Noether Theorems and Discrete Variational Integrators in Field Theory

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The discrete analogue of the Noether-type identities in field theory is investigated by means of the difference discrete variational principle in which the difference is regarded as an entire geometric object. The discrete counterparts of the Noether theorems are obtained. It is proved that there exists the discrete version of the Noether conservation law in field theory. The discretization for the nonlinear Schrödinger equation is presented to illustrate the results.

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1. Introduction

The discretizations in continuous mechanics provide a powerful way in the course of numerical calculations and simulations. The integration theory for difference equations is potentially useful for analysis of the solutions of the discrete systems. The discrete Noether theorem is one of the most important methods of solving difference equations by obtaining the first integrals from a certain kind of system symmetries. Birkhoff first discussed the difference equation in 1913 [1]. Cadzow presented a discrete variational principle and obtained the discrete Euler–Lagrange equation in his paper “Discrete Calculus of Variations” [2], and the most important contribution is the discrete Noether theorem which is obtained via the invariance of the Lagrange equation. The Noether theorem reveals that conserved quantities can be obtained by the corresponding Noether symmetries [3]. The discrete Noether theorem provides a simple and clear approach to find the first integrals of difference equations. Maeda studied the symmetries of difference systems, applied the difference variational principle to the mechanical systems, and generalized the discrete Noether theorem [4]. Dorodnitsyn and Kozlov also gave the Noether theorem of difference canonical Hamiltonian equations based on the Hamiltonian identity. This connection between symmetries and first integrals of difference Hamiltonian equations can be used to conserve the structural properties of Hamiltonian equations under discretization [5]. Fu et al. extended discrete Noether symmetries to mechanico-electrical dynamical systems with nonconservative forces and a dissipation function [6, 7]. In Ref. [8] the Noether symmetries

and conserved quantities are studied to construct the exact solutions with continuous groups of transformations in discrete nonconservative systems. Particularly, a new method was given to investigate Noether symmetries and conservation laws of nonconservative and nonholonomic mechanical systems on time scales [9]. All the new results attract extensive attention in physics.

Noether symmetries are extensively investigated in classical mechanics, such as Djukic and Vujanovic [10], Sarlet and Cantrijn [11] and Mei [12]. Based on Noether symmetries in classical mechanics, the Noether symmetries were generalized in field theories [13]. Li derived the generalization of Noether’s first theorem in phase space for an invariant system with a singular Lagrangian in field theories, and deduced Noether’s second theorems in phase space for a noninvariant system in field theories [14]. Arens obtained certain Noetherian currents based on translations in field space, and gave the conclusion that certain well-known conserved currents are not the result of any symmetries of the Maxwellian (electromagnetic) field [15].

However, all aforementioned researches on Noether symmetry in field theories are confined to continuous mechanical systems. There has been no research on discrete ones in field theory. Marsden et al. generalized the Veselov discretization to multisymplectic field theory [16]. Guo et al. discussed firstly the difference discrete variational principle in multi-parameter differential approach for discrete classical field theory, and deduced the difference discrete Euler–Lagrange equations and canonical ones for the difference discrete versions of classical field theory [17]. Chen developed a total variation calculus in discrete multisymplectic field theory and obtained multisymplectic-energy-momentum integrators by using this discrete total variation calculus [18]. Although the discrete variational principle in field theory plays a crucial role in the research of the Noether

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theorems in discrete mechanical systems, there has been no work on discrete Noether theorems in field theory. To address the lack of this aspect, this paper deals with the Noether theorem and the variational integrator in field theory.

In this work, the Noether theorem and the variational integrator in field theory are investigated. The paper is structured as follows. In Sect. 2, the discrete energy evolution equation and the discrete momentum evolution equation in field theory are presented based on the discrete difference variational principle. In Sect. 3, the definitions of the Noether symmetry and quasi-symmetry in field theory are given. Based on the invariance of the discrete Hamiltonian action of the field theory under the infinitesimal transformation with respect to the coordinates, the discrete counterparts of the Noether-type identity in field theory are given. The discrete version of the Noether theorems and the Noether conservation laws in field theory are constructed. In Sect. 4, the nonlinear Schrödinger equation is treated to illustrate the theory. Section 5 draws the conclusion of the present paper.

