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Supporting information for article:

Combinatorial aspects of the Löwenstein avoidance rule. Part III. The relational system of configurations

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Supplementary Material for Combinatorial aspects of the Löwenstein avoidance rule - Part III: the relational system of configurations

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The analysis of R(sdh) using the full symmetry group pg of the net **sdh** was started in the main text of the paper, where it was shown that the independence ratio associated to homomorphisms restricted to the subsystems $(1, 2, 13)$ and $(1, 2, 3, 13)$ was $7/18$. This supplementary material shows that inclusion of the empty configuration to these subsystems does not increase the independence ratio. For the sake of simplicity, we consider the two subsystems separately.

1 The subsystem $(1, 2, 13, \emptyset)$

Figures 1 and 2 show all the possible neighbourhoods of an empty configuration in a quotient of the relational system $\overrightarrow{\mathbf{pg}}$ partitioned into unsaturated and saturated neighbourhoods, respectively, excluding vertex 3. Thus, in Figures 1(a) and 1(b) one may substitute a configuration 1 or 2, respectively (green arrow), for the empty configuration with the effect of increasing the independence number of the quotient. In contrast, Figure 2 shows five different saturated neighbourhoods and the respective substitutions with the effect of replacing the central empty configuration by configurations 1 or 2, while substituting also one or two neighbours without decreasing the independence ratio. In Figure 2(a), the red arc $(13, \emptyset)$ is substituted by the red arc (1,1). In Figure 2(e) three changes are needed: substitution of configuration 2 for the central empty configuration should be accompanied by the substitution of the blue arc (13,2) by a blue arc (1,1). No substitution of the third vertex is needed

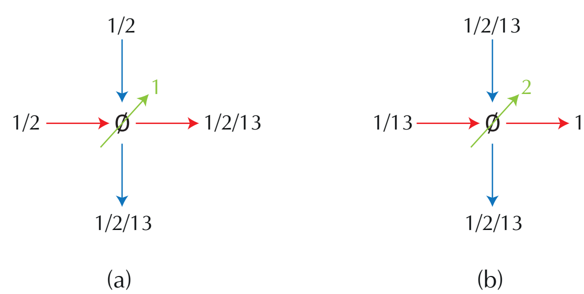

Figure 1: Unsaturated neighbourhoods of an empty configuration in a quotient of the relational system $\overrightarrow{\mathbf{pg}}$ excluding vertex 3; the notation 1/2 means that any of the two vertices 1 or 2 are allowed in the respective position.

