Serret–Frenet Formulae for Null Quaternionic Curves in Semi Euclidean 4-Space $\mathbb{R}_1^4$

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In this study, we construct the Cartan frame of a null quaternionic curves in semi Euclidean spaces $\mathbb{R}_1^4$ for an arbitrary parameter.

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1. Introduction

In a semi-Euclidean spaces, there exist three families of quaternionic curves, that is, spacelike, time-like, and null or lightlike quaternionic curves, according to their causal characters. The theory of Frenet frames for a semi-real quaternionic curve has been studied and developed by several researchers in this field (cf. [1] and [2]). In Bharathi and Nagaraj studied a Frenet frame for a quaternionic curve in $\mathbb{R}^3$ and $\mathbb{R}^4$, and gave the Frenet formulas for it. Coken and A. Tuna [2, 3] studied the differential geometry of a quaternionic curve in a 4-dimensional semi-Euclidean space $\mathbb{R}_2^4$. And then, we study Frenet formulae for null quaternionic curve in $\mathbb{R}_1^3$ and $\mathbb{R}_2^3$. But, to our knowledge, there has been no study on the Serret Frenet formulae for null quaternionic curves in $\mathbb{R}_1^4$.

In this paper we discuss Cartan frame for null quaternionic curves in the 4-dimensional semi Euclidean space $\mathbb{R}_1^4$. In this section, following [2] and [4], we use the proposition [4] and Cartan Frame for a null quaternionic curve in $\mathbb{R}_1^3$ [5].

Let h denote the quaternion metric on $\mathbb{R}_1^4$. A quaternionic curve $\beta(s)$ in $\mathbb{R}_1^4$ is called a null quaternionic curve if $h(\beta'(s), \beta'(s)) = 0$ and $\beta'(s) \neq 0$ for all s . We note that a null curve $\beta(s)$ in $\mathbb{R}_1^4$ satisfies $h(\beta''(s), \beta''(s)) \geq 0$ (cf. [4, Ch. 1]). We say that a null quaternionic curve $\beta(s)$ in $\mathbb{R}_1^4$ is parametrized by the pseudo-arc if $h(\beta''(s), \beta''(s)) = 1$, [5].

2. Preliminaries

Let we give basic concept about the semi-real quaternions. The set of the semi real quaternions

$$\mathbb{Q}_\nu = \{q | q = ae_1 + be_2 + ce_3 + d; a, b, c, d \in \mathbb{R},$$

$$e_1, e_2, e_3 \in \mathbb{R}_\nu^3; (\nu = 1, 2), h_\nu(e_i, e_i) = \varepsilon(e_i),$$

$$1 \leq i \leq 3\},$$

where

$$e_i \times e_i = -\varepsilon(e_i), 1 \leq i \leq 3$$

$$e_i \times e_j = \varepsilon(e_i)\varepsilon(e_j)e_k \text{ in } \mathbb{R}_1^3$$

$$e_i \times e_j = -\varepsilon(e_i)\varepsilon(e_j)e_k \text{ in } \mathbb{R}_2^4$$

and (ijk) is an even permutation of (123) . The multiplication of two semi real quaternions p and q is defined by

$$p \times q = S_p S_q + S_p V_q + S_q V_p + h(V_p, V_q) + V_p \wedge V_q$$

for every $p, q \in \mathbb{Q}_\nu$,

where we have used the inner and cross products in semi-Euclidean space $\mathbb{R}_\nu^3$. For a semi real quaternion $q = ae_1 + be_2 + ce_3 + d \in \mathbb{Q}_\nu$ the conjugate αq of q is defined by $\alpha q = -ae_1 - be_2 - ce_3 + d$. Thus, we define symmetric, non-degenerate valued bilinear form h as follows:

$$h_\nu : \mathbb{Q}_\nu \times \mathbb{Q}_\nu \rightarrow \mathbb{R}$$

by

$$h_1(p, q) = \frac{1}{2}[\varepsilon(p)\varepsilon(\alpha q)(p \times \alpha q) + \varepsilon(q)\varepsilon(\alpha p)(q \times \alpha p)]$$