2. The discrete variational integrator in field theory

For simplicity, let $X^{1,1}$ be a $1 + 1$ or 2-dimensional Minkowskian space as the base manifold with coordinates x and t , M the configuration space on $X^{1,1}$ with a set of generic field u . For a discrete version of $X = (x, t)$ and $X \times U = (x, t, u)$, we take $X = Z \times Z = \{(i, j)\}$ and $X \times U = (i, j, u_{ij})$, respectively. A point of $(i, j) \in X$ is touched by a triangle if it is vertex of that triangle. The discretization of $L(x, t, u, u_x, u_t) dx \wedge dt$ is

$$L(x_i, t_j, u_{ij}, \Delta_x u_{ij}, \Delta_t u_{ij}) \Delta x_i \Delta t_j, \quad (1)$$

where $\Delta_x u_{ij} = \frac{u_{i+1,j} - u_{ij}}{x_{i+1} - x_i}$ and $\Delta_t u_{ij} = \frac{u_{ij+1} - u_{ij}}{t_{j+1} - t_j}$ are the expressions of the difference with variable step-length, the variable step-lengths are $\Delta x_i = x_{i+1} - x_i$ and $\Delta t_j = t_{j+1} - t_j$.

Corresponding to the discrete Lagrangian density, the discrete action is defined to be:

$$S_D = \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} L_D^{ij}(x_i, t_j, u_{ij}, \Delta_x u_{ij}, \Delta_t u_{ij}) \times (x_{i+1} - x_i)(t_{j+1} - t_j).$$

The discrete action S_D now reads

$$S_D = L_D^{ij} \Delta x_i \Delta t_j. \quad (2)$$

The total variation of the discrete action (2) is

$$\delta S_D = \delta(\Delta x_i) \Delta t_j L_D^{ij} + \Delta x_i \delta(\Delta t_j) L_D^{ij} + \Delta x_i \Delta t_j \delta L_D^{ij}.$$

Let us firstly consider the coordinate t_j , the step-length $\Delta t_j = t_{j+1} - t_j$. And

$$\delta \Delta t_j = \delta(t_{j+1} - t_j). \quad (3)$$

The difference $\Delta_t \delta t_j$ of δt_j at t_k is

$$\Delta_t \delta t_j = \frac{\delta t_{j+1} - \delta t_j}{t_{k+1} - t_k}. \quad (4)$$

Equation (3) can be written as

$$\delta \Delta t_j = \delta(t_{j+1} - t_j) = \frac{\delta t_{j+1} - \delta t_j}{t_{j+1} - t_j} (t_{j+1} - t_j) = \Delta_t \delta t_j \Delta t_j. \quad (5)$$

So for the coordinates x_i and t_j , there exist

$$\delta \Delta x_i = \Delta_x \delta x_i \Delta x_i, \quad \delta \Delta t_j = \Delta_t \delta t_j \Delta t_j. \quad (6)$$

The total variation of discrete action (2) is

$$\begin{aligned} \delta S_D &= \delta \Delta x_i \Delta t_j L_D^{ij} + \Delta x_i \delta \Delta t_j L_D^{ij} + \Delta x_i \Delta t_j \delta L_D^{ij} = \\ &\Delta_x \delta x_i \Delta x_i \Delta t_j L_D^{ij} + \Delta x_i \Delta_t \delta t_j \Delta t_j L_D^{ij} + \Delta x_i \Delta t_j \delta L_D^{ij} = \\ &\Delta x_i \Delta t_j \left\{ \Delta_x \delta x_i L_D^{ij} + \Delta_t \delta t_j L_D^{ij} + \delta L_D^{ij} \right\}, \end{aligned} \quad (7)$$

where δL_D^{ij} can be written as

$$\begin{aligned} \delta L_D^{ij} &= \frac{\partial L_D^{ij}}{\partial x_i} \delta x_i + \frac{\partial L_D^{ij}}{\partial t_j} \delta t_j + \frac{\partial L_D^{ij}}{\partial u_{ij}} \delta u_{ij} \\ &+ \frac{\partial L_D^{ij}}{\partial (\Delta_x u_{ij})} \delta (\Delta_x u_{ij}) + \frac{\partial L_D^{ij}}{\partial (\Delta_t u_{ij})} \delta (\Delta_t u_{ij}). \end{aligned}$$