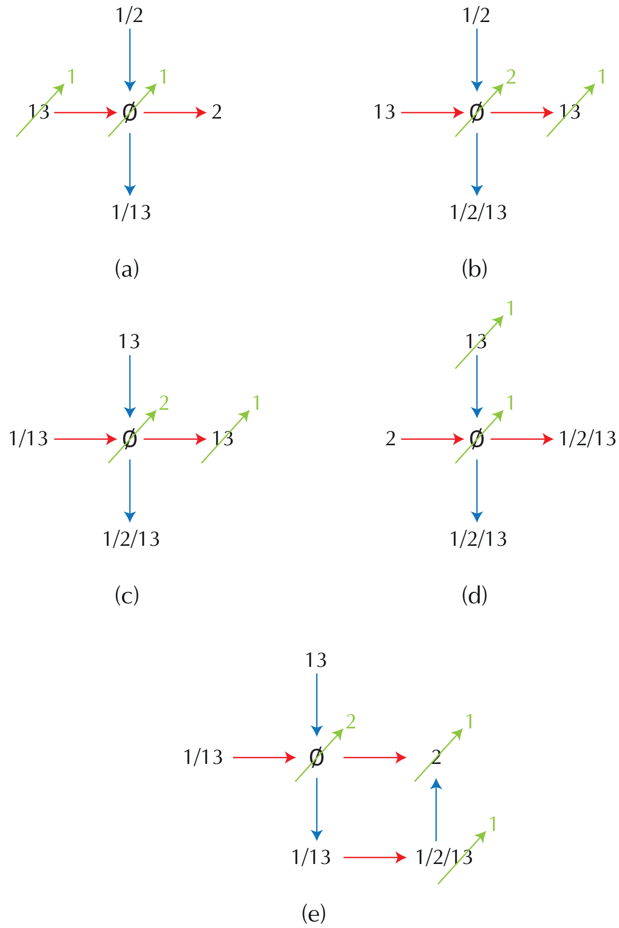

Figure 2: Saturated neighbourhoods of an empty configuration in a quotient of the relational system $\overrightarrow{\mathbf{pg}}$ excluding vertex 3.

if the tail of this blue arc belongs to $\theta^{-1}(1)$ or $\theta^{-1}(2)$. In the five situations, the result is elimination of the central empty configuration without decreasing the independence number. It should also be clear that all described changes do not affect the complementary part of the quotient. For instance, vertex 13 is submitted to the restrictions of the two vertices 1 and 3, so that replacement of a vertex 13 by a vertex 1 is always possible. Such substitution steps can be repeated until complete elimination of the empty configuration, since we only consider finite quotients. This shows that the independence ratio of the system (1,2,13,∅) cannot be larger than that of the subsystem (1,2,13).

2 The full system $(1, 2, 3, 13, \emptyset)$

The argument is similar to the previous one but for the presence of vertices from $\theta^{-1}(3)$ which requires a different analysis. Figures 3 and 4 show all the possible neighbourhoods of an empty configuration in a quotient of the relational system $\overrightarrow{\mathbf{pg}}$ partitioned into unsaturated and saturated neighbourhoods, respectively, this time including also vertex 3. Thus, in Figures 3(a) to 3(d) one may substitute a configuration 1, 2, 3 or 13, respectively, for

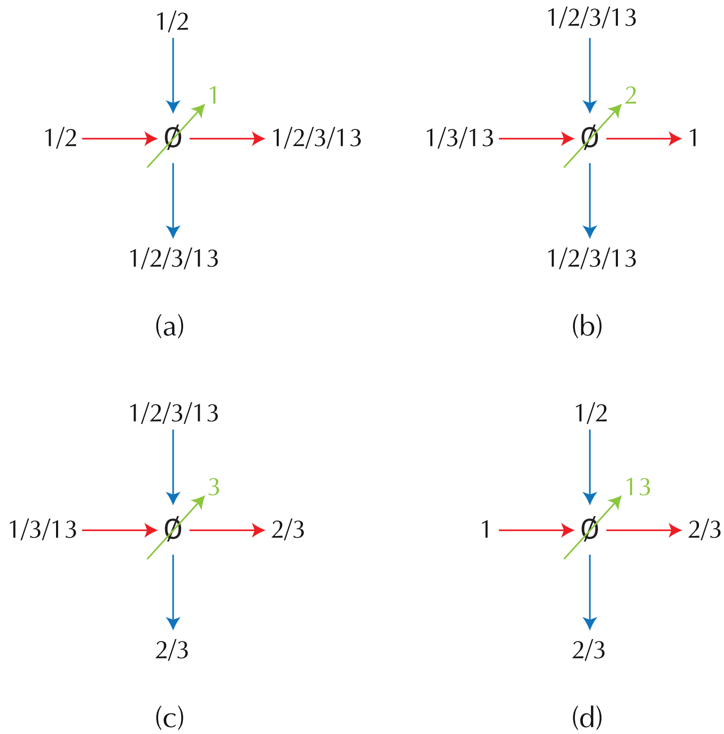

Figure 3: Unsaturated neighbourhoods of an empty configuration in a quotient of the relational system $\overrightarrow{\mathbf{pg}}$.

