice. A *positive cone* of L is one of the two connected components of

$$\{ x \in L \otimes \mathbb{R} \mid x^2 > 0 \}.$$

Let $\mathcal{P}_L$ be a positive cone of L . We denote by $O^+(L)$ the group of isometries of L that preserve $\mathcal{P}_L$. We have $O(L) = O^+(L) \times \{\pm 1\}$. For a vector $v \in L \otimes \mathbb{R}$ with $v^2 < 0$, we put

$$(v)^\perp := \{ x \in \mathcal{P}_L \mid \langle x, v \rangle = 0 \},$$

which is a real hyperplane of $\mathcal{P}_L$. An isometry $g \in O^+(L)$ is called a *reflection with respect to v* or a *reflection into $(v)^\perp$* if g is of order two and fixes each point of $(v)^\perp$. An element r of $\mathcal{R}(L)$ defines a reflection

$$s_r : x \mapsto x + \langle x, r \rangle r$$

with respect to r . We denote by $W(L)$ the subgroup of $O^+(L)$ generated by the set of these reflections $\{s_r \mid r \in \mathcal{R}(L)\}$. It is obvious that $W(L)$ is normal in $O^+(L)$.

2.2 K3 surfaces

Let Y be a K3 surface, and let S_Y denote the Néron–Severi lattice of Y . A smooth rational curve on Y is called a *(-2)-curve*. We denote by $\mathcal{P}(Y) \subset S_Y \otimes \mathbb{R}$ the positive cone containing an ample class of Y . Recall that the *nef cone* $\text{Nef}(Y)$ of Y is defined by

$$\text{Nef}(Y) := \{ x \in S_Y \otimes \mathbb{R} \mid \langle x, [C] \rangle \geq 0 \text{ for any curve } C \text{ on } Y \},$$

where $[C] \in S_Y$ is the class of a curve $C \subset Y$. Then $\text{Nef}(Y)$ is contained in the closure $\overline{\mathcal{P}}(Y)$ of $\mathcal{P}(Y)$ in $S_Y \otimes \mathbb{R}$. We put

$$\text{Nef}^\circ(Y) := \text{Nef}(Y) \cap \mathcal{P}(Y) = \{ x \in \text{Nef}(Y) \mid x^2 > 0 \}.$$

The following is well known. See Piatetski-Shapiro and Shafarevich [PSS71], Rudakov and Shafarevich [RS81, Section 3].

PROPOSITION 2.1. (1) We have

$$\text{Nef}(Y) = \{ x \in S_Y \otimes \mathbb{R} \mid \langle x, [C] \rangle \geq 0 \text{ for any } (-2)\text{-curve } C \text{ on } Y \}.$$

(2) If $v \in S_Y$ is contained in $\overline{\mathcal{P}}(Y)$, then there exists an element $g \in W(S_Y)$ such that $v^g \in \text{Nef}(Y)$.

3. Néron–Severi lattices of supersingular $K3$ surfaces

Let $X_{p,\sigma}$ be a supersingular $K3$ surface with Artin invariant σ in characteristic $p > 0$. Then the isomorphism class of the Néron–Severi lattice $S_{p,\sigma}$ of $X_{p,\sigma}$ depends only on p and σ , and is characterized as follows (see Rudakov and Shafarevich [RS81, Sections 3,4 and 5] for the proof).

THEOREM 3.1 [RS81]. (1) *The lattice $S_{p,\sigma}$ is an even hyperbolic p -elementary lattice of rank 22 with discriminant $-p^{2\sigma}$. Moreover, $S_{2,\sigma}$ is of type I.*

(2) *Suppose that N is an even hyperbolic p -elementary lattice of rank 22 with discriminant $-p^{2\sigma}$. When $p = 2$, we further assume that N is of type I. Then N is isomorphic to $S_{p,\sigma}$.*

Using this theorem, we can prove Lemma 1.1 easily.

Proof of Lemma 1.1. It is enough to show that $S_{p,\sigma}^\vee(p)$ is an even p -elementary lattice of discriminant $-p^{2\sigma'}$, and that $S_{2,\sigma}^\vee(2)$ is of type I. Since $S_{p,\sigma}$ is p -elementary, we have $pS_{p,\sigma}^\vee \subset S_{p,\sigma}$. Therefore $S_{p,\sigma}^\vee(p)$ is a lattice. Let $G_{p,\sigma}$ be a Gram matrix of $S_{p,\sigma}$. Then the determinant of the Gram matrix $pG_{p,\sigma}^{-1}$ of $S_{p,\sigma}^\vee(p)$ is equal to $p^{22} \cdot \det(G_{p,\sigma})^{-1} = -p^{2\sigma'}$. Therefore the discriminant of $S_{p,\sigma}^\vee(p)$ is $-p^{2\sigma'}$. Since $p(pG_{p,\sigma}^{-1})^{-1} = G_{p,\sigma}$ is an integer matrix, $S_{p,\sigma}^\vee(p)$ is p -elementary. Suppose that p is odd. Then, for any $\xi \in S_{p,\sigma}^\vee$, we have $p\xi \in S_{p,\sigma}$ and hence $\langle p\xi, p\xi \rangle_{S_{p,\sigma}} = p\langle \xi, \xi \rangle_{S_{p,\sigma}^\vee(p)}$ is even. Therefore $S_{p,\sigma}^\vee(p)$ is even. Suppose that $p = 2$. Then, for any $\xi \in S_{2,\sigma}^\vee$, we have $\langle \xi, \xi \rangle_{S_{2,\sigma}^\vee} \in \mathbb{Z}$, because $S_{2,\sigma}$ is of type I. Therefore $S_{2,\sigma}^\vee(2)$ is even. Moreover, for any $\eta \in (S_{2,\sigma}^\vee(2))^\vee = S_{2,\sigma}(1/2)$, we have $\langle \eta, \eta \rangle_{S_{2,\sigma}(1/2)} \in \mathbb{Z}$, because $S_{2,\sigma}$ is even. Therefore $S_{2,\sigma}^\vee(2)$ is of type I. $\square$

COROLLARY 3.2. *Suppose that $\sigma + \sigma' = 11$. Then there exists an embedding of $\mathbb{Z}$ -modules $j : S_{p,\sigma} \hookrightarrow S_{p,\sigma'}$ that induces an isomorphism of lattices $S_{p,\sigma}^\vee(p) \cong S_{p,\sigma'}^\vee(p)$. This embedding induces an isomorphism*

$$j_* : \mathcal{O}(S_{p,\sigma}) \xrightarrow{\sim} \mathcal{O}(S_{p,\sigma'}).$$

Moreover, such an embedding j is unique up to compositions with elements of $\mathcal{O}(S_{p,\sigma'})$.

Remark 3.3. Suppose that $v \in S_{p,\sigma}$ satisfies $v^2 \geq 0$. Then, by Proposition 2.1(2), we can choose $j : S_{p,\sigma} \hookrightarrow S_{p,\sigma'}$ in Corollary 3.2 in such a way that $j(v) \in \text{Nef}(X_{p,\sigma'})$.

4. Genus one fibrations

Let Y be a $K3$ surface defined over an algebraically closed field of arbitrary characteristic. Recall that $f_\phi \in S_Y$ is the class of a fiber of a genus one fibration $\phi : Y \rightarrow \mathbb{P}^1$, $\mathbb{E}(Y)$ is the set of lattice equivalence classes of genus one fibrations on Y , and $[\phi] \in \mathbb{E}(Y)$ is the class containing ϕ . We summarize properties of a genus one fibration $\phi : Y \rightarrow \mathbb{P}^1$ that depend only on the class $[\phi]$. See Piatetski-Shapiro and Shafarevich [PSS71], Rudakov and Shafarevich [RS81, Sections 3 and 4], and Shioda [Shi90] for the proof.

(1) The fibration ϕ admits a section if and only if there exists a (-2) -vector $z \in S_Y$ such that $\langle f_\phi, z \rangle = 1$.

(2) Note that $f_\phi \in S_Y$ is primitive of norm zero, and that $\langle f_\phi \rangle^\perp / \langle f_\phi \rangle$ is an even negative definite lattice, where $\langle f_\phi \rangle^\perp$ is the orthogonal complement in S_Y of the submodule $\langle f_\phi \rangle$ of rank one generated by f_ϕ . The ADE -type of the reducible fibers of ϕ is equal to the ADE -type of the set $\mathcal{R}(\langle f_\phi \rangle^\perp / \langle f_\phi \rangle)$ of (-2) -vectors in $\langle f_\phi \rangle^\perp / \langle f_\phi \rangle$.

(3) Suppose that ϕ admits a section $Z \subset Y$. Then f_ϕ and $[Z] \in S_Y$ generate an even unimodular hyperbolic lattice U_ϕ of rank two in S_Y . Let K_ϕ denote the orthogonal complement of U_ϕ in S_Y . We have an orthogonal direct-sum decomposition

$$S_Y = U_\phi \oplus K_\phi,$$

and the lattice $\langle f_\phi \rangle^\perp / \langle f_\phi \rangle$ is isomorphic to K_ϕ . Then the Mordell–Weil group of ϕ is isomorphic to $K_\phi / \langle \mathcal{R}(K_\phi) \rangle$, where $\langle \mathcal{R}(K_\phi) \rangle$ is the root sublattice of K_ϕ generated by the (-2) -vectors in K_ϕ .

(4) In characteristic $p = 2$ or 3 , ϕ is quasi-elliptic if and only if $\langle \mathcal{R}(K_\phi) \rangle$ is p -elementary of rank 20.

As a corollary, we obtain the following result.