Let us consider the coordinates of nodes on the lattice subject to the infinitesimal deformations $\kappa_\theta \rightarrow \kappa'_\theta(x_i, t_j) = \kappa_\theta + \delta \kappa_\theta$, $\kappa_\theta = (x_i, t_j)$, the general variation of $u_{ij} \rightarrow u'_{ij}(\kappa'_\theta) = u_{ij}(\kappa_\theta) + \delta u_{ij}(\kappa_\theta)$. Here $\delta u_{ij}(\kappa)$ denotes the total variation that can be divided into two parts

$$\begin{aligned} \delta u_{ij}(\kappa_\theta) &= \delta_v u_{ij}(\kappa_\theta) + \delta_h u_{ij}(\kappa_\theta), \\ \delta_v u_{ij}(\kappa_\theta) &= u'_{ij}(\kappa_\theta) - u_{ij}(\kappa_\theta), \\ \delta_h u_{ij}(\kappa_\theta) &= u'_{ij}(\kappa'_\theta) - u'_{ij}(\kappa_\theta) = \\ u_{ij}(\kappa'_\theta) - u_{ij}(\kappa_\theta) + O(\delta^2) &= \delta \kappa_\theta \Delta_\theta u_{ij} \end{aligned}$$

where $\delta_v u_{ij}(\kappa_\theta)$ denotes the equal time variation or the vertical one and $\delta_h u_{ij}(\kappa_\theta)$ the horizontal part along the fiber induced by the reparametrization of x_i and t_j . The total variation of $\Delta_\theta u_{ij}$ (where $\Delta_\theta u_{ij} = \Delta_\theta u_{ij}(\kappa_\theta)$) can be calculated as

$$\begin{aligned} \delta \Delta_\theta u_{ij} &= \delta_v \Delta_\theta u_{ij} + \delta_h \Delta_\theta u_{ij} = \\ \Delta_\theta u'_{ij} - \Delta_\theta u_{ij} + \delta_h \Delta_\theta u_{ij} &= \\ \Delta_\theta \delta_v u_{ij} + \delta \kappa_\theta \Delta_\theta \Delta_\theta u_{ij} &= \\ \Delta_\theta \delta_v u_{ij} + \Delta_\theta (\delta \kappa_\theta \Delta_\theta u_{ij}) - (\Delta_\theta \delta \kappa_\theta) \Delta_\theta u_{ij} &= \\ \Delta_\theta \delta_v u_{ij} + \Delta_\theta \delta_h u_{ij} - (\Delta_\theta \delta \kappa_\theta) \Delta_\theta u_{ij} &= \\ \Delta_\theta \delta u_{ij} - (\Delta_\theta \delta \kappa_\theta) \Delta_\theta u_{ij}, \end{aligned}$$

where $\Delta_\theta u_{ij} = (\Delta_x u_{ij}, \Delta_t u_{ij})$. So the relations are

$$\begin{aligned} \delta \Delta_t u_{ij} &= \Delta_t \delta u_{ij} - (\Delta_t \delta t_j) \Delta_t u_{ij}, \\ \delta \Delta_x u_{ij} &= \Delta_x \delta u_{ij} - (\Delta_x \delta x_i) \Delta_x u_{ij}, \end{aligned} \quad (8)$$

where $\delta \Delta_t u_{ij}$ and $\delta \Delta_x u_{ij}$ are the discrete total variation for $\Delta_t u_{ij}$ and $\Delta_x u_{ij}$. One can derive the following equations:

$$\begin{aligned} \delta L_D^{ij} &= \frac{\partial L_D^{ij}}{\partial x_i} \delta x_i + \frac{\partial L_D^{ij}}{\partial t_j} \delta t_j + \frac{\partial L_D^{ij}}{\partial u_{ij}} \delta u_{ij} \\ &+ \frac{\partial L_D^{ij}}{\partial (\Delta_x u_{ij})} (\Delta_x \delta u_{ij} - (\Delta_x \delta x_i) \Delta_x u_{ij}) \\ &+ \frac{\partial L_D^{ij}}{\partial (\Delta_t u_{ij})} (\Delta_t \delta u_{ij} - (\Delta_t \delta t_j) \Delta_t u_{ij}) = \end{aligned}$$