the empty configuration with the effect of increasing the independence number of the quotient. In contrast, Figure 4 shows five different saturated neighbourhoods and the respective substitutions with the effect of replacing the central empty configuration by configurations 1 or 2, while substituting also one or two neighbours without decreasing the independence ratio. Figure 4(a) describes two situations where i) a red arc $(3, \emptyset)$ is substituted by a red arc $(\emptyset, 1)$ and ii) a red arc $(13, \emptyset)$ is replaced by a red arc (1,1). In Figure 2(e) three changes are needed: substitution of configuration 2 for the central empty configuration should be accompanied by the substitution of a blue arc (13,2/3) by a blue arc (1,1) or of a blue arc (3,2/3) by a blue arc $(\emptyset, 1)$. In the five situations, the result is decreasing the number of vertices in the union set $\theta^{-1}(3) \cup \theta^{-1}(\emptyset)$ without decreasing the independence number. Similar substitution steps can then be repeated until complete elimination of the empty configuration, or of configurations 3, whichever first occurs, since we only consider finite quotients. After the last step, we are left with a homomorphism to one of the subsystems $(1, 2, 3, 13)$ or $(1, 2, 13, \emptyset)$. This shows that the independence ratio of the system (1,2,3,13,∅) cannot be larger than that of any of these subsystems.

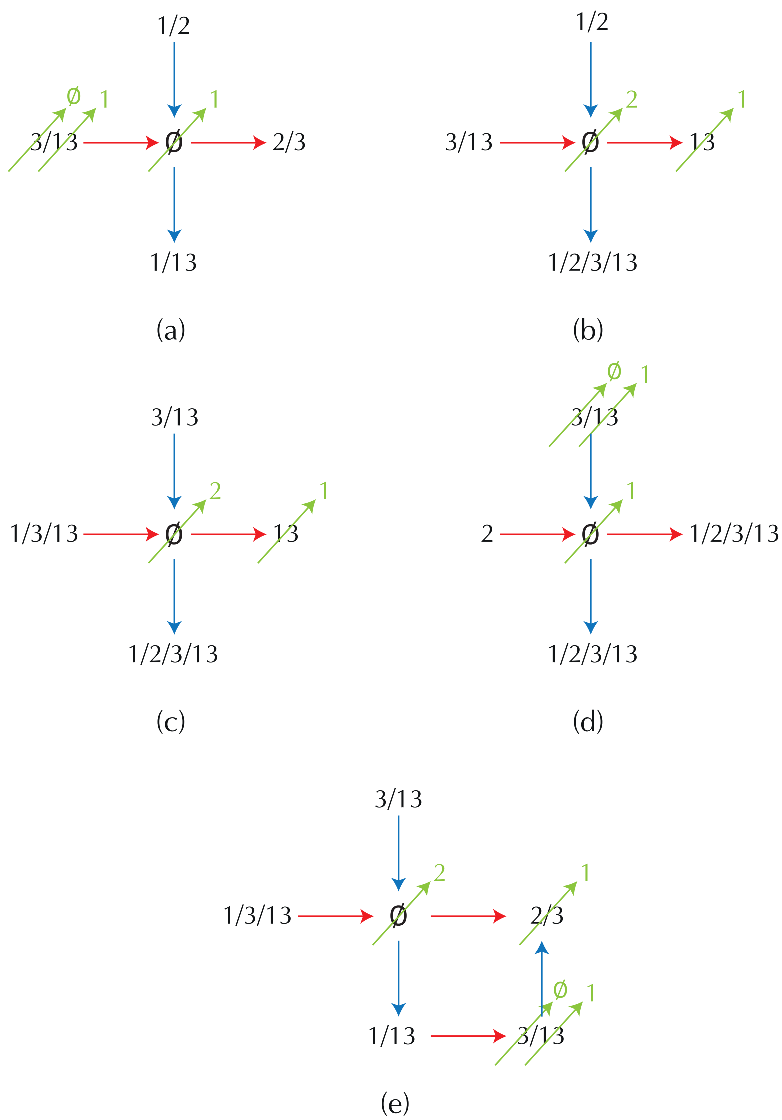

Figure 4: Saturated neighbourhoods of an empty configuration in a quotient of the relational system $\overrightarrow{\mathbf{pg}}$.