PROPOSITION 4.1. *Suppose that genus one fibrations $\phi : Y \rightarrow \mathbb{P}^1$ and $\psi : Y \rightarrow \mathbb{P}^1$ on Y are lattice equivalent. Then the following hold.*

- (1) *The fibration ϕ admits a section if and only if ψ does.*
- (2) *The ADE-type of the reducible fibers of ϕ is equal to that of ψ .*
- (3) *Suppose that ϕ and ψ admit a section. Then the Mordell–Weil groups for ϕ and for ψ are isomorphic.*
- (4) *In characteristic 2 or 3, the fibration ϕ is quasi-elliptic if and only if ψ is.*

DEFINITION 4.2. For a hyperbolic lattice S , we put

$$\tilde{\mathcal{E}}(S) := \{v \in S \otimes \mathbb{Q} \mid v \neq 0, v^2 = 0\} / \mathbb{Q}^\times \quad \text{and} \quad \mathcal{E}(S) := \tilde{\mathcal{E}}(S) / \mathcal{O}(S).$$

Remark 4.3. Let a positive cone $\mathcal{P}_S$ of S be fixed. Then each element of $\tilde{\mathcal{E}}(S)$ is represented by a unique nonzero primitive vector $v \in S$ of norm zero that is contained in the closure $\overline{\mathcal{P}}_S$ of $\mathcal{P}_S$ in $S \otimes \mathbb{R}$.

In Piatetski-Shapiro and Shafarevich [PSS71], and in Rudakov and Shafarevich [RS81, Sections 3 and 4], the following result is proved.

PROPOSITION 4.4. *Let v be a nonzero vector of S_Y . Then there exists a genus one fibration $\phi : Y \rightarrow \mathbb{P}^1$ such that $v = f_\phi$ if and only if v is primitive, $v^2 = 0$, and $v \in \text{Nef}(Y)$.*

Combining Propositions 2.1, 4.4 and Remark 4.3, we obtain the following result.

COROLLARY 4.5. *The map $\phi \mapsto f_\phi$ induces a bijection from $\mathbb{E}(Y)$ to $\mathcal{E}(S_Y)$.*

From now on, we work over an algebraically closed field of characteristic $p > 0$.

Proof of Theorem 1.3. Consider the embedding $j : S_{p,\sigma} \hookrightarrow S_{p,\sigma'}$ in Corollary 3.2. Then j is unique up to $\mathcal{O}(S_{p,\sigma'})$, induces a bijection from $\tilde{\mathcal{E}}(S_{p,\sigma})$ to $\tilde{\mathcal{E}}(S_{p,\sigma'})$, and induces an isomorphism $\mathcal{O}(S_{p,\sigma}) \cong \mathcal{O}(S_{p,\sigma'})$. Hence it induces a canonical bijection from $\mathcal{E}(S_{p,\sigma})$ to $\mathcal{E}(S_{p,\sigma'})$. $\square$

We denote this canonical one-to-one correspondence from $\mathbb{E}(X_{p,\sigma})$ to $\mathbb{E}(X_{p,\sigma'})$ by $[\phi] \mapsto [\phi']$.

Remark 4.6. Let a genus one fibration $\phi : X_{p,\sigma} \rightarrow \mathbb{P}^1$ be given, and let $\phi' : X_{p,\sigma'} \rightarrow \mathbb{P}^1$ be a representative of $[\phi']$. Then we can choose the embedding $j : S_{p,\sigma} \hookrightarrow S_{p,\sigma'}$ inducing $S_{p,\sigma}^\vee(p) \cong S_{p,\sigma'}^\vee(p)$ in such a way that $j(f_\phi)$ is a scalar multiple of $f_{\phi'}$ by a positive integer.

THEOREM 4.7. *Suppose that a genus one fibration $\phi : X_{p,\sigma} \rightarrow \mathbb{P}^1$ admits a section. Then the corresponding genus one fibration $\phi' : X_{p,\sigma'} \rightarrow \mathbb{P}^1$ does not admit a section. Moreover, the ADE-type of the reducible fibers of ϕ' is equal to the ADE-type of $\mathcal{R}(K_\phi^\vee(p))$.*

No.	R_N	$\sigma = 1$			$\sigma = 10$
		$R_{[\phi]}$	MW_{tor}	$\text{rank}(\text{MW})$	$R_{[\phi']}$
1	$4A_5 + D_4$	$4A_5$	$[3, 6]$	0	0
2	$6D_4$	$5D_4$	$[2, 2, 2, 2]$	0	0
3	$2A_7 + 2D_5$	$2A_7 + D_5$	$[8]$	1	A_1
4	$2A_9 + D_6$	$2A_1 + 2A_9$	$[10]$	0	$2A_1$
5	$4D_6$	$2A_1 + 3D_6$	$[2, 2, 2]$	0	$2A_1$
6	$A_{11} + D_7 + E_6$	$A_{11} + D_7$	$[4]$	2	A_2
7	$A_{11} + D_7 + E_6$	$A_3 + A_{11} + E_6$	$[6]$	0	$3A_1$
8	$4E_6$	$3E_6$	$[3]$	2	A_2
9	$3D_8$	$D_4 + 2D_8$	$[2, 2]$	0	$4A_1$
10	$A_{15} + D_9$	$A_{15} + D_5$	$[4]$	0	$5A_1$
11	$A_{17} + E_7$	$3A_1 + A_{17}$	$[6]$	0	A_3
12	$D_{10} + 2E_7$	$3A_1 + D_{10} + E_7$	$[2, 2]$	0	A_3
13	$D_{10} + 2E_7$	$D_6 + 2E_7$	$[2]$	0	$6A_1$
14	$2D_{12}$	$D_8 + D_{12}$	$[2]$	0	$8A_1$
15	$D_{16} + E_8$	$D_4 + D_{16}$	$[2]$	0	D_4
16	$D_{16} + E_8$	$D_{12} + E_8$	$[1]$	0	$12A_1$
17	$3E_8$	$D_4 + 2E_8$	$[1]$	0	D_4
18	D_{24}	D_{20}	$[1]$	0	$20A_1$

 TABLE 4.1. Genus one fibrations on $X_{2,1}$ and $X_{2,10}$

Proof. Let $z \in S_{p,\sigma}$ be the class of a section of ϕ . We choose $j : S_{p,\sigma} \hookrightarrow S_{p,\sigma'}$ as in Remark 4.6. Since $U_\phi^\vee = U_\phi$, we see that $j(f_\phi)$ is primitive in $S_{p,\sigma'}$ and hence $j(f_\phi) = f_{\phi'}$. We have an isomorphism $S_{p,\phi'} \cong U_\phi(p) \oplus K_\phi^\vee(p)$ such that $f_{\phi'}$ and $j(z)$ form a basis of $U_\phi(p)$. Since there are no vectors $v \in U_\phi(p) \oplus K_\phi^\vee(p)$ with $\langle v, f_{\phi'} \rangle = 1$, the fibration ϕ' does not admit a section. Moreover, the lattice $\langle f_{\phi'} \rangle^\perp / \langle f_{\phi'} \rangle$ is isomorphic to $K_\phi^\vee(p)$. $\square$

The list of lattice equivalence classes of genus one fibrations on $X_{2,1}$ and $X_{3,1}$ have been obtained by Elkies and Schütt [ES12] and by Sengupta [Sen11], respectively. From their results, we obtain the following results on supersingular $K3$ surfaces with Artin invariant 10.

THEOREM 4.8. *There exist 18 lattice equivalence classes of genus one fibrations on $X_{2,10}$. The ADE-type $R_{[\phi']}$ of the reducible fibers of each $[\phi'] \in \mathbb{E}(X_{2,10})$ is given in the last column of Table 4.1.*

THEOREM 4.9. *There exist 52 lattice equivalence classes of genus one fibrations on $X_{3,10}$. The ADE-type $R_{[\phi']}$ of the reducible fibers of each $[\phi'] \in \mathbb{E}(X_{3,10})$ is given in the last column of Table 4.2.*

In Table 4.1 (resp. Table 4.2), the lists $\mathbb{E}(X_{2,1})$ and $\mathbb{E}(X_{2,10})$ (resp. $\mathbb{E}(X_{3,1})$ and $\mathbb{E}(X_{3,10})$) are presented. Two lattice equivalence classes in the same row are the pair of $[\phi] \in \mathbb{E}(X_{p,1})$ and its corresponding partner $[\phi'] \in \mathbb{E}(X_{p,10})$. The ADE-type $R_{[\phi]}$ of the reducible fibers of ϕ , the torsion MW_{tor} , and the rank of the Mordell–Weil group of ϕ are also given. In the column MW_{tor} , $[3, 6]$ denotes the finite abelian group $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$, for example. (Recall that ϕ is Jacobian for any $[\phi] \in \mathbb{E}(X_{p,1})$ by Elkies and Schütt [ES12].) The meaning of the entry R_N is explained in the proof of Theorems 4.8 and 4.9.