3. Noether theory of the discrete system in field theory

In continuous field theories, early works were based on the infinitesimal transformation to investigate the Yang-Mill field theory [13] and the electromagnetic field [14]. For discrete systems in field theory, the case of independent variables x_i , t_j and dependent variable u_{ij} will be considered

$$x_i^* = x_i + \delta x_i, \quad t_j^* = t_j + \delta t_j, \quad u_{ij}^* = u_{ij} + \delta u_{ij}, \quad (15)$$

where $\delta t_j = \varepsilon \tau_j$, $\delta x_i = \varepsilon \xi_i$, $\delta u_{ij} = \varepsilon \eta_{ij}$, and ε is a small parameter, the infinitesimal generators $\tau_j = \tau_j(x_i, t_j, u_{ij}, \Delta_x u_{ij}, \Delta_t u_{ij})$, $\xi_i = \xi_i(x_i, t_j, u_{ij}, \Delta_x u_{ij}, \Delta_t u_{ij})$ and $\eta_{ij} = \eta_{ij}(x_i, t_j, u_{ij}, \Delta_x u_{ij}, \Delta_t u_{ij})$ constitute the Lie group of the infinitesimal transformations. The generators of generalized invariant transformations are solutions of the Noether equations [19]. The discrete transformations were applied to systems of Lagrangian systems [20], the Hamiltonian systems [5], the non-conservative dynamical systems [21] and the field theories [13].

Definition 1. If the discrete Hamiltonian elementary action (2) is invariant with respect to the infinitesimal transformations (15) that is

$$\Delta x_i \Delta t_j \left\{ \Delta_x \delta x_i L_D^{ij} + \Delta_t \delta t_j L_D^{ij} + \delta L_D^{ij} \right\} = 0, \quad (16)$$

then the infinitesimal transformations are the Noether symmetrical transformations.

Criterion 1. Under the transformations (15), if the equation

$$\begin{aligned} & -\Delta_x \delta x_i H_{xD}^{ij} - \Delta_t \delta t_j H_{tD}^{ij} + \frac{\partial L_D^{ij}}{\partial x_i} \delta x_i + \frac{\partial L_D^{ij}}{\partial t_j} \delta t_j \\ & + \frac{\partial L_D^{ij}}{\partial u_{ij}} \delta u_{ij} + \frac{\partial L_D^{ij}}{\partial (\Delta_x u_{ij})} \Delta_x \delta u_{ij} + \frac{\partial L_D^{ij}}{\partial (\Delta_t u_{ij})} \Delta_t \delta u_{ij} = 0 \end{aligned} \quad (17)$$

is satisfied, the transformations (15) are the Noether symmetrical transformations.

Theorem 1. If the infinitesimal transformations (15) are the Noether symmetrical transformation, then the systems (12), (13) and (14) possess the discrete conserved quantity

$$\begin{aligned} I_{ij} &= \frac{\partial L_D^{i-1j}}{\partial (\Delta_x u_{i-1j})} \delta u_{ij} \\ &+ \left(L_D^{i-1j} - \frac{\partial L_D^{i-1j}}{\partial (\Delta_x u_{i-1j})} \Delta_x u_{i-1j} \right) \delta x_i + \frac{\partial L_D^{ij-1}}{\partial (\Delta_t u_{ij-1})} \delta u_{ij} \\ &+ \left(L_D^{ij-1} - \frac{\partial L_D^{ij-1}}{\partial (\Delta_t u_{ij-1})} \Delta_t u_{ij-1} \right) \delta t_j = \text{const.} \end{aligned} \quad (18)$$

Proof. Since the difference discrete Hamiltonian is

$$H_{xD}^{ij} = \frac{\partial L_D^{ij}}{\partial (\Delta_x u_{ij})} \Delta_x u_{ij} - L_D^{ij},$$

$$H_{tD}^{ij} = \frac{\partial L_D^{ij}}{\partial (\Delta_t u_{ij})} \Delta_t u_{ij} - L_D^{ij},$$

