EVALUATION OF THE CLD AND CF APPROXIMATIONS FOR AN OCTAHEDRON APPLYING THE FORMULAE OF THE FILE "ANALYTIC_APPRXMTN_FNL_BB.nb"

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The octahedron CF and its first two derivatives have been copied from the file contained in the folder "/STOCHASTICS/PLATONIC_CF"
stored in the memory of 1 Tb

In[1]:=
```
SetDirectory["/Users/salvino/Desktop/WORK_IN_PRGS/PLATONIC_SOLD_CF_APPRXMTN"];
Directory[]
```

Out[2]=
```
/Users/salvino/Desktop/WORK_IN_PRGS/PLATONIC_SOLD_CF_APPRXMTN
```

OCTAHEDRON CF

```
Theta[x_] := If[x > 0, 1, 0];
OctDist = {0, √(2 / 3), √3 / 2, 1, √2};
αOct = ArcCos[-1 / 3]; SOct = 8 * (1 / 2) * 1 * √3 / 2; VOct = √2 / 3;
(* 0 < r < √(2/3) *)
```

$$\text{OctCFAA[r\_] := } 1-\frac{3}{2}\sqrt{\frac{3}{2}}\,r+\frac{3r^2\left(2\sqrt{2}-\pi+\text{ArcCos}\left[-\frac{1}{3}\right]\right)}{2\pi}-\frac{\left(3-3\pi+\sqrt{3}\,\pi\right)r^3}{4\sqrt{2}\,\pi};$$

```
(* OctDCFAA[r_]:=Simplify[D[OctCFAA[r],r],Assumptions→{0 < r < √(2/3)}]; *)
OctDCFAA[r_] :=
```

$$-\frac{1}{8\pi}\,3\left(\pi\left(2\sqrt{6}+8r+\sqrt{2}\left(-3+\sqrt{3}\right)r^2\right)+r\left(3\sqrt{2}\,r-8\left(2\sqrt{2}+\text{ArcCos}\left[-\frac{1}{3}\right]\right)\right)\right);$$

```
(* OctDDCFAA[r_]:=Simplify[D[OctDCFAA[r],r],Assumptions→{0 < r < √(2/3)}]; *)
```

$$\text{OctDDCFAA[r\_] := } -\frac{3\left(3\sqrt{2}\,r+\pi\left(4+\sqrt{2}\left(-3+\sqrt{3}\right)r\right)-4\left(2\sqrt{2}+\text{ArcCos}\left[-\frac{1}{3}\right]\right)\right)}{4\pi};$$

```
(* √(2/3) < r < √3/2 *)
```

$$\text{OctCFBB[r\_] := } -3+\frac{2}{r}\sqrt{\frac{2}{3}}-\frac{1}{2}\sqrt{\frac{3}{2}}\,r+\frac{3r^2\left(2\sqrt{2}+\pi+\text{ArcCos}\left[-\frac{1}{3}\right]\right)}{2\pi}-\frac{\left(3-3\pi+7\sqrt{3}\,\pi\right)r^3}{4\sqrt{2}\,\pi};$$

```
(* OctDCFBB[r_]:=Simplify[D[OctCFBB[r],r],Assumptions→{√(2/3) < r < √3/2}]; *)
OctDCFBB[r_] :=
```

$$\frac{1}{24}\left(-6\sqrt{6}-\frac{16\sqrt{6}}{r^2}-\frac{9\sqrt{2}\left(3+\left(-3+7\sqrt{3}\right)\pi\right)r^2}{\pi}+\frac{72\,r\left(2\sqrt{2}+\pi+\text{ArcCos}\left[-\frac{1}{3}\right]\right)}{\pi}\right);$$

```
(* OctDDCFBB[r_]:=Simplify[D[OctDCFBB[r],r],Assumptions→{√(2/3) < r < √3/2}]; *)
```

$$\text{OctDDCFBB[r\_] := } \frac{1}{24}\left(\frac{32\sqrt{6}}{r^3}-\frac{18\sqrt{2}\left(3+\left(-3+7\sqrt{3}\right)\pi\right)r}{\pi}+\frac{72\left(2\sqrt{2}+\pi+\text{ArcCos}\left[-\frac{1}{3}\right]\right)}{\pi}\right);$$

```
(* √3/2 < r < 1 *)
```

$$\text{OctCFCC}[r\_] := -3 + \frac{2\sqrt{\frac{2}{3}}}{r} + 2\sqrt{6}\, r - \frac{\left(36 - 36\,\pi + 23\sqrt{3}\,\pi\right) r^3}{48\sqrt{2}\,\pi} - \frac{\sqrt{-3+4\,r^2}\,\left(3+17\,r^2\right)}{2\sqrt{2}\,\pi\, r} +$$

$$\frac{3\,r^2\left(2\sqrt{2} + \text{ArcCos}\left[-\frac{1}{3}\right]\right)}{2\,\pi} + \frac{3\,r^2\,\text{ArcTan}\left[\frac{r}{\sqrt{2}*\sqrt{-3+4\,r^2}}\right]}{\pi} -$$

$$\frac{\sqrt{\frac{3}{2}}\, r\left(9\,\text{ArcTan}\left[\frac{\sqrt{3}}{\sqrt{-3+4\,r^2}}\right] - \text{ArcTan}\left[\sqrt{3} * \sqrt{-3+4\,r^2}\right]\right)}{\pi} + \frac{1}{8\sqrt{6}\,\pi} r^3$$

$$\left(18\,\text{ArcTan}\left[\frac{\sqrt{3}}{\sqrt{-3+4\,r^2}}\right] - 90\,\text{ArcTan}\left[\frac{1}{\sqrt{3} * \sqrt{-3+4\,r^2}}\right] + 8\,\text{ArcTan}\left[\frac{-3+2\,r^2}{\sqrt{3} * \sqrt{-3+4\,r^2}}\right] -\right.$$

$$24\,\text{ArcTan}\left[\frac{-5+6\,r^2}{\sqrt{3} * \sqrt{-3+4\,r^2}}\right] + \text{ArcTan}\left[\frac{-9+12\,r^2-2\,r^4}{\sqrt{3} * \left(-3+2\,r^2\right) * \sqrt{-3+4\,r^2}}\right] +$$

$$\left.6\,\text{ArcTan}\left[\frac{\sqrt{3}\,\left(27-90\,r^2+96\,r^4-34\,r^6+2\,r^8\right)}{\sqrt{-3+4\,r^2}\,\left(-27+72\,r^2-54\,r^4+10\,r^6\right)}\right]\right);$$

$$(*\ \text{OctDCFCC}[r\_]:=\text{Simplify}\left[\text{D}[\text{OctCFCC}[r],r],\text{Assumptions}\to\left\{\sqrt{3}\big/2 < r < 1\right\}\right];\ *)$$

$$\text{OctDCFCC}[r\_] :=$$

$$\frac{1}{96}\left(192\sqrt{6} - \frac{64\sqrt{6}}{r^2} + \frac{576\sqrt{2}\, r}{\pi} + 108\sqrt{2}\, r^2 - 69\sqrt{6}\, r^2 - \frac{108\sqrt{2}\, r^2}{\pi} + \frac{2520\sqrt{2}}{\pi\sqrt{-3+4\,r^2}} -\right.$$

$$\frac{216\sqrt{2}}{\pi\, r^2\sqrt{-3+4\,r^2}} - \frac{2976\sqrt{2}\, r^2}{\pi\sqrt{-3+4\,r^2}} + \frac{288\, r\,\text{ArcCos}\left[-\frac{1}{3}\right]}{\pi} + \frac{108\sqrt{6}\,\left(-4+r^2\right)\text{ArcTan}\left[\frac{\sqrt{3}}{\sqrt{-3+4\,r^2}}\right]}{\pi} +$$

$$\frac{576\, r\,\text{ArcTan}\left[\frac{r}{\sqrt{-6+8\,r^2}}\right]}{\pi} - \frac{540\sqrt{6}\, r^2\,\text{ArcTan}\left[\frac{1}{\sqrt{-9+12\,r^2}}\right]}{\pi} + \frac{48\sqrt{6}\, r^2\,\text{ArcTan}\left[\frac{-3+2\,r^2}{\sqrt{-9+12\,r^2}}\right]}{\pi} -$$

$$\frac{144\sqrt{6}\, r^2\,\text{ArcTan}\left[\frac{-5+6\,r^2}{\sqrt{-9+12\,r^2}}\right]}{\pi} + \frac{48\sqrt{6}\,\text{ArcTan}\left[\sqrt{-9+12\,r^2}\right]}{\pi} +$$

$$\frac{6\sqrt{6}\ r^2\ \text{ArcTan}\left[\frac{-9+12\ r^2-2\ r^4}{\left(-3+2\ r^2\right)\sqrt{-9+12\ r^2}}\right]}{\pi}+\frac{36\sqrt{6}\ r^2\ \text{ArcTan}\left[\frac{\sqrt{3}\ \left(27-90\ r^2+96\ r^4-34\ r^6+2\ r^8\right)}{\sqrt{-3+4\ r^2}\ \left(-27+72\ r^2-54\ r^4+10\ r^6\right)}\right]}{\pi}\Bigg);$$

$$\text{(* OctDDCFCC[r\_]:=Simplify}\left[\text{D[OctDCFCC[r],r],Assumptions}\to\left\{\sqrt{3}\big/2<r<1\right\}\right];\ \text{*)}$$

$$\text{OctDDCFCC[r\_]} :=$$

$$\frac{1}{48}\Bigg(\frac{288\sqrt{2}}{\pi}+\frac{64\sqrt{6}}{r^3}+108\sqrt{2}\ r-69\sqrt{6}\ r-\frac{108\sqrt{2}\ r}{\pi}+\frac{216\sqrt{2}}{\pi\ r^3\sqrt{-3+4\ r^2}}+\frac{504\sqrt{2}}{\pi\ r\sqrt{-3+4\ r^2}}-$$

$$\frac{1056\sqrt{2}\ r}{\pi\sqrt{-3+4\ r^2}}+\frac{144\ \text{ArcCos}\left[-\frac{1}{3}\right]}{\pi}+\frac{108\sqrt{6}\ r\ \text{ArcTan}\left[\frac{\sqrt{3}}{\sqrt{-3+4\ r^2}}\right]}{\pi}+\frac{288\ \text{ArcTan}\left[\frac{r}{\sqrt{-6+8\ r^2}}\right]}{\pi}-$$

$$\frac{540\sqrt{6}\ r\ \text{ArcTan}\left[\frac{1}{\sqrt{-9+12\ r^2}}\right]}{\pi}+\frac{48\sqrt{6}\ r\ \text{ArcTan}\left[\frac{-3+2\ r^2}{\sqrt{-9+12\ r^2}}\right]}{\pi}-\frac{144\sqrt{6}\ r\ \text{ArcTan}\left[\frac{-5+6\ r^2}{\sqrt{-9+12\ r^2}}\right]}{\pi}+$$

$$\frac{6\sqrt{6}\ r\ \text{ArcTan}\left[\frac{-9+12\ r^2-2\ r^4}{\left(-3+2\ r^2\right)\sqrt{-9+12\ r^2}}\right]}{\pi}+\frac{36\sqrt{6}\ r\ \text{ArcTan}\left[\frac{\sqrt{3}\ \left(27-90\ r^2+96\ r^4-34\ r^6+2\ r^8\right)}{\sqrt{-3+4\ r^2}\ \left(-27+72\ r^2-54\ r^4+10\ r^6\right)}\right]}{\pi}\Bigg);$$

$$\text{(* }1<r<\sqrt{2}\text{ *)}$$

$$\text{OctCFDD[r\_]} := 3-\frac{3+2\sqrt{3}\ \pi}{3\sqrt{2}\ \pi\ r}+\frac{\left(-9+4\sqrt{3}\ \pi\right)r}{3\sqrt{2}\ \pi}+\frac{3\ r^2}{4}-$$

$$\frac{\left(6-3\ \pi+4\sqrt{3}\ \pi\right)r^3}{8\sqrt{2}\ \pi}+\frac{\sqrt{2}\sqrt{-1+r^2}\ \left(1+2\ r^2\right)}{\pi\ r}+\frac{2\sqrt{6}\ \text{ArcTan}\left[\frac{1}{\sqrt{3}\sqrt{-1+r^2}}\right]}{\pi\ r}-$$

$$\frac{2\left(6\ \text{ArcTan}\left[\frac{r}{\sqrt{2}\sqrt{-1+r^2}}\right]+\sqrt{6}\ r\ \text{ArcTan}\left[\sqrt{3}\sqrt{-1+r^2}\right]\right)}{\pi}+\frac{3\ r^2\ \text{ArcTan}\left[\frac{-4-4\ r^2+7\ r^4}{4\sqrt{2}\ r\ \left(-2+r^2\right)\sqrt{-1+r^2}}\right]}{2\ \pi}-$$

$$\frac{1}{2\sqrt{2}\ \pi}\ r^3\left(2\sqrt{3}\ \text{ArcTan}\left[\frac{-10+9\ r^2}{3\sqrt{3}\ \left(-2+r^2\right)\sqrt{-1+r^2}}\right]+3\ \text{ArcTan}\left[\frac{-2+r^2+2\sqrt{-1+r^2}}{2-r^2+2\sqrt{-1+r^2}}\right]\right);$$

(* OctDCFDD[r_]:=Simplify[D[OctCFDD[r],r],Assumptions→{1 < r < $\sqrt{2}$}]; *)

$$\text{OctDCFDD[r\_] := } 2\sqrt{\frac{2}{3}} - \frac{3}{\sqrt{2}\,\pi} + \frac{\sqrt{\frac{2}{3}}}{r^2} + \frac{1}{\sqrt{2}\,\pi\,r^2} + \frac{3\,r}{2} - \frac{3}{2}\sqrt{\frac{3}{2}}\,r^2 + \frac{9\,r^2}{8\sqrt{2}} - \frac{9\,r^2}{4\sqrt{2}\,\pi} - \frac{5\sqrt{2}}{\pi\sqrt{-1+r^2}} + \frac{\sqrt{2}}{\pi\,r^2\sqrt{-1+r^2}} + \frac{4\sqrt{2}\,r^2}{\pi\sqrt{-1+r^2}} - \frac{2\sqrt{6}\,\text{ArcTan}\left[\frac{1}{\sqrt{3}\,\sqrt{-1+r^2}}\right]}{\pi\,r^2} - \frac{2\sqrt{6}\,\text{ArcTan}\left[\sqrt{3}\,\sqrt{-1+r^2}\right]}{\pi} - \frac{3\sqrt{\frac{3}{2}}\,r^2\,\text{ArcTan}\left[\frac{-10+9\,r^2}{3\sqrt{3}\,(-2+r^2)\sqrt{-1+r^2}}\right]}{\pi} + \frac{3\,r\,\text{ArcTan}\left[\frac{-4-4\,r^2+7\,r^4}{4\sqrt{2}\,r\,(-2+r^2)\sqrt{-1+r^2}}\right]}{\pi} - \frac{9\,r^2\,\text{ArcTan}\left[\frac{-2+r^2+2\sqrt{-1+r^2}}{2-r^2+2\sqrt{-1+r^2}}\right]}{2\sqrt{2}\,\pi};$$

(* OctDDCFDD[r_]:=Simplify[D[OctDCFDD[r],r],Assumptions→{1 < r < $\sqrt{2}$}]; *)

$$\text{OctDDCFDD[r\_] := } \frac{3}{2} - \frac{2\sqrt{\frac{2}{3}}}{r^3} - \frac{\sqrt{2}}{\pi\,r^3} - 3\sqrt{\frac{3}{2}}\,r + \frac{9\,r}{4\sqrt{2}} - \frac{9\,r}{2\sqrt{2}\,\pi} - \frac{2\sqrt{2}}{\pi\,r^3\sqrt{-1+r^2}} - \frac{2\sqrt{2}}{\pi\,r\sqrt{-1+r^2}} + \frac{4\sqrt{2}\,r}{\pi\sqrt{-1+r^2}} + \frac{4\sqrt{6}\,\text{ArcTan}\left[\frac{1}{\sqrt{3}\,\sqrt{-1+r^2}}\right]}{\pi\,r^3} - \frac{3\sqrt{6}\,r\,\text{ArcTan}\left[\frac{-10+9\,r^2}{3\sqrt{3}\,(-2+r^2)\sqrt{-1+r^2}}\right]}{\pi} + \frac{3\,\text{ArcTan}\left[\frac{-4-4\,r^2+7\,r^4}{4\sqrt{2}\,r\,(-2+r^2)\sqrt{-1+r^2}}\right]}{\pi} - \frac{9\,r\,\text{ArcTan}\left[\frac{-2+r^2+2\sqrt{-1+r^2}}{2-r^2+2\sqrt{-1+r^2}}\right]}{\sqrt{2}\,\pi};$$

```
OctCFTot[r_] :=
  Theta[√2 / 3 - r] * OctCFAA[r] + Theta[r - √2 / 3] * Theta[√3 / 2 - r] * OctCFBB[r] +
   Theta[r - √3 / 2] * Theta[1 - r] * OctCFCC[r] + Theta[r - 1] * Theta[√2 - r] * OctCFDD[r];
OctDCFTot[r_] := Theta[√2 / 3 - r] * OctDCFAA[r] +
   Theta[r - √2 / 3] * Theta[√3 / 2 - r] * OctDCFBB[r] +
   Theta[r - √3 / 2] * Theta[1 - r] * OctDCFCC[r] + Theta[r - 1] * Theta[√2 - r] * OctDCFDD[r];
OctDDCFTot[r_] := Theta[√2 / 3 - r] * OctDDCFAA[r] +
```

```
Theta[r - √(2/3)] * Theta[√3 / 2 - r] * OctDDCFBB[r] + Theta[r - √3 / 2] *
  Theta[1 - r] * OctDDCFCC[r] + Theta[r - 1] * Theta[√2 - r] * OctDDCFDD[r];
OctDDCFTot[r_] := If[r < √(2/3), OctDDCFAA[r], If[r < √3 / 2, OctDDCFBB[r],
   If[r < 1, OctDDCFCC[r], If[r < √2, OctDDCFDD[r]]]]];
```

We scale the variable r so as its full range is [0, 1].

```
FullSimplify[OctDist / OctDist[[5]] - {0, 1/√3, √(3/8), 1/√2, 1}];

N[{0, 1/√3, √(3/8), 1/√2, 1}]

FullSimplify[OctCFAA[r * OctDist[[5]]] -
  (1 - (3 √3 r)/2 - ((3 + (-3 + √3) π) r^3)/(2 π) + (3 r^2 (2 √2 - π + ArcSec[-3]))/π)] ;
FullSimplify[OCTCFAA[OCTDst[[1]]] - OctCFAA[OctDist[[2]]]] ;

D[OCTCFAA[r], r]
D[D[OCTCFAA[r], r] , r]

FullSimplify[OctCFBB[r * OctDist[[5]]] - OCTCFBB[r]]

Simplify[OctCFCC[r * OctDist[[5]]] - OCTCFCC[r]]

D[OCTCFBB[r], r]
D[D[OCTCFBB[r], r] , r]

Simplify[D[OCTCFCC[r], r], Assumptions → {OCTDst[[2]] < r < OCTDst[[3]]}]
Simplify[D[D[OCTCFCC[r], r] , r]  , Assumptions → {OCTDst[[2]] < r < OCTDst[[3]]}]

Simplify[
 (Simplify[OctCFDD[r * OctDist[[5]]]]) /. {ArcTan[r/√(-1 + 2 r^2)] → ArcSin[r/√(-1 + 3 r^2)],
   ArcTan[1/√(-3 + 6 r^2)] → ArcSin[1/(√2 √(3 r^2 - 1))],
   ArcTan[√(-3 + 6 r^2)] → ArcSin[√(3/2) √((2 r^2 - 1)/(3 r^2 - 1))], ArcTan[(-5 + 9 r^2)/(3 (-1 + r^2) √(-3 + 6 r^2))] →
    ArcSin[(5 - 9 r^2)/(√2 (-1 + 3 r^2)^(3/2))], ArcTan[(-1 - 2 r^2 + 7 r^4)/(4 r (-1 + r^2) √(-1 + 2 r^2))] → ArcSin[(1 + 2 r^2 - 7 r^4)/(1 - 3 r^2)^2],
   ArcTan[(-1 + r^2 + √(-1 + 2 r^2))/(1 - r^2 + √(-1 + 2 r^2))] → ArcSin[(-1 + r^2 + √(-1 + 2 r^2))/(√2 r^2)]},
 Assumptions → {OCTDst[[3]] < r < OCTDst[[4]]}]
```

```
OCTCFDD[r_] := -1/(12 π r)
  (6 + 4 Sqrt[3] π - 36 π r + 36 r^2 - 16 Sqrt[3] π r^2 - 18 π r^3 + 18 r^4 - 9 π r^4 + 12 Sqrt[3] π r^4 - 12 Sqrt[-1 + 2 r^2] -
    48 r^2 Sqrt[-1 + 2 r^2] + 48 Sqrt[3] r^2 ArcSin[Sqrt[(3 - 6 r^2)/(2 - 6 r^2)]] + 24 Sqrt[3] r^4 ArcSin[(5 - 9 r^2)/(Sqrt[2] (-1 + 3 r^2)^(3/2))] +
    144 r ArcSin[r/Sqrt[-1 + 3 r^2]] - 24 Sqrt[3] ArcSin[1/Sqrt[-2 + 6 r^2]] -
    36 r^3 ArcSin[(1 + 2 r^2 - 7 r^4)/(1 - 3 r^2)^2] + 36 r^4 ArcSin[(-1 + r^2 + Sqrt[-1 + 2 r^2])/(Sqrt[2] r^2)]);

(r^4 + 2 Sqrt[-1 + 2 r^2] - 2 r^2 Sqrt[-1 + 2 r^2])

Reduce[{Sqrt[-1 + 2 r^2] - (r^2 - 1) > 0 && OCTDst[[3]] < r < OCTDst[[4]]}, r, Reals]

Expand[(Sqrt[-1 + 2 r^2] - (r^2 - 1))^2]

Simplify[(Simplify[D[OCTCFDD[r], r], Assumptions → {OCTDst[[3]] < r < OCTDst[[4]]}]) /.
  {72 r^4/Sqrt[r^4 + 2 Sqrt[-1 + 2 r^2] - 2 r^2 Sqrt[-1 + 2 r^2]] → 72 r^4/(Sqrt[-1 + 2 r^2] - (r^2 - 1)),
   1/Sqrt[(-1 + 2 r^2) (r^4 + 2 Sqrt[-1 + 2 r^2] - 2 r^2 Sqrt[-1 + 2 r^2])] → 1/((Sqrt[-1 + 2 r^2] - (r^2 - 1)) Sqrt[-1 + 2 r^2])}, ,
  Assumptions → {OCTDst[[3]] < r < OCTDst[[4]]}]

OCTDCFDD[r_] := 4/Sqrt[3] - 3/π + 1/(Sqrt[3] r^2) + 1/(2 π r^2) + 3 r + (9 r^2)/4 - 3 Sqrt[3] r^2 - (9 r^2)/(2 π) - 10/(π Sqrt[-1 + 2 r^2]) +
   1/(π r^2 Sqrt[-1 + 2 r^2]) + (16 r^2)/(π Sqrt[-1 + 2 r^2]) - (4 Sqrt[3] ArcSin[Sqrt[(3 - 6 r^2)/(2 - 6 r^2)]])/π - (6 Sqrt[3] r^2 ArcSin[(5 - 9 r^2)/(Sqrt[2] (-1 + 3 r^2)^(3/2))])/π -
   (2 Sqrt[3] ArcSin[1/Sqrt[-2 + 6 r^2]])/(π r^2) + (6 r ArcSin[(1 + 2 r^2 - 7 r^4)/(1 - 3 r^2)^2])/π - (9 r^2 ArcSin[(-1 + r^2 + Sqrt[-1 + 2 r^2])/(Sqrt[2] r^2)])/π;

Simplify[(Simplify[D[OCTDCFDD[r], r], Assumptions → {OCTDst[[3]] < r < OCTDst[[4]]}]) /.
  {1/Sqrt[(-1 + 2 r^2) (r^4 + 2 Sqrt[-1 + 2 r^2] - 2 r^2 Sqrt[-1 + 2 r^2])] → 1/((Sqrt[-1 + 2 r^2] - (r^2 - 1)) Sqrt[-1 + 2 r^2])},
  Assumptions → {OCTDst[[3]] < r < OCTDst[[4]]}]
```