SUPERSINGULAR K3 SURFACES

No.	R_N	$\sigma = 1$			$\sigma = 10$
		$R_{[\phi]}$	MW_{tor}	$\text{rank}(\text{MW})$	$R_{[\phi']}$
1	$12A_2$	$10A_2$	$[3, 3, 3, 3]$	0	0
2	$8A_3$	$6A_3$	$[4, 4]$	2	0
3	$6A_4$	$2A_1 + 4A_4$	$[5]$	2	0
4	$6D_4$	$4D_4$	$[2, 2]$	4	0
5	$4A_5 + D_4$	$A_2 + 3A_5$	$[3]$	3	0
6	$4A_5 + D_4$	$3A_5 + D_4$	$[2, 6]$	1	A_1
7	$4A_5 + D_4$	$2A_2 + 2A_5 + D_4$	$[2]$	2	0
8	$4A_6$	$3A_6$	$[7]$	2	A_1
9	$4A_6$	$2A_3 + 2A_6$	$[1]$	2	0
10	$2A_7 + 2D_5$	$4A_1 + 2A_7$	$[2, 4]$	2	0
11	$2A_7 + 2D_5$	$A_1 + A_7 + 2D_5$	$[4]$	2	A_1
12	$2A_7 + 2D_5$	$2A_1 + A_4 + A_7 + D_5$	$[2]$	2	0
13	$2A_7 + 2D_5$	$2A_4 + 2D_5$	$[1]$	2	0
14	$3A_8$	$A_2 + 2A_8$	$[3]$	2	A_1
15	$3A_8$	$2A_5 + A_8$	$[1]$	2	0
16	$4D_6$	$3D_6$	$[2, 2]$	2	$2A_1$
17	$4D_6$	$2A_3 + 2D_6$	$[2, 2]$	2	0
18	$2A_9 + D_6$	$2A_9$	$[5]$	2	$2A_1$
19	$2A_9 + D_6$	$A_3 + A_9 + D_6$	$[2]$	2	A_1
20	$2A_9 + D_6$	$A_3 + A_6 + A_9$	$[1]$	2	0
21	$2A_9 + D_6$	$2A_6 + D_6$	$[1]$	2	0
22	$4E_6$	$A_2 + 3E_6$	$[3]$	0	A_2
23	$4E_6$	$4A_2 + 2E_6$	$[3, 3]$	0	0
24	$A_{11} + D_7 + E_6$	$A_2 + A_{11} + D_7$	$[4]$	0	A_2
25	$A_{11} + D_7 + E_6$	$A_{11} + E_6$	$[3]$	3	$2A_1$
26	$A_{11} + D_7 + E_6$	$2A_2 + A_{11} + D_4$	$[6]$	1	0
27	$A_{11} + D_7 + E_6$	$A_5 + D_7 + E_6$	$[1]$	2	A_1
28	$A_{11} + D_7 + E_6$	$2A_2 + A_8 + D_7$	$[1]$	1	0
29	$A_{11} + D_7 + E_6$	$A_8 + D_4 + E_6$	$[1]$	2	0
30	$2A_{12}$	$A_6 + A_{12}$	$[1]$	2	A_1
31	$2A_{12}$	$2A_9$	$[1]$	2	0
32	$3D_8$	$2A_1 + 2D_8$	$[2, 2]$	2	$2A_1$
33	$3D_8$	$2D_5 + D_8$	$[2]$	2	0
34	$A_{15} + D_9$	$A_3 + A_{15}$	$[4]$	2	$2A_1$
35	$A_{15} + D_9$	$A_9 + D_9$	$[1]$	2	A_1
36	$A_{15} + D_9$	$A_{12} + D_6$	$[1]$	2	0
37	$A_{17} + E_7$	$A_2 + A_{17}$	$[3]$	1	$A_1 + A_2$
38	$A_{17} + E_7$	$A_{11} + E_7$	$[1]$	2	A_1
39	$A_{17} + E_7$	$A_5 + A_{14}$	$[1]$	1	0
40	$D_{10} + 2E_7$	$A_2 + D_{10} + E_7$	$[2]$	1	$A_1 + A_2$
41	$D_{10} + 2E_7$	$2A_5 + D_{10}$	$[2, 2]$	0	0
42	$D_{10} + 2E_7$	$D_4 + 2E_7$	$[2]$	2	$2A_1$
43	$D_{10} + 2E_7$	$A_5 + D_7 + E_7$	$[2]$	1	0
44	$2D_{12}$	$D_6 + D_{12}$	$[2]$	2	$2A_1$
45	$2D_{12}$	$2D_9$	$[1]$	2	0
46	$3E_8$	$2A_2 + 2E_8$	$[1]$	0	$2A_2$
47	$3E_8$	$2E_6 + E_8$	$[1]$	0	0
48	$D_{16} + E_8$	$2A_2 + D_{16}$	$[2]$	0	$2A_2$
49	$D_{16} + E_8$	$D_{10} + E_8$	$[1]$	2	$2A_1$
50	$D_{16} + E_8$	$D_{13} + E_6$	$[1]$	1	0
51	A_{24}	A_{18}	$[1]$	2	A_1
52	D_{24}	D_{18}	$[1]$	2	$2A_1$

 TABLE 4.2. Genus one fibrations on $X_{3,1}$ and $X_{3,10}$

Proof of Theorems 4.8 and 4.9. By Theorem 4.7, it is enough to calculate the ADE -type of $\mathcal{R}(K_\phi^\vee(p))$ for $p = 2, 3$ and $[\phi] \in \mathbb{E}(X_{p,1})$. The lattices K_ϕ are calculated in Elkies and Schütt [ES12] and Sengupta [Sen11]. We put

$$T := \text{the root lattice of type } \begin{cases} D_4 & \text{if } p = 2, \\ 2A_2 & \text{if } p = 3. \end{cases}$$

Then, for each $[\phi] \in \mathbb{E}(X_{p,1})$, there exist a Niemeier lattice N_ϕ and a primitive embedding of T into N_ϕ such that K_ϕ is isomorphic to the orthogonal complement of T in N_ϕ . The entry R_N in Tables 4.1 and 4.2 indicates the ADE -type of $\mathcal{R}(N_\phi)$. From a Gram matrix of K_ϕ , we can calculate the ADE -type of $\mathcal{R}(K_\phi^\vee(p))$ by the algorithm described in [Shi04c, Section 4] or [Shi14, Section 3]. $\square$

COROLLARY 4.10. *There exist no quasi-elliptic fibrations on $X_{3,10}$.*

Remark 4.11. Rudakov and Shafarevich [RS81, Section 5] showed that there exists a quasi-elliptic fibration on $X_{2,\sigma}$ for any σ . The quasi-elliptic fibration on $X_{2,10}$ (no. 18 in Table 4.1) was discovered by Rudakov and Shafarevich [RS78, Section 4].

5. Chamber decomposition of a positive cone

Let S be an even hyperbolic lattice, and let $\mathcal{P}_S \subset S \otimes \mathbb{R}$ be a positive cone. In this section, we review a general method to find a set of generators of a subgroup of $O^+(S)$ by means of a chamber decomposition of $\mathcal{P}_S$, which was developed by Vinberg [Vin75], Conway [Con83], and Borcherds [Bor87].

Any real hyperplane in $\mathcal{P}_S$ can be written in the form $(v)^\perp$ for some vector $v \in S \otimes \mathbb{R}$ with negative norm. We denote by $\mathcal{H}_S$ the set of real hyperplanes in $\mathcal{P}_S$, which is canonically identified with

$$\{ v \in S \otimes \mathbb{R} \mid v^2 < 0 \} / \mathbb{R}^\times.$$

For a subset V of $\{v \in S \otimes \mathbb{R} \mid v^2 < 0\}$, we denote by $V^* \subset \mathcal{H}_S$ the image of V by $v \mapsto (v)^\perp$. A closed subset D of $\mathcal{P}_S$ is called a *chamber* if the interior D° of D is nonempty and there exists a set Δ_D of vectors $v \in S \otimes \mathbb{R}$ with $v^2 < 0$ such that

$$D = \{ x \in \mathcal{P}_S \mid \langle x, v \rangle \geq 0 \text{ for all } v \in \Delta_D \}.$$

A hyperplane $(v)^\perp$ of $\mathcal{P}_S$ is called a *wall* of D if $D^\circ \cap (v)^\perp = \emptyset$ and $D \cap (v)^\perp$ contains an open subset of $(v)^\perp$. When D is a chamber, we always assume that the set Δ_D is minimal in the sense that, for any $v \in \Delta_D$, there exists a point $x \in \mathcal{P}_S$ such that $\langle x, v \rangle < 0$ and $\langle x, v' \rangle \geq 0$ for any $v' \in \Delta_D \setminus \{v\}$, that is, the projection $\Delta_D \rightarrow \Delta_D^*$ is bijective and every hyperplane $(v)^\perp \in \Delta_D^*$ is a wall of D .

For a chamber D , we put

$$\text{Aut}(D) := \{ g \in O^+(S) \mid D^g = D \}.$$

A chamber D is said to be *fundamental* if the following hold:

- (i) $\mathcal{P}_S$ is the union of all D^g , where g runs through $O^+(S)$, and
- (ii) if $D^\circ \cap D^g \neq \emptyset$, then $g \in \text{Aut}(D)$.

Let $\mathcal{F}$ be a family of hyperplanes in $\mathcal{P}_S$ with the following properties:

- (a) $\mathcal{F}$ is locally finite in $\mathcal{P}_S$, and

(b) $\mathcal{F}$ is invariant under the action of $O^+(S)$ on $\mathcal{H}_S$.

Then the closure of each connected component of

$$\mathcal{P}_S \setminus \bigcup_{\mathcal{F}} (v)^\perp$$

is a chamber, which we call an $\mathcal{F}$ -chamber.

Suppose that D is an $\mathcal{F}$ -chamber. Then D^g is also an $\mathcal{F}$ -chamber for any $g \in O^+(S)$ by property (b) of $\mathcal{F}$, and D satisfies property (ii) in the definition of fundamental chambers. Moreover, D satisfies property (i) if and only if every $\mathcal{F}$ -chamber is equal to D^g for some $g \in O^+(S)$.

For each wall $(v)^\perp \in \Delta_D^*$ of an $\mathcal{F}$ -chamber D , there exists a unique $\mathcal{F}$ -chamber D' distinct from D such that $D \cap D' \cap (v)^\perp$ contains an open subset of $(v)^\perp$. We say that D' is *adjacent to D along $(v)^\perp$* , and that $(v)^\perp$ is the *wall between the adjacent chambers D and D'* .

PROPOSITION 5.1. *An $\mathcal{F}$ -chamber D is fundamental if and only if, for each $v \in \Delta_D$, there exists an element $g_v \in O^+(S)$ such that D^{g_v} is adjacent to D along $(v)^\perp$.*

Proof. The ‘only if’ part is obvious. We prove the ‘if’ part. It is enough to show that, for an arbitrary $\mathcal{F}$ -chamber D' , there exists an element $g \in O^+(S)$ such that $D' = D^g$. Since the family $\mathcal{F}$ of hyperplanes is locally finite in $\mathcal{P}_S$, there exists a finite chain of $\mathcal{F}$ -chambers $D_0 = D, D_1, \dots, D_N = D'$ such that D_i and D_{i+1} are adjacent. We show, by induction on N , that there exists a sequence of vectors $v_1, \dots, v_N$ in Δ_D such that $D_i = D^{g_{v_i} \cdots g_{v_1}}$ holds for $i = 1, \dots, N$. The case $N = 0$ is trivial. Suppose that $N > 0$. Let $(w)^\perp$ be the wall between D_{N-1} and D_N , and let $v_N \in \Delta_D$ be the vector such that the wall $(v_N)^\perp$ of D is mapped to the wall $(w)^\perp$ of D_{N-1} by $g_{v_{N-1}} \cdots g_{v_1}$. Then we have $D_N = D^{g_{v_N} \cdots g_{v_1}}$. $\square$

Remark 5.2. If an $\mathcal{F}$ -chamber is fundamental, then any $\mathcal{F}$ -chamber is fundamental.