for $\mathbb{R}_1^3$

and

$$h_2(p, q) = \frac{1}{2}[-\varepsilon(p)\varepsilon(\alpha q)(p \times \alpha q) - \varepsilon(q)\varepsilon(\alpha p)(q \times \alpha p)]$$

for $\mathbb{R}_2^4$,

where

$$h_\nu(q, q) = a^2\varepsilon(e_1) + b^2\varepsilon(e_2) + c^2\varepsilon(e_3) + d^2$$

for $\nu = \{1, 2\}$

is called the semi-real quaternion inner product. The vector product of two semi real quaternions $p = a_1e_1 + b_1e_2 + c_1e_3 + d_1$ and $q = ae_1 + be_2 + ce_3 + d$ is defined as

$$V_p \wedge V_q = \varepsilon(e_2)\varepsilon(e_3)(b_1c - bc_1)e_1 - \varepsilon(e_1)\varepsilon(e_3)$$

$$\times (a_1c - ac_1)e_2 + \varepsilon(e_1)\varepsilon(e_2)(a_1b - ab_1)e_3$$

for $\nu = 1$

and

$$V_p \wedge V_q = -\varepsilon(e_2)\varepsilon(e_3)(b_1c - bc_1)e_1 + \varepsilon(e_1)\varepsilon(e_3)$$

$$\times (a_1c - ac_1)e_2 - \varepsilon(e_1)\varepsilon(e_2)(a_1b - ab_1)e_3$$

for $\nu = 2$.

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The norm of semi real quaternion q is denoted by

$$\|q\|^2 = |h_\nu(q, q)| = |a^2\varepsilon(e_1) + b^2\varepsilon(e_2) + c^2\varepsilon(e_3) + d^2|$$

for $\nu = \{1, 2\}$.

The concept of a spatial quaternion will be made use throughout our work. q is called a spatial quaternion whenever $q + \alpha q = 0$. It is a temporal quaternion whenever $q - \alpha q = 0$ [1–3, 5].

Theorem I: (Let e_1 be a time-like vector.)

Let $\mathbb{R}_1^3$ denote a 3-dimensional semi-Euclidean space with quaternionic metric h . Let $\gamma : I \rightarrow \mathbb{R}_1^3$ be a null quaternionic Cartan curve such that $\{\gamma'(s), \gamma''(s), \gamma'''(s)\}$ is positively or negatively oriented for all $s \in I$. Let us consider its Cartan frame $\{l = \gamma'(s), n, u\}$ where $h(l, l) = h(n, n) = h(l, u) = h(n, u) = 0$, $h(l, n) = h(u, u) = 1$. The Frenet equations are

$$l' = ku, \quad n' = \tau u, \quad u' = -\tau l - kn \quad (2.1)$$

and

$$l' = ku, \quad n' = -\tau u, \quad u' = \tau l - kn, \quad (2.2)$$

where τ denotes covariant derivative and k is the curvature of the curve. In case γ is a null quaternionic Cartan curve, labeling $k(s) = 1$, then the Frenet formula of $\gamma(s)$ with respect to $\{l, n, u\}$ becomes [5]

$$l' = u, \quad n' = \tau u, \quad u' = -\tau l - n$$

and

$$l' = u, \quad n' = -\tau u, \quad u' = \tau l - n.$$

Theorem II: (Let e_2 be a time-like vector.)

Let $\mathbb{R}_1^3$ denote a 3-dimensional semi-Euclidean space with quaternionic metric h . Let $\gamma : I \rightarrow \mathbb{R}_1^3$ be a null quaternionic Cartan curve such that $\{\gamma'(s), \gamma''(s), \gamma'''(s)\}$ is positively or negatively oriented for all $s \in I$. Let us consider its Cartan frame $\{l = \gamma'(s), n, u\}$ where $h(l, l) = h(n, n) = h(l, u) = h(n, u) = 0$, $h(l, n) = -1$, $h(u, u) = 1$. The Frenet equations are

$$l' = ku, \quad n' = -\tau u, \quad u' = -\tau l + kn$$

and

$$l' = ku, \quad n' = \tau u, \quad u' = \tau l + kn, \quad (2.3)$$

where τ denotes covariant derivative and k is the curvature of the curve. In case γ is a null quaternionic Cartan curve, labeling $k(s) = 1$, then the Frenet formula of $\gamma(s)$ with respect to $\{l, n, u\}$ becomes [5]

$$l' = un' = -\tau u, \quad u' = -\tau l + n$$

and

$$l' = u, \quad n' = \tau u, \quad u' = \tau l + n.$$

Now let us begin to set up the Frenet equations of a null quaternionic curve by using the main results of [4, proposition 2.3, page 7] and Theorems I and II.