$$3-\frac{2}{\sqrt{3}\ r^3}-\frac{1}{\pi r^3}+\frac{9 r}{2}-6\sqrt{3}\ r-\frac{9 r}{\pi}+\text{Factor}\left[-\frac{2}{\pi r^3\sqrt{-1+2 r^2}}-\frac{4}{\pi r\sqrt{-1+2 r^2}}+\frac{16 r}{\pi\sqrt{-1+2 r^2}}\right]+$$

$$\text{Factor}\left[-\frac{12\sqrt{3}\ r\,\text{ArcSin}\left[\frac{5-9 r^2}{\sqrt{2}\ \left(-1+3 r^2\right)^{3/2}}\right]}{\pi}+\frac{4\sqrt{3}\ \text{ArcSin}\left[\frac{1}{\sqrt{-2+6 r^2}}\right]}{\pi r^3}+\frac{6\,\text{ArcSin}\left[\frac{1+2 r^2-7 r^4}{\left(1-3 r^2\right)^2}\right]}{\pi}-\frac{18 r\,\text{ArcSin}\left[\frac{-1+r^2+\sqrt{-1+2 r^2}}{\sqrt{2}\ r^2}\right]}{\pi}\right]$$

$$\text{OCTDDCFDD[r\_]} := 3-\frac{2}{\sqrt{3}\ r^3}-\frac{1}{\pi r^3}+\frac{9 r}{2}-6\sqrt{3}\ r-\frac{9 r}{\pi}+\frac{2\sqrt{-1+2 r^2}\ \left(1+4 r^2\right)}{\pi r^3}-$$
$$\frac{1}{\pi r^3}\,2\left(6\sqrt{3}\ r^4\,\text{ArcSin}\left[\frac{5-9 r^2}{\sqrt{2}\ \left(-1+3 r^2\right)^{3/2}}\right]-2\sqrt{3}\ \text{ArcSin}\left[\frac{1}{\sqrt{-2+6 r^2}}\right]-\right.$$
$$\left.3 r^3\,\text{ArcSin}\left[\frac{1+2 r^2-7 r^4}{\left(1-3 r^2\right)^2}\right]+9 r^4\,\text{ArcSin}\left[\frac{-1+r^2+\sqrt{-1+2 r^2}}{\sqrt{2}\ r^2}\right]\right);$$

```
Plot[OCTCFDD[r] - OCTCFDDOld[r], {r, OCTDst[[3]], OCTDst[[4]]}]
Plot[OCTDCFDD[r] - OCTDCFDDOld[r], {r, OCTDst[[3]], OCTDst[[4]]}]
Plot[OCTDDCFDD[r] - OCTDDCFDDOld[r], {r, OCTDst[[3]], OCTDst[[4]]}]
```

$$\text{Simplify}\left[\text{TrigExpand}\left[\text{Sin}\left[\text{ArcTan}\left[\frac{r}{\sqrt{-1+2 r^2}}\right]\right]\right]-\frac{r}{\sqrt{-1+3 r^2}},\ \text{Assumptions}\to\{\text{OCTDst[[3]]}<r<\text{OCTDst[[4]]}\}\right]$$

$$\text{Simplify}\left[\text{TrigExpand}\left[\text{Sin}\left[\text{ArcTan}\left[\frac{1}{\sqrt{-3+6 r^2}}\right]\right]\right]-\frac{1}{\sqrt{2}\ \sqrt{3 r^2-1}},\ \text{Assumptions}\to\{\text{OCTDst[[3]]}<r<\text{OCTDst[[4]]}\}\right]$$

$$\text{Simplify}\left[\text{TrigExpand}\left[\text{Sin}\left[\text{ArcTan}\left[\sqrt{-3+6 r^2}\right]\right]\right]-\sqrt{\frac{3}{2}}\ \sqrt{\frac{2 r^2-1}{3 r^2-1}},\ \text{Assumptions}\to\{\text{OCTDst[[3]]}<r<\text{OCTDst[[4]]}\}\right]$$

$$\text{Simplify}\left[\text{TrigExpand}\left[\text{Sin}\left[\text{ArcTan}\left[\frac{-5+9 r^2}{3\left(-1+r^2\right)\sqrt{-3+6 r^2}}\right]\right]\right]-\frac{5-9 r^2}{\sqrt{2}\ \left(-1+3 r^2\right)^{3/2}},\ \text{Assumptions}\to\{\text{OCTDst[[3]]}<r<\text{OCTDst[[4]]}\}\right]$$

$$\text{Simplify}\left[\text{TrigExpand}\left[\text{Sin}\left[\text{ArcTan}\left[\frac{-1-2 r^2+7 r^4}{4 r\left(-1+r^2\right)\sqrt{-1+2 r^2}}\right]\right]\right]-\frac{1+2 r^2-7 r^4}{\left(1-3 r^2\right)^2},\ \text{Assumptions}\to\{\text{OCTDst[[3]]}<r<\text{OCTDst[[4]]}\}\right]$$

$$\text{Simplify}\left[\text{TrigExpand}\left[\text{Sin}\left[\text{ArcTan}\left[\frac{-1+r^2+\sqrt{-1+2\,r^2}}{1-r^2+\sqrt{-1+2\,r^2}}\right]\right]\right]-\frac{-1+r^2+\sqrt{-1+2\,r^2}}{\sqrt{2}\,r^2},\right.$$
$$\left.\text{Assumptions}\to\{\text{OCTDst}[[3]]<r<\text{OCTDst}[[4]]\}\right]$$

```
Simplify[OctCFDD[r * OctDist[[5]]] - OCTCFDD[r]]

Simplify[D[OCTCFDD[r], r], Assumptions → {OCTDst[[3]] < r < OCTDst[[4]]}]
Simplify[D[D[OCTCFDD[r], r] , r] , Assumptions → {OCTDst[[3]] < r < OCTDst[[4]]}]
```

OCTCFDDOld[r_] :=

$$-\frac{1}{12\,\pi\,r}\Bigg(6+4\sqrt{3}\,\pi-36\,\pi\,r+36\,r^2-16\sqrt{3}\,\pi\,r^2-18\,\pi\,r^3+18\,r^4-9\,\pi\,r^4+12\sqrt{3}\,\pi\,r^4-$$
$$12\sqrt{-1+2\,r^2}-48\,r^2\sqrt{-1+2\,r^2}+144\,r\,\text{ArcTan}\left[\frac{r}{\sqrt{-1+2\,r^2}}\right]-24\sqrt{3}\,\text{ArcTan}\left[\frac{1}{\sqrt{-3+6\,r^2}}\right]+$$
$$48\sqrt{3}\,r^2\,\text{ArcTan}\left[\sqrt{-3+6\,r^2}\right]+24\sqrt{3}\,r^4\,\text{ArcTan}\left[\frac{-5+9\,r^2}{3\left(-1+r^2\right)\sqrt{-3+6\,r^2}}\right]-$$
$$36\,r^3\,\text{ArcTan}\left[\frac{-1-2\,r^2+7\,r^4}{4\,r\left(-1+r^2\right)\sqrt{-1+2\,r^2}}\right]+36\,r^4\,\text{ArcTan}\left[\frac{-1+r^2+\sqrt{-1+2\,r^2}}{1-r^2+\sqrt{-1+2\,r^2}}\right]\Bigg);$$

$$\text{OCTDCFDDOld}[r\_]:=\frac{4}{\sqrt{3}}-\frac{3}{\pi}+\frac{1}{\sqrt{3}\,r^2}+\frac{1}{2\,\pi\,r^2}+3\,r+\frac{9\,r^2}{4}-3\sqrt{3}\,r^2-\frac{9\,r^2}{2\,\pi}-\frac{10}{\pi\sqrt{-1+2\,r^2}}+$$
$$\frac{1}{\pi\,r^2\sqrt{-1+2\,r^2}}+\frac{16\,r^2}{\pi\sqrt{-1+2\,r^2}}-\frac{2\sqrt{3}\,\text{ArcTan}\left[\frac{1}{\sqrt{-3+6\,r^2}}\right]}{\pi\,r^2}-\frac{4\sqrt{3}\,\text{ArcTan}\left[\sqrt{-3+6\,r^2}\right]}{\pi}-$$
$$\frac{6\sqrt{3}\,r^2\,\text{ArcTan}\left[\frac{-5+9\,r^2}{3\left(-1+r^2\right)\sqrt{-3+6\,r^2}}\right]}{\pi}+\frac{6\,r\,\text{ArcTan}\left[\frac{-1-2\,r^2+7\,r^4}{4\,r\left(-1+r^2\right)\sqrt{-1+2\,r^2}}\right]}{\pi}-\frac{9\,r^2\,\text{ArcTan}\left[\frac{-1+r^2+\sqrt{-1+2\,r^2}}{1-r^2+\sqrt{-1+2\,r^2}}\right]}{\pi};$$

$$\text{OCTDDCFDDOld}[r\_]:=3-\frac{2}{\sqrt{3}\,r^3}-\frac{1}{\pi\,r^3}+\frac{9\,r}{2}-6\sqrt{3}\,r-\frac{9\,r}{\pi}-\frac{2}{\pi\,r^3\sqrt{-1+2\,r^2}}-$$
$$\frac{4}{\pi\,r\sqrt{-1+2\,r^2}}+\frac{16\,r}{\pi\sqrt{-1+2\,r^2}}+\frac{4\sqrt{3}\,\text{ArcTan}\left[\frac{1}{\sqrt{-3+6\,r^2}}\right]}{\pi\,r^3}-$$
$$\frac{12\sqrt{3}\,r\,\text{ArcTan}\left[\frac{-5+9\,r^2}{3\left(-1+r^2\right)\sqrt{-3+6\,r^2}}\right]}{\pi}+\frac{6\,\text{ArcTan}\left[\frac{-1-2\,r^2+7\,r^4}{4\,r\left(-1+r^2\right)\sqrt{-1+2\,r^2}}\right]}{\pi}-\frac{18\,r\,\text{ArcTan}\left[\frac{-1+r^2+\sqrt{-1+2\,r^2}}{1-r^2+\sqrt{-1+2\,r^2}}\right]}{\pi};$$

```
N[ArcSec[-3] - (π - ArcSec[3])]
```

$$N\left[\text{ArcCos}\left[-\frac{1}{3}\right]-(\pi-\text{ArcSec}[3])\right]$$

```
Simplify[(OCTCFAA[r]) /. {ArcSec[-3] → π - ArcSec[3]}]
```

```
FullSimplify[OCTDst[[2]] - OCTDst[[1]] - ((3 - 2 Sqrt[2])/(2 Sqrt[6]))]

FullSimplify[OCTDst[[3]] - OCTDst[[2]] - ((2 - Sqrt[3])/(2 Sqrt[2]))]

OCTDst[[4]] - OCTDst[[3]] - ((2 - Sqrt[3])/(2 Sqrt[2]))

FullSimplify[OCTDst[[4]] - OCTDst[[3]] - ((Sqrt[2] - 1)/Sqrt[2])]

ΔΔ = {1/Sqrt[3], (3 - 2 Sqrt[2])/(2 Sqrt[6]), (2 - Sqrt[3])/(2 Sqrt[2]), (Sqrt[2] - 1)/Sqrt[2]}
```

$$\left\{\frac{27-180r^2+384r^4-272r^6+32r^8}{\sqrt{-1+\frac{8r^2}{3}}\left(-27+144r^2-216r^4+80r^6\right)}\to\frac{\sqrt{3}\left(27-180r^2+384r^4-272r^6+32r^8\right)}{\sqrt{-3+8r^2}\left(-27+144r^2-216r^4+80r^6\right)},\right.$$
$$\frac{1}{\sqrt{-1+\frac{8r^2}{3}}}\to\frac{\sqrt{3}}{\sqrt{-3+8r^2}},\ \frac{-3+4r^2}{\sqrt{-9+24r^2}}\to\frac{-3+4r^2}{\sqrt{3}\sqrt{-3+8r^2}},$$
$$\left.\frac{1}{\sqrt{-9+24r^2}}\to\frac{1}{\sqrt{3}\sqrt{-3+8r^2}},\ \sqrt{-9+24r^2}\to\sqrt{3}\sqrt{-3+8r^2}\right\}$$

```
ccccf = Simplify[CoefficientList[
   Expand[27 - 180 r^2 + 384 r^4 - 272 r^6 + 32 r^8 - (a + b r^2) (-27 + 144 r^2 - 216 r^4 + 80 r^6)], r^2]]
```

{27 (1 + a), -9 (20 + 16 a - 3 b), 24 (16 + 9 a - 6 b), -8 (34 + 10 a - 27 b), 32 - 80 b}

```
Solve[{ccccf[[1]] == 0 && ccccf[[2]] == 0 && ccccf[[3]] == 0 && ccccf[[4]] == 0 && ccccf[[5]] == 0},
  {a, b}]
```

$\{27+27a+27br^2,\ 0,\ -180-144a-144br^2,\ 0,\ 384+216a+216br^2,\ 0,\ -272-80a-80br^2,\ 0,\ 32\}$

```
Simplify[D[OCTDCFCC[r], r], Assumptions → {OCTDst[[2]] < r < OCTDst[[3]]}]

D[D[Simplify[OCTCFBB[r]], r], r]
```

In[3]:=

$$\texttt{Theta[x\_] := If[x > 0, 1, 0];}$$

$$\texttt{OCTDst} = \left\{\frac{1}{\sqrt{3}}, \sqrt{\frac{3}{8}}, \frac{1}{\sqrt{2}}, 1\right\};$$

$$\Delta\Delta = \left\{\frac{1}{\sqrt{3}}, \frac{3-2\sqrt{2}}{2\sqrt{6}}, \frac{2-\sqrt{3}}{2\sqrt{2}}, \frac{\sqrt{2}-1}{\sqrt{2}}\right\};\ \texttt{VOcth = VOct * }\left(1\Big/\sqrt{2}\right)\texttt{\^{}3;}$$

$$\texttt{OCTCFAA[r\_] := } 1 - \frac{3\sqrt{3}\,r}{2} + \frac{6\sqrt{2}\,r^2}{\pi} - \frac{\left(3-3\pi+\sqrt{3}\,\pi\right)r^3}{2\pi} - \frac{3\,r^2\,\text{ArcSec}[3]}{\pi};$$

$$\texttt{OCTDCFAA[r\_] := } -\frac{3\sqrt{3}}{2} + \frac{12\sqrt{2}\,r}{\pi} - \frac{3\left(3-3\pi+\sqrt{3}\,\pi\right)r^2}{2\pi} - \frac{6\,r\,\text{ArcSec}[3]}{\pi};$$

$$\texttt{OCTDDCFAA[r\_] := } \frac{6\left(2\sqrt{2}-\text{ArcSec}[3]\right)}{\pi} - \frac{3\left(3-3\pi+\sqrt{3}\,\pi\right)r}{\pi};$$

$$\texttt{OCTCFBB[r\_] := } -3 + \frac{2}{\sqrt{3}\,r} - \frac{\sqrt{3}\,r}{2} - \frac{\left(3+\left(-3+7\sqrt{3}\right)\pi\right)r^3}{2\pi} + \frac{3\,r^2\left(2\sqrt{2}+2\pi-\text{ArcSec}[3]\right)}{\pi};$$

$$\texttt{OCTDCFBB[r\_] := } -\frac{\sqrt{3}}{2} - \frac{2}{\sqrt{3}\,r^2} - \frac{3\left(3+\left(-3+7\sqrt{3}\right)\pi\right)r^2}{2\pi} + \frac{6\,r\left(2\sqrt{2}+2\pi-\text{ArcSec}[3]\right)}{\pi};$$

$$\texttt{OCTDDCFBB[r\_] := } \frac{4}{\sqrt{3}\,r^3} - \frac{3\left(3+\left(-3+7\sqrt{3}\right)\pi\right)r}{\pi} + \frac{6\left(2\sqrt{2}+2\pi-\text{ArcSec}[3]\right)}{\pi};$$

$$\texttt{OCTCFCC[r\_] := } -3 + \frac{2}{\sqrt{3}\,r} + 4\sqrt{3}\,r + 3\,r^2 + \frac{6\sqrt{2}\,r^2}{\pi} + \frac{3\,r^3}{2} - \frac{23\,r^3}{8\sqrt{3}} - \frac{3\,r^3}{2\pi} -$$
$$\frac{3\sqrt{-3+8\,r^2}}{4\pi r} - \frac{17\,r\sqrt{-3+8\,r^2}}{2\pi} - \frac{3\,r^2\,\text{ArcCos}\left[\frac{1}{3}\right]}{\pi} - \frac{15\sqrt{3}\,r^3\,\text{ArcTan}\left[\frac{1}{\sqrt{3}\sqrt{-3+8r^2}}\right]}{2\pi} -$$
$$\frac{9\sqrt{3}\,r\,\text{ArcTan}\left[\frac{\sqrt{3}}{\sqrt{-3+8r^2}}\right]}{\pi} + \frac{3\sqrt{3}\,r^3\,\text{ArcTan}\left[\frac{\sqrt{3}}{\sqrt{-3+8r^2}}\right]}{2\pi} + \frac{6\,r^2\,\text{ArcTan}\left[\frac{r}{\sqrt{-3+8r^2}}\right]}{\pi} +$$
$$\frac{2\,r^3\,\text{ArcTan}\left[\frac{-3+4r^2}{\sqrt{3}\sqrt{-3+8r^2}}\right]}{\sqrt{3}\,\pi} + \frac{\sqrt{3}\,r\,\text{ArcTan}\left[\sqrt{3}\sqrt{-3+8\,r^2}\right]}{\pi} - \frac{2\sqrt{3}\,r^3\,\text{ArcTan}\left[\frac{-5+12r^2}{\sqrt{3}\sqrt{-3+8r^2}}\right]}{\pi} +$$

$$\frac{r^3\,\text{ArcTan}\left[\frac{-9+24r^2-8r^4}{\sqrt{3}\,(-3+4r^2)\sqrt{-3+8r^2}}\right]}{4\sqrt{3}\,\pi}+\frac{\sqrt{3}\,r^3\,\text{ArcTan}\left[\frac{\sqrt{3}\,(27-180r^2+384r^4-272r^6+32r^8)}{\sqrt{-3+8r^2}\,(-27+144r^2-216r^4+80r^6)}\right]}{2\pi};$$

$$\text{OCTDCFCC}[r\_]:=4\sqrt{3}-\frac{2}{\sqrt{3}\,r^2}+6r+\frac{12\sqrt{2}\,r}{\pi}+\frac{9r^2}{2}-\frac{23\sqrt{3}\,r^2}{8}-\frac{9r^2}{2\pi}+\frac{105}{2\pi\sqrt{-3+8r^2}}-\frac{9}{4\pi r^2\sqrt{-3+8r^2}}-\frac{124r^2}{\pi\sqrt{-3+8r^2}}-\frac{6r\,\text{ArcCos}\left[\frac{1}{3}\right]}{\pi}+\frac{9\sqrt{3}\,(-2+r^2)\,\text{ArcTan}\left[\frac{\sqrt{3}}{\sqrt{-3+8r^2}}\right]}{2\pi}+\frac{12r\,\text{ArcTan}\left[\frac{r}{\sqrt{-3+8r^2}}\right]}{\pi}-\frac{45\sqrt{3}\,r^2\,\text{ArcTan}\left[\frac{1}{\sqrt{-9+24r^2}}\right]}{2\pi}+\frac{2\sqrt{3}\,r^2\,\text{ArcTan}\left[\frac{-3+4r^2}{\sqrt{-9+24r^2}}\right]}{\pi}-\frac{6\sqrt{3}\,r^2\,\text{ArcTan}\left[\frac{-5+12r^2}{\sqrt{-9+24r^2}}\right]}{\pi}+\frac{\sqrt{3}\,\text{ArcTan}\left[\sqrt{-9+24r^2}\right]}{\pi}+\frac{\sqrt{3}\,r^2\,\text{ArcTan}\left[\frac{-9+24r^2-8r^4}{(-3+4r^2)\sqrt{-9+24r^2}}\right]}{4\pi}+\frac{3\sqrt{3}\,r^2\,\text{ArcTan}\left[\frac{\sqrt{3}\,(27-180r^2+384r^4-272r^6+32r^8)}{\sqrt{-3+8r^2}\,(-27+144r^2-216r^4+80r^6)}\right]}{2\pi};$$