Let $\mathcal{G}$ be a subset of $\mathcal{F}$ that is invariant under the action of $O^+(S)$. Then $\mathcal{G}$ is locally finite, and any $\mathcal{G}$ -chamber is a union of $\mathcal{F}$ -chambers. If an $\mathcal{F}$ -chamber is fundamental, then any $\mathcal{G}$ -chamber is also fundamental.

PROPOSITION 5.3. *Let D be an $\mathcal{F}$ -chamber and let C be a $\mathcal{G}$ -chamber such that $D \subset C$. Suppose that D is fundamental. For $v \in \Delta_D$, let $g_v \in O^+(S)$ be an isometry such that D^{g_v} is adjacent to D along $(v)^\perp$. We put*

$$\Gamma := \{ g_v \mid v \in \Delta_D, (v)^\perp \notin \mathcal{G} \}.$$

Then $\text{Aut}(C)$ is generated by $\text{Aut}(D)$ and Γ .

Proof. If $g_v \in \Gamma$, then D^{g_v} is contained in C because the wall $(v)^\perp$ between D and D^{g_v} does not belong to $\mathcal{G}$, and hence $g_v \in \text{Aut}(C)$. Therefore the subgroup $\langle \text{Aut}(D), \Gamma \rangle$ of $O^+(S)$ generated by $\text{Aut}(D)$ and Γ is contained in $\text{Aut}(C)$. To prove that $\text{Aut}(C) \subset \langle \text{Aut}(D), \Gamma \rangle$, it is enough to show that, for any $g \in \text{Aut}(C)$, there exists a sequence $\gamma_1, \dots, \gamma_N$ of elements of Γ such that $D^g = D^{\gamma_N \cdots \gamma_1}$. There exists a sequence of $\mathcal{F}$ -chambers $D_0 = D, D_1, \dots, D_N = D^g$ such that each D_i is contained in C and that D_{i+1} is adjacent to D_i for $i = 0, \dots, N-1$. Suppose that we have constructed $\gamma_1, \dots, \gamma_i \in \Gamma$ such that $D_i = D^{\gamma_i \cdots \gamma_1}$ holds. The wall $(w)^\perp$ between D_i and D_{i+1} does not belong to $\mathcal{G}$. Let v_{i+1} be an element of Δ_D such that the wall $(v_{i+1})^\perp$ of D is mapped to the wall $(w)^\perp$ of D_i by $\gamma_i \cdots \gamma_1$. Since $\mathcal{G}$ is invariant under the action of $O^+(S)$, we have $(v_{i+1})^\perp \notin \mathcal{G}$ and hence $\gamma_{i+1} := g_{v_{i+1}}$ is an element of Γ . Then $D_{i+1} = D^{\gamma_{i+1} \gamma_i \cdots \gamma_1}$ holds. $\square$

Remark 5.4. Let D and C be as in Proposition 5.3. Let v and v' be elements of Δ_D . Suppose that the wall $(v)^\perp$ of D is mapped to the wall $(v')^\perp$ of D by $h \in \text{Aut}(D)$. Then $D^{hg_v'h^{-1}}$ is adjacent to D along $(v)^\perp$. Let Δ'_D be a subset of Δ_D such that the subset Δ'^*_D of Δ^*_D is a complete set of representatives of the orbit decomposition of Δ^*_D by the action of $\text{Aut}(D)$. Then $\text{Aut}(C)$ is generated by $\text{Aut}(D)$ and $\{g_v \mid v \in \Delta'_D, (v)^\perp \notin \mathcal{G}\}$.

Considering the case $\mathcal{G} = \emptyset$, we obtain the following result.

COROLLARY 5.5. *Let D be an $\mathcal{F}$ -chamber. If D is fundamental, then $\text{O}^+(S)$ is generated by $\text{Aut}(D)$ and the isometries g_v that map D to its adjacent chambers.*

EXAMPLE 5.6. Recall that $W(S) \subset \text{O}^+(S)$ is the subgroup generated by $\{s_r \mid r \in \mathcal{R}(S)\}$. Any $\mathcal{R}(S)^*$ -chamber is fundamental, because every $r \in \mathcal{R}(S)$ defines a reflection s_r . It follows that $\text{O}^+(S)$ is equal to the semi-direct product of $W(S)$ and the automorphism group $\text{Aut}(D)$ of an $\mathcal{R}(S)^*$ -chamber D . In particular, we have

$$\text{Aut}(D) \cong \text{O}^+(S)/W(S).$$

Let L be an even *unimodular* hyperbolic lattice, and let $\iota : S \hookrightarrow L$ be a primitive embedding. Let $\mathcal{P}_L$ be the positive cone of L that contains $\iota(\mathcal{P}_S)$. We put

$$T_\iota := (\iota(S))^\perp \subset L$$

and denote by

$$v \mapsto v_S$$

the orthogonal projection $L \otimes \mathbb{R} \rightarrow S \otimes \mathbb{R}$. Since L is a submodule of $S^\vee \oplus T_\iota^\vee$, the image of L by $v \mapsto v_S$ is contained in $S^\vee$. We assume the following:

$$\text{the natural homomorphism } \text{O}(T_\iota) \rightarrow \text{O}(q_{T_\iota}) \text{ is surjective.} \quad (5.1)$$

Then we have the following result.

PROPOSITION 5.7. *For any $g \in \text{O}^+(S)$, there exists an element $\tilde{g} \in \text{O}^+(L)$ such that $\iota(v^g) = \iota(v)^{\tilde{g}}$ holds for any $v \in S \otimes \mathbb{R}$.*

Proof. See Nikulin [Nik79, Proposition 1.6.1]. □

A hyperplane $(r)^\perp$ of $\mathcal{P}_L$ defined by a (-2) -vector $r \in \mathcal{R}(L)$ intersects $\iota(\mathcal{P}_S)$ if and only if $r_S^2 < 0$. We put

$$\mathcal{R}(L, \iota) := \{ r_S \mid r \in \mathcal{R}(L) \text{ and } r_S^2 < 0 \} \subset S^\vee.$$

Since T_ι is negative definite, we have $-2 \leq r_S^2$ for any $r \in \mathcal{R}(L)$. Since $S^\vee$ is discrete in $S \otimes \mathbb{R}$, the family of hyperplanes $\mathcal{R}(L, \iota)^*$ is locally finite in $\mathcal{P}_S$. By Proposition 5.7, if $r \in \mathcal{R}(L)$ satisfies $r_S \in \mathcal{R}(L, \iota)$, then, for any $g \in \text{O}^+(S)$, we have $r_S^g = (r^{\tilde{g}})_S \in \mathcal{R}(L, \iota)$. Therefore $\mathcal{R}(L, \iota)$ is invariant under the action of $\text{O}^+(S)$. Note that $\mathcal{R}(L, \iota)$ contains $\mathcal{R}(S)$, and that $\mathcal{R}(S)$ is obviously invariant under the action of $\text{O}^+(S)$. Therefore, by Proposition 5.3, we can obtain a set of generators of the automorphism group $\text{Aut}(C)$ of an $\mathcal{R}(S)^*$ -chamber C if we find an $\mathcal{R}(L, \iota)^*$ -chamber D contained in C , show that D is fundamental, calculate the group $\text{Aut}(D)$, and find isometries of S that map D to its adjacent chambers.

Let L_{26} denote an even hyperbolic unimodular lattice of rank 26, which is unique up to isomorphisms. The walls of an $\mathcal{R}(L_{26})^*$ -chamber $\mathcal{D} \subset L_{26} \otimes \mathbb{R}$ and the group $\text{Aut}(\mathcal{D}) \subset \text{O}^+(L_{26})$ were determined by Conway [Con83]. Then Borchers [Bor87] determined the structure of $\text{O}^+(S)$

for some even hyperbolic lattices S of rank < 26 by embedding S into L_{26} in such a way that T_i is a root lattice.

Kondo [Kon98] first applied the Borcherds method to the study of the automorphism group of a generic Jacobian Kummer surface. Later Dolgachev and Kondo [DK03] applied the Borcherds method to the study of the automorphism group of $X_{2,1}$, and Kondo and Shimada [KS12] applied it to $X_{3,1}$.

We say that an even hyperbolic lattice S is *2-reflective* if the index of $W(S)$ in $O^+(S)$ is finite, or equivalently, if the automorphism group of an $\mathcal{R}(S)^*$ -chamber is finite (see Example 5.6).

Nikulin [Nik81] classified all 2-reflective lattices of rank ≥ 5 . It turns out that there are no 2-reflective lattices of rank > 19 .

Let Y be a K3 surface with Néron–Severi lattice S_Y and positive cone $\mathcal{P}(Y)$ containing an ample class. Then the closed subset $\text{Nef}^\circ(Y) = \text{Nef}(Y) \cap \mathcal{P}(Y)$ of $\mathcal{P}(Y)$ is an $\mathcal{R}(S_Y)^*$ -chamber by Proposition 2.1(1), and hence we have

$$\text{Aut}(\text{Nef}(Y)) = \text{Aut}(\text{Nef}^\circ(Y)) \cong O^+(S_Y)/W(S_Y).$$

Therefore $\text{Aut}(\text{Nef}(X_{p,\sigma}))$ is infinite for any supersingular K3 surface $X_{p,\sigma}$.