Eq. (17) may be expressed as

$$\begin{aligned} & -\Delta_x \delta x_i H_{xD}^{ij} - \Delta_t \delta t_j H_{tD}^{ij} + \frac{\partial L_D^{ij}}{\partial x_i} \delta x_i + \frac{\partial L_D^{ij}}{\partial t_j} \delta t_j \\ & + \frac{\partial L_D^{ij}}{\partial u_{ij}} \delta u_{ij} + \frac{\partial L_D^{ij}}{\partial (\Delta_x u_{ij})} \Delta_x \delta u_{ij} + \frac{\partial L_D^{ij}}{\partial (\Delta_t u_{ij})} \Delta_t \delta u_{ij} = \\ & \Delta_x \delta x_i L_D^{ij} + \Delta_t \delta t_j L_D^{ij} + \frac{\partial L_D^{ij}}{\partial x_i} \delta x_i + \frac{\partial L_D^{ij}}{\partial t_j} \delta t_j + \frac{\partial L_D^{ij}}{\partial u_{ij}} \delta u_{ij} \\ & + \frac{\partial L_D^{ij}}{\partial (\Delta_x u_{ij})} (\Delta_x \delta u_{ij} - (\Delta_x \delta x_i) \Delta_x u_{ij}) \\ & + \frac{\partial L_D^{ij}}{\partial (\Delta_t u_{ij})} (\Delta_t \delta u_{ij} - (\Delta_t \delta t_j) \Delta_t u_{ij}) = \\ & \left[\frac{\partial L_D^{ij}}{\partial x_i} + \Delta_x \left(\frac{\partial L_D^{i-1j}}{\partial (\Delta_x u_{i-1j})} \Delta_x u_{i-1j} - L_D^{i-1j} \right) \right] \delta x_i \\ & + \left[\frac{\partial L_D^{ij}}{\partial t_j} + \Delta_t \left(\frac{\partial L_D^{ij-1}}{\partial (\Delta_t u_{ij-1})} \Delta_t u_{ij-1} - L_D^{ij-1} \right) \right] \delta t_j \\ & + \left[\frac{\partial L_D^{ij}}{\partial u_{ij}} - \Delta_x \left(\frac{\partial L_D^{i-1j}}{\partial (\Delta_x u_{i-1j})} \right) - \Delta_t \left(\frac{\partial L_D^{ij-1}}{\partial (\Delta_t u_{i,j-1})} \right) \right] \delta u_{ij} \\ & + \Delta_x \left(\frac{\partial L_D^{i-1j}}{\partial (\Delta_x u_{i-1j})} \delta u_{ij} \right. \\ & \left. + \left(L_D^{i-1j} - \frac{\partial L_D^{i-1j}}{\partial (\Delta_x u_{i-1j})} \Delta_x u_{i-1j} \right) \delta x_i \right) \\ & + \Delta_t \left(\frac{\partial L_D^{ij-1}}{\partial (\Delta_t u_{ij-1})} \delta u_{ij} \right. \\ & \left. + \left(L_D^{ij-1} - \frac{\partial L_D^{ij-1}}{\partial (\Delta_t u_{ij-1})} \Delta_t u_{ij-1} \right) \delta t_j \right) = 0. \end{aligned} \quad (19)$$

Substituting Eqs. (12), (13) and (14) into (19), one can derive the following equation:

$$\begin{aligned} & \Delta_x \left(\frac{\partial L_D^{i-1j}}{\partial (\Delta_x u_{i-1j})} \delta u_{ij} \right. \\ & \left. + \left(L_D^{i-1j} - \frac{\partial L_D^{i-1j}}{\partial (\Delta_x u_{i-1j})} \Delta_x u_{i-1j} \right) \delta x_i \right) \\ & + \Delta_t \left(\frac{\partial L_D^{ij-1}}{\partial (\Delta_t u_{ij-1})} \delta u_{ij} \right. \\ & \left. + \left(L_D^{ij-1} - \frac{\partial L_D^{ij-1}}{\partial (\Delta_t u_{ij-1})} \Delta_t u_{ij-1} \right) \delta t_j \right) = 0, \end{aligned}$$

that is $\Delta I_{ij} = \Delta_x I_{ij} + \Delta_t I_{ij} = 0$, so Eq. (18) is conserved quantity.