$$\text{OCTDDCFCC}[r\_]:=6+\frac{12\sqrt{2}}{\pi}+\frac{4}{\sqrt{3}\,r^3}+9r-\frac{23\sqrt{3}\,r}{4}-\frac{9r}{\pi}+\frac{9}{2\pi r^3\sqrt{-3+8r^2}}+\frac{21}{\pi r\sqrt{-3+8r^2}}-\frac{88r}{\pi\sqrt{-3+8r^2}}-\frac{6\,\text{ArcCos}\left[\frac{1}{3}\right]}{\pi}+\frac{9\sqrt{3}\,r\,\text{ArcTan}\left[\frac{\sqrt{3}}{\sqrt{-3+8r^2}}\right]}{\pi}+\frac{12\,\text{ArcTan}\left[\frac{r}{\sqrt{-3+8r^2}}\right]}{\pi}-\frac{45\sqrt{3}\,r\,\text{ArcTan}\left[\frac{1}{\sqrt{-9+24r^2}}\right]}{\pi}+\frac{4\sqrt{3}\,r\,\text{ArcTan}\left[\frac{-3+4r^2}{\sqrt{-9+24r^2}}\right]}{\pi}-\frac{12\sqrt{3}\,r\,\text{ArcTan}\left[\frac{-5+12r^2}{\sqrt{-9+24r^2}}\right]}{\pi}+\frac{\sqrt{3}\,r\,\text{ArcTan}\left[\frac{-9+24r^2-8r^4}{(-3+4r^2)\sqrt{-9+24r^2}}\right]}{2\pi}+\frac{3\sqrt{3}\,r\,\text{ArcTan}\left[\frac{\sqrt{3}\,(27-180r^2+384r^4-272r^6+32r^8)}{\sqrt{-3+8r^2}\,(-27+144r^2-216r^4+80r^6)}\right]}{\pi};$$

$$\text{OCTCFDD}[r\_]:=-\frac{1}{12\pi r}\left(6+4\sqrt{3}\,\pi-36\pi r+36r^2-16\sqrt{3}\,\pi r^2-18\pi r^3+18r^4-9\pi r^4+12\sqrt{3}\,\pi r^4-12\sqrt{-1+2r^2}-48r^2\sqrt{-1+2r^2}+48\sqrt{3}\,r^2\,\text{ArcSin}\left[\sqrt{\frac{3-6r^2}{2-6r^2}}\right]+\right.$$

$$24\sqrt{3}\,r^4\,\text{ArcSin}\left[\frac{5-9r^2}{\sqrt{2}\,(-1+3r^2)^{3/2}}\right]+144\,r\,\text{ArcSin}\left[\frac{r}{\sqrt{-1+3r^2}}\right]-24\sqrt{3}$$

$$\left.\text{ArcSin}\left[\frac{1}{\sqrt{-2+6r^2}}\right]-36\,r^3\,\text{ArcSin}\left[\frac{1+2r^2-7r^4}{(1-3r^2)^2}\right]+36\,r^4\,\text{ArcSin}\left[\frac{-1+r^2+\sqrt{-1+2r^2}}{\sqrt{2}\,r^2}\right]\right);$$

$$\text{OCTDCFDD}[r\_]:=\frac{4}{\sqrt{3}}-\frac{3}{\pi}+\frac{1}{\sqrt{3}\,r^2}+\frac{1}{2\pi r^2}+3r+\frac{9r^2}{4}-3\sqrt{3}\,r^2-\frac{9r^2}{2\pi}-\frac{10}{\pi\sqrt{-1+2r^2}}+$$

$$\frac{1}{\pi r^2\sqrt{-1+2r^2}}+\frac{16r^2}{\pi\sqrt{-1+2r^2}}-\frac{4\sqrt{3}\,\text{ArcSin}\left[\sqrt{\frac{3-6r^2}{2-6r^2}}\right]}{\pi}-\frac{6\sqrt{3}\,r^2\,\text{ArcSin}\left[\frac{5-9r^2}{\sqrt{2}\,(-1+3r^2)^{3/2}}\right]}{\pi}-$$

$$\frac{2\sqrt{3}\,\text{ArcSin}\left[\frac{1}{\sqrt{-2+6r^2}}\right]}{\pi r^2}+\frac{6\,r\,\text{ArcSin}\left[\frac{1+2r^2-7r^4}{(1-3r^2)^2}\right]}{\pi}-\frac{9r^2\,\text{ArcSin}\left[\frac{-1+r^2+\sqrt{-1+2r^2}}{\sqrt{2}\,r^2}\right]}{\pi};$$

$$\text{OCTDDCFDD}[r\_]:=3-\frac{2}{\sqrt{3}\,r^3}-\frac{1}{\pi r^3}+\frac{9r}{2}-6\sqrt{3}\,r-\frac{9r}{\pi}+\frac{2\sqrt{-1+2r^2}\,(1+4r^2)}{\pi r^3}-$$

$$\frac{1}{\pi r^3}2\left(6\sqrt{3}\,r^4\,\text{ArcSin}\left[\frac{5-9r^2}{\sqrt{2}\,(-1+3r^2)^{3/2}}\right]-2\sqrt{3}\,\text{ArcSin}\left[\frac{1}{\sqrt{-2+6r^2}}\right]-\right.$$

$$\left.3r^3\,\text{ArcSin}\left[\frac{1+2r^2-7r^4}{(1-3r^2)^2}\right]+9r^4\,\text{ArcSin}\left[\frac{-1+r^2+\sqrt{-1+2r^2}}{\sqrt{2}\,r^2}\right]\right);$$

```
OCTAxctTotalCF[r_] := If[r < OCTDst[[1]], OCTCFAA[r],
  If[r < OCTDst[[2]], OCTCFBB[r], If[r < OCTDst[[3]], OCTCFCC[r]], OCTCFDD[r]]];
OCTAxctTotalDCF[r_] := If[r < OCTDst[[1]], OCTDCFAA[r],
  If[r < OCTDst[[2]], OCTDCFBB[r], If[r < OCTDst[[3]], OCTDCFCC[r]], OCTDCFDD[r]]];
OCTAxctTotalDDCF[r_] := If[r < OCTDst[[1]], OCTDDCFAA[r],
  If[r < OCTDst[[2]], OCTDDCFBB[r], If[r < OCTDst[[3]], OCTDDCFCC[r]], OCTDDCFDD[r]]];
```

```
Plot[{OCTAxctTotalCF[r] - OctCFTot[r * √2]}, {r, 0, OCTDst[[4]]}]

Plot[{OCTAxctTotalDCF[r] - √2 OctDCFTot[r * √2]}, {r, 0, OCTDst[[4]]}]

Plot[{OCTAxctTotalDDCF[r] - 2 OctDDCFTot[r * √2],
   If[r > OCTDst[[1]], If[r < OCTDst[[2]], 0]], If[r > OCTDst[[2]], If[r < OCTDst[[3]], 0]],
   If[r > OCTDst[[3]], If[r < OCTDst[[4]], 0]]}, {r, 0, OCTDst[[4]]},
 PlotRange → {{0, 1}, {-10^(-13), 10^(-13)}}, PlotStyle →
  {Directive[Blue, Thin], Directive[Blue, Thin], Directive[Red, Thickness[0.008]],
   Directive[Blue, Thickness[0.008]], Directive[Magenta, Thickness[0.008]]}]
```

```
In[14]:= Plot[{If[r < OCTDst[[1]], OCTDDCFAA[r]],
   If[r > OCTDst[[1]], If[r < OCTDst[[2]], OCTDDCFBB[r]]],
   If[r > OCTDst[[2]], If[r < OCTDst[[3]], OCTDDCFCC[r]]],
   If[r > OCTDst[[3]], If[r < OCTDst[[4]], OCTDDCFDD[r]]]}, {r, 0, OCTDst[[4]]},
 PlotStyle → {Directive[Blue, Thickness[0.004]], Directive[Red, Thickness[0.004]],
   Directive[Cyan, Thickness[0.004]], Directive[Magenta, Thickness[0.004]]},
 PlotRange → {{0, 1.02}, {-0.05, 10}}]
```

```
Plot[{OCTAxctTotalDDCF[r], If[r > OCTDst[[3]], If[r < OCTDst[[4]], OCTDDCFDD[r]]],
   If[r > OCTDst[[1]], If[r < OCTDst[[2]], 0]], If[r > OCTDst[[2]], If[r < OCTDst[[3]], 0]],
   If[r > OCTDst[[3]], If[r < OCTDst[[4]], 0]]}, {r, 0, OCTDst[[4]]},
 PlotStyle → {Directive[Blue, Thickness[0.005]], Directive[Blue, Thickness[0.005]],
   Directive[Blue, Thick], Directive[Red, Thickness[0.008]],
   Directive[Blue, Thickness[0.008]], Directive[Magenta, Thickness[0.008]]}]
```

EVALUATION OF THE BEHAVIOUR AROUND D1$^+$, D2$^-$, D2$^+$, D3$^-$, D3$^+$, D4$^-$

```
cfD1Minus =
 FullSimplify[(CoefficientList[Series[OCTDDCFAA[OCTDst[[1]] - x^2], {x, 0, 6}], x]) /.
   {ArcSec[-3] → π - ArcSec[3]}]
```

In[15]:=

$$\text{cfD1Minus} = \left\{\frac{3\left(4\sqrt{2}-\sqrt{3}+\left(-1+\sqrt{3}\right)\pi-2\,\text{ArcSec}[3]\right)}{\pi},\ 0,\ -9+3\sqrt{3}+\frac{9}{\pi},\ 0,\ 0,\ 0,\ 0\right\};$$

```
cfD1Plus =
 FullSimplify[(CoefficientList[Series[OCTDDCFBB[OCTDst[[1]] + x^2], {x, 0, 6}], x]) /.
   {ArcSec[-3] → π - ArcSec[3]}]
```

$$\text{cfD1Plus} = \left\{\frac{3\left(4\sqrt{2}-\sqrt{3}+\pi+\sqrt{3}\,\pi-2\,\text{ArcSec}[3]\right)}{\pi},\ 0,\ 9-57\sqrt{3}-\frac{9}{\pi},\ 0,\ 216,\ 0,\ -360\sqrt{3}\right\};$$

```
cfD2Minus =
 FullSimplify[(CoefficientList[Series[OCTDDCFBB[OCTDst[[2]] - x^2], {x, 0, 6}], x]) /.
   {ArcSec[-3] → π - ArcSec[3]}]
```

$$\text{cfD2Minus} = \left\{\frac{1}{36}\left(432-311\sqrt{2}+81\sqrt{6}-\frac{27\left(-16\sqrt{2}+3\sqrt{6}+8\,\text{ArcSec}[3]\right)}{\pi}\right),\ 0,\ -9+\frac{445}{3\sqrt{3}}+\frac{9}{\pi},\ 0,\ \frac{1024\sqrt{2}}{9},\ 0,\ \frac{20\,480}{27\sqrt{3}}\right\};$$

```
FullSimplify[Series[OCTDDCFCC[OCTDst[[2]] + x], {x, 0, 3}], Assumptions → {x > 0}]
```

$$\text{FullSimplify}\Big[\text{CoefficientList}\Big[\text{Simplify}\Big[\left(\frac{1}{36}\left(432-311\sqrt{2}+81\sqrt{6}-\frac{27\left(-16\sqrt{2}+3\sqrt{6}+8\,\text{ArcSec}[3]\right)}{\pi}\right)+\left(9-\frac{445}{3\sqrt{3}}-\frac{9}{\pi}\right)x+\frac{1024\sqrt{2}\,x^2}{9}+\frac{16\,384\times 2^{3/4}\,x^{5/2}}{5\times 3^{1/4}\,\pi}-\frac{20\,480\,x^3}{27\sqrt{3}}\right)\ /.\ \{x\to y\,\hat{}\,2\},\ \text{Assumptions}\to\{y>0\}\Big],\ y\Big]\Big]$$

$$\text{cfD2Plus} = \left\{\frac{1}{36}\left(432-311\sqrt{2}+81\sqrt{6}-\frac{27\left(-16\sqrt{2}+3\sqrt{6}+8\,\text{ArcSec}[3]\right)}{\pi}\right),\ 0,\ 9-\frac{445}{3\sqrt{3}}-\frac{9}{\pi},\ 0,\ \frac{1024\sqrt{2}}{9},\ \frac{16\,384\times 2^{3/4}}{5\times 3^{1/4}\,\pi},\ -\frac{20\,480}{27\sqrt{3}}\right\};$$

```
Simplify[cfD2Plus - cfD2PlusOLD]
```

{0, 0, 0, 0, 0, 0, 0}

$$\text{cfold} = \left\{\frac{\left(216 - 311\sqrt{2} + 81\sqrt{6}\right)\pi + 27\left(16\sqrt{2} - 3\sqrt{6} + 8\,\text{ArcSec}[-3]\right)}{72\pi},\right.$$
$$\left. 0,\ -\frac{\left(81 + \left(-81 + 445\sqrt{3}\right)\pi\right)}{18\left(\sqrt{2}\,\pi\right)},\ 0,\ +\frac{256}{9}\sqrt{2},\ +\frac{4096\sqrt{2}}{5\times 3^{1/4}\pi},\ -\frac{2560}{27}\sqrt{\frac{2}{3}}\right\};$$

```
FullSimplify[(Simplify[2 * cfold[[1]] - cfD2Plus[[1]]]) /. {ArcSec[-3] → π - ArcSec[3]}]
```

0

```
FullSimplify[
 (Simplify[cfold[[2]] (2) ^ (5 / 4) - cfD2Plus[[2]]]) /. {ArcSec[-3] → π - ArcSec[3]}]
```

0

```
FullSimplify[
 (Simplify[cfold[[3]] (2) ^ (6 / 4) - cfD2Plus[[3]]]) /. {ArcSec[-3] → π - ArcSec[3]}]
```

0

```
FullSimplify[
 (Simplify[cfold[[4]] (2) ^ (7 / 4) - cfD2Plus[[4]]]) /. {ArcSec[-3] → π - ArcSec[3]}]
```

0

```
FullSimplify[
 (Simplify[cfold[[5]] (2) ^ (8 / 4) - cfD2Plus[[5]]]) /. {ArcSec[-3] → π - ArcSec[3]}]
```

0

```
FullSimplify[
 (Simplify[cfold[[6]] (2) ^ (9 / 4) - cfD2Plus[[6]]]) /. {ArcSec[-3] → π - ArcSec[3]}]
```

0

```
FullSimplify[
 (Simplify[cfold[[7]] (2) ^ (10 / 4) - cfD2Plus[[7]]]) /. {ArcSec[-3] → π - ArcSec[3]}]
```

0

```
cfD3Minus = FullSimplify[
  (Simplify[CoefficientList[Series[OCTDDCFCC[OCTDst[[3]] - x^2], {x, 0, 6}], x]])]
```

$$\text{cfD3Minus} = \left\{6 - 10\sqrt{\frac{2}{3}} + \frac{9}{\sqrt{2}} - \frac{13}{\sqrt{2}\,\pi},\ 0,\ -9 + 28\sqrt{3} - \frac{3}{\pi},\ 0,\ \frac{8\sqrt{2}\left(15 + 4\sqrt{3}\,\pi\right)}{\pi},\ 0,\ \frac{320}{\sqrt{3}} + \frac{688}{\pi}\right\};$$

The evaluation requires that OCTDDCFDD[r] be split into a sum and that we expand each term

```
FullSimplify[Series[Simplify[OCTDDCFDD[x + OCTDst[[3]]], Assumptions → {x > 0}], {x, 0, 3}],
 Assumptions → {x > 0}]
```

$$\text{cfD3Plus} = \text{FullSimplify}\Big[\text{CoefficientList}\Big[\text{FullSimplify}\Big[\left(\left(6 - 10\sqrt{\frac{2}{3}} + \frac{9}{\sqrt{2}} - \frac{13}{\sqrt{2}\,\pi}\right) + \left(9 - 28\sqrt{3} + \frac{3}{\pi}\right) x + \frac{192 \times 2^{3/4}\, x^{3/2}}{\pi} + \frac{8\sqrt{2}\left(-3 + 4\sqrt{3}\,\pi\right) x^2}{\pi} - \frac{6624 \times 2^{1/4}\, x^{5/2}}{5\pi} + \left(-\frac{320}{\sqrt{3}} - \frac{56\,752}{\pi}\right) x^3\right) \text{/. }\{x \to y\text{^}2\}, \text{Assumptions} \to \{y > 0\}\Big], y\Big]\Big]$$

$$\text{cfD3Plus} = \left\{6 - 10\sqrt{\frac{2}{3}} + \frac{9}{\sqrt{2}} - \frac{13}{\sqrt{2}\,\pi}, 0, 9 - 28\sqrt{3} + \frac{3}{\pi}, \frac{192 \times 2^{3/4}}{\pi}, \frac{8\sqrt{2}\left(-3 + 4\sqrt{3}\,\pi\right)}{\pi}, -\frac{6624 \times 2^{1/4}}{5\pi}, -\frac{320}{\sqrt{3}} - \frac{56\,752}{\pi}\right\};$$

```
Simplify[OCTDDCFDD[x + OCTDst[[3]]], Assumptions → {x > 0}]
```

$$\text{ausausaa}[x\_] := 3 - \frac{2}{\sqrt{3}\left(\frac{1}{\sqrt{2}} + x\right)^3} - \frac{1}{\pi\left(\frac{1}{\sqrt{2}} + x\right)^3} + \frac{9}{2}\left(\frac{1}{\sqrt{2}} + x\right) - 6\sqrt{3}\left(\frac{1}{\sqrt{2}} + x\right) - \frac{9\left(\frac{1}{\sqrt{2}} + x\right)}{\pi} + \frac{2\sqrt{2}\sqrt{x\left(\sqrt{2} + x\right)}\left(1 + \left(\sqrt{2} + 2x\right)^2\right)}{\pi\left(\frac{1}{\sqrt{2}} + x\right)^3};$$

$$\text{ausausbb}[x\_] := -\frac{1}{\pi\left(\frac{1}{\sqrt{2}} + x\right)^3}\, 2\left(6\sqrt{3}\left(\frac{1}{\sqrt{2}} + x\right)^4 \text{ArcSin}\left[\frac{1 - 18\sqrt{2}\,x - 18x^2}{\left(1 + 6\sqrt{2}\,x + 6x^2\right)^{3/2}}\right]\right);$$

$$\text{ausauscc}[x\_] := -\frac{1}{\pi\left(\frac{1}{\sqrt{2}} + x\right)^3}\, 2\left(-2\sqrt{3}\,\text{ArcSin}\left[\frac{1}{\sqrt{1 + 6\sqrt{2}\,x + 6x^2}}\right]\right);$$

$$\text{ausausdd}[x\_] := -\frac{1}{\pi\left(\frac{1}{\sqrt{2}} + x\right)^3}\, 2\left(-3\left(\frac{1}{\sqrt{2}} + x\right)^3 \text{ArcSin}\left[\frac{1 + 2\left(\frac{1}{\sqrt{2}} + x\right)^2 - 7\left(\frac{1}{\sqrt{2}} + x\right)^4}{\left(1 - 3\left(\frac{1}{\sqrt{2}} + x\right)^2\right)^2}\right]\right);$$

$$\text{ausausee}[x\_] := -\frac{1}{\pi\left(\frac{1}{\sqrt{2}} + x\right)^3}\, 2\left(9\left(\frac{1}{\sqrt{2}} + x\right)^4 \text{ArcSin}\left[\frac{-1 + \left(\frac{1}{\sqrt{2}} + x\right)^2 + \sqrt{2}\sqrt{x\left(\sqrt{2} + x\right)}}{\sqrt{2}\left(\frac{1}{\sqrt{2}} + x\right)^2}\right]\right);$$

```
Simplify[(ausausaa[x] + ausausbb[x] + ausauscc[x] + ausausdd[x] + ausausee[x]) -
  Simplify[OCTDDCFDD[x + OCTDst[[3]]], Assumptions → {x > 0}]]
```