6. The groups $\text{Aut}(\text{Nef}(X_{2,10}))$ and $\text{Aut}(\text{Nef}(X_{3,10}))$

6.1 The group $\text{Aut}(\text{Nef}(X_{2,10}))$

By Lemma 1.1, the result of Dolgachev and Kondo [DK03], and the method of the previous section, we obtain a set of generators of $\text{Aut}(\text{Nef}(X_{2,10}))$. First we recall the results of [DK03]. As a projective model of $X_{2,1}$, we consider the minimal resolution X of the inseparable double cover $Y \rightarrow \mathbb{P}^2$ of $\mathbb{P}^2$ defined by

$$w^2 = x_0x_1x_2(x_0^3 + x_1^3 + x_2^3).$$

Note that the projective plane $\mathbb{P}^2(\mathbb{F}_4)$ defined over $\mathbb{F}_4$ contains 21 points $p_1, \dots, p_{21}$ and 21 lines $\ell_1, \dots, \ell_{21}$. The inseparable double cover Y has 21 ordinary nodes over the 21 points in $\mathbb{P}^2(\mathbb{F}_4)$ and hence X has 21 disjoint (-2) -curves. We denote by $e_1, \dots, e_{21} \in S_{2,1}$ the classes of these (-2) -curves, by $h \in S_{2,1}$ the class of the pullback of a line on $\mathbb{P}^2$, and by $f_1, \dots, f_{21} \in S_{2,1}$ the classes of the proper transforms of the 21 lines in $\mathbb{P}^2(\mathbb{F}_4)$. Then $S_{2,1}$ is generated by the (-2) -vectors $e_1, \dots, e_{21}, f_1, \dots, f_{21}$. The vector

$$w_M := \frac{1}{3} \sum_{i=1}^{21} (e_i + f_i)$$

has the properties

$$w_M \in S_X, \quad w_M^2 = 14, \quad \langle w_M, e_i \rangle = \langle w_M, f_i \rangle = 1.$$

The complete linear system associated with the line bundle corresponding to w_M defines an embedding of X into $\mathbb{P}^2 \times \mathbb{P}^2$, and its image $X_M \subset \mathbb{P}^2 \times \mathbb{P}^2$ is defined by

$$\begin{cases} x_0y_0^2 + x_1y_1^2 + x_2y_2^2 &= 0, \\ x_0^2y_0 + x_1^2y_1 + x_2^2y_2 &= 0. \end{cases}$$

Six points on $\mathbb{P}^2(\mathbb{F}_4)$ are said to be *in general position* if no three points of them are collinear. There exist 168 sets of six points in general position in $\mathbb{P}^2(\mathbb{F}_4)$. If $I = \{p_{i_1}, \dots, p_{i_6}\}$ is a set of six

points in general position, then the (-1) -vector

$$c_I := h - \frac{1}{2}(e_{i_1} + \cdots + e_{i_6})$$

is contained in $S_{2,1}^\vee$. Note that each c_I defines a reflection

$$x \mapsto x + 2\langle x, c_I \rangle c_I$$

in $O^+(S_{2,1})$ because $c_I \in S_{2,1}^\vee$. Let $\mathcal{P}(X_{2,1})$ be the positive cone of $S_{2,1}$ containing an ample class, and let $\Delta(X_{2,1})$ be the set consisting of $e_1, \dots, e_{21}, f_1, \dots, f_{21}$ and the (-1) -vectors c_I defined above. We define a chamber $D(X_{2,1})$ in $P(X_{2,1})$ by

$$D(X_{2,1}) := \{ x \in P(X_{2,1}) \mid \langle x, v \rangle \geq 0 \text{ for all } v \in \Delta(X_{2,1}) \}.$$

Then, for each $v \in \Delta(X_{2,1})$, the hyperplane $(v)^\perp$ is a wall of $D(X_{2,1})$. Moreover, the ample class w_M is contained in the interior of $D(X_{2,1})$. Recall that L_{26} is the even unimodular hyperbolic lattice of rank 26. There exists a primitive embedding $\iota : S_{2,1} \hookrightarrow L_{26}$, which is unique up to $O(L_{26})$. The orthogonal complement T_ι of $S_{2,1}$ in L_{26} is isomorphic to the root lattice of type D_4 . We can easily check by direct computation that T_ι satisfies hypothesis (5.1).

PROPOSITION 6.1. *The chamber $D(X_{2,1})$ is an $\mathcal{R}(L_{26}, \iota)^*$ -chamber contained in the $\mathcal{R}(S_{2,1})^*$ -chamber $\text{Nef}^\circ(X_{2,1})$. An isometry $g \in O^+(S_{2,1})$ belongs to $\text{Aut}(D(X_{2,1}))$ if and only if $w_M^g = w_M$.*

Thus we can apply Proposition 5.3 to the pair of chambers $D(X_{2,1})$ and $\text{Nef}^\circ(X_{2,1})$ for the study of $\text{Aut}(\text{Nef}(X_{2,1}))$ and $\text{Aut}(X_{2,1})$.

We have the following elements in $\text{Aut}(X_{2,1})$ and $O^+(S_{2,1})$. Since $\text{Aut}(X_{2,1})$ is naturally embedded in $O^+(S_{2,1})$, we use the same letter to denote an element of $\text{Aut}(X_{2,1})$ and its image in $O^+(S_{2,1})$.

- The action of $\text{PGL}(3, \mathbb{F}_4)$ on $\mathbb{P}^2$ induces automorphisms of the inseparable double cover Y of $\mathbb{P}^2$, and hence automorphisms of $X_{2,1}$. Their action on $S_{2,1}$ preserves $D(X_{2,1})$.
- The interchange of the two factors of $\mathbb{P}^2 \times \mathbb{P}^2$ preserves $X_M \subset \mathbb{P}^2 \times \mathbb{P}^2$, and hence induces an involution $\text{sw} \in \text{Aut}(X_{2,1})$, which we call the *switch*. Its action on $S_{2,1}$ preserves $D(X_{2,1})$.
- For each set I of six points in general position in $\mathbb{P}^2(\mathbb{F}_4)$, the linear system of plane curves of degree five that pass through the points of I and are singular at each point of I defines a birational involution of $\mathbb{P}^2$, and this involution lifts to an involution of Y . Hence we obtain an involution $\text{Cr}_I \in \text{Aut}(X_{2,1})$, which we call a *Cremona automorphism* of $X_{2,1}$. The action of Cr_I on $S_{2,1}$ is the reflection with respect to $c_I \in S_{2,1}^\vee$.
- The Frobenius action of $\text{Gal}(\mathbb{F}_4/\mathbb{F}_2)$ on X_M induces an isometry Fr of $S_{2,1}$ that preserves $D(X_{2,1})$.
- The (-2) -vectors e_i and f_i define reflections s_{e_i} and s_{f_i} .

It now follows from Proposition 5.1 that the chamber $D(X_{2,1})$ is fundamental.

THEOREM 6.2 [DK03]. (1) *The projective automorphism group $\text{Aut}(X_{2,1}, w_M)$ of $X_M \subset \mathbb{P}^2 \times \mathbb{P}^2$ is generated by $\text{PGL}(3, \mathbb{F}_4)$ and the switch sw .*

(2) *The group $\text{Aut}(D(X_{2,1}))$ is generated by $\text{Aut}(X_{2,1}, w_M)$ and Fr .*

(3) *The automorphism group $\text{Aut}(X_{2,1})$ is generated by $\text{Aut}(X_{2,1}, w_M)$ and the 168 Cremona automorphisms Cr_I .*

(4) *The group $\text{Aut}(\text{Nef}(X_{2,1}))$ is generated by $\text{Aut}(X_{2,1})$ and Fr .*

(5) The group $O^+(S_{2,1})$ is generated by $\text{Aut}(\text{Nef}(X_{2,1}))$ and the 21 + 21 reflections s_{e_i} and s_{f_i} .

We then study $\text{Aut}(\text{Nef}(X_{2,10}))$. By Corollary 3.2, we have an embedding

$$j : S_{2,1} \hookrightarrow S_{2,10}$$

that induces $S_{2,1}^\vee(2) \cong S_{2,10}$. Composing j with some element of $W(S_{2,10}) \times \{\pm 1\}$, we can assume that $j(w_M)$ is contained in $\text{Nef}(X_{2,10})$ (Proposition 2.1(2)). The isomorphism $j_* : O^+(S_{2,1}) \xrightarrow{\sim} O^+(S_{2,10})$ induced by j is denoted by

$$g \mapsto g'.$$

The $j(\mathcal{R}(L_{26}, \iota))^*$ -chamber $j(D(X_{2,1}))$ is fundamental, and we have

$$\text{Aut}(j(D(X_{2,1}))) = \text{Aut}(D(X_{2,1}))'.$$

LEMMA 6.3. *The set $j(\mathcal{R}(L_{26}, \iota))$ contains $\mathcal{R}(S_{2,10})$. Hence the $j(\mathcal{R}(L_{26}, \iota))^*$ -chamber $j(D(X_{2,1}))$ is contained in the $\mathcal{R}(S_{2,10})^*$ -chamber $\text{Nef}^\circ(X_{2,10})$.*

Proof. It is enough to show that, if $v \in S_{2,1}^\vee$ satisfies $v^2 = -1$, then $v \in \mathcal{R}(L_{26}, \iota)$, that is, there exists a vector $u \in T_\iota^\vee$ such that $u^2 = -1$ and that $u + v$ is contained in the submodule L_{26} of $S_{2,1}^\vee \oplus T_\iota^\vee$. By Nikulin [Nik79, Proposition 1.4.1], the submodule $L_{26}/(S_{2,1} \oplus T_\iota)$ of $(S_{2,1}^\vee \oplus T_\iota^\vee)/(S_{2,1} \oplus T_\iota) = A_{S_{2,1}} \oplus A_{T_\iota}$ is the graph of an isomorphism

$$q_{S_{2,1}} \cong -q_{T_\iota}.$$

Hence it is enough to show that, for any $\bar{u} \in A_{T_\iota}$ with $q_{T_\iota}(\bar{u}) = 1$, there exists a vector $u \in T_\iota^\vee$ such that $u^2 = -1$ and $u \bmod T_\iota = \bar{u}$. Since T_ι is a root lattice of type D_4 , we can confirm this fact by direct computation. The set of (-1) -vectors in $T_\iota^\vee$ consists of 24 vectors, and its image by the natural projection $T_\iota^\vee \rightarrow A_{T_\iota}$ is the set of all nonzero elements of $A_{T_\iota} \cong \mathbb{F}_2^2$. $\square$

The set of walls of $j(D(X_{2,1}))$ is equal to

$$\begin{aligned} & \{(j(e_i))^\perp \mid i = 1, \dots, 21\} \cup \{(j(f_i))^\perp \mid i = 1, \dots, 21\} \cup \\ & \{(j(c_I))^\perp \mid I \text{ is a set of six points in general position}\}. \end{aligned}$$

Note that the 21 + 21 vectors $j(e_i)$ and $j(f_i)$ are of norm -4 and the 168 vectors $j(c_I)$ are of norm -2 . Note also that neither $(j(e_i))^\perp$ nor $(j(f_i))^\perp$ is contained in $\mathcal{R}(S_{2,10})^*$, because there are no rational numbers λ such that $(-4)\lambda^2 = -2$. By Proposition 5.3, Theorem 6.2, and Lemma 6.3, we obtain the following result.