Definition 2. If the infinitesimal transformations (15) and gauge function $G_{ij} = G_{ij}(x_i, t_j, u_{ij}, \Delta_x u_{ij}, \Delta_t u_{ij})$ satisfy

$$\delta S_D = -(x_{i+1} - x_i)(t_{j+1} - t_j) \Delta(\varepsilon G_{ij}), \quad (20)$$

where $\Delta(\varepsilon G_{ij}) = \Delta_x(\varepsilon G_{ij}) + \Delta_t(\varepsilon G_{ij})$, then the

infinitesimal transformations are the Noether quasi-symmetrical transformations.

Criterion 2. Under the transformations (15), if the equation

$$\begin{aligned} & \Delta_x \delta x_i H_D^{ij} + \Delta_t \delta t_j H_D^{ij} + \frac{\partial L_D^{ij}}{\partial x_i} \delta x_i + \frac{\partial L_D^{ij}}{\partial t_j} \delta t_j + \frac{\partial L_D^{ij}}{\partial u_{ij}} \delta u_{ij} \\ & + \frac{\partial L_D^{ij}}{\partial (\Delta_x u_{ij})} \Delta_t \delta u_{ij} + \frac{\partial L_D^{ij}}{\partial (\Delta_t u_{ij})} \Delta_x \delta u_{ij} = -\Delta (\varepsilon G_{ij}) \end{aligned} \quad (21)$$

is satisfied, the transformations (15) are the quasi-symmetrical transformations.

Theorem 2. If the infinitesimal transformations (15) are the quasi-symmetrical transformation, then the Lagrangian systems in field theory possess the discrete conserved quantities

$$\begin{aligned} I_{ij} = & \frac{\partial L_D^{i-1j}}{\partial (\Delta_x u_{i-1j})} \delta u_{ij} + \frac{\partial L_D^{ij-1}}{\partial (\Delta_t u_{ij-1})} \delta u_{ij} \\ & + \left(L_D^{i-1j} - \frac{\partial L_D^{i-1j}}{\partial (\Delta_x u_{i-1j})} \Delta_x u_{i-1j} \right) \delta x_i \\ & + \left(L_D^{ij-1} - \frac{\partial L_D^{ij-1}}{\partial (\Delta_t u_{ij-1})} \Delta_t u_{ij-1} \right) \delta t_j + \varepsilon G_{ij} = \text{const}. \end{aligned} \quad (22)$$

Proof. According to the definition 2, Eq. (21) may be expressed as

$$\begin{aligned} & -\Delta_x \delta x_i H_{xD}^{ij} - \Delta_t \delta t_j H_{tD}^{ij} + \frac{\partial L_D^{ij}}{\partial x_i} \delta x_i + \frac{\partial L_D^{ij}}{\partial t_j} \delta t_j \\ & + \frac{\partial L_D^{ij}}{\partial u_{ij}} \delta u_{ij} + \frac{\partial L_D^{ij}}{\partial (\Delta_x u_{ij})} \Delta_x \delta u_{ij} + \frac{\partial L_D^{ij}}{\partial (\Delta_t u_{ij})} \Delta_t \delta u_{ij} \\ & + \Delta (\varepsilon G_{ij}) = 0. \end{aligned} \quad (23)$$

Using the same method in the proof of theorem 1, one can have $\Delta I_{ij} = 0$. The conserved quantity I_{ij} can be obtained by using the discrete Leibnitz law and Eqs. (12), (13) and (14). Therefore, Eq. (22) is satisfied.