0

```
Series[ausausaa[x], {x, 0, 3}]
```

```
ccffax = Simplify[
  CoefficientList[Simplify[(((3 - 13 Sqrt[2/3 + 9/(2 Sqrt[2]) - 13/(Sqrt[2] π)]) + (24 × 2^(1/4) Sqrt[x])/π + (9/2 + 2 Sqrt[3] + 3/π) x -
      (34 × 2^(3/4) x^(3/2))/π + (-16 Sqrt[6] - (24 Sqrt[2])/π) x^2 + (213 x^(5/2))/(2^(3/4) π) + (160/Sqrt[3] + 80/π) x^3) /.
     {x → y^2}, Assumptions → {y > 0}], y]];
FullSimplify[Series[ausausbb[x], {x, 0, 3}] , Assumptions → {x > 0}]
ccffbx = Simplify[CoefficientList[
  FullSimplify[(-3 Sqrt[6] + (108 × 2^(1/4) Sqrt[x])/π - 6 Sqrt[3] x - (81 × 2^(3/4) x^(3/2))/π + (16 929 x^(5/2))/(10 × 2^(3/4) π) + o x^3) /.
    {x → y^2}, Assumptions → {y > 0}], y]]
FullSimplify[Series[ausauscc[x], {x, 0, 3}] , Assumptions → {x > 0}]
ccffcx =
 Simplify[CoefficientList[FullSimplify[(4 Sqrt[6] - (48 × 2^(1/4) Sqrt[x])/π - 24 Sqrt[3] x + (228 × 2^(3/4) x^(3/2))/π +
      48 Sqrt[6] x^2 - (8121 × 2^(1/4) x^(5/2))/(5 π) - 160 Sqrt[3] x^3) /. {x → y^2}, Assumptions → {y > 0}], y]]
FullSimplify[Series[ausausdd[x], {x, 0, 3}] , Assumptions → {x > 0}]
ccffdx = Simplify[CoefficientList[
  FullSimplify[(3 - (48 × 2^(1/4) Sqrt[x])/π + (100 × 2^(3/4) x^(3/2))/π - (3321 × 2^(1/4) x^(5/2))/(5 π) + o x^3) /. {x → y^2},
   Assumptions → {y > 0}], y]]
FullSimplify[Series[ausausee[x], {x, 0, 3}] , Assumptions → {x > 0}]
ccffex = Simplify[CoefficientList[
  FullSimplify[(9/(2 Sqrt[2]) - (36 × 2^(1/4) Sqrt[x])/π + (9 x)/2 - (21 × 2^(3/4) x^(3/2))/π + (213 x^(5/2))/(10 × 2^(3/4) π) - (56 832 x^3)/π) /. {x → y^2},
   Assumptions → {y > 0}], y]]
```

$$\left\{\frac{9}{2\sqrt{2}},\ -\frac{36\times 2^{1/4}}{\pi},\ \frac{9}{2},\ -\frac{21\times 2^{3/4}}{\pi},\ 0,\ \frac{213}{10\times 2^{3/4}\,\pi},\ -\frac{56\,832}{\pi}\right\}$$

```
((Simplify[ccffax + ccffbx + ccffcx + ccffdx + ccffex]) /. {o → 0}) - cfD3Plus
```

we similarly proceed to determine cfD4Minus

```
FullSimplify[
 Series[Simplify[(OCTDDCFDD[OCTDst[[4]] - x]), Assumptions → {x > 0}], {x, 0, 3}],
 Assumptions → {x > 0}]
Simplify[CoefficientList[(((24 x^3)/π) /. {x → y^2}), y]]
```

$$\text{cfD4Minus} = \left\{0, 0, 0, 0, 0, 0, \frac{24}{\pi}\right\};$$

```
Simplify[(OCTDDCFDD[OCTDst[[4]] - x]), Assumptions → {x > 0}]
```

$$\text{Aus44AA[x\_]} := 3 + \frac{2}{\sqrt{3}\,(-1+x)^3} + \frac{1}{\pi\,(-1+x)^3} - \frac{9}{2}(-1+x) + 6\sqrt{3}\,(-1+x) + \frac{9\,(-1+x)}{\pi} - \frac{2\sqrt{1-4x+2x^2}\,\left(5-8x+4x^2\right)}{\pi\,(-1+x)^3};\ \text{Aus44BB[x\_]} :=$$

$$-\frac{1}{\pi\,(1-x)^3}\,2\left(6\sqrt{3}\,(-1+x)^4\,\text{ArcSin}\left[\frac{-4+18x-9x^2}{\sqrt{2}\,\left(2-6x+3x^2\right)^{3/2}}\right] - 2\sqrt{3}\,\text{ArcSin}\left[\frac{1}{\sqrt{4-12x+6x^2}}\right]\right);$$

$$\text{Aus44CC[x\_]} := -\frac{1}{\pi\,(1-x)^3}\,2\left(3\,(-1+x)^3\,\text{ArcSin}\left[\frac{1+2\,(-1+x)^2-7\,(-1+x)^4}{\left(2-6x+3x^2\right)^2}\right] + 9\,(-1+x)^4\,\text{ArcSin}\left[\frac{-2x+x^2+\sqrt{1-4x+2x^2}}{\sqrt{2}\,(-1+x)^2}\right]\right);$$

```
Simplify[Simplify[(OCTDDCFDD[OCTDst[[4]] - x]), Assumptions → {x > 0}] -
  (Aus44AA[x] + Aus44BB[x] + Aus44CC[x])]

Simplify[Series[Aus44AA[x], {x, 0, 3}], Assumptions → {x > 0}]

FullSimplify[Series[Aus44BB[x], {x, 0, 3}], Assumptions → {x > 0}]

FullSimplify[Series[Aus44CC[x], {x, 0, 3}], Assumptions → {x > 0}]

Simplify[Simplify[Series[Aus44AA[x], {x, 0, 3}], Assumptions → {x > 0}] +
  FullSimplify[Series[Aus44BB[x], {x, 0, 3}], Assumptions → {x > 0}] +
  FullSimplify[Series[Aus44CC[x], {x, 0, 3}], Assumptions → {x > 0}]]
```

The discontinuities at the points joining two different r-subintervals are

```
ΔCfD1 = Simplify[cfD1Plus - cfD1Minus]

ΔCfD2 = Simplify[cfD2Plus - cfD2Minus]

ΔCfD3 = Simplify[cfD3Plus - cfD3Minus]

ΔCfD4 = Simplify[-cfD4Minus]
```

The lowest order singularities are:
- a discontinuity of the CLD at D1, a discontinuity of the 1st derivative of the CLD at D1, D2 and D3,
- a singular $\sqrt{r - \text{D3}}$ behaviour of 1st derivative of the CLD at D3,
- a disconitnuity of 2nd derivative of the CLD at D1 and D3

We construct now the polynomial approximations in the variables $\sqrt{r - \text{D1}}$ and $\sqrt{\text{D2} - r}$ using the following formulae worked out in the file "FNL_ANALYTIC_APPROXMTN_CF_CLD_BB.nb "

THE FINAL FORMULAE ARE:

```
GLftNw[ξ_, K_] :=
  If[K < 4, 4 * ξ^4 * Sum[(cflft[[i]] / ((3 + i) (1 + i))) * (ξ)^(i-1), {i, 1, 2 K + 1}]];
GRgtNw[η_, K_] := If[K < 4,
   4 * η^4 * Sum[(cfrgt[[i]] / ((3 + i) (1 + i))) * (η)^(i-1), {i, 1, 2 K + 1}]];
```

K = 0

```
LeftCLD00[r_, D1_, D2_] := a0 (1 + (Sqrt[-D1 + r] (2 D1 - 3 D2 + r))/(2 (-D1 + D2)^(3/2)));
RgtCLD00[r_, D1_, D2_] := b0 (1 - (Sqrt[D2 - r] (-3 D1 + 2 D2 + r))/(2 (-D1 + D2)^(3/2)));
CLDApprx00[r_, D1_, D2_] := LeftCLD00[r, D1, D2] + RgtCLD00[r, D1, D2 ] ;
LEFTCFApprx00[r_, D1_, D2_] := b0 (D2^2/2 - D2 r + r^2/2 - (2 (D2 - r)^(5/2) (-7 D1 + 6 D2 + r))/(35 (-D1 + D2)^(3/2))) +
   1/(70 (D1 - D2)) a0 (-24 D1^3 Sqrt[(D1 - r)/(D1 - D2)] + 28 D1^2 D2 Sqrt[(D1 - r)/(D1 - D2)] + 44 D1^2 Sqrt[(D1 - r)/(D1 - D2)] r -
     56 D1 D2 Sqrt[(D1 - r)/(D1 - D2)] r - 16 D1 Sqrt[(D1 - r)/(D1 - D2)] r^2 + 28 D2 Sqrt[(D1 - r)/(D1 - D2)] r^2 -
     4 Sqrt[(D1 - r)/(D1 - D2)] r^3 + (D1 - D2) (24 D1^2 + 8 D1 D2 + 3 D2^2 - 14 (4 D1 + D2) r + 35 r^2));
RGHTCFApprx00[r_, D1_, D2_] := a0 (D1^2/2 - D1 r + r^2/2 - 2/35 (D1 - r) ((D1 - r)/(D1 - D2))^(3/2) (6 D1 - 7 D2 + r)) -
   1/(70 (-D1 + D2)^(3/2)) b0 (3 D1^3 Sqrt[-D1 + D2] - 24 D2^3 (Sqrt[-D1 + D2] - Sqrt[D2 - r]) +
     D1^2 Sqrt[-D1 + D2] (5 D2 - 14 r) + D2^2 (56 Sqrt[-D1 + D2] - 44 Sqrt[D2 - r]) r +
     D2 (-35 Sqrt[-D1 + D2] + 16 Sqrt[D2 - r]) r^2 + 4 Sqrt[D2 - r] r^3 + D1 (4 D2^2 (4 Sqrt[-D1 + D2] - 7 Sqrt[D2 - r]) +
        D2 (-42 Sqrt[-D1 + D2] + 56 Sqrt[D2 - r]) r + 7 (5 Sqrt[-D1 + D2] - 4 Sqrt[D2 - r]) r^2));
EXTRCNTRBLft00[r_, D1_, D2_, a_, b_] := 1/60 (D2 - r)^3
  (5 a (-2 D1 + D2 + r) + b (10 D1^2 + 3 D2^2 + 4 D2 r + 3 r^2 - 10 D1 (D2 + r))) ;
EXTRCNTRBRgt00[r_, D1_, D2_, a_, b_] :=
  -1/60 (D1 - r)^3 (-5 a (D1 - 2 D2 + r) + b (D1 - r) (2 D1 - 5 D2 + 3 r));
```

K = 1

```
LeftCLD11[r_, D1_, D2_] := (a0 + a1 Sqrt[-D1 + r] + a2 (-D1 + r))
   (1 - ((-D1 + r)^(3/2) (8 D1^2 + 35 D2^2 - 42 D2 r + 15 r^2 + 4 D1 (-7 D2 + 3 r)))/(8 (-D1 + D2)^(7/2)));
RgtCLD11[r_, D1_, D2_] := (b0 + b1 Sqrt[D2 - r] + b2 (D2 - r))
   (1 - ((D2 - r)^(3/2) (35 D1^2 + 8 D2^2 + 12 D2 r + 15 r^2 - 14 D1 (2 D2 + 3 r)))/(8 (-D1 + D2)^(7/2)));
CLDApprx11[r_, D1_, D2_] := LeftCLD11[r, D1, D2] + RgtCLD11[r, D1, D2 ] ;
LEFTCFApprx11[r_, D1_, D2_] :=
 -1/(205 920 (-D1 + D2)^(7/2)) (429 b1 (D2 - r)^2 (175 D1^2 (D2 - r)^2 + 128 D1^3 Sqrt[(-D1 + D2) (D2 - r)] -
        384 D1^2 D2 Sqrt[(-D1 + D2) (D2 - r)] + 384 D1 D2^2 Sqrt[(-D1 + D2) (D2 - r)] - 128 D2^3
         Sqrt[(-D1 + D2) (D2 - r)] - 14 D1 (D2 - r)^2 (16 D2 + 9 r) + (D2 - r)^2 (79 D2^2 + 66 D2 r + 30 r^2)) +
     429 a1 (128 D1^6 - 21 D2^6 + 70 D2^5 r + 175 D2^2 r^4 - 126 D2 r^5 + 30 r^6 -
        128 D2^3 r^2 Sqrt[(-D1 + D2) (-D1 + r)] + 64 D1^5 (-7 D2 - 5 r + 2 Sqrt[(-D1 + D2) (-D1 + r)]) - 16 D1^4
         (-35 D2^2 - 70 D2 r - 15 r^2 + 24 D2 Sqrt[(-D1 + D2) (-D1 + r)] + 16 r Sqrt[(-D1 + D2) (-D1 + r)]) +
        D1^2 (35 D2^4 - 5 r^4 + 6 D2^2 r (175 r - 128 Sqrt[(-D1 + D2) (-D1 + r)]) +
           D2^3 (700 r - 128 Sqrt[(-D1 + D2) (-D1 + r)]) + 4 D2 r^2 (35 r - 96 Sqrt[(-D1 + D2) (-D1 + r)])) +
        8 D1^3 (-35 D2^3 + r^2 (-5 r + 16 Sqrt[(-D1 + D2) (-D1 + r)]) +
           3 D2 r (-35 r + 32 Sqrt[(-D1 + D2) (-D1 + r)]) + D2^2 (-175 r + 48 Sqrt[(-D1 + D2) (-D1 + r)])) +
        D1 (56 D2^5 - 350 D2^4 r + 256 D2^3 Sqrt[(D1 - D2) (D1 - r)] r + 280 D2 r^4 - 54 r^5 +
           4 D2^2 r^2 (-175 r + 96 Sqrt[(-D1 + D2) (-D1 + r)]))) - 80
     ((D2 - r)^2 (-39 b0 (33 D1^3 Sqrt[-D1 + D2] + D2^3 (-33 Sqrt[-D1 + D2] + 16 Sqrt[D2 - r]) - 4 D2^2 Sqrt[D2 - r] r -
              7 D2 Sqrt[D2 - r] r^2 - 5 Sqrt[D2 - r] r^3 - 33 D1^2 (3 D2 Sqrt[-D1 + D2] - D2 Sqrt[D2 - r] + Sqrt[D2 - r] r) +
              11 D1 (D2^2 (9 Sqrt[-D1 + D2] - 4 Sqrt[D2 - r]) + 2 D2 Sqrt[D2 - r] r + 2 Sqrt[D2 - r] r^2)) +
           b2 (D2 - r) (-429 D1^3 Sqrt[-D1 + D2] + D2^3 (429 Sqrt[-D1 + D2] - 304 Sqrt[D2 - r]) +
              28 D2^2 Sqrt[D2 - r] r + 141 D2 Sqrt[D2 - r] r^2 + 135 Sqrt[D2 - r] r^3 +
              143 D1^2 (9 D2 Sqrt[-D1 + D2] - 5 D2 Sqrt[D2 - r] + 5 Sqrt[D2 - r] r) -
              13 D1 (D2^2 (99 Sqrt[-D1 + D2] - 68 Sqrt[D2 - r]) + 26 D2 Sqrt[D2 - r] r + 42 Sqrt[D2 - r] r^2))) -
        39 a0 (-5 D2^5 Sqrt[-D1 + D2] + 22 D2^4 Sqrt[-D1 + D2] r - 33 D2^3 Sqrt[-D1 + D2] r^2 + 33 D2^2 r^3 Sqrt[-D1 + r] -
           22 D2 r^4 Sqrt[-D1 + r] + 5 r^5 Sqrt[-D1 + r] + 16 D1^5 (Sqrt[-D1 + D2] - Sqrt[-D1 + r]) +
           4 D1^4 (-9 D2 Sqrt[-D1 + D2] - 11 Sqrt[-D1 + D2] r + 11 D2 Sqrt[-D1 + r] + 9 r Sqrt[-D1 + r]) +
```

```
        D1^3 (D2^2 (17 Sqrt[-D1 + D2] - 33 Sqrt[-D1 + r]) + r^2 (33 Sqrt[-D1 + D2] - 17 Sqrt[-D1 + r]) +
          110 D2 r (Sqrt[-D1 + D2] - Sqrt[-D1 + r])) + D1 (3 D2^4 Sqrt[-D1 + D2] - 22 D2^3 Sqrt[-D1 + D2] r +
          22 D2 r^3 Sqrt[-D1 + r] - 3 r^4 Sqrt[-D1 + r] + 99 D2^2 r^2 (Sqrt[-D1 + D2] - Sqrt[-D1 + r])) +
        D1^2 (5 D2^3 Sqrt[-D1 + D2] - 5 r^3 Sqrt[-D1 + r] + 33 D2 r^2 (-3 Sqrt[-D1 + D2] + 2 Sqrt[-D1 + r]) +
          33 D2^2 r (-2 Sqrt[-D1 + D2] + 3 Sqrt[-D1 + r]))) +
      a2 (70 D2^6 Sqrt[-D1 + D2] - 195 D2^5 Sqrt[-D1 + D2] r + 429 D2^3 Sqrt[-D1 + D2] r^3 - 715 D2^2 r^4 Sqrt[-D1 + r] +
        546 D2 r^5 Sqrt[-D1 + r] - 135 r^6 Sqrt[-D1 + r] + 304 D1^6 (Sqrt[-D1 + D2] - Sqrt[-D1 + r]) +
        D1^5 (-732 D2 Sqrt[-D1 + D2] - 1092 Sqrt[-D1 + D2] r + 884 D2 Sqrt[-D1 + r] + 940 r Sqrt[-D1 + r]) +
        D1^4 (D2^2 (387 Sqrt[-D1 + D2] - 715 Sqrt[-D1 + r]) +
          26 D2 r (111 Sqrt[-D1 + D2] - 115 Sqrt[-D1 + r]) + 9 r^2 (143 Sqrt[-D1 + D2] - 95 Sqrt[-D1 + r])) +
        D1^2 (75 D2^4 Sqrt[-D1 + D2] - 663 D2^3 Sqrt[-D1 + D2] r - 10 r^4 Sqrt[-D1 + r] + 13 D2 r^3
           (99 Sqrt[-D1 + D2] - 20 Sqrt[-D1 + r]) + 429 D2^2 r^2 (9 Sqrt[-D1 + D2] - 10 Sqrt[-D1 + r])) +
        D1 (-225 D2^5 Sqrt[-D1 + D2] + 975 D2^4 Sqrt[-D1 + D2] r - 1287 D2^3 Sqrt[-D1 + D2] r^2 - 1300
           D2 r^4 Sqrt[-D1 + r] + 264 r^5 Sqrt[-D1 + r] + 143 D2^2 r^3 (-9 Sqrt[-D1 + D2] + 20 Sqrt[-D1 + r])) +
        D1^3 (121 D2^3 Sqrt[-D1 + D2] + 39 D2 r^2 (-99 Sqrt[-D1 + D2] + 80 Sqrt[-D1 + r]) + r^3 (-429
              Sqrt[-D1 + D2] + 100 Sqrt[-D1 + r]) + 13 D2^2 r (-147 Sqrt[-D1 + D2] + 220 Sqrt[-D1 + r]))));
RGHTCFApprx11[r_, D1_, D2_] := 1 / (205920 (-D1 + D2)^(7/2))
   (429 b1
      (21 D1^6 - 128 D2^6 - 35 D1^4 D2 (D2 - 10 r) + 5 D2^2 r^4 + 54 D2 r^5 -
        30 r^6 - 14 D1^5 (4 D2 + 5 r) + 64 D2^5 (2 Sqrt[(-D1 + D2) (D2 - r)] + 5 r) +
        8 D2^3 r^2 (16 Sqrt[(-D1 + D2) (D2 - r)] + 5 r) - 16 D2^4 r (16 Sqrt[(-D1 + D2) (D2 - r)] + 15 r) -
        2 D1 (-224 D2^5 + 140 D2 r^4 - 63 r^5 + 16 D2^4 (12 Sqrt[(-D1 + D2) (D2 - r)] + 35 r) - 12 D2^3 r
             (32 Sqrt[(-D1 + D2) (D2 - r)] + 35 r) + 2 D2^2 r^2 (96 Sqrt[(-D1 + D2) (D2 - r)] + 35 r)) -
        4 D1^3 (-70 D2^3 - 64 D2 Sqrt[(-D1 + D2) (D2 - r)] r + 32 Sqrt[(-D1 + D2) (D2 - r)] r^2 +
           D2^2 (32 Sqrt[(-D1 + D2) (D2 - r)] + 175 r)) +
        D1^2 (-560 D2^4 - 175 r^4 + 8 D2^3 (48 Sqrt[(-D1 + D2) (D2 - r)] + 175 r) + 4 D2 r^2
             (96 Sqrt[(-D1 + D2) (D2 - r)] + 175 r) - 6 D2^2 r (128 Sqrt[(-D1 + D2) (D2 - r)] + 175 r))) +
     80 (-39 b0 (5 D1^5 Sqrt[-D1 + D2] + 16 D2^5 (-Sqrt[-D1 + D2] + Sqrt[D2 - r]) +
           4 D2^4 (11 Sqrt[-D1 + D2] - 9 Sqrt[D2 - r]) r + D2^3 (-33 Sqrt[-D1 + D2] + 17 Sqrt[D2 - r]) r^2 +
```

```
5 D2^2 Sqrt[D2 - r] r^3 + 3 D2 Sqrt[D2 - r] r^4 - 5 Sqrt[D2 - r] r^5 -
D1^4 Sqrt[-D1 + D2] (3 D2 + 22 r) + D1^3 Sqrt[-D1 + D2] (-5 D2^2 + 22 D2 r + 33 r^2) +
D1^2 (D2^3 (-17 Sqrt[-D1 + D2] + 33 Sqrt[D2 - r]) + 33 D2^2 (2 Sqrt[-D1 + D2] - 3 Sqrt[D2 - r]) r +
   99 D2 (-Sqrt[-D1 + D2] + Sqrt[D2 - r]) r^2 - 33 Sqrt[D2 - r] r^3) +
D1 (4 D2^4 (9 Sqrt[-D1 + D2] - 11 Sqrt[D2 - r]) + 110 D2^3 (-Sqrt[-D1 + D2] + Sqrt[D2 - r]) r +
   33 D2^2 (3 Sqrt[-D1 + D2] - 2 Sqrt[D2 - r]) r^2 - 22 D2 Sqrt[D2 - r] r^3 + 22 Sqrt[D2 - r] r^4)) +
b2 (70 D1^6 Sqrt[-D1 + D2] + 304 D2^6 (Sqrt[-D1 + D2] - Sqrt[D2 - r]) + 4 D2^5
   (-273 Sqrt[-D1 + D2] + 235 Sqrt[D2 - r]) r + 9 D2^4 (143 Sqrt[-D1 + D2] - 95 Sqrt[D2 - r]) r^2 +
  D2^3 (-429 Sqrt[-D1 + D2] + 100 Sqrt[D2 - r]) r^3 - 10 D2^2 Sqrt[D2 - r] r^4 + 264 D2 Sqrt[D2 - r] r^5 -
  135 Sqrt[D2 - r] r^6 + 75 D1^4 D2 Sqrt[-D1 + D2] (D2 + 13 r) - 15 D1^5 Sqrt[-D1 + D2] (15 D2 + 13 r) +
  D1^3 Sqrt[-D1 + D2] (121 D2^3 - 663 D2^2 r - 1287 D2 r^2 + 429 r^3) +
  D1^2 (D2^4 (387 Sqrt[-D1 + D2] - 715 Sqrt[D2 - r]) + 13 D2^3 (-147 Sqrt[-D1 + D2] + 220 Sqrt[D2 - r]) r +
     429 D2^2 (9 Sqrt[-D1 + D2] - 10 Sqrt[D2 - r]) r^2 + 143 D2 (-9 Sqrt[-D1 + D2] + 20 Sqrt[D2 - r]) r^3 -
     715 Sqrt[D2 - r] r^4) + D1 (D2^5 (-732 Sqrt[-D1 + D2] + 884 Sqrt[D2 - r]) + 26 D2^4
      (111 Sqrt[-D1 + D2] - 115 Sqrt[D2 - r]) r + 39 D2^3 (-99 Sqrt[-D1 + D2] + 80 Sqrt[D2 - r]) r^2 + 13
     D2^2 (99 Sqrt[-D1 + D2] - 20 Sqrt[D2 - r]) r^3 - 1300 D2 Sqrt[D2 - r] r^4 + 546 Sqrt[D2 - r] r^5)))) -
1/(205920 (-D1 + D2)^(7/2)) (D1 - r)^2 (429 a1 (128 D1^3 Sqrt[(-D1 + D2) (-D1 + r)] -
    384 D1^2 D2 Sqrt[(-D1 + D2) (-D1 + r)] +
    384 D1 D2^2 Sqrt[(-D1 + D2) (-D1 + r)] - 128 D2^3 Sqrt[(-D1 + D2) (-D1 + r)] +
    (D1 - r)^2 (79 D1^2 - 224 D1 D2 + 175 D2^2 + 66 D1 r - 126 D2 r + 30 r^2)) -
  80 (-39 a0 (-33 D2^3 Sqrt[-D1 + D2] + 33 D2^2 r Sqrt[-D1 + r] - 22 D2 r^2 Sqrt[-D1 + r] +
      5 r^3 Sqrt[-D1 + r] + D1^3 (33 Sqrt[-D1 + D2] - 16 Sqrt[-D1 + r]) +
      D1^2 (-99 D2 Sqrt[-D1 + D2] + 44 D2 Sqrt[-D1 + r] + 4 r Sqrt[-D1 + r]) +
      D1 (-22 D2 r Sqrt[-D1 + r] + 7 r^2 Sqrt[-D1 + r] + 33 D2^2 (3 Sqrt[-D1 + D2] - Sqrt[-D1 + r]))) +
    a2 (D1 - r) (-429 D2^3 Sqrt[-D1 + D2] + 715 D2^2 r Sqrt[-D1 + r] - 546 D2 r^2 Sqrt[-D1 + r] +
      135 r^3 Sqrt[-D1 + r] + D1^3 (429 Sqrt[-D1 + D2] - 304 Sqrt[-D1 + r]) +
      D1 (-338 D2 r Sqrt[-D1 + r] + 141 r^2 Sqrt[-D1 + r] + 143 D2^2 (9 Sqrt[-D1 + D2] - 5 Sqrt[-D1 + r])) +
      D1^2 (28 r Sqrt[-D1 + r] + 13 D2 (-99 Sqrt[-D1 + D2] + 68 Sqrt[-D1 + r])))));
EXTRCNTRBLft11[r_, D1_, D2_, a_, b_] :=
 1/420
```

```
   (D2 - r)^4
   (7
       a
       (5 D1^2 - 4 D1 D2 + D2^2 - 6 D1 r + 2 D2 r + 2 r^2) +
     b (-35 D1^3 + 4 D2^3 + 9 D2^2 r + 12 D2 r^2 + 10 r^3 + 21 D1^2 (2 D2 + 3 r) -
         21 D1 (D2^2 + 2 D2 r + 2 r^2)));
EXTRCNTRBRgt11[r_, D1_, D2_, a_, b_] :=
  -  1
    ---
    420
   (D1 - r)^4
   (b
       (D1 - r)
       (3 D1^2 - 14 D1 D2 + 21 D2^2 + 8 D1 r - 28 D2 r + 10 r^2) -
     7 a (D1^2 + 5 D2^2 - 6 D2 r + 2 r^2 + 2 D1 (-2 D2 + r)));
```