THEOREM 6.4. *The group $\text{Aut}(\text{Nef}(X_{2,10}))$ is generated by $\text{PGL}(3, \mathbb{F}_4)'$, sw' , Fr' , s'_{e_i} , and s'_{f_i} .*

6.2 The group $\text{Aut}(\text{Nef}(X_{3,10}))$

By the same argument as above, we obtain a set of generators of $\text{Aut}(\text{Nef}(X_{3,10}))$ from the result of Kondo and Shimada [KS12].

We consider the Fermat quartic surface

$$X_{\text{FQ}} : x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$$

in characteristic 3. Then X_{FQ} is isomorphic to $X_{3,1}$. The surface X_{FQ} contains 112 lines, and their classes $l_1, \dots, l_{112}$ span $S_{3,1}$. We denote by $h_{\text{FQ}} \in S_{3,1}$ the class of a hyperplane section of X_{FQ} .

There exists a primitive embedding $\iota : S_{3,1} \hookrightarrow L_{26}$, which is unique up to $O(L_{26})$. The orthogonal complement T_ι is isomorphic to the root lattice of type $2A_2$. We can easily check

by direct computation that T_ι satisfies hypothesis (5.1). We calculated an $\mathcal{R}(L_{26}, \iota)^*$ -chamber $D(X_{3,1})$ that contains h_{FQ} in its interior, and found

$$648 \text{ vectors } u_j \in S_{3,1}^\vee \text{ of norm } -4/3 \quad \text{and} \quad 5184 \text{ vectors } w_k \in S_{3,1}^\vee \text{ of norm } -2/3$$

such that the walls of $D(X_{3,1})$ consist of the 112 hyperplanes $(l_i)^\perp$, the 648 hyperplanes $(u_j)^\perp$, and the 5184 hyperplanes $(w_k)^\perp$. Note that the $\mathcal{R}(L_{26}, \iota)^*$ -chamber $D(X_{3,1})$ is contained in the $\mathcal{R}(S_{3,1})^*$ -chamber $\text{Nef}^\circ(X_{3,1})$, because $h_{\text{FQ}} \in D(X_{3,1})^\circ$. Moreover, since $28 h_{\text{FQ}} = \sum l_i$, the following holds.

PROPOSITION 6.5. *An isometry $g \in O^+(S_{3,1})$ belongs to $\text{Aut}(D(X_{3,1}))$ if and only if $h_{\text{FQ}}^g = h_{\text{FQ}}$.*

We have the following elements in $\text{Aut}(X_{3,1})$ and $O^+(S_{3,1})$. Note that, for a polarization $h \in S_{3,1}$ of degree two, we have the deck transformation $\tau(h) \in \text{Aut}(X_{3,1})$ of the generically finite morphism $X_{3,1} \rightarrow \mathbb{P}^2$ of degree two induced by the complete linear system associated with h .

- The subgroup $\text{PGU}(4, \mathbb{F}_9)$ of $\text{PGL}(4, k) = \text{Aut}(\mathbb{P}^3)$ acts on X_{FQ} . Its action on $S_{3,1}$ preserves $D(X_{3,1})$. Moreover, the action of $\text{PGU}(4, \mathbb{F}_9)$ on $S_{3,1}^\vee$ is transitive on each of the set of 112 vectors l_i , the set of 648 vectors u_j , and the set of 5184 vectors w_k .
- There exists a polarization $h_{648} \in S_{3,1}$ of degree two such that the deck transformation $\tau(h_{648}) \in \text{Aut}(X_{3,1})$ maps $D(X_{3,1})$ to an $\mathcal{R}(L_{26}, \iota)^*$ -chamber adjacent to $D(X_{3,1})$ along one of the 648 walls $(u_j)^\perp$.
- There exists a polarization $h_{5184} \in S_{3,1}$ of degree two such that the deck transformation $\tau(h_{5184}) \in \text{Aut}(X_{3,1})$ maps $D(X_{3,1})$ to an $\mathcal{R}(L_{26}, \iota)^*$ -chamber adjacent to $D(X_{3,1})$ along one of the 5184 walls $(w_k)^\perp$.
- The Frobenius action of $\text{Gal}(\mathbb{F}_9/\mathbb{F}_3)$ on X_{FQ} gives rise to an element $\text{Fr} \in \text{Aut}(D(X_{3,1}))$ of order two.
- We have the reflections s_{l_i} with respect to the classes l_i of the 112 lines on X_{FQ} .

Thus $D(X_{3,1})$ is fundamental, and hence we have the following result.

THEOREM 6.6 [KS12]. (1) *The projective automorphism group $\text{Aut}(X, h_{\text{FQ}})$ of the Fermat quartic surface $X_{\text{FQ}} \subset \mathbb{P}^3$ is equal to $\text{PGU}(4, \mathbb{F}_9)$.*

(2) *The group $\text{Aut}(D(X_{3,1}))$ is generated by $\text{Aut}(X, h_{\text{FQ}})$ and Fr .*

(3) *The automorphism group $\text{Aut}(X_{3,1})$ is generated by $\text{Aut}(X, h_{\text{FQ}})$ and the two involutions $\tau(h_{648})$ and $\tau(h_{5184})$.*

(4) *The group $\text{Aut}(\text{Nef}(X_{3,1}))$ is generated by $\text{Aut}(X_{3,1})$ and Fr .*

(5) *The group $O^+(S_{3,1})$ is generated by $\text{Aut}(\text{Nef}(X_{3,1}))$ and the 112 reflections s_{l_i} .*

By Corollary 3.2, we have an embedding

$$j : S_{3,1} \hookrightarrow S_{3,10}$$

that induces $S_{3,1}^\vee(3) \cong S_{3,10}$. By Proposition 2.1(2), we can assume that $j(h_{\text{FQ}})$ is contained in $\text{Nef}(X_{3,10})$. The isomorphism $j_* : O^+(S_{3,1}) \xrightarrow{\sim} O^+(S_{3,10})$ induced by j is denoted by $g \mapsto g'$. The $j(\mathcal{R}(L_{26}, \iota))^*$ -chamber $j(D(X_{3,1}))$ is fundamental, and the group $\text{Aut}(j(D(X_{3,1})))$ is equal to $\text{Aut}(D(X_{3,1}))'$.

LEMMA 6.7. *The set $j(\mathcal{R}(L_{26}, \iota))$ contains $\mathcal{R}(S_{3,10})$. Hence the $j(\mathcal{R}(L_{26}, \iota))^*$ -chamber $j(D(X_{3,1}))$ is contained in the $\mathcal{R}(S_{3,10})^*$ -chamber $\text{Nef}^\circ(X_{3,10})$.*

Proof. It is enough to show that, if $v \in S_{3,1}^\vee$ satisfies $v^2 = -2/3$, then there exists a vector $u \in T_\ell^\vee$ such that $u^2 = -4/3$ and that $u + v$ is contained in $L_{26} \subset S_{3,1}^\vee \oplus T_\ell^\vee$. For this, it suffices to prove that, for any $\bar{u} \in A_{T_\ell}$ with $q_{T_\ell}(\bar{u}) = -4/3$, there exists a vector $u \in T_\ell^\vee$ such that $u^2 = -4/3$ and $u \bmod T_\ell = \bar{u}$. Since T_ℓ is a root lattice of type $2A_2$, we can confirm this fact by direct computation. $\square$

The set of walls of $j(D(X_{3,1}))$ is equal to

$$\{(j(l_i))^\perp \mid i = 1, \dots, 112\} \cup \{(j(u_j))^\perp \mid j = 1, \dots, 648\} \cup \{(j(w_k))^\perp \mid k = 1, \dots, 5184\}.$$

Note that the vectors $j(l_i)$ are of norm -6 , the vectors $j(u_j)$ are of norm -4 , and the vectors $j(w_k)$ are of norm -2 . Note also that neither $(j(l_i))^\perp$ nor $(j(u_j))^\perp$ is contained in $\mathcal{R}(S_{3,10})^*$. By Proposition 5.3, Theorem 6.6, and Lemma 6.7, we obtain the following result.

THEOREM 6.8. *The group $\text{Aut}(\text{Nef}(X_{3,10}))$ is generated by $\text{PGU}(4, \mathbb{F}_9)'$, Fr' , s'_{l_i} , and $\tau(h_{648})'$.*

7. The Torelli theorem for supersingular K3 surfaces

We review the theory of period mapping and the Torelli theorem for supersingular K3 surfaces in odd characteristics by Ogus [Ogu79], [Ogu83]. Throughout this section, we assume that p is odd.