4. An illustrative example

We consider the nonlinear Schrödinger equation written in the form (with $i = \sqrt{-1}$) [18]:

$$i\psi_t + \psi_{xx} + 2|\psi|^2\psi = 0. \quad (24)$$

The nonlinear Schrödinger equation is an example of first order field theories. Using $\psi = p + iq$ the discrete Lagrangian density for (24) is

$$\begin{aligned} L_D^{ij} = & \frac{1}{2} \left[(\Delta_x p_{ij})^2 + (\Delta_x q_{ij})^2 + p_{ij} \Delta_t q_{ij} - q_{ij} \Delta_t p_{ij} \right. \\ & \left. - (p_{ij}^2 + q_{ij}^2)^2 \right]. \end{aligned} \quad (25)$$

According to the integrators (12), (13) and (14), the discrete Euler-Lagrange equations are

$$\frac{1}{2} (\Delta_t p_{ij} + \Delta_t p_{ij-1}) + \Delta_x (\Delta_x q_{ij-1})$$

$$+ 2(p_{ij}^2 + q_{ij}^2) q_{ij} = 0, \quad (26)$$

$$\begin{aligned} & \frac{1}{2} (\Delta_t q_{ij} + \Delta_t q_{ij-1}) - \Delta_x (\Delta_x p_{i-1j}) \\ & - 2(p_{ij}^2 + q_{ij}^2) p_{ij} = 0. \end{aligned} \quad (27)$$

The discrete energy evolution equation is

$$\begin{aligned} & \sum_{i=0}^{M-1} \Delta_t \left[(p_{ij-1}^2 + q_{ij-1}^2)^2 - (\Delta_x p_{ij-1})^2 \right. \\ & \left. - (\Delta_x q_{ij-1})^2 \right] = 0. \end{aligned} \quad (28)$$

The discrete momentum evolution equation is

$$\begin{aligned} & \sum_{j=0}^{N-1} \Delta_x \left[(p_{i-1j}^2 + q_{i-1j}^2)^2 + (\Delta_x p_{i-1j})^2 + (\Delta_x q_{i-1j})^2 \right. \\ & \left. - p_{i-1j} \Delta_t q_{i-1j} + q_{i-1j} \Delta_x p_{i-1j} \right] = 0. \end{aligned} \quad (29)$$

Direct verification shows that Eq. (17) of this system is difference invariant under the infinitesimal transformations

$$t_j^* = t_j + \varepsilon, x_i^* = x_i + \varepsilon, u_{ij}^* = u_{ij}. \quad (30)$$

It is the Noether symmetry according to Criterion 1. From the Noether theorem, the Noether symmetry of the Lagrangian system in field theory can possess the discrete conserved quantities directly. According to Theorem 1, the discrete system possesses the discrete conserved quantities

$$\begin{aligned} I_{ij} = & L_D^{i-1j} - \frac{\partial L_D^{i-1j}}{\partial (\Delta_x u_{i-1j})} \Delta_x u_{i-1j} + L_D^{ij-1} \\ & - \frac{\partial L_D^{ij-1}}{\partial (\Delta_t u_{ij-1})} \Delta_t u_{ij-1} = \text{const}. \end{aligned} \quad (31)$$

The difference conserved quantity (31) is the energy of the system. It means the energy is constant.

Choose the infinitesimal transformation and the gauge function

$$t_j^* = t_j + \varepsilon, x_i^* = x_i - \varepsilon, u_{ij}^* = u_{ij}, G_{ij} =$$

$$(p_{ij-1}^2 + q_{ij-1}^2)^2 - (\Delta_x p_{i-1j})^2 - (\Delta_x q_{i-1j-1})^2. \quad (32)$$

Equation (17) is satisfied. The discrete conserved quantity is

$$\begin{aligned} I_{ij} = & (p_{ij-1}^2 + q_{ij-1}^2)^2 - (\Delta_x p_{i-1j})^2 \\ & - (\Delta_x q_{i-1j})^2 = \text{const}. \end{aligned} \quad (33)$$

Noticing the discrete energy evolution Eq. (28), one can also derive that Eq. (33) is conserved. The conserved quantities are consistent to the properties of the nonlinear Schrödinger equation [22].

5. Conclusion

This paper investigates the Noether theorem of discrete Lagrangian systems in field theory. The main results are the integrator (12)–(14), the determining Eqs. (17) and (21) and the conserved quantities (18) and (22). The calculation of the nonlinear Schrödinger equation demonstrates that some conserved quantities can be obtained when the proper symmetries are given.

The difficulty is how to derive the infinitesimal generators and gauge function to satisfy the determining equations.

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