K = 2

```
LeftCLD22[r_, D1_, D2_] := (1 + 1/(16 (-D1 + D2)^(11/2))
       (16 (D1 - D2)^3 - 40 (D1 - D2)^2 (D2 - r) + 70 (D1 - D2) (D2 - r)^2 - 105 (D2 - r)^3) (-D1 + r)^(5/2))
   (a0 + a4 (D1 - r)^2 + a2 (-D1 + r) + Sqrt[-D1 + r] (a1 + a3 (-D1 + r)));
RgtCLD22[r_, D1_, D2_] := (b0 + (b1 + b3 (D2 - r)) Sqrt[D2 - r] + b2 (D2 - r) + b4 (D2 - r)^2)
   (1 - 1/(16 (-D1 + D2)^(11/2)) (D2 - r)^(5/2) (-231 D1^3 + 16 D2^3 + 40 D2^2 r +
         70 D2 r^2 + 105 r^3 + 99 D1^2 (2 D2 + 5 r) - 11 D1 (8 D2^2 + 20 D2 r + 35 r^2)));
CLDApprx22[r_, D1_, D2_] := (1 + 1/(16 (-D1 + D2)^(11/2))
        (16 (D1 - D2)^3 - 40 (D1 - D2)^2 (D2 - r) + 70 (D1 - D2) (D2 - r)^2 - 105 (D2 - r)^3) (-D1 + r)^(5/2))
    (a0 + a4 (D1 - r)^2 + a2 (-D1 + r) + Sqrt[-D1 + r] (a1 + a3 (-D1 + r))) +
   (b0 + (b1 + b3 (D2 - r)) Sqrt[D2 - r] + b2 (D2 - r) + b4 (D2 - r)^2)
    (1 - 1/(16 (-D1 + D2)^(11/2)) (D2 - r)^(5/2) (-231 D1^3 + 16 D2^3 + 40 D2^2 r + 70 D2 r^2 + 105 r^3 +
          99 D1^2 (2 D2 + 5 r) - 11 D1 (8 D2^2 + 20 D2 r + 35 r^2))); LEFTCFApprx22[r_, D1_, D2_] :=
```

```
(29 393 a1 (-512 D1^8 + 2816 D1^7 D2 - 6336 D1^6 D2^2 + 7392 D1^5 D2^3 - 4620 D1^4 D2^4 + 1386 D1^3 D2^5 +
      462 D1^2 D2^6 - 462 D1 D2^7 + 99 D2^8 + 1280 D1^7 r - 7040 D1^6 D2 r + 15 840 D1^5 D2^2 r -
      18 480 D1^4 D2^3 r + 11 550 D1^3 D2^4 r - 6930 D1^2 D2^5 r + 2310 D1 D2^6 r - 330 D2^7 r - 960 D1^6 r^2 +
      5280 D1^5 D2 r^2 - 11 880 D1^4 D2^2 r^2 + 13 860 D1^3 D2^3 r^2 + 160 D1^5 r^3 - 880 D1^4 D2 r^3 + 1980 D1^3
       D2^2 r^3 - 13 860 D1^2 D2^3 r^3 + 20 D1^4 r^4 - 110 D1^3 D2 r^4 + 8910 D1^2 D2^2 r^4 + 6930 D1 D2^3 r^4 +
      6 D1^3 r^5 - 3498 D1^2 D2 r^5 - 7722 D1 D2^2 r^5 - 1386 D2^3 r^5 + 580 D1^2 r^6 + 3740 D1 D2 r^6 +
      1980 D2^2 r^6 - 700 D1 r^7 - 1100 D2 r^7 + 225 r^8 + 512 (-D1 + D2)^(11/2) (-D1 + r)^(5/2)) -
   4199 a3 (1225 D1^9 + 275 (D1 - D2)^9 + 693 (D1 - D2)^8 r + 6468 (-D1 + D2)^3 r^6 -
      9900 (D1 - D2)^2 r^7 + 5775 (-D1 + D2) r^8 - 1225 r^9 - 1536 (-D1 + D2)^(11/2) (-D1 + r)^(7/2) -
      525 D1^8 (11 D1 - 11 D2 + 21 r) + 300 D1^7 (33 (D1 - D2)^2 + 154 (D1 - D2) r + 147 r^2) -
      420 D1^2 r^4 (231 (D1 - D2)^3 + 495 (D1 - D2)^2 r + 385 (D1 - D2) r^2 + 105 r^3) +
      420 D1^3 r^3 (308 (D1 - D2)^3 + 825 (D1 - D2)^2 r + 770 (D1 - D2) r^2 + 245 r^3) -
      210 D1^4 r^2 (462 (D1 - D2)^3 + 1650 (D1 - D2)^2 r + 1925 (D1 - D2) r^2 + 735 r^3) -
      84 D1^6 (77 (D1 - D2)^3 + 825 (D1 - D2)^2 r + 1925 (D1 - D2) r^2 + 1225 r^3) + 42 D1^5 r
        (924 (D1 - D2)^3 + 4950 (D1 - D2)^2 r + 7700 (D1 - D2) r^2 + 3675 r^3) + 21 D1 (-33 (D1 - D2)^8 +
          1848 (D1 - D2)^3 r^5 + 3300 (D1 - D2)^2 r^6 + 2200 (D1 - D2) r^7 + 525 r^8)) - 1120
    (323 (a0 (143 (-D1 + D2)^3 + 195 (D1 - D2)^2 (D1 - r) + 105 (-D1 + D2) (D1 - r)^2 + 21 (D1 - r)^3)
          (-D1 + r)^(9/2) - 2 a0 (-D1 + D2)^(11/2) (16 D1^2 + 7 D2^2 + 16 D1 (D2 - 3 r) - 30 D2 r + 39 r^2)) +
      (-D1 + D2)^(11/2) (-693 a4 (D1 - D2)^4 + 1596 (D1 - D2)^3 (a2 + a4 (D1 - r)) -
          4522 a2 (D1 - D2)^2 (D1 - r) + 4199 (D1 - r)^3 (2 a2 + a4 (-D1 + r))) -
      (-D1 + r)^(11/2) (2907 (D1 - D2)^2 (-15 a2 + 11 a4 (D1 - r)) (D1 - r) +
          1463 (-D1 + D2) (-17 a2 + 13 a4 (D1 - r)) (D1 - r)^2 +
          273 (-19 a2 + 15 a4 (D1 - r)) (D1 - r)^3 - 2261 (-D1 + D2)^3 (13 a2 + 9 a4 (-D1 + r)))) +
   (D2 - r)^2 (29 393 b1 (-281 (D1 - D2)^6 + 388 (D1 - D2)^5 (D1 - r) + 97 (D1 - D2)^4 (D1 - r)^2 +
          34 (-D1 + D2)^3 (D1 - r)^3 - 145 (D1 - D2)^2 (D1 - r)^4 +
          250 (-D1 + D2) (D1 - r)^5 + 225 (D1 - r)^6 + 512 (-D1 + D2)^(11/2) Sqrt[D2 - r]) +
      4199 b3 (-1118 (D1 - D2)^6 + 1329 (D1 - D2)^5 (D1 - r) + 621 (D1 - D2)^4 (D1 - r)^2 +
          118 (D1 - D2)^3 (D1 - r)^3 - 600 (D1 - D2)^2 (D1 - r)^4 + 1575 (-D1 + D2) (D1 - r)^5 +
          1225 (D1 - r)^6 + 1536 (-D1 + D2)^(11/2) Sqrt[D2 - r]) (D2 - r) + 1120
        (323 b0 (78 (-D1 + D2)^(11/2) + (32 (D1 - D2)^3 + 48 (D1 - D2)^2 (D1 - r) + 42 (D1 - D2) (D1 - r)^2 +
                21 (D1 - r)^3) (D2 - r)^(5/2)) + 4199 (-D1 + D2)^(11/2) (2 b2 + b4 (D2 - r)) (D2 - r) +
          (D2 - r)^(7/2) (-3296 b4 (D1 - D2)^4 - 7 (-D1 + D2) (1330 b2 + 377 b4 (D1 - r)) (D1 - r)^2 +
              273 (19 b2 + 15 b4 (D1 - r)) (D1 - r)^3 - 48 (-D1 + D2)^3 (114 b2 + 61 b4 (-D1 + r)) +
              2 (D1 - D2)^2 (D1 - r) (4712 b2 + 255 b4 (-D1 + r)))))) /
 (56 434 560 (-D1 + D2)^(11/2)); RGHTCFApprx22[r_, D1_, D2_] :=
(29 393 b1 (99 D1^8 - 462 D1^7 D2 + 462 D1^6 D2^2 + 1386 D1^5 D2^3 - 4620 D1^4 D2^4 + 7392 D1^3 D2^5 -
      6336 D1^2 D2^6 + 2816 D1 D2^7 - 512 D2^8 + 512 (-D1 + D2)^(11/2) (D2 - r)^(5/2) - 330 D1^7 r +
      2310 D1^6 D2 r - 6930 D1^5 D2^2 r + 11 550 D1^4 D2^3 r - 18 480 D1^3 D2^4 r + 15 840 D1^2 D2^5 r -
      7040 D1 D2^6 r + 1280 D2^7 r + 13 860 D1^3 D2^3 r^2 - 11 880 D1^2 D2^4 r^2 + 5280 D1 D2^5 r^2 -
      960 D2^6 r^2 - 13 860 D1^3 D2^2 r^3 + 1980 D1^2 D2^3 r^3 - 880 D1 D2^4 r^3 + 160 D2^5 r^3 + 6930 D1^3 D2 r^4 +
```

```
        8910 D1^2 D2^2 r^4 - 110 D1 D2^3 r^4 + 20 D2^4 r^4 - 1386 D1^3 r^5 - 7722 D1^2 D2 r^5 - 3498 D1 D2^2 r^5 +
        6 D2^3 r^5 + 1980 D1^2 r^6 + 3740 D1 D2 r^6 + 580 D2^2 r^6 - 1100 D1 r^7 - 700 D2 r^7 + 225 r^8) +
    4199 b3 (-275 D1^9 + 1782 D1^8 D2 - 4356 D1^7 D2^2 + 3696 D1^6 D2^3 + 4158 D1^5 D2^4 -
        13 860 D1^4 D2^5 + 22 176 D1^3 D2^6 - 19 008 D1^2 D2^7 + 8448 D1 D2^8 - 1536 D2^9 +
        1536 (-D1 + D2)^(11/2) (D2 - r)^(7/2) + 693 D1^8 r - 5544 D1^7 D2 r + 19 404 D1^6 D2^2 r -
        38 808 D1^5 D2^3 r + 48 510 D1^4 D2^4 r - 77 616 D1^3 D2^5 r + 66 528 D1^2 D2^6 r - 29 568 D1 D2^7 r +
        5376 D2^8 r + 97 020 D1^3 D2^4 r^2 - 83 160 D1^2 D2^5 r^2 + 36 960 D1 D2^6 r^2 - 6720 D2^7 r^2 -
        129 360 D1^3 D2^3 r^3 + 41 580 D1^2 D2^4 r^3 - 18 480 D1 D2^5 r^3 + 3360 D2^6 r^3 + 97 020 D1^3 D2^2 r^4 +
        55 440 D1^2 D2^3 r^4 + 2310 D1 D2^4 r^4 - 420 D2^5 r^4 - 38 808 D1^3 D2 r^5 - 91 476 D1^2 D2^2 r^5 -
        24 024 D1 D2^3 r^5 - 42 D2^4 r^5 + 6468 D1^3 r^6 + 49 896 D1^2 D2 r^6 + 42 504 D1 D2^2 r^6 + 4032 D2^3 r^6 -
        9900 D1^2 r^7 - 26 400 D1 D2 r^7 - 7800 D2^2 r^7 + 5775 D1 r^8 + 5250 D2 r^8 - 1225 r^9) -
    1120 (b4 (-D1 + D2)^(11/2) (-3296 (D1 - D2)^4 + 15 200 (D1 - D2)^3 (D1 - r) -
            25 194 (D1 - D2)^2 (D1 - r)^2 + 16 796 (D1 - D2) (D1 - r)^3 - 4199 (D1 - r)^4) -
        19 b2 (288 (D1 - D2)^3 + 496 (D1 - D2)^2 (D1 - r) + 490 (D1 - D2) (D1 - r)^2 + 273 (D1 - r)^3)
          (D2 - r)^(11/2) - b4 (3296 (D1 - D2)^3 + 6224 (D1 - D2)^2 (D1 - r) +
            6734 (D1 - D2) (D1 - r)^2 + 4095 (D1 - r)^3) (D2 - r)^(13/2) + 38 b2 (-D1 + D2)^(11/2)
          (144 (D1 - D2)^3 + 663 (D1 - D2) (D1 - r)^2 - 221 (D1 - r)^3 + 544 (D1 - D2)^2 (-D1 + r)) +
        323 (-b0 (32 (D1 - D2)^3 + 48 (D1 - D2)^2 (D1 - r) + 42 (D1 - D2) (D1 - r)^2 + 21 (D1 - r)^3)
              (D2 - r)^(9/2) - 2 b0 (-D1 + D2)^(11/2) (7 D1^2 + 16 D1 D2 + 16 D2^2 - 30 D1 r - 48 D2 r + 39 r^2))) -
    (D1 - r)^2 (-29 393 a1 ((1386 (-D1 + D2)^3 + 1980 (D1 - D2)^2 (D1 - r) + 1100 (-D1 + D2) (D1 - r)^2 +
              225 (D1 - r)^3) (D1 - r)^3 + 512 (-D1 + D2)^(11/2) Sqrt[-D1 + r]) + 4199 a3 (D1 - r)
          ((6468 (-D1 + D2)^3 + 9900 (D1 - D2)^2 (D1 - r) + 5775 (-D1 + D2) (D1 - r)^2 + 1225 (D1 - r)^3)
              (D1 - r)^3 + 1536 (-D1 + D2)^(11/2) Sqrt[-D1 + r]) +
        1120 (-4199 (-D1 + D2)^(11/2) (6 a0 + (-2 a2 + a4 (D1 - r)) (D1 - r)) +
            (-D1 + r)^(5/2) (969 (D1 - D2)^2 (65 a0 + 3 (-15 a2 + 11 a4 (D1 - r)) (D1 - r)) (D1 - r) +
                133 (-D1 + D2) (255 a0 + 11 (-17 a2 + 13 a4 (D1 - r)) (D1 - r)) (D1 - r)^2 +
                21 (323 a0 + 13 (-19 a2 + 15 a4 (D1 - r)) (D1 - r)) (D1 - r)^3 + 323 (-D1 + D2)^3
                 (143 a0 - 7 (D1 - r) (13 a2 + 9 a4 (-D1 + r))))))) / (56 434 560 (-D1 + D2)^(11/2));

EXTRCNTRBLft22[r_, D1_, D2_, a_, b_] := 1/2520 (D2 - r)^5
   (-9 a (14 D1^3 - D2^3 - 3 D2^2 r - 5 D2 r^2 - 5 r^3 - 14 D1^2 (D2 + 2 r) + 2 D1 (3 D2^2 + 8 D2 r + 10 r^2)) +
     b (126 D1^4 + 5 D2^4 + 16 D2^3 r + 30 D2^2 r^2 + 40 D2 r^3 + 35 r^4 - 168 D1^3 (D2 + 2 r) +
        36 D1^2 (3 D2^2 + 8 D2 r + 10 r^2) - 36 D1 (D2^3 + 3 D2^2 r + 5 D2 r^2 + 5 r^3))) ;

EXTRCNTRBRgt22[r_, D1_, D2_, a_, b_] := 1/2520 (D1 - r)^5
   (9 a (D1^3 - 6 D1^2 D2 + 14 D1 D2^2 - 14 D2^3 + (3 D1^2 - 16 D1 D2 + 28 D2^2) r + 5 (D1 - 4 D2) r^2 + 5 r^3) -
     b (D1 - r) (4 D1^3 - 27 D1^2 D2 + 72 D1 D2^2 - 84 D2^3 +
        15 (D1^2 - 6 D1 D2 + 12 D2^2) r + 15 (2 D1 - 9 D2) r^2 + 35 r^3));
```

THE CASE $K = 0$

$$\mathtt{LeftCLD00[r\_,\ D1\_,\ D2\_]} := \mathrm{a0}\left(1+\frac{\sqrt{-D1+r}\ (2\,D1-3\,D2+r)}{2\ (-D1+D2)^{3/2}}\right);$$

$$\mathtt{RgtCLD00[r\_,\ D1\_,\ D2\_]} := \mathrm{b0}\left(1-\frac{\sqrt{D2-r}\ (-3\,D1+2\,D2+r)}{2\ (-D1+D2)^{3/2}}\right);$$

```
CLDApprx00[r_, D1_, D2_] := LeftCLD00[r, D1, D2] + RgtCLD00[r, D1, D2 ] ;
LEFTCFApprx00[r_, D1_, D2_] := (* ∫_r^D2 F[x] dx *)
```

$$\mathrm{b0}\left(\frac{D2^2}{2}-D2\,r+\frac{r^2}{2}-\frac{2\ (D2-r)^{5/2}\ (-7\,D1+6\,D2+r)}{35\ (-D1+D2)^{3/2}}\right)+$$
$$\frac{1}{70\ (D1-D2)}\ \mathrm{a0}\left(-24\,D1^3\sqrt{\frac{D1-r}{D1-D2}}+28\,D1^2\,D2\sqrt{\frac{D1-r}{D1-D2}}+44\,D1^2\sqrt{\frac{D1-r}{D1-D2}}\ r-\right.$$
$$56\,D1\,D2\sqrt{\frac{D1-r}{D1-D2}}\ r-16\,D1\sqrt{\frac{D1-r}{D1-D2}}\ r^2+28\,D2\sqrt{\frac{D1-r}{D1-D2}}\ r^2-$$
$$\left.4\sqrt{\frac{D1-r}{D1-D2}}\ r^3+(D1-D2)\ \left(24\,D1^2+8\,D1\,D2+3\,D2^2-14\ (4\,D1+D2)\ r+35\,r^2\right)\right);$$

```
RGHTCFApprx00[r_, D1_, D2_] := (* ∫_D1^x F[x] dx *)
```

$$\mathrm{a0}\left(\frac{D1^2}{2}-D1\,r+\frac{r^2}{2}-\frac{2}{35}\ (D1-r)\ \left(\frac{D1-r}{D1-D2}\right)^{3/2}\ (6\,D1-7\,D2+r)\right)-$$
$$\frac{1}{70\ (-D1+D2)^{3/2}}\ \mathrm{b0}\left(3\,D1^3\sqrt{-D1+D2}-24\,D2^3\left(\sqrt{-D1+D2}-\sqrt{D2-r}\right)+\right.$$
$$D1^2\sqrt{-D1+D2}\ (5\,D2-14\,r)+D2^2\left(56\sqrt{-D1+D2}-44\sqrt{D2-r}\right)r+$$
$$D2\left(-35\sqrt{-D1+D2}+16\sqrt{D2-r}\right)r^2+4\sqrt{D2-r}\ r^3+D1\left(4\,D2^2\left(4\sqrt{-D1+D2}-7\sqrt{D2-r}\right)+\right.$$
$$\left.\left.D2\left(-42\sqrt{-D1+D2}+56\sqrt{D2-r}\right)r+7\left(5\sqrt{-D1+D2}-4\sqrt{D2-r}\right)r^2\right)\right);$$

$$\mathtt{EXTRCNTRBLft00[r\_,\ D1\_,\ D2\_,\ a\_,\ b\_]} := \frac{1}{60}\ (D2-r)^3$$
$$\left(5\,a\ (-2\,D1+D2+r)+b\left(10\,D1^2+3\,D2^2+4\,D2\,r+3\,r^2-10\,D1\ (D2+r)\right)\right);$$

$$\mathtt{EXTRCNTRBRgt00[r\_,\ D1\_,\ D2\_,\ a\_,\ b\_]} :=$$
$$-\frac{1}{60}\ (D1-r)^3\ (-5\,a\ (D1-2\,D2+r)+b\ (D1-r)\ (2\,D1-5\,D2+3\,r));$$

```
Limit[LEFTCFApprx00[r, D1, D2], r → D2, Direction → 1]
```

```
Simplify[Series[EXTRCNTRBLft00[r, D1, D2, a, b], {r, D1, 4}]]
Simplify[Series[EXTRCNTRBLft00[r, D1, D2, a, b], {r, D2, 4}]]
Simplify[Series[EXTRCNTRBRgt00[r, D1, D2, a, b], {r, D1, 4}]]
Simplify[Series[EXTRCNTRBRgt00[r, D1, D2, a, b], {r, D2, 4}]]
```

The approximation is determined in this way:
1A) we first determine OcthCFApprxPrtlyMatchd00AA[r] equating it to OCTACFAA[r];
2A) we determine OcthCFApprxPrtlyMatchd00BB[r] using first LEFTCFApprxnn[..] within the subinterval [D1,D2] and then matching this expression to OCTCFAA[r] at r=D1. The error [OcthCFApprxFnl00BB[r] -OCTCFBB[r]] at r=D2 is ~0.005.
3A) we determine OcthCFApprxFnl00DD[r] by RGHTCFApprxnn[..]. The error (OcthCFApprxFnl00DD[r] -OCTCFDD[r]) is ~0.002. It cannot be cured and propagates to the left subinterval;
4A) we determine OcthCFApprxFnl00CC[r] by the above formula. It automatically matches OcthCFApprxFnl00DD[r] at r=D3 and is equal to 0 at r=D2.
5A) finally, OcthCFApprx00BB[r] is determined matching OcthCFApprxPrtlyMatchd00BB[r] to OcthCFApprxFnl00CC[r] at r=D2 adding the extracontribution EXTRCNTRB00[r].
The resulting inconsistencies are:
a small negativity on the left of D2; the vanishing at r=D2, r=D3 and r=D4.