We summarize results on quadratic spaces over finite fields. Let $\mathbb{F}_q$ be a finite extension of $\mathbb{F}_p$. There exist exactly two isomorphism classes of nondegenerate quadratic forms in 2σ variables $x_1, \dots, x_{2\sigma}$ over $\mathbb{F}_q$. They are represented by

$$f_+ := x_1x_2 + \dots + x_{2\sigma-1}x_{2\sigma}, \quad \text{and} \quad (7.1)$$

$$f_- := x_1^2 + cx_1x_2 + x_2^2 + x_3x_4 + \dots + x_{2\sigma-1}x_{2\sigma}, \quad (7.2)$$

where c is an element of $\mathbb{F}_q$ such that $t^2 + ct + 1 \in \mathbb{F}_q[t]$ is irreducible. The quadratic form f_+ (resp. f_-) is called *neutral* (resp. *nonneutral*). We denote by $\text{O}(\mathbb{F}_q^{2\sigma}, f_\epsilon)$ the group of the self-isometries of the quadratic space $(\mathbb{F}_q^{2\sigma}, f_\epsilon)$.

Let N be an even hyperbolic p -elementary lattice of rank 22 with discriminant $-p^{2\sigma}$. We define a quadratic space (N_0, q_0) over $\mathbb{F}_p$ by (1.2). It is known that q_0 is nondegenerate and *nonneutral*. We denote by $\text{O}(N_0, q_0)$ the group of the self-isometries of (N_0, q_0) . Note that the only scalar multiplications in $\text{O}(N_0, q_0)$ are those by ± 1 .

Let k be a field of characteristic p . We put

$$\varphi := \text{id}_{N_0} \otimes F_k : N_0 \otimes k \rightarrow N_0 \otimes k,$$

where F_k is the Frobenius map of k .

DEFINITION 7.1. A subspace K of $N_0 \otimes k$ with $\dim K = \sigma$ is said to be a *characteristic subspace* of (N_0, q_0) if K is totally isotropic with respect to the quadratic form $q_0 \otimes k$ and $\dim(K \cap \varphi(K)) = \sigma - 1$ holds.

Suppose that k is algebraically closed. Let X be a supersingular K3 surface with Artin invariant σ defined over k . An isomorphism

$$\eta : N \xrightarrow{\sim} S_X$$

of lattices is called a *marking* of X . We fix a marking η of X . The composite of the marking η and the Chern class map $S_X \rightarrow H_{\text{DR}}^2(X/k)$ defines a linear homomorphism

$$\bar{\eta} : N \otimes k \rightarrow H_{\text{DR}}^2(X/k).$$

It is known that $\text{Ker } \bar{\eta}$ is contained in $N_0 \otimes k$ and is totally isotropic with respect to $q_0 \otimes k$. We put

$$K_{(X,\eta)} := \varphi^{-1}(\text{Ker } \bar{\eta}),$$

and call $K_{(X,\eta)}$ the *period* of the marked supersingular $K3$ surface (X, η) . Then it was proved by Ogus [Ogu79], [Ogu83] that $K_{(X,\eta)}$ is a characteristic subspace of (N_0, q_0) . We denote by $\eta^* : \text{O}(S_X) \xrightarrow{\sim} \text{O}(N)$ the isomorphism induced by the marking η , and let

$$\bar{\eta}^* : \text{O}(S_X) \rightarrow \text{O}(N_0, q_0)$$

be the composite of η^* with the natural homomorphism $\text{O}(N) \rightarrow \text{O}(N_0, q_0)$. As a corollary of the Torelli theorem by Ogus [Ogu83, Corollary of Theorem II''], we have the following result.

COROLLARY 7.2. *Let η be a marking of X . Then, as a subgroup of $\text{O}^+(S_X)$, the automorphism group $\text{Aut}(X)$ of X is equal to*

$$\{ g \in \text{Aut}(\text{Nef}(X)) \mid K_{(X,\eta)}^{\bar{\eta}^*(g)} = K_{(X,\eta)} \}.$$

In particular, the index of $\text{Aut}(X)$ in $\text{Aut}(\text{Nef}(X))$ is at most $|\text{O}(N_0, q_0)/\{\pm 1\}|$.

Since $\text{Aut}(\text{Nef}(X))$ is infinite, we obtain the following result.

COROLLARY 7.3. *The automorphism group $\text{Aut}(X)$ is infinite.*

DEFINITION 7.4. We say that a supersingular $K3$ surface X with Artin invariant σ is *generic* if there exists a marking η for X such that the subgroup

$$\{ \gamma \in \text{O}(N_0, q_0) \mid K_{(X,\eta)}^\gamma = K_{(X,\eta)} \} \quad (7.3)$$

of $\text{O}(N_0, q_0)$ consists of only the scalar multiplications by ± 1 .

If X is generic, then the subgroup (7.3) consists of only scalar multiplications for any marking η . The existence of generic supersingular $K3$ surfaces with Artin invariant > 1 (Theorem 1.7) is proved in the next section.

Recall that A_{S_X} is the discriminant group of S_X , and $q_{S_X} : A_{S_X} \rightarrow \mathbb{Q}/2\mathbb{Z}$ is the discriminant quadratic form. We will regard A_{S_X} as a 2σ -dimensional vector space over $\mathbb{F}_p$. Note that the image of q_{S_X} is contained in $(2/p)\mathbb{Z}/2\mathbb{Z}$. We define $\bar{q}_{S_X} : A_{S_X} \rightarrow \mathbb{F}_p$ by

$$\bar{q}_{S_X}(x \bmod S_X) := p \cdot q_{S_X}(x) \bmod p.$$

Then we obtain a quadratic space $(A_{S_X}, \bar{q}_{S_X})$ over $\mathbb{F}_p$. Note that we can recover q_{S_X} from $\bar{q}_{S_X}$. We have natural homomorphisms

$$\text{O}(S_X) \rightarrow \text{O}(q_{S_X}) \cong \text{O}(A_{S_X}, \bar{q}_{S_X}) \twoheadrightarrow \text{PO}(A_{S_X}, \bar{q}_{S_X}) := \text{O}(A_{S_X}, \bar{q}_{S_X})/\{\pm 1\}. \quad (7.4)$$

Let $\eta : N^\vee \xrightarrow{\sim} S_X^\vee$ be the isomorphism induced by a marking η . Then the map

$$px \bmod pN \in N_0 \mapsto \eta(x) \bmod S_X \in A_{S_X} \quad (x \in N^\vee)$$

induces an isomorphism of quadratic spaces from (N_0, q_0) to $(A_{S_X}, \bar{q}_{S_X})$. By Corollary 7.2, we obtain the following result.

COROLLARY 7.5. *Suppose that X is generic. Then $\text{Aut}(X)$ is equal to the kernel of the homomorphism*

$$\Phi : \text{Aut}(\text{Nef}(X)) \rightarrow \text{PO}(A_{S_X}, \bar{q}_{S_X})$$

obtained by restricting (7.4) to $\text{Aut}(\text{Nef}(X)) \subset \text{O}(S_X)$.

Remark 7.6. Suppose that $X_{3,10}$ is generic. By the standard algorithm of combinatorial group theory, we can obtain a finite set of generators of $\text{Aut}(X_{3,10})$ from the generators of $\text{Aut}(\text{Nef}(X_{3,10}))$ given in Theorem 6.8. However, a naive application of the algorithm would be inexecutable, because, when $p = 3$ and $\sigma = 10$, the order of $\text{O}(N_0, q_0)$ is

$$2^{36} \cdot 3^{90} \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 37 \cdot 41^2 \cdot 61 \cdot 73 \cdot 193 \cdot 547 \cdot 757 \cdot 1093 \cdot 1181,$$

which is about 7.886×10^{90} . In fact, we can show that there exists a genus one fibration ϕ' on $X_{3,10}$ whose lattice equivalence class contains at least 6531840 Aut-equivalence classes, by giving the class f_ϕ of a Jacobian fibration on $X_{3,1} = X_{\text{FQ}}$ explicitly.

8. Existence of generic supersingular K3 surfaces

We prove Theorem 1.7. For the proof, we recall the construction by Ogus [Ogu79] of the scheme $\mathcal{M}$ parameterizing characteristic subspaces of the 2σ -dimensional quadratic space (N_0, q_0) over $\mathbb{F}_p$. This scheme $\mathcal{M}$ plays the role of the period domain for supersingular K3 surfaces. We continue to assume that p is odd.

Let $\text{Grass}(\nu, N_0)$ denote the Grassmannian variety of ν -dimensional subspaces of N_0 , and let $\text{Isot}(\nu, q_0)$ be the subscheme of $\text{Grass}(\nu, N_0)$ parameterizing ν -dimensional totally isotropic subspaces of (N_0, q_0) . We put

$$\text{Gen} := \text{Isot}(\sigma, q_0),$$

where Gen is for “generatrix”. Note that $\text{Isot}(\nu, q_0)$ is defined over $\mathbb{F}_p$ for any ν . Let k be a field of characteristic p . For a subspace L of $N_0 \otimes k$ with dimension ν , we denote by $[L]$ the k -valued point of $\text{Grass}(\nu, N_0)$ corresponding to L . We then have the following.

- (1) If $\nu < \sigma$, then $\text{Isot}(\nu, q_0)$ is geometrically connected.
- (2) The scheme $\text{Gen} \otimes \mathbb{F}_{p^2}$ has two connected components Gen_+ and Gen_- , each of which is geometrically connected. Since q_0 is nonneutral, the action of $\text{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$ interchanges the two connected components.
- (3) Let K and K' be two σ -dimensional totally isotropic subspaces of $(N_0, q_0) \otimes k$. Suppose that $\dim(K \cap K') = \sigma - 1$. Then the k -valued points $[K]$ and $[K']$ belong to distinct connected components of Gen.
- (4) Suppose that k is algebraically closed. Then, for each k -valued point $[L]$ of the scheme $\text{Isot}(\sigma - 1, q_0)$, there exist exactly two σ -dimensional totally isotropic subspaces of $(N_0, q_0) \otimes k$ that contain L .
- (5) Let P be the subscheme of $\text{Gen} \times \text{Gen}$ parameterizing pairs (K, K') of linear subspaces such that $\dim(K \cap K') = \sigma - 1$. Then the scheme $P \otimes \mathbb{F}_{p^2}$ has two connected components, each of which is isomorphic to $\text{Isot}(\sigma - 1, q_0)$ over $\mathbb{F}_{p^2}$. The action of $\text{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$ interchanges the two connected components.