3. Serret–Frenet formulae for null quaternionic curves in the semi-Euclidean spaces $\mathbb{R}_1^4$

Case I: (Let e_1 be a time-like vector.)

An orthonormal basis $\{e_1, e_2, e_3, e_4 = 1\}$ of $\mathbb{R}_1^4$. Construct vectors

$$L = \frac{1}{\sqrt{2}}(e_2 + e_1), \quad N = \frac{1}{\sqrt{2}}(e_2 - e_1), \quad U = e_3, \quad W = 1$$

which satisfy

$$h(L, L) = h(N, N) = h(L, U) = h(N, U) = h(W, U) = h(N, W) = h(L, W) = 0$$

and

$$h(L, N) = h(U, U) = h(W, W) = +1.$$

Thus $\{L, N, U, W\}$ is a basis of $\mathbb{R}_1^4$ which contains L and N null vectors and U and W spacelike vectors. Let $\mathbb{R}_1^4$ denote a 4-dimensional semi-Euclidean space with quaternionic metric h . The 4-dimensional semi-Euclidean spaces in $\mathbb{R}_1^4$ are identified with the spaces of unit quaternion. Let $\beta : I \subset \mathbb{R} \rightarrow \mathbb{Q}_\nu, \beta = \gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_3 + \gamma_4 e_4$ be a null quaternionic curve in $\mathbb{R}_1^4$ defined over the interval I . Let the distinguished parameter s be chosen such that the tangent $L(s) = \sum_{i=1}^4 \gamma'_i(s) e_i$ has zero length.

Let $\{L(s), N(s), U(s), W(s)\}$ be the Frenet apparatus of the differentiable semi-Euclidean space curve in the semi-Euclidean spaces $\mathbb{R}_1^4$. We denote the Frenet frame and Frenet equations of this particular class of β by [4, proposition 2.3, Case I]. Then Frenet equations are

$$L' = KW N' = -(\tau + p)U + pW$$

$$U' = (\tau + p)L W' = -pL - KN$$

and

$$L' = KW N' = (\tau - p)U + pW$$

$$U' = -(\tau - p)L W' = -pL - KN.$$

Let β be a null quaternionic curve, with the tangent $L(s) = \beta'(s) = \sum_{i=1}^4 \gamma'_i(s) e_i$ has zero length. Then we have

$$h(\beta'(s), \beta'(s)) = \beta'(s) \times \alpha \beta'(s) = 0, \quad \beta'(s) \neq 0. \quad (3.1)$$

Since $\beta'(s) = L$ is a null, we have $h(L, L) = 0$. But we do not allow $\beta''(s) = W$ to be null, so it is always spacelike. We also have

$$h(W, W) = h(\beta''(s), \beta''(s)) = 1. \quad (3.2)$$

Differentiating of equation (3.1) with respect to s ,

$$\beta''(s) \times \alpha \beta'(s) + \beta'(s) \times \alpha \beta''(s) = 0$$

is obtained. Then, this equation follows that

$$(i) \quad \beta''(s) \times \alpha \beta'(s) \text{ is a semi spatial quaternion,}$$

$$(ii) \quad h(\beta'(s), \beta''(s)) = 0 \Rightarrow h(L, W) = 0 \\ h(L, L') = 0.$$

If we use the differentiation of (ii), we obtained that $h(LW') = -1$. Now, L' is a vector and can be written in terms of any basis. With the orthonormal basis $\{L, N, U, W, L'\}$ becomes,

$$L' = aL + bN + cU + dW, \quad a, b, c, d \in \mathbb{R}. \quad (3.3)$$

Taking the scalar product of equation (3.3) with L, N, U and W , respectively. Since L' itself is a semi-spatial

quaternion, we define the semi-spatial quaternion W and the scalar function K by means of the following equation

$$L' = KW, h(L', W) = K = 1 \quad (3.4)$$

If we use the differentiation of $\|L\| = H(L, L) = 0$ and (3.4) in the resulting equation, we obtain that

$$W \times \alpha L + L \times \alpha W = 0$$

because of $K(s) \neq 0$.