Approximation within the interval [D0, D1]

```
Expand[OCTCFAA[r]]
Expand[Simplify[D[D[OCTCFAA[r], r], r]]]
```

$$\text{OcthCFApprxFnl00AA[r\_]} := 1 - \frac{3\sqrt{3}\, r}{2} + \frac{6\sqrt{2}\, r^2}{\pi} + \frac{3 r^3}{2} - \frac{\sqrt{3}\, r^3}{2} - \frac{3 r^3}{2\pi} - \frac{3 r^2 \,\text{ArcSec}[3]}{\pi};$$

$$\text{OcthCLDApprx00AA[r\_]} := \frac{12\sqrt{2}}{\pi} + 9 r - 3\sqrt{3}\, r - \frac{9 r}{\pi} - \frac{6\,\text{ArcSec}[3]}{\pi};$$

```
Plot[{OCTCFAA[r], OcthCFApprxFnl00AA[r]}, {r, 0, OCTDst[[1]]}]
```

Approximation within the interval [D1, D2]

```
Clear[a0]; Clear[a1]; Clear[a2]; Clear[a3]; Clear[a4]; Clear[a5]; Clear[a6];
Clear[b0]; Clear[b1]; Clear[b2]; Clear[b3]; Clear[b4]; Clear[b5]; Clear[b6];

cfD1Plus
cfD2Minus

FullSimplify[((CLDApprx00[r, D1, D2]) /.
   {a0 → cfD1Plus[[1]], b0 → cfD2Minus[[1]], D1 → OCTDst[[1]], D2 → OCTDst[[2]]}),
 Assumptions → {OCTDst[[1]] < r < OCTDst[[2]]}]

FullSimplify[((CLDApprx00[r, D1, D2]) /.
    {a0 → cfD1Plus[[1]], b0 → cfD2Minus[[1]], D1 → OCTDst[[1]], D2 → OCTDst[[2]]}) -
  OcthCLDApprx00BB[r], Assumptions → {OCTDst[[1]] < r < OCTDst[[2]]}]
```

the K=0 approximation of the CLD

```
OcthCLDApprx00BB[r_] :=
```

$$\frac{1}{36}\left(\frac{108\left(1+\frac{\sqrt{-\sqrt{3}+3\,r}\,\left(8\sqrt{3}-9\sqrt{6}+12\,r\right)}{3^{3/4}\left(-4+3\sqrt{2}\right)^{3/2}}\right)\left(4\sqrt{2}-\sqrt{3}+\pi+\sqrt{3}\,\pi-2\,\text{ArcSec}[3]\right)}{\pi}+\left(1-\frac{3^{3/4}\sqrt{\sqrt{6}-4\,r}\,\left(-2\sqrt{3}+\sqrt{6}+2\,r\right)}{\left(-4+3\sqrt{2}\right)^{3/2}}\right)\left(432-311\sqrt{2}+81\sqrt{6}-\frac{27\left(-16\sqrt{2}+3\sqrt{6}+8\,\text{ArcSec}[3]\right)}{\pi}\right)\right);$$

```
Plot[{OCTDDCFBB[r], OcthCLDApprx00BB[r]}, {r, OCTDst[[1]], OCTDst[[2]]},
 PlotStyle → {Directive[Blue, Thickness[0.003]], Directive[Red, Thickness[0.006]]},
 AxesLabel → {"r", "γ0''[r]"}, PlotLabel → " CLD:  D1 < r < D2"]
```

Approximation (not yet matched) of the CLD

```
Simplify[((LEFTCFApprx00[r, D1, D2]) /.
   {a0 → cfD1Plus[[1]], b0 → cfD2Minus[[1]], D1 → OCTDst[[1]], D2 → OCTDst[[2]]}),
 Assumptions → {OCTDst[[1]] < r < OCTDst[[2]]}]

Simplify[((LEFTCFApprx00[r, D1, D2]) /.
    {a0 → cfD1Plus[[1]], b0 → cfD2Minus[[1]], D1 → OCTDst[[1]], D2 → OCTDst[[2]]}) -
  ApprxOcthCF00NotMatchBB[r], Assumptions → {OCTDst[[1]] < r < OCTDst[[2]]}]
```

```
ApprxOcthCF00NotMatchBB[r_] :=
```

$$\frac{1}{1260}\left(-\frac{1}{4\left(-4+3\sqrt{2}\right)^{3/2}\pi}3\times 3^{3/4}\left(588\times 3^{1/4}\sqrt{-4+3\sqrt{2}}-465\times 3^{1/4}\sqrt{-8+6\sqrt{2}}-1536\sqrt{-\sqrt{3}+3\,r}+1344\sqrt{-2\sqrt{3}+6\,r}-768\,r^3\sqrt{-3\sqrt{3}+9\,r}-24\,r^2\left(-140\,3^{1/4}\sqrt{-4+3\sqrt{2}}+105\times 3^{1/4}\sqrt{-8+6\sqrt{2}}+8\left(16-21\sqrt{2}\right)\sqrt{-\sqrt{3}+3\,r}\right)+8\,r\left(-161\,3^{3/4}\sqrt{-4+3\sqrt{2}}+126\times 3^{3/4}\sqrt{-8+6\sqrt{2}}+16\left(22-21\sqrt{2}\right)\sqrt{-3\sqrt{3}+9\,r}\right)\right)\left(4\sqrt{2}-\sqrt{3}+\pi+\sqrt{3}\,\pi-2\,\mathrm{ArcSec}[3]\right)+35\left(\frac{3}{16}-\frac{1}{2}\sqrt{\frac{3}{2}}\,r+\frac{r^2}{2}-\frac{3^{3/4}\left(\sqrt{6}-4\,r\right)^{5/2}\left(3\sqrt{\frac{3}{2}}-\frac{7}{\sqrt{3}}+r\right)}{70\left(-4+3\sqrt{2}\right)^{3/2}}\right)\left(432-311\sqrt{2}+81\sqrt{6}-\frac{27\left(-16\sqrt{2}+3\sqrt{6}+8\,\mathrm{ArcSec}[3]\right)}{\pi}\right)\right);$$

```
Plot[{OCTCFBB[r], ApprxOcthCF00NotMatchBB[r]}, {r, OCTDst[[1]], OCTDst[[2]]},
 PlotStyle → {Directive[Blue, Thickness[0.003]], Directive[Red, Thickness[0.006]]},
 AxesLabel → {"r", "γT[r]"}]
```

The approximation is matched at r=D1

```
Solve[{Limit[ApprxOcthCF00NotMatchBB[r] + a + b r, r → OCTDst[[1]], Direction → -1] -
      Limit[OcthCFApprxFnl00AA[r], r → OCTDst[[1]], Direction → 1] == 0 &&
    Limit[D[ApprxOcthCF00NotMatchBB[r] + a + b r, r], r → OCTDst[[1]], Direction → -1] -
      Limit[D[OcthCFApprxFnl00AA[r], r], r → OCTDst[[1]], Direction → 1] == 0}, {a, b}]
```

```
Simplify[(Simplify[
```

$$(\mathrm{ApprxOcthCF00NotMatchBB}[r]+a+b\,r)\ /.\ \left\{a\to-\frac{1}{20\,160\,\pi}\left(42\,336\sqrt{2}-1932\sqrt{3}-3753\sqrt{6}-28\,484\,\pi-12\,959\sqrt{2}\,\pi+1932\sqrt{3}\,\pi+3753\sqrt{6}\,\pi+20\,160\,\mathrm{ArcSec}[-3]-1008\,\mathrm{ArcSec}[3]\right),\ b\to-\frac{1}{1080\,\pi}\left(1701-2592\sqrt{3}-2592\sqrt{6}-1701\,\pi+6333\sqrt{3}\,\pi-1432\sqrt{6}\,\pi-2160\sqrt{3}\,\mathrm{ArcSec}[-3]-864\sqrt{3}\,\mathrm{ArcSec}[3]+648\sqrt{6}\,\mathrm{ArcSec}[3]\right)\right\},$$

```
  Assumptions → {OCTDst[[1]] < r < OCTDst[[2]]}]) /. {ArcSec[-3] → π - ArcSec[3]}]
```

```
Simplify[Simplify[
  (ApprxOcthCF00NotMatchBB[r] + a + b r) /. {a -> -1/(20160 π) (42336 √2 - 1932 √3 - 3753 √6 -
        28484 π - 12959 √2 π + 1932 √3 π + 3753 √6 π + 20160 ArcSec[-3] - 1008 ArcSec[3]),
    b -> -1/(1080 π) (1701 - 2592 √3 - 2592 √6 - 1701 π + 6333 √3 π - 1432 √6 π -
        2160 √3 ArcSec[-3] - 864 √3 ArcSec[3] + 648 √6 ArcSec[3])},
  Assumptions -> {OCTDst[[1]] < r < OCTDst[[2]]}] - ApprxOcthCF00PrtlyMatchBB[r]]
```

$$\text{ApprxOcthCF00PrtlyMatchBB[r\_]} := \frac{1}{20160\,\pi}\left(\left(8324 + 12959\sqrt{2} - 1932\sqrt{3} - 3753\sqrt{6}\right)\pi + 3\left(-14112\sqrt{2} + 644\sqrt{3} + 1251\sqrt{6} + 7056\,\text{ArcSec}[3]\right)\right) + \frac{r\left(\left(1701 - 4173\sqrt{3} + 1432\sqrt{6}\right)\pi - 81\left(21 - 32\sqrt{3} - 32\sqrt{6} + 8\sqrt{3}\left(2+\sqrt{2}\right)\text{ArcSec}[3]\right)\right)}{1080\,\pi} + \frac{1}{1260}\left(-\frac{1}{4\left(-4+3\sqrt{2}\right)^{3/2}\pi}\, 3\times 3^{3/4}\left(588\times 3^{1/4}\sqrt{-4+3\sqrt{2}} - 465\times 3^{1/4}\sqrt{-8+6\sqrt{2}} - 1536\sqrt{-\sqrt{3}+3r} + 1344\sqrt{-2\sqrt{3}+6r} - 768\,r^3\sqrt{-3\sqrt{3}+9r} - 24\,r^2\left(-140\,3^{1/4}\sqrt{-4+3\sqrt{2}} + 105\times 3^{1/4}\sqrt{-8+6\sqrt{2}} + 8\left(16-21\sqrt{2}\right)\sqrt{-\sqrt{3}+3r}\right) + 8\,r\left(-161\,3^{3/4}\sqrt{-4+3\sqrt{2}} + 126\times 3^{3/4}\sqrt{-8+6\sqrt{2}} + 16\left(22-21\sqrt{2}\right)\sqrt{-3\sqrt{3}+9r}\right)\right)\left(4\sqrt{2} - \sqrt{3} + \pi + \sqrt{3}\,\pi - 2\,\text{ArcSec}[3]\right) + 35\left(\frac{3}{16} - \frac{1}{2}\sqrt{\frac{3}{2}}\,r + \frac{r^2}{2} - \frac{3^{3/4}\left(\sqrt{6}-4r\right)^{5/2}\left(3\sqrt{\frac{3}{2}} - \frac{7}{\sqrt{3}} + r\right)}{70\left(-4+3\sqrt{2}\right)^{3/2}}\right)\left(432 - 311\sqrt{2} + 81\sqrt{6} - \frac{27\left(-16\sqrt{2} + 3\sqrt{6} + 8\,\text{ArcSec}[3]\right)}{\pi}\right)\right);$$

```
Plot[{OCTCFBB[r], ApprxOcthCF00PrtlyMatchBB[r], ApprxOcthCF00NotMatchBB[r]},
 {r, OCTDst[[1]], OCTDst[[2]]},
 PlotStyle -> {Directive[Blue, Thickness[0.003]], Directive[Red, Thickness[0.004], Dashed],
   Directive[Purple, Thickness[0.004], Dotted]}, AxesLabel -> {"r", "γ0[r]"}]
```

The error at r=D2 is ~0.00305039, the relative error is 15%.

```
N[Limit[OCTCFBB[r] - ApprxOcthCF00PrtlyMatchBB[r], r -> OCTDst[[2]], Direction -> 1]]
N[Limit[OCTCFBB[r] - ApprxOcthCF00PrtlyMatchBB[r], r -> OCTDst[[2]], Direction -> 1] /
  (Limit[OCTCFBB[r], r -> OCTDst[[2]], Direction -> 1])]
```

```
ausfiga = Plot[{If[r < OCTDst[[1]], OCTCFAA[r], OCTCFBB[r] ],
   If[r < OCTDst[[1]], OcthCFApprxFnl00AA[r], ApprxOcthCF00PrtlyMatchBB[r]],
   10 (If[r < OCTDst[[1]], OCTCFAA[r], OCTCFBB[r] ] -
      If[r < OCTDst[[1]], OcthCFApprxFnl00AA[r], ApprxOcthCF00PrtlyMatchBB[r]])},
  {r, 0, OCTDst[[2]]}, PlotRange → {{0, 1.05}, {-0.05, 1.05}}, PlotStyle →
   {Directive[Blue, Thickness[0.003]], Directive[Red, Thickness[0.004], Dashed],
    Directive[Black, Thickness[0.003], Dashed]}, AxesLabel → {"r", "γ₀[r],10xΔγ"}]
```

We start now from D4 and proceed towards the left

Approximation within the interval [D3, D4]

```
cfD3Plus
cfD4Minus
```

evaluation of the CLD approximation

```
Simplify[(CLDApprx00[r, D1, D2]) /.
  {a0 → cfD3Plus[[1]], b0 → cfD4Minus[[1]], D1 → OCTDst[[3]], D2 → OCTDst[[4]]},
 Assumptions → {OCTDst[[3]] < r < OCTDst[[4]]}]

Simplify[Simplify[(CLDApprx00[r, D1, D2]) /.
    {a0 → cfD3Plus[[1]], b0 → cfD4Minus[[1]], D1 → OCTDst[[3]], D2 → OCTDst[[4]]},
   Assumptions → {OCTDst[[3]] < r < OCTDst[[4]]}] - OcthCLDApprx00DD[r]]
```

$$\text{OcthCLDApprx00DD[r\_]} := \left(6 - 10\sqrt{\frac{2}{3}} + \frac{9}{\sqrt{2}} - \frac{13}{\sqrt{2}\,\pi}\right)\left(1 + \frac{\left(-3 + \sqrt{2} + r\right)\sqrt{-\sqrt{2} + 2\,r}}{\left(2 - \sqrt{2}\right)^{3/2}}\right);$$

```
Plot[{OCTDDCFDD[r], OcthCLDApprx00DD[r]},
 {r, OCTDst[[3]], OCTDst[[4]]}, AxesLabel → {"r", "γ''[r]"},
 PlotStyle → {Directive[Blue, Thickness[0.003]], Directive[Red, Thickness[0.006]]},
 PlotRange → {{OCTDst[[3]], OCTDst[[4]]}, {0, 1.5}},
 AxesLabel → {"r", "γ₀''[r],10xΔγ"}, PlotLabel → " CLD:  D₃ < r < D₄"]

Plot[{OCTDDCFDD[r] - OcthCLDApprx00DD[r]}, {r, OCTDst[[3]], OCTDst[[4]]}]
```

The largest errors are ~0.1 amd -0.15 around 0.71 and 0.79

Evaluation of the approximation of the CF

```
Simplify[(Simplify[Simplify[(LEFTCFApprx00[r, D1, D2]) /.
     {a0 → cfD3Plus[[1]], b0 → cfD4Minus[[1]], D1 → OCTDst[[3]], D2 → OCTDst[[4]]},
    Assumptions → {OCTDst[[3]] < r < OCTDst[[4]]}]]) /.
```

$$\left\{\sqrt{\frac{\sqrt{2} - 2\,r}{-2 + \sqrt{2}}} \to \frac{\sqrt{r\sqrt{2} - 1}}{\sqrt{\sqrt{2} - 1}}\right\}, \text{Assumptions} \to \{\text{OCTDst[[3]]} < r < \text{OCTDst[[4]]}\}\Big]$$

$$\text{FullSimplify}\left[\sqrt{\frac{\sqrt{2} - 2\,r}{-2 + \sqrt{2}}} - \frac{\sqrt{r\sqrt{2} - 1}}{\sqrt{\sqrt{2} - 1}}, \text{Assumptions} \to \{\text{OCTDst[[3]]} < r < \text{OCTDst[[4]]}\}\right]$$

0

```
FullSimplify[Simplify[(LEFTCFApprx00[r, D1, D2]) /.
    {a0 → cfD3Plus[[1]], b0 → cfD4Minus[[1]], D1 → OCTDst[[3]], D2 → OCTDst[[4]]},
   Assumptions → {OCTDst[[3]] < r < OCTDst[[4]]}] - OcthCFApprxFnl00DD[r],
 Assumptions → {OCTDst[[3]] < r < OCTDst[[4]]}]
```

0

```
OcthCFApprxFnl00DD[r_] :=
```

$$\left(\left(39\sqrt{2}+\left(-36-27\sqrt{2}+20\sqrt{6}\right)\pi\right)\left(22\sqrt{-1+\sqrt{2}}-7\sqrt{2\left(-1+\sqrt{2}\right)}-28\sqrt{-1+\sqrt{2}}\,r+8r^3\sqrt{-1+\sqrt{2}}\,r+12\sqrt{-2+2\sqrt{2}}\,r+r^2\left(70\sqrt{-1+\sqrt{2}}-35\sqrt{2\left(-1+\sqrt{2}\right)}+8\left(-7+2\sqrt{2}\right)\sqrt{-1+\sqrt{2}}\,r\right)+r\left(28\sqrt{-1+\sqrt{2}}-42\sqrt{2\left(-1+\sqrt{2}\right)}+4\left(-11+14\sqrt{2}\right)\sqrt{-1+\sqrt{2}}\,r\right)\right)\right)\Big/\left(420\left(-2+\sqrt{2}\right)\sqrt{-1+\sqrt{2}}\,\pi\right);$$

The error of the CF-approximation is -0.00223423 at D_3, i.e. ~90%

```
N[Limit[OCTCFDD[r] - OcthCFApprxFnl00DD[r], r → OCTDst[[3]], Direction → -1]]
N[Limit[OCTCFDD[r] - OcthCFApprxFnl00DD[r], r → OCTDst[[3]], Direction → -1] /
  Limit[OCTCFDD[r], r → OCTDst[[3]], Direction → -1]]
```

```
-0.00223423
```

```
-0.913454
```

```
Plot[{OCTCFDD[r], OcthCFApprxFnl00DD[r]}, {r, OCTDst[[3]], OCTDst[[4]]},
 PlotRange → {{OCTDst[[3]], OCTDst[[4]]}, {0, 0.005}},
 PlotStyle → {Directive[Blue, Thickness[0.003]], Directive[Red, Thickness[0.006]]},
 AxesLabel → {"r", "γ0[r],10xΔγ"}]
```

Approximation within the interval [D2, D3]

```
cfD2Plus
cfD3Minus
```

```
cfD2Plus[[1]]
cfD3Minus[[1]]
N[cfD2Plus[[1]]]
N[cfD3Minus[[1]]]
```

approximation of the CLD

```
Simplify[(CLDApprx00[r, D1, D2]) /.
  {a0 → cfD2Plus[[1]], b0 → cfD3Minus[[1]], D1 → OCTDst[[2]], D2 → OCTDst[[3]]},
 Assumptions → {OCTDst[[2]] < r < OCTDst[[3]]}]
```

```
OCTDst[[2]]
OCTDst[[3]]
```

FullSimplify$\left[\left(\sqrt{-\sqrt{6}+4r}\right)\text{^}2-\left(2^{1/4}\sqrt{-\sqrt{3}+2\sqrt{2}\,r}\right)\text{^}2,\right.$ Assumptions → {OCTDst[[2]] < r < OCTDst[[3]]}]