Consider the graph

$$\text{id} \times \varphi : \text{Gen} \rightarrow \text{Gen} \times \text{Gen}$$

of the Frobenius morphism $\text{Gen} \rightarrow \text{Gen}$ given by $K \mapsto \varphi(K)$. The subscheme $\mathcal{M}$ of Gen that parameterizes the characteristic subspaces of (N_0, q_0) is defined by the fiber product

$$\begin{array}{ccc} \mathcal{M} & \hookrightarrow & \text{Gen} \\ \downarrow & \square & \downarrow \text{id} \times \varphi \\ P & \hookrightarrow & \text{Gen} \times \text{Gen}. \end{array}$$

Ogus [Ogu79] proved the following result.

THEOREM 8.1. *The scheme $\mathcal{M}$ defined over $\mathbb{F}_p$ is smooth and projective of dimension $\sigma - 1$. The scheme $\mathcal{M} \otimes \mathbb{F}_{p^2}$ has two connected components $\mathcal{M}_+ = \mathcal{M} \cap \text{Gen}_+$ and $\mathcal{M}_- = \mathcal{M} \cap \text{Gen}_-$, each of which is geometrically connected. The action of $\text{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$ interchanges $\mathcal{M}_+$ and $\mathcal{M}_-$.*

Proof of Theorem 1.7. Let κ be an algebraic closure of the function field of the scheme $\mathcal{M}_+$ over $\mathbb{F}_{p^2}$, and let $[K_\kappa]$ be the geometric generic point of $\mathcal{M}_+$. By the surjectivity of the period mapping for supersingular $K3$ surfaces (Ogus [Ogu83, Theorem III']), there exist a supersingular $K3$ surface X of Artin invariant σ defined over κ and a marking $\eta : N \xrightarrow{\sim} S_X$ such that $K_{(X, \eta)} = K_\kappa$. We prove that this X is generic, that is,

$$G_\kappa := \{ \gamma \in \text{O}(N_0, q_0) \mid K_\kappa^\gamma = K_\kappa \}$$

is equal to $\{\pm 1\}$. Note that the closure of the point $[K_\kappa]$ coincides with $\mathcal{M}_+$. Therefore we have the following: If a field k contains $\mathbb{F}_{p^2}$, then the action of G_κ leaves K invariant for any k -valued point $[K]$ of $\mathcal{M}_+$.

Suppose that $\sigma \geq 3$. Let u be an arbitrary nonzero isotropic vector of N_0 . We will prove that u is an eigenvector of G_κ . Let

$$b_0 : N_0 \times N_0 \rightarrow \mathbb{F}_p$$

denote the symmetric bilinear form obtained from q_0 . There exists a vector $v \in N_0$ such that $q_0(v) = 0$ and $b_0(u, v) = 1$, and hence (N_0, q_0) has an orthogonal direct-sum decomposition

$$N_0 = U^\perp \oplus U,$$

where U is the subspace spanned by u and v . Repeating this procedure and noting that q_0 is nonneutral, we obtain a basis $a_1, \dots, a_{2\sigma}$ of N_0 with $u = a_{2\sigma}$ such that $q_0(x_1 a_1 + \dots + x_{2\sigma} a_{2\sigma})$ is equal to the quadratic polynomial f_- in (7.2). Let α and $\bar{\alpha} = \alpha^p$ be the roots in $\mathbb{F}_{p^2}$ of the irreducible polynomial $t^2 + ct + 1 \in \mathbb{F}_p[t]$. We consider the basis

$$\begin{aligned} b_1^{(-1)} &:= \alpha a_1 + a_2, & b_1^{(1)} &:= \bar{\alpha} a_1 + a_2, & \text{and} \\ b_i^{(-1)} &:= a_{2i-1}, & b_i^{(1)} &:= a_{2i} & (i = 2, \dots, \sigma) \end{aligned} \tag{8.1}$$

of $N_0 \otimes \mathbb{F}_{p^2}$. Note that each $b_i^{(\pm 1)}$ is isotropic, and that

$$b_0(b_i^{(\alpha)}, b_j^{(\beta)}) = 0 \quad \text{if } i \neq j, \quad b_0(b_i^{(1)}, b_i^{(-1)}) = \begin{cases} (4 - c^2)/2 & \text{if } i = 1, \\ 1/2 & \text{if } i \geq 2. \end{cases}$$

We put

$$\mathcal{E} := \{1, -1\}^\sigma.$$

For $e = (\varepsilon_1, \dots, \varepsilon_\sigma) \in \mathcal{E}$, we denote by K_e the linear subspace of $N_0 \otimes \mathbb{F}_{p^2}$ spanned by

$$b_1^{(\varepsilon_1)}, \dots, b_\sigma^{(\varepsilon_\sigma)}.$$

It is obvious that K_e is isotropic. Moreover, since

$$\varphi(b_1^{(\varepsilon)}) = b_1^{(-\varepsilon)} \quad \text{and} \quad \varphi(b_i^{(\varepsilon)}) = b_i^{(\varepsilon)} \quad \text{if } i \geq 2,$$

we have $\dim(K_e \cap \varphi(K_e)) = \sigma - 1$. Therefore K_e is a characteristic subspace of (N_0, q_0) . Suppose that e and $e' \in \mathcal{E}$ differ only at one component. Then we have $\dim(K_e \cap K_{e'}) = \sigma - 1$, and hence the $\mathbb{F}_{p^2}$ -valued points $[K_e]$ and $[K_{e'}]$ of $\mathcal{M}$ belong to distinct connected components. We put

$$\mathcal{E}_+ := \{ e \in \mathcal{E} \mid \text{the number of } -1 \text{ in } e \text{ is even} \}, \quad \mathbf{1} := (1, \dots, 1) \in \mathcal{E}_+.$$

Interchanging α and $\bar{\alpha}$ if necessary, we can assume that $[K_1]$ is an $\mathbb{F}_{p^2}$ -valued point of $\mathcal{M}_+$, and hence $[K_e]$ is an $\mathbb{F}_{p^2}$ -valued point of $\mathcal{M}_+$ for any $e \in \mathcal{E}_+$. It follows that K_e is invariant under the action of G_κ for any $e \in \mathcal{E}_+$. Let $b_i^{(\alpha)}$ be an arbitrary element of the basis (8.1). Recall that we have assumed that $\sigma \geq 3$. Therefore, for each element $b_j^{(\beta)}$ of the basis (8.1) that is distinct from $b_i^{(\alpha)}$, there exists an element $e(j, \beta) = (\varepsilon_1, \dots, \varepsilon_\sigma) \in \mathcal{E}_+$ such that $\varepsilon_i = \alpha$ and $\varepsilon_j \neq \beta$. Since

$$\bigcap_{(j, \beta) \neq (i, \alpha)} K_{e(j, \beta)} = \langle b_i^{(\alpha)} \rangle$$

is invariant under the action of G_κ , we see that $b_i^{(\alpha)}$ is an eigenvector of G_κ . In particular, the isotropic vector $u = a_{2\sigma} = b_\sigma^{(1)}$ given at the beginning is an eigenvector of G_κ .

Let

$$\lambda_i^{(\alpha)} : G_\kappa \rightarrow \mathbb{F}_{p^2}^\times$$

be the character defined by $b_i^{(\alpha)}$. Suppose that $i, j \geq 2$ and $i \neq j$. Then $b_i^{(\alpha)} + b_j^{(\beta)}$ is an isotropic vector of N_0 for any choice of $\alpha, \beta \in \{\pm 1\}$, and hence is an eigenvector of G_κ . Therefore we have

$$\lambda_i^{(\alpha)} = \lambda_j^{(\beta)} \quad \text{if } i, j \geq 2 \text{ and } i \neq j. \quad (8.2)$$

Since the cardinality of $\{x^2 \mid x \in \mathbb{F}_p\}$ is $(p+1)/2$, there exist $\xi, \eta \in \mathbb{F}_p$ such that

$$(4 - c^2) + \xi^2 + \eta^2 = 0.$$

Then

$$b_1^{(1)} + b_1^{(-1)} + \xi(b_2^{(1)} + b_2^{(-1)}) + \eta(b_3^{(1)} + b_3^{(-1)})$$

is also an isotropic vector of N_0 , and hence is an eigenvector of G_κ . Therefore we have

$$\lambda_1^{(1)} = \lambda_1^{(-1)} = \lambda_2^{(1)} = \lambda_2^{(-1)} \quad \text{or} \quad \lambda_1^{(1)} = \lambda_1^{(-1)} = \lambda_3^{(1)} = \lambda_3^{(-1)}. \quad (8.3)$$

Combining (8.2) and (8.3), we see that all the characters $\lambda_i^{(\alpha)}$ are equal to each other. Thus G_κ consists of only scalar multiplications.

Suppose that $\sigma = 2$. In this case, the scheme $\mathcal{M}$ coincides with $\text{Isot}(2, q_0)$, which is the scheme parameterizing lines on the smooth quadratic surface $Q_0 = \{q_0 = 0\}$ in the projective space $\mathbb{P}_* N_0 = \text{Grass}(1, N_0)$. Hence $\mathcal{M}_+$ and $\mathcal{M}_-$ correspond to the two rulings of Q_0 . Let g be an element of G_κ . Then g leaves every line in the ruling of Q_0 corresponding to $\mathcal{M}_+$ invariant. Since g is defined over $\mathbb{F}_p$ and $\text{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$ interchanges $\mathcal{M}_+$ and $\mathcal{M}_-$, we see that g also leaves every line in the other ruling of Q_0 invariant. Therefore g fixes every point of Q_0 , and hence every point of $\mathbb{P}_* N_0$. $\square$

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