Then, this equation follows that

- (i) W is h -orthogonal to L , $h(L, W) = 0$
- (ii) Since l is a semi-spatial quaternion, $L \times \alpha W$ is a semi-spatial quaternion.

Then, it is obvious that l is null and W is spacelike unity. Let $L = l \times W$ be null quaternionic vector, along β , we can now write L as follows

$$l = L \times \alpha W \text{ and } L = l \times W \quad (3.5)$$

Using differentiation of Eq. (3.5) and Eqs. (3.4) and (2.1) in the resulting equation, we obtain that

$$U = u \times W \text{ and } l \times W' = KW - kU. \quad (3.6)$$

The properties of U are,

- (i) $h(U, U) = U \times \alpha U = 1$
- (ii) LU and W are mutually h -orthogonal since l and u are so.

By differentiating $U = u \times W$ given by Eq. (3.6), using the Eqs. (2.1) and (3.5) in the resulting equation, we have

$$U' = \tau L - kN + u \times W' \text{ where } N = n \times W \quad (3.7)$$

The properties of N are,

- (i) $|N|^2 = h(N, N) = 0$
- (ii) N, U and W are mutually h -orthogonal since n is so. Then L and N are mutually h -orthogonal since $h(L, N) = 1$ is so.

By differentiating $N = n \times W$ given by Eq. (3.7), using the Eqs. (2.1) and (3.6) in the resulting equation, we have

$$N' = -\tau U + n \times W' \quad (3.8)$$

Consequently, we get

$$l \times W' = KW - kU, \quad U = u \times W$$

$$U' = \tau L - kN + u \times W', \quad N = n \times W,$$

$$N' = -\tau U + n \times W'. \quad (3.9)$$

With the way viewed up here, among the Frenet equataions, just the (3.4) equation has been obtained. In order to get Frenet equations by the method below;

$$L' = a_{11}L + a_{12}N + a_{13}U + a_{14}W$$

$$N' = a_{21}L + a_{22}N + a_{23}U + a_{24}W$$

$$U' = a_{31}L + a_{32}N + a_{33}U + a_{34}W$$

$$W' = a_{41}L + a_{42}N + a_{43}U + a_{44}W$$

Taking the scalar product of these equations with L, N, U and W , respectively, we find that

$$\begin{bmatrix} L' \\ N' \\ U' \\ W' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & K \\ 0 & 0 & m & p \\ -m & 0 & 0 & r \\ -p & -K & -r & 0 \end{bmatrix} \begin{bmatrix} L \\ N \\ U \\ W \end{bmatrix} \quad (3.10)$$

We obtain W' from this matrix as follows:

$$W' = -pL - KN - rU$$

and if we have the quaternion multiplication of W' vector by u, n and l vectors, we get

$$l \times W' = KW + KU - rL,$$

$$n \times W' = pW - pU + rN,$$

$$u \times W' = pL - KN + rW. \quad (3.11)$$

Hence, substituting Eq. (3.11) in the Eq. (3.9), we obtain that

$$(K + k)U + rL = 0,$$

$$N' = (-\tau - p)U + pW + rN,$$

$$U' = (\tau + p)L - (K + k)N + rW. \quad (3.12)$$

Hence we have equation (3.13) of the following form

$$\begin{bmatrix} L' \\ N' \\ U' \\ W' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & K \\ 0 & 0 & -\tau-p & p \\ \tau+p & -(K+k) & 0 & r \\ -p & -K & -r & 0 \end{bmatrix} \begin{bmatrix} L \\ N \\ U \\ W \end{bmatrix} \quad (3.13)$$

If the matrices in Eqs. (3.13) and (3.10) are equaled, we have $r = 0$ because $K + k = 0$. Consequently, we get

$$\begin{bmatrix} L' \\ N' \\ U' \\ W' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & K \\ 0 & 0 & -\tau-p & p \\ \tau+p & 0 & 0 & 0 \\ -p & -K & 0 & 0 \end{bmatrix} \begin{bmatrix} L \\ N \\ U \\ W \end{bmatrix} \quad (3.14)$$