FullSimplify$\left[\left(\sqrt{\sqrt{2}-2r}\right)\text{^}2-\left(2^{1/4}\sqrt{1-\sqrt{2}\,r}\right)\text{^}2,\right.$ Assumptions → {OCTDst[[2]] < r < OCTDst[[3]]}]

```
(Together[Simplify[(CLDApprx00[r, D1, D2]) /.
    {a0 → cfD2Plus[[1]], b0 → cfD3Minus[[1]], D1 → OCTDst[[2]], D2 → OCTDst[[3]]},
   Assumptions → {OCTDst[[2]] < r < OCTDst[[3]]}]]) /.
```

$$\left\{\sqrt{\sqrt{2}-2r}\to 2^{1/4}\sqrt{1-\sqrt{2}\,r},\ \sqrt{-\sqrt{6}+4r}\to 2^{1/4}\sqrt{-\sqrt{3}+2\sqrt{2}\,r}\right\}$$

```
FullSimplify[(Together[Simplify[(CLDApprx00[r, D1, D2]) /.
      {a0 → cfD2Plus[[1]], b0 → cfD3Minus[[1]], D1 → OCTDst[[2]], D2 → OCTDst[[3]]},
     Assumptions → {OCTDst[[2]] < r < OCTDst[[3]]}]] - OcthCLDApprx00CC[r]) /.
  {Sqrt[Sqrt[2] - 2 r] → 2^(1/4) Sqrt[1 - Sqrt[2] r], Sqrt[-Sqrt[6] + 4 r] → 2^(1/4) Sqrt[-Sqrt[3] + 2 Sqrt[2] r]},
 Assumptions → {OCTDst[[2]] < r < OCTDst[[3]]}]
```

$$\text{OcthCLDApprx00CC[r\_]} := \frac{1}{36\sqrt{2-\sqrt{3}}\left(-2+\sqrt{3}\right)\pi}\Big(-639\sqrt{2\left(2-\sqrt{3}\right)}+360\sqrt{6\left(2-\sqrt{3}\right)}-1296\sqrt{2-\sqrt{3}}\,\pi+181\sqrt{2\left(2-\sqrt{3}\right)}\,\pi+648\sqrt{3\left(2-\sqrt{3}\right)}\,\pi-71\sqrt{6\left(2-\sqrt{3}\right)}\,\pi-936\sqrt{1-\sqrt{2}\,r}+702\sqrt{3}\sqrt{1-\sqrt{2}\,r}+1728\pi\sqrt{1-\sqrt{2}\,r}+432\sqrt{2}\,\pi\sqrt{1-\sqrt{2}\,r}-966\sqrt{3}\,\pi\sqrt{1-\sqrt{2}\,r}-324\sqrt{6}\,\pi\sqrt{1-\sqrt{2}\,r}-468\sqrt{2}\,r\sqrt{1-\sqrt{2}\,r}+432\pi r\sqrt{1-\sqrt{2}\,r}+324\sqrt{2}\,\pi r\sqrt{1-\sqrt{2}\,r}-240\sqrt{6}\,\pi r\sqrt{1-\sqrt{2}\,r}+1539\sqrt{2}\sqrt{-\sqrt{3}+2\sqrt{2}\,r}-675\sqrt{6}\sqrt{-\sqrt{3}+2\sqrt{2}\,r}+1296\pi\sqrt{-\sqrt{3}+2\sqrt{2}\,r}-1176\sqrt{2}\,\pi\sqrt{-\sqrt{3}+2\sqrt{2}\,r}-432\sqrt{3}\,\pi\sqrt{-\sqrt{3}+2\sqrt{2}\,r}+554\sqrt{6}\,\pi\sqrt{-\sqrt{3}+2\sqrt{2}\,r}-864r\sqrt{-\sqrt{3}+2\sqrt{2}\,r}+162\sqrt{3}\,r\sqrt{-\sqrt{3}+2\sqrt{2}\,r}+622\pi r\sqrt{-\sqrt{3}+2\sqrt{2}\,r}-432\sqrt{2}\,\pi r\sqrt{-\sqrt{3}+2\sqrt{2}\,r}-162\sqrt{3}\,\pi r\sqrt{-\sqrt{3}+2\sqrt{2}\,r}+432\sqrt{2-\sqrt{3}}\,\text{ArcSec[3]}-216\sqrt{3\left(2-\sqrt{3}\right)}\,\text{ArcSec[3]}-648\sqrt{-\sqrt{3}+2\sqrt{2}\,r}\,\text{ArcSec[3]}+216\sqrt{3}\sqrt{-\sqrt{3}+2\sqrt{2}\,r}\,\text{ArcSec[3]}+216\sqrt{2}\,r\sqrt{-\sqrt{3}+2\sqrt{2}\,r}\,\text{ArcSec[3]}\Big);$$

```
Plot[{OCTDDCFCC[r], OcthCLDApprx00CC[r]},
 {r, OCTDst[[2]], OCTDst[[3]]}, AxesLabel → {"r", "γ''[r]"}, PlotStyle →
  {Directive[Blue, Thickness[0.003]], Directive[Red, Thickness[0.004], Dashed]},
 PlotRange → {{OCTDst[[2]], OCTDst[[3]]}, {0, 7}},
 AxesLabel → {"r", "γT''[r],10xΔγ"}, PlotLabel → " CLD: K=0; D2< r < D3"]
```

The largest error is about 60% around 0.64

```
Plot[{OCTDDCFCC[r] - OcthCLDApprx00CC[r]}, {r, OCTDst[[2]], OCTDst[[3]]}]

N[(OCTDDCFCC[64 / 100] - OcthCLDApprx00CC[64 / 100]) / OCTDDCFCC[64 / 100]]

0.60603

Plot[{If[r < OCTDst[[2]], OCTDDCFBB[r], If[r < OCTDst[[3]], OCTDDCFCC[r], OCTDDCFDD[r]]],
  If[r < OCTDst[[2]], OcthCLDApprx00BB[r],
   If[r < OCTDst[[3]], OcthCLDApprx00CC[r], OcthCLDApprx00DD[r]]]},
 {r, OCTDst[[1]], OCTDst[[4]]}, AxesLabel → {"r", "γ''[r]"}, PlotStyle →
  {Directive[Blue, Thickness[0.003]], Directive[Red, Thickness[0.004], Dashed]},
 PlotRange → {{OCTDst[[1]], OCTDst[[4]]}, {0, 10}}, AxesLabel → {"r", "γT''[r],10xΔγ"}]
```

Overall the CLD approximation is not very accurate.

Approximation (not matched) of the CF

```
Simplify[(Simplify[((LEFTCFApprx00[r, D1, D2]) /.
      {a0 → cfD2Plus[[1]], b0 → cfD3Minus[[1]], D1 → OCTDst[[2]], D2 → OCTDst[[3]]}),
    Assumptions → {OCTDst[[2]] < r < OCTDst[[3]]}]) /. {Sqrt[(Sqrt[6] - 4 r)/(-2 + Sqrt[3])] → (2^(1/4) Sqrt[-Sqrt[3] + 2 Sqrt[2] r])/Sqrt[2 - Sqrt[3]],
   Sqrt[3/(2 - Sqrt[3])] Sqrt[-Sqrt[3] + 2 Sqrt[2] r] → Sqrt[3]/Sqrt[2 - Sqrt[3]] Sqrt[-Sqrt[3] + 2 Sqrt[2] r]}]

(FullSimplify[(((LEFTCFApprx00[r, D1, D2]) /.
        {a0 → cfD2Plus[[1]], b0 → cfD3Minus[[1]], D1 → OCTDst[[2]], D2 → OCTDst[[3]]}) -
      ApprxOcthCF00NotMatchCC[r]) /. {Sqrt[(Sqrt[6] - 4 r)/(-2 + Sqrt[3])] → (2^(1/4) Sqrt[-Sqrt[3] + 2 Sqrt[2] r])/Sqrt[2 - Sqrt[3]],
     Sqrt[3/(2 - Sqrt[3])] Sqrt[-Sqrt[3] + 2 Sqrt[2] r] → Sqrt[3]/Sqrt[2 - Sqrt[3]] Sqrt[-Sqrt[3] + 2 Sqrt[2] r]},
   Assumptions → {OCTDst[[2]] < r < OCTDst[[3]]}]) /.
  {Sqrt[-Sqrt[6] + 4 r] → 2^(1/4) Sqrt[-Sqrt[3] + 2 Sqrt[2] r]}
```

0

```
ApprxOcthCF00NotMatchCC[r_] := 1/2520
  (2520 (6 - 10 Sqrt[2/3] + 9/Sqrt[2] - 13/(Sqrt[2] π)) (1/4 - r/Sqrt[2] + r^2/2 -
        (2^(3/4) (Sqrt[2] - 2 r)^(5/2) (-(7 Sqrt[3/2])/2 + 3 Sqrt[2] + r))/(35 (2 - Sqrt[3])^(3/2))) -
    1/(2 (2 - Sqrt[3])^(3/2) π) (13 Sqrt[6 - 3 Sqrt[3]] - 30 Sqrt[2 - Sqrt[3]] + 42 Sqrt[-Sqrt[3] + 2 Sqrt[2] r] -
       16 r^3 Sqrt[-2 Sqrt[3] + 4 Sqrt[2] r] - 18 Sqrt[-3 Sqrt[3] + 6 Sqrt[2] r] +
       2 Sqrt[2] r (21 Sqrt[6 - 3 Sqrt[3]] - 28 Sqrt[2 - Sqrt[3]] + (33 - 28 Sqrt[3]) Sqrt[-Sqrt[3] + 2 Sqrt[2] r]) +
       2 r^2 (35 Sqrt[6 - 3 Sqrt[3]] - 70 Sqrt[2 - Sqrt[3]] + 8 (7 - 2 Sqrt[3]) Sqrt[-Sqrt[3] + 2 Sqrt[2] r]))
    ((432 - 311 Sqrt[2] + 81 Sqrt[6]) π - 27 (-16 Sqrt[2] + 3 Sqrt[6] + 8 ArcSec[3])));
```

```
Plot[{OCTCFCC[r], ApprxOcthCF00NotMatchCC[r]},
 {r, OCTDst[[2]], OCTDst[[3]]}, AxesLabel → {"r", "γ[r]"},
 PlotStyle → {Directive[Blue, Thickness[0.003]], Directive[Red, Thickness[0.006]]},
 AxesLabel → {"r", "γT[r],10xΔγ"}]
```

The approximation is partly matched at r=D3

```
Simplify[
 Solve[{Simplify[Limit[ApprxOcthCF00NotMatchCC[r] + a + b r, r → OCTDst[[3]], Direction → 1] -
       Limit[OcthCFApprxFnl00DD[r], r → OCTDst[[3]], Direction → -1]] == 0 &&
    Simplify[Limit[D[ApprxOcthCF00NotMatchCC[r] + a + b r, r],
        r → OCTDst[[3]], Direction → 1] -
       Limit[D[OcthCFApprxFnl00DD[r], r], r → OCTDst[[3]], Direction → -1]] == 0}, {a, b}]]
```

```
Simplify[(Simplify[Simplify[(ApprxOcthCF00NotMatchCC[r] + a + b r) /.
      {a → -(-819 + 507 √2 + (99 + 27 √2 - 420 √3 + 260 √6) π)/(420 (-2 + √2) π),
       b → (39 (4 - 3 √2) + (9 √2 + 80 √3 - 60 √6) π)/(30 (-2 + √2) π)},
     Assumptions → {OCTDst[[2]] < r < OCTDst[[3]]}]]) /.
  {√(-√6 + 4 r) → 2^(1/4) √(-√3 + 2 √2 r), √(6 - 3 √3) → √3 √(2 - √3),
   √(-2 √3 + 4 √2 r) → √2 √(-√3 + 2 √2 r),
   -16 r^3 √(-2 √3 + 4 √2 r) → -16 √2 r^3 √(-√3 + 2 √2 r)}]

Simplify[(Simplify[(ApprxOcthCF00NotMatchCC[r] + a + b r) /.
      {a → -(-819 + 507 √2 + (99 + 27 √2 - 420 √3 + 260 √6) π)/(420 (-2 + √2) π),
       b → (39 (4 - 3 √2) + (9 √2 + 80 √3 - 60 √6) π)/(30 (-2 + √2) π)},
     Assumptions → {OCTDst[[2]] < r < OCTDst[[3]]}] - ApprxOcthCF00PrtlyMatchCC[r]) /.
  {√(-√6 + 4 r) → 2^(1/4) √(-√3 + 2 √2 r), √(6 - 3 √3) → √3 √(2 - √3),
   √(-2 √3 + 4 √2 r) → √2 √(-√3 + 2 √2 r),
   -16 r^3 √(-2 √3 + 4 √2 r) → -16 √2 r^3 √(-√3 + 2 √2 r)}]
```

0

```
ApprxOcthCF00PrtlyMatchCC[r_] :=
```

$$\frac{1}{2520}\left(-\frac{6\left(-819+507\sqrt{2}+\left(99+27\sqrt{2}-420\sqrt{3}+260\sqrt{6}\right)\pi\right)}{\left(-2+\sqrt{2}\right)\pi}+\frac{84\left(39\left(4-3\sqrt{2}\right)+\left(9\sqrt{2}+80\sqrt{3}-60\sqrt{6}\right)\pi\right)r}{\left(-2+\sqrt{2}\right)\pi}+2520\left(6-10\sqrt{\frac{2}{3}}+\frac{9}{\sqrt{2}}-\frac{13}{\sqrt{2}\,\pi}\right)\left(\frac{1}{4}-\frac{r}{\sqrt{2}}+\frac{r^2}{2}-\frac{2^{3/4}\left(\sqrt{2}-2r\right)^{5/2}\left(-\frac{7\sqrt{\frac{3}{2}}}{2}+3\sqrt{2}+r\right)}{35\left(2-\sqrt{3}\right)^{3/2}}\right)-\frac{1}{2\left(2-\sqrt{3}\right)^{3/2}\pi}\left(13\sqrt{6-3\sqrt{3}}-30\sqrt{2-\sqrt{3}}+42\sqrt{-\sqrt{3}+2\sqrt{2}\,r}-16r^3\sqrt{-2\sqrt{3}+4\sqrt{2}\,r}-18\sqrt{-3\sqrt{3}+6\sqrt{2}\,r}+2\sqrt{2}\,r\left(21\sqrt{6-3\sqrt{3}}-28\sqrt{2-\sqrt{3}}+\left(33-28\sqrt{3}\right)\sqrt{-\sqrt{3}+2\sqrt{2}\,r}\right)+2r^2\left(35\sqrt{6-3\sqrt{3}}-70\sqrt{2-\sqrt{3}}+8\left(7-2\sqrt{3}\right)\sqrt{-\sqrt{3}+2\sqrt{2}\,r}\right)\right)\left(\left(432-311\sqrt{2}+81\sqrt{6}\right)\pi-27\left(-16\sqrt{2}+3\sqrt{6}+8\,\mathrm{ArcSec}[3]\right)\right)\right);$$

```
Plot[{If[r < OCTDst[[3]], OCTCFCC[r], OCTCFDD[r]],
   If[r < OCTDst[[3]], ApprxOcthCF00PrtlyMatchCC[r], OcthCFApprxFnl00DD[r]],
   If[r < OCTDst[[3]], OCTCFCC[r], OCTCFDD[r]] -
    If[r < OCTDst[[3]], ApprxOcthCF00PrtlyMatchCC[r], OcthCFApprxFnl00DD[r]]},
  {r, OCTDst[[2]], OCTDst[[4]]}, AxesLabel → {"r", "γ[r]"},
  PlotRange → {{OCTDst[[2]], OCTDst[[4]]}, {-0.003, 0.02}},
  PlotStyle → {Directive[Blue, Thickness[0.003]],
    Directive[Red, Thickness[0.004]], Directive[Black, Thickness[0.003], Dotted]}]
```

The situation around D2 is shown in the figure.

```
Plot[{If[r < OCTDst[[2]], OCTCFBB[r], OCTCFCC[r]],
   If[r < OCTDst[[2]], ApprxOcthCF00PrtlyMatchBB[r], ApprxOcthCF00PrtlyMatchCC[r]]},
  {r, OCTDst[[1]], OCTDst[[3]]}, AxesLabel → {"r", "γ[r]"},
  PlotStyle → {Directive[Blue, Thickness[0.003]], Directive[Red, Thickness[0.004]]}]
```

```
N[OCTDst[[2]] - OCTDst[[1]]]
N[OCTDst[[3]] - OCTDst[[2]]]
```

0.0350222

0.0947343

The figure shows that end-value of ApprxOcthCF00PrtlyMatchCC[r] is farther from the exact one than ApprxOcthCF00PrtlyMatchCC[r]. Since [D2, D3] is wider than [D1, D2] we presume that the error of ApprxOcthCF00PrtlyMatchCC[r] is larger.
Thus, the match at r=D2 is made adding to ApprxOcthCF00PrtlyMatchCC[r] the contribution EXTRCNTRBLft00[r, D2, D3, a, b] and matching the result to ApprxOcthCF00PrtlyMatchBB[r] that will be assumed eqaul to OcthCFApprxFnl00BB[r].

```
ApprxOcthCF00PrtlyMatchBB[r]
```

$$\text{OcthCFApprxFnl00BB[r\_]} := \frac{1}{20160\,\pi}\Big(\big(8324+12959\sqrt{2}-1932\sqrt{3}-3753\sqrt{6}\big)\pi + 3\big(-14112\sqrt{2}+644\sqrt{3}+1251\sqrt{6}+7056\,\text{ArcSec}[3]\big)\Big) + \frac{r\Big(\big(1701-4173\sqrt{3}+1432\sqrt{6}\big)\pi-81\big(21-32\sqrt{3}-32\sqrt{6}+8\sqrt{3}\,(2+\sqrt{2})\,\text{ArcSec}[3]\big)\Big)}{1080\,\pi} + \frac{1}{1260}\Bigg(-\frac{1}{4\left(-4+3\sqrt{2}\right)^{3/2}\pi}\,3\times 3^{3/4}\Big(588\times 3^{1/4}\sqrt{-4+3\sqrt{2}}-465\times 3^{1/4}\sqrt{-8+6\sqrt{2}}-1536\sqrt{-\sqrt{3}+3r}+1344\sqrt{-2\sqrt{3}+6r}-768\,r^3\sqrt{-3\sqrt{3}+9r}-24\,r^2\Big(-140\;3^{1/4}\sqrt{-4+3\sqrt{2}}+105\times 3^{1/4}\sqrt{-8+6\sqrt{2}}+8\big(16-21\sqrt{2}\big)\sqrt{-\sqrt{3}+3r}\Big)+8\,r\Big(-161\;3^{3/4}\sqrt{-4+3\sqrt{2}}+126\times 3^{3/4}\sqrt{-8+6\sqrt{2}}+16\big(22-21\sqrt{2}\big)\sqrt{-3\sqrt{3}+9r}\Big)\Big)\Big(4\sqrt{2}-\sqrt{3}+\pi+\sqrt{3}\,\pi-2\,\text{ArcSec}[3]\Big) + 35\left(\frac{3}{16}-\frac{1}{2}\sqrt{\frac{3}{2}}\,r+\frac{r^2}{2}-\frac{3^{3/4}\left(\sqrt{6}-4r\right)^{5/2}\left(3\sqrt{\frac{3}{2}}-\frac{7}{\sqrt{3}}+r\right)}{70\left(-4+3\sqrt{2}\right)^{3/2}}\right)\left(432-311\sqrt{2}+81\sqrt{6}-\frac{27\left(-16\sqrt{2}+3\sqrt{6}+8\,\text{ArcSec}[3]\right)}{\pi}\right)\Bigg);$$

```
Simplify[(EXTRCNTRBLft00[r, D2, D3, a, b]) /. {D2 → OCTDst[[2]], D3 → OCTDst[[3]]},
 Assumptions → {OCTDst[[2]] < r < OCTDst[[3]]}]
```

```
EXTRCNTRB00CC[r_, a_, b_] :=
```

$$\frac{1}{60}\left(\frac{1}{\sqrt{2}}-r\right)^3\left(5\,a\left(-\sqrt{\frac{3}{2}}+\frac{1}{\sqrt{2}}+r\right)+b\left(\frac{21}{4}+2\sqrt{2}\,r+3\,r^2-5\sqrt{\frac{3}{2}}\left(\frac{1}{\sqrt{2}}+r\right)\right)\right);$$

```
Simplify[Solve[
  {Simplify[Limit[ApprxOcthCF00PrtlyMatchCC[r] + EXTRCNTRB00CC[r, a, b], r → OCTDst[[2]],
       Direction → -1] - Limit[OcthCFApprxFnl00BB[r],
       r → OCTDst[[2]], Direction → 1]] == 0 &&
    Simplify[Limit[D[ApprxOcthCF00PrtlyMatchCC[r] + EXTRCNTRB00CC[r, a, b], r],
       r → OCTDst[[2]], Direction → -1] -
      Limit[D[OcthCFApprxFnl00BB[r], r], r → OCTDst[[2]], Direction → 1]] == 0}, {a, b}]]
```

Simplify[Simplify[(ApprxOcthCF00PrtlyMatchCC[r] + EXTRCNTRB00CC[r, a, b]) /.

$$\left\{a\to\frac{1}{105\,\pi}8\left(\left(-14\,556\,942+16\,833\,121\sqrt{2}-8\,404\,210\sqrt{3}+9\,718\,610\sqrt{6}\right)\pi-3\left(5\,887\,008-268\,875\sqrt{2}+3\,398\,906\sqrt{3}-155\,202\sqrt{6}+3360\left(-485+795\sqrt{2}-280\sqrt{3}+459\sqrt{6}\right)\text{ArcSec}[-3]+12\left(-127\,337+89\,040\sqrt{2}-73\,516\sqrt{3}+51\,408\sqrt{6}\right)\text{ArcSec}[3]\right)\right),\right.$$

$$\left.b\to\frac{1}{63\,\pi}64\left(\left(-37\,158\,831+16\,174\,533\sqrt{2}-21\,453\,639\sqrt{3}+9\,338\,357\sqrt{6}\right)\pi-9\left(232\,650-2\,173\,797\sqrt{2}+134\,319\sqrt{3}-1\,255\,042\sqrt{6}+840\left(-2340+724\sqrt{2}-1351\sqrt{3}+418\sqrt{6}\right)\text{ArcSec}[-3]+24\left(-32\,760+23\,381\sqrt{2}-18\,914\sqrt{3}+13\,499\sqrt{6}\right)\text{ArcSec}[3]\right)\right)\right\},$$

Assumptions → {OCTDst[[2]] < r < OCTDst[[3]]}]]