By the same method using Eq. (2.2) in $\mathbb{R}_1^3$ and $K + k = 0$ in $\mathbb{R}_1^4$, we get the matrix as follows;

$$\begin{bmatrix} L' \\ N' \\ U' \\ W' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & K \\ 0 & 0 & \tau-p & p \\ -(\tau-p) & 0 & 0 & 0 \\ -p & -K & 0 & 0 \end{bmatrix} \begin{bmatrix} L \\ N \\ U \\ W \end{bmatrix} \quad (3.15)$$

Here, K is the first curvature of β curve in $\mathbb{R}_1^4$, k is both the first curvature of γ curve in $\mathbb{R}_1^3$ and the opposite signed second curvature of β curve in $\mathbb{R}_1^4$. However, this is not possible. Because, $K = 1$ and $k = 1$. Whereas from $K + k = 0$, it should have been $K = -k$. Thus, when we label $K = 1$ and $k = 1$, we can not prove the Frenet equations. While e_1 is time-like in $\mathbb{R}_1^4$, we do not construct Frenet frame for a null quaternionic curve in $\mathbb{R}_1^4$.

Case II: (Let e_2 be a time-like vector.)

An ortonormal basis $\{e_1, e_2, e_3, e_4 = 1\}$ of $\mathbb{R}_1^4$ Let e_2 be a time-like vector. Construct vectors

$$L = \frac{1}{\sqrt{2}}(e_2 + e_1), \quad N = \frac{1}{\sqrt{2}}(e_2 - e_1), \quad U = e_3, \quad W = 1$$

which satisfy

$$\begin{aligned} h(L, L) &= h(N, N) = h(L, U) = h(N, U) = h(W, U) = \\ h(N, W) &= h(L, W) = 0, \\ h(U, U) &= h(W, W) = +1 \end{aligned}$$

and

$$h(L, N) = -1.$$

Thus $\{L, N, U, W\}$ is a basis of $\mathbb{R}_1^4$ which contains L and N null vectors and U and W spacelike vectors. Let $\mathbb{R}_1^4$ denote a 4-dimensional semi-Euclidean space with quaternionic metric h . The four-dimensional semi-Euclidean spaces in $\mathbb{R}_1^4$ are identified with the spaces of unit quaternion. Let $\beta : I \subset \mathbb{R} \rightarrow \mathbb{Q}_\nu$, $\beta = \gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_3 + \gamma_4 e_4$ be a null quaternionic curve in $\mathbb{R}_1^4$ defined over the interval I . Let the distinguished parameter s be chosen such that the tangent $L(s) = \beta'(s) = \sum_{i=1}^4 \gamma'_i(s) e_i$ has zero length. Let $\{L(s), N(s), U(s), W(s)\}$ be the Frenet apparatus of the differentiable semi-Euclidean space curve in the semi-Euclidean spaces $\mathbb{R}_1^4$. We denote the Frenet frame and Frenet equations of this particular class of β by proposition (Case I). Then Frenet equations are

$$\begin{aligned} L' &= KW, \quad N' = (\tau + p)U + pW, \\ U' &= (\tau + p)L, \quad W' = pL + KN \end{aligned}$$

and

$$\begin{aligned} L' &= KW, \quad N' = -(\tau - p)U + pW, \\ U' &= -(\tau - p)L, \quad W' = pL + KN. \end{aligned}$$

Here, K is the first curvature of β curve in $\mathbb{R}_1^4$, k is both the first curvature of γ curve in $\mathbb{R}_1^3$ and second curvature of β curve in $\mathbb{R}_1^4$. We find that $K = 1$ and $k = 1$ truly and these values provide $(K - k) = 0$. Using the method of theorem 3.2 [2], while e_2 is time-like in $\mathbb{R}_1^4$, we construct Frenet frame for a null quaternionic curve in $\mathbb{R}_1^4$.

4. Conclusions

While e_1 is time-like in $\mathbb{R}_1^4$, we do not construct Frenet frame for a null quaternionic curve in $\mathbb{R}_1^4$. While e_2 is time-like in $\mathbb{R}_1^4$, we construct Frenet frame for a null quaternionic curve in $\mathbb{R}_1^4$.

Acknowledgments

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