OcthCFApprxFnl00CC[r_] :=

$$\frac{1}{2520}\left(-\frac{6\left(-819+507\sqrt{2}+\left(99+27\sqrt{2}-420\sqrt{3}+260\sqrt{6}\right)\pi\right)}{\left(-2+\sqrt{2}\right)\pi}+\frac{84\left(39\left(4-3\sqrt{2}\right)+\left(9\sqrt{2}+80\sqrt{3}-60\sqrt{6}\right)\pi\right)r}{\left(-2+\sqrt{2}\right)\pi}+2520\left(6-10\sqrt{\frac{2}{3}}+\frac{9}{\sqrt{2}}-\frac{13}{\sqrt{2}\,\pi}\right)\right.$$

$$\left(\frac{1}{4}-\frac{r}{\sqrt{2}}+\frac{r^2}{2}-\frac{2^{3/4}\left(\sqrt{2}-2\,r\right)^{5/2}\left(-\frac{7\sqrt{\frac{3}{2}}}{2}+3\sqrt{2}+r\right)}{35\left(2-\sqrt{3}\right)^{3/2}}\right)-$$

$$\frac{1}{2\left(2-\sqrt{3}\right)^{3/2}\pi}\left(13\sqrt{6-3\sqrt{3}}-30\sqrt{2-\sqrt{3}}+42\sqrt{-\sqrt{3}+2\sqrt{2}\,r}-\right.$$
$$16\,r^3\sqrt{-2\sqrt{3}+4\sqrt{2}\,r}-18\sqrt{-3\sqrt{3}+6\sqrt{2}\,r}+$$
$$2\sqrt{2}\,r\left(21\sqrt{6-3\sqrt{3}}-28\sqrt{2-\sqrt{3}}+\left(33-28\sqrt{3}\right)\sqrt{-\sqrt{3}+2\sqrt{2}\,r}\right)+$$
$$\left.2\,r^2\left(35\sqrt{6-3\sqrt{3}}-70\sqrt{2-\sqrt{3}}+8\left(7-2\sqrt{3}\right)\sqrt{-\sqrt{3}+2\sqrt{2}\,r}\right)\right)$$

$$\left(\left(432-311\sqrt{2}+81\sqrt{6}\right)\pi-27\left(-16\sqrt{2}+3\sqrt{6}+8\,\text{ArcSec}[3]\right)\right)\Bigg)+$$

$$\frac{1}{945\,\pi}2\left(\frac{1}{\sqrt{2}}-r\right)^3\left(8\left(\frac{21}{4}+2\sqrt{2}\,r+3\,r^2-5\sqrt{\frac{3}{2}}\left(\frac{1}{\sqrt{2}}+r\right)\right)\right.$$
$$\left(\left(-37\,158\,831+16\,174\,533\sqrt{2}-21\,453\,639\sqrt{3}+9\,338\,357\sqrt{6}\right)\pi-9\left(232\,650-2\,173\,797\right.\right.$$
$$\sqrt{2}+134\,319\sqrt{3}-1\,255\,042\sqrt{6}+840\left(-2340+724\sqrt{2}-1351\sqrt{3}+418\sqrt{6}\right)$$
$$\left.\left.\text{ArcSec}[-3]+24\left(-32\,760+23\,381\sqrt{2}-18\,914\sqrt{3}+13\,499\sqrt{6}\right)\text{ArcSec}[3]\right)\right)+$$
$$3\left(-\sqrt{\frac{3}{2}}+\frac{1}{\sqrt{2}}+r\right)\left(\left(-14\,556\,942+16\,833\,121\sqrt{2}-8\,404\,210\sqrt{3}+9\,718\,610\sqrt{6}\right)\pi-\right.$$
$$3\left(5\,887\,008-268\,875\sqrt{2}+3\,398\,906\sqrt{3}-155\,202\sqrt{6}+\right.$$
$$3360\left(-485+795\sqrt{2}-280\sqrt{3}+459\sqrt{6}\right)\text{ArcSec}[-3]+$$
$$\left.\left.\left.12\left(-127\,337+89\,040\sqrt{2}-73\,516\sqrt{3}+51\,408\sqrt{6}\right)\text{ArcSec}[3]\right)\right)\right);$$

```
Plot[{If[r < OCTDst[[2]], OCTCFBB[r], OCTCFCC[r]], If[r < OCTDst[[2]],
   OcthCFApprxFnl00BB[r], OcthCFApprxFnl00CC[r]]}, {r, OCTDst[[1]], OCTDst[[3]]},
  PlotStyle → {Directive[Blue, Thickness[0.003]], Directive[Red, Thickness[0.004]]}]
```

```
In[18]:= Plot[{If[r < OCTDst[[1]], OCTCFAA[r],
     If[r < OCTDst[[2]], OCTCFBB[r], If[r < OCTDst[[3]], OCTCFCC[r], OCTCFDD[r]]]],
   If[r < OCTDst[[1]], OcthCFApprxFnl00AA[r], If[r < OCTDst[[2]], OcthCFApprxFnl00BB[r],
     If[r < OCTDst[[3]], OcthCFApprxFnl00CC[r], OcthCFApprxFnl00DD[r]]]],
   10 * If[r < OCTDst[[1]], (OCTCFAA[r] - OcthCFApprxFnl00AA[r]),
     If[r < OCTDst[[2]], (OCTCFBB[r] - OcthCFApprxFnl00BB[r]), If[r < OCTDst[[3]],
       (OCTCFCC[r] - OcthCFApprxFnl00CC[r]), (OCTCFDD[r] - OcthCFApprxFnl00DD[r])]]]},
  {r, 0, 1}, PlotStyle → {Directive[Blue, Thickness[0.005]],
    Directive[Red, Thickness[0.005], Dashed], Directive[Black, Thickness[0.003], Dotted]},
  AxesLabel → {"r", " γo(r), 10xΔγ"}]

In[19]:= Plot[{If[r < OCTDst[[1]], OCTDDCFAA[r],
     If[r < OCTDst[[2]], OCTDDCFBB[r], If[r < OCTDst[[3]], OCTDDCFCC[r], OCTDDCFDD[r]]]],
   If[r < OCTDst[[1]], OcthCLDApprx00AA[r], If[r < OCTDst[[2]], OcthCLDApprx00BB[r],
     If[r < OCTDst[[3]], OcthCLDApprx00CC[r], OcthCLDApprx00DD[r]]]],
   If[r < OCTDst[[1]], (OCTDDCFAA[r] - OcthCLDApprx00AA[r]),
    If[r < OCTDst[[2]], (OCTDDCFBB[r] - OcthCLDApprx00BB[r]),
     If[r < OCTDst[[3]], (OCTDDCFCC[r] - OcthCLDApprx00CC[r]),
      (OCTDDCFDD[r] - OcthCLDApprx00DD[r])]]]}, {r, -0, 1},
  PlotRange → {{0, 1}, {-1, 10}}, PlotStyle → {Directive[Blue, Thickness[0.003]],
    Directive[Red, Thickness[0.005], Dashed], Directive[Black, Thickness[0.003], Dotted]},
  AxesLabel → {"r", " γo''(r), Δγ''"}]
```

The final formulae of the tetrahedron-CF approximation for the case K=0 and with the BB procedure B are

In[16]:=

$$\text{OcthCLDApprx00AA}[r\_] := \frac{12\sqrt{2}}{\pi} + 9\,r - 3\sqrt{3}\,r - \frac{9\,r}{\pi} - \frac{6\,\text{ArcSec}[3]}{\pi};\ \text{OcthCLDApprx00BB}[r\_] :=$$

$$\frac{1}{36}\left(\frac{108\left(1 + \frac{\sqrt{-\sqrt{3}+3\,r}\,\left(8\sqrt{3}-9\sqrt{6}+12\,r\right)}{3^{3/4}\left(-4+3\sqrt{2}\right)^{3/2}}\right)\left(4\sqrt{2}-\sqrt{3}+\pi+\sqrt{3}\,\pi-2\,\text{ArcSec}[3]\right)}{\pi} + \left(1 - \frac{3^{3/4}\sqrt{\sqrt{6}-4\,r}\,\left(-2\sqrt{3}+\sqrt{6}+2\,r\right)}{\left(-4+3\sqrt{2}\right)^{3/2}}\right)\left(432 - 311\sqrt{2} + 81\sqrt{6} - \frac{27\left(-16\sqrt{2}+3\sqrt{6}+8\,\text{ArcSec}[3]\right)}{\pi}\right)\right);$$

$$\text{OcthCLDApprx00CC}[r\_] := \frac{1}{36\sqrt{2-\sqrt{3}}\,\left(-2+\sqrt{3}\right)\pi}$$

$$\Big(-639\sqrt{2\left(2-\sqrt{3}\right)} + 360\sqrt{6\left(2-\sqrt{3}\right)} - 1296\sqrt{2-\sqrt{3}}\,\pi + 181\sqrt{2\left(2-\sqrt{3}\right)}\,\pi +$$
$$648\sqrt{3\left(2-\sqrt{3}\right)}\,\pi - 71\sqrt{6\left(2-\sqrt{3}\right)}\,\pi - 936\sqrt{1-\sqrt{2}\,r} + 702\sqrt{3}\sqrt{1-\sqrt{2}\,r} +$$
$$1728\,\pi\sqrt{1-\sqrt{2}\,r} + 432\sqrt{2}\,\pi\sqrt{1-\sqrt{2}\,r} - 966\sqrt{3}\,\pi\sqrt{1-\sqrt{2}\,r} - 324\sqrt{6}\,\pi\sqrt{1-\sqrt{2}\,r} -$$
$$468\sqrt{2}\,r\sqrt{1-\sqrt{2}\,r} + 432\,\pi\,r\sqrt{1-\sqrt{2}\,r} + 324\sqrt{2}\,\pi\,r\sqrt{1-\sqrt{2}\,r} -$$
$$240\sqrt{6}\,\pi\,r\sqrt{1-\sqrt{2}\,r} + 1539\sqrt{2}\sqrt{-\sqrt{3}+2\sqrt{2}\,r} - 675\sqrt{6}\sqrt{-\sqrt{3}+2\sqrt{2}\,r} +$$
$$1296\,\pi\sqrt{-\sqrt{3}+2\sqrt{2}\,r} - 1176\sqrt{2}\,\pi\sqrt{-\sqrt{3}+2\sqrt{2}\,r} - 432\sqrt{3}\,\pi\sqrt{-\sqrt{3}+2\sqrt{2}\,r} +$$
$$554\sqrt{6}\,\pi\sqrt{-\sqrt{3}+2\sqrt{2}\,r} - 864\,r\sqrt{-\sqrt{3}+2\sqrt{2}\,r} + 162\sqrt{3}\,r\sqrt{-\sqrt{3}+2\sqrt{2}\,r} +$$
$$622\,\pi\,r\sqrt{-\sqrt{3}+2\sqrt{2}\,r} - 432\sqrt{2}\,\pi\,r\sqrt{-\sqrt{3}+2\sqrt{2}\,r} - 162\sqrt{3}\,\pi\,r\sqrt{-\sqrt{3}+2\sqrt{2}\,r} +$$
$$432\sqrt{2-\sqrt{3}}\,\text{ArcSec}[3] - 216\sqrt{3\left(2-\sqrt{3}\right)}\,\text{ArcSec}[3] - 648\sqrt{-\sqrt{3}+2\sqrt{2}\,r}\,\text{ArcSec}[3] +$$
$$216\sqrt{3}\sqrt{-\sqrt{3}+2\sqrt{2}\,r}\,\text{ArcSec}[3] + 216\sqrt{2}\,r\sqrt{-\sqrt{3}+2\sqrt{2}\,r}\,\text{ArcSec}[3]\Big);$$

$$\text{OcthCLDApprx00DD}[r\_] := \left(6 - 10\sqrt{\frac{2}{3}} + \frac{9}{\sqrt{2}} - \frac{13}{\sqrt{2}\,\pi}\right)\left(1 + \frac{\left(-3+\sqrt{2}+r\right)\sqrt{-\sqrt{2}+2\,r}}{\left(2-\sqrt{2}\right)^{3/2}}\right);$$

In[17]:=

```
OcthCFApprxFnl00AA[r_] := 1 - (3 Sqrt[3] r)/2 + (6 Sqrt[2] r^2)/π + (3 r^3)/2 - (Sqrt[3] r^3)/2 - (3 r^3)/(2 π) - (3 r^2 ArcSec[3])/π;
OcthCFApprxFnl00BB[r_] := 1/(20160 π) ((8324 + 12959 Sqrt[2] - 1932 Sqrt[3] - 3753 Sqrt[6]) π +
     3 (-14112 Sqrt[2] + 644 Sqrt[3] + 1251 Sqrt[6] + 7056 ArcSec[3])) +
   (r ((1701 - 4173 Sqrt[3] + 1432 Sqrt[6]) π - 81 (21 - 32 Sqrt[3] - 32 Sqrt[6] + 8 Sqrt[3] (2 + Sqrt[2]) ArcSec[3])))/(1080 π) +
   1/1260 (-1/(4 (-4 + 3 Sqrt[2])^(3/2) π) 3 × 3^(3/4) (588 × 3^(1/4) Sqrt[-4 + 3 Sqrt[2]] - 465 × 3^(1/4) Sqrt[-8 + 6 Sqrt[2]] -
        1536 Sqrt[-Sqrt[3] + 3 r] + 1344 Sqrt[-2 Sqrt[3] + 6 r] - 768 r^3 Sqrt[-3 Sqrt[3] + 9 r] -
        24 r^2 (-140 3^(1/4) Sqrt[-4 + 3 Sqrt[2]] + 105 × 3^(1/4) Sqrt[-8 + 6 Sqrt[2]] + 8 (16 - 21 Sqrt[2]) Sqrt[-Sqrt[3] + 3 r]) +
        8 r (-161 3^(3/4) Sqrt[-4 + 3 Sqrt[2]] + 126 × 3^(3/4) Sqrt[-8 + 6 Sqrt[2]] + 16 (22 - 21 Sqrt[2]) Sqrt[-3 Sqrt[3] + 9 r]))
      (4 Sqrt[2] - Sqrt[3] + π + Sqrt[3] π - 2 ArcSec[3]) +
     35 (3/16 - 1/2 Sqrt[3/2] r + r^2/2 - (3^(3/4) (Sqrt[6] - 4 r)^(5/2) (3 Sqrt[3/2 - 7/Sqrt[3] + r]))/(70 (-4 + 3 Sqrt[2])^(3/2)))
      (432 - 311 Sqrt[2] + 81 Sqrt[6] - (27 (-16 Sqrt[2] + 3 Sqrt[6] + 8 ArcSec[3]))/π));

OcthCFApprxFnl00CC[r_] := 1/2520 (-(6 (-819 + 507 Sqrt[2] + (99 + 27 Sqrt[2] - 420 Sqrt[3] + 260 Sqrt[6]) π))/((-2 + Sqrt[2]) π) +
```

$$\frac{84\left(39\left(4-3\sqrt{2}\right)+\left(9\sqrt{2}+80\sqrt{3}-60\sqrt{6}\right)\pi\right)r}{\left(-2+\sqrt{2}\right)\pi}+$$

$$2520\left(6-10\sqrt{\frac{2}{3}}+\frac{9}{\sqrt{2}}-\frac{13}{\sqrt{2}\,\pi}\right)\left(\frac{1}{4}-\frac{r}{\sqrt{2}}+\frac{r^2}{2}-\frac{2^{3/4}\left(\sqrt{2}-2r\right)^{5/2}\left(-\frac{7\sqrt{\frac{3}{2}}}{2}+3\sqrt{2}+r\right)}{35\left(2-\sqrt{3}\right)^{3/2}}-\right.$$

$$\frac{1}{2\left(2-\sqrt{3}\right)^{3/2}\pi}\left(13\sqrt{6-3\sqrt{3}}-30\sqrt{2-\sqrt{3}}+42\sqrt{-\sqrt{3}+2\sqrt{2}\,r}-\right.$$

$$16r^3\sqrt{-2\sqrt{3}+4\sqrt{2}\,r}-18\sqrt{-3\sqrt{3}+6\sqrt{2}\,r}+$$

$$2\sqrt{2}\,r\left(21\sqrt{6-3\sqrt{3}}-28\sqrt{2-\sqrt{3}}+\left(33-28\sqrt{3}\right)\sqrt{-\sqrt{3}+2\sqrt{2}\,r}\right)+$$

$$\left.2r^2\left(35\sqrt{6-3\sqrt{3}}-70\sqrt{2-\sqrt{3}}+8\left(7-2\sqrt{3}\right)\sqrt{-\sqrt{3}+2\sqrt{2}\,r}\right)\right)$$

$$\left.\left(\left(432-311\sqrt{2}+81\sqrt{6}\right)\pi-27\left(-16\sqrt{2}+3\sqrt{6}+8\,\text{ArcSec}[3]\right)\right)\right)+$$

$$\frac{1}{945\pi}2\left(\frac{1}{\sqrt{2}}-r\right)^3\left(8\left(\frac{21}{4}+2\sqrt{2}\,r+3r^2-5\sqrt{\frac{3}{2}}\left(\frac{1}{\sqrt{2}}+r\right)\right)\right.$$

$$\left(\left(-37158831+16174533\sqrt{2}-21453639\sqrt{3}+9338357\sqrt{6}\right)\pi-9\left(232650-2173797\right.\right.$$

$$\sqrt{2}+134319\sqrt{3}-1255042\sqrt{6}+840\left(-2340+724\sqrt{2}-1351\sqrt{3}+418\sqrt{6}\right)$$

$$\left.\left.\text{ArcSec}[-3]+24\left(-32760+23381\sqrt{2}-18914\sqrt{3}+13499\sqrt{6}\right)\text{ArcSec}[3]\right)\right)+$$

$$3\left(-\sqrt{\frac{3}{2}}+\frac{1}{\sqrt{2}}+r\right)\left(\left(-14556942+16833121\sqrt{2}-8404210\sqrt{3}+9718610\sqrt{6}\right)\pi-\right.$$

$$3\left(5887008-268875\sqrt{2}+3398906\sqrt{3}-155202\sqrt{6}+\right.$$

$$3360\left(-485+795\sqrt{2}-280\sqrt{3}+459\sqrt{6}\right)\text{ArcSec}[-3]+$$

```
12 (-127 337 + 89 040 √2 - 73 516 √3 + 51 408 √6) ArcSec[3]))|;

OcthCFApprxFnl00DD[r_] := ((39 √2 + (-36 - 27 √2 + 20 √6) π)
  (22 √(-1 + √2) - 7 √(2 (-1 + √2)) - 28 √(-1 + √2 r) + 8 r^3 √(-1 + √2 r) + 12 √(-2 + 2 √2 r) +
    r^2 (70 √(-1 + √2) - 35 √(2 (-1 + √2)) + 8 (-7 + 2 √2) √(-1 + √2 r)) +
    r (28 √(-1 + √2) - 42 √(2 (-1 + √2)) + 4 (-11 + 14 √2) √(-1 + √2 r))))/
  (420 (-2 + √2) √(-1 + √2) π);
```

Check of the sum rule $4\pi \int_0^\infty r^2\,\gamma(r)\,dr = V_p$

In our case $V_p =$ VOcth = 1 / 6.

We find 0.11777 with an error ~ -0.000077 (~ 0.04%) that will appear in the FT, i.e. ΔI(0) ~- 0.000077.

$$\frac{1}{6}$$

```
4 π (NIntegrate[r^2 * OcthCFApprxFnl00AA[r], {r, 0, OCTDst[[1]]},
     WorkingPrecision → 30, PrecisionGoal → 15] + NIntegrate[r^2 * OcthCFApprxFnl00BB[r],
     {r, OCTDst[[1]], OCTDst[[2]]}, WorkingPrecision → 30, PrecisionGoal → 15] +
   NIntegrate[r^2 * OcthCFApprxFnl00CC[r], {r, OCTDst[[2]], OCTDst[[3]]},
     WorkingPrecision → 30, PrecisionGoal → 15 ] + NIntegrate[r^2 * OcthCFApprxFnl00DD[r],
     {r, OCTDst[[3]], OCTDst[[4]]}, WorkingPrecision → 30, PrecisionGoal → 15 ])
```

0.166743866063037712813689460960

```
N[VOcth - 0.166743866063037712813]
N[(VOcth - 0.166743866063037712813) / VOcth]
```

-0.0000771994

-0.000463196

THE CASE K = 1

THE FORMULAE ARE:

```
LeftCLD11[r_, D1_, D2_] := (a0 + a1 Sqrt[-D1 + r] + a2 (-D1 + r))
   (1 - ((-D1 + r)^(3/2) (8 D1^2 + 35 D2^2 - 42 D2 r + 15 r^2 + 4 D1 (-7 D2 + 3 r)))/(8 (-D1 + D2)^(7/2)));
RgtCLD11[r_, D1_, D2_] := (b0 + b1 Sqrt[D2 - r] + b2 (D2 - r))
   (1 - ((D2 - r)^(3/2) (35 D1^2 + 8 D2^2 + 12 D2 r + 15 r^2 - 14 D1 (2 D2 + 3 r)))/(8 (-D1 + D2)^(7/2)));
CLDApprx11[r_, D1_, D2_] := LeftCLD11[r, D1, D2] + RgtCLD11[r, D1, D2 ] ;
LEFTCFApprx11[r_, D1_, D2_] :=
 -1/(205 920 (-D1 + D2)^(7/2)) (429 b1 (D2 - r)^2 (175 D1^2 (D2 - r)^2 + 128 D1^3 Sqrt[(-D1 + D2) (D2 - r)] -
        384 D1^2 D2 Sqrt[(-D1 + D2) (D2 - r)] + 384 D1 D2^2 Sqrt[(-D1 + D2) (D2 - r)] - 128 D2^3
         Sqrt[(-D1 + D2) (D2 - r)] - 14 D1 (D2 - r)^2 (16 D2 + 9 r) + (D2 - r)^2 (79 D2^2 + 66 D2 r + 30 r^2)) +
     429 a1 (128 D1^6 - 21 D2^6 + 70 D2^5 r + 175 D2^2 r^4 - 126 D2 r^5 + 30 r^6 -
        128 D2^3 r^2 Sqrt[(-D1 + D2) (-D1 + r)] + 64 D1^5 (-7 D2 - 5 r + 2 Sqrt[(-D1 + D2) (-D1 + r)]) - 16 D1^4
         (-35 D2^2 - 70 D2 r - 15 r^2 + 24 D2 Sqrt[(-D1 + D2) (-D1 + r)] + 16 r Sqrt[(-D1 + D2) (-D1 + r)]) +
        D1^2 (35 D2^4 - 5 r^4 + 6 D2^2 r (175 r - 128 Sqrt[(-D1 + D2) (-D1 + r)]) +
           D2^3 (700 r - 128 Sqrt[(-D1 + D2) (-D1 + r)]) + 4 D2 r^2 (35 r - 96 Sqrt[(-D1 + D2) (-D1 + r)])) +
        8 D1^3 (-35 D2^3 + r^2 (-5 r + 16 Sqrt[(-D1 + D2) (-D1 + r)]) +
           3 D2 r (-35 r + 32 Sqrt[(-D1 + D2) (-D1 + r)]) + D2^2 (-175 r + 48 Sqrt[(-D1 + D2) (-D1 + r)])) +
        D1 (56 D2^5 - 350 D2^4 r + 256 D2^3 Sqrt[(D1 - D2) (D1 - r)] r + 280 D2 r^4 - 54 r^5 +
           4 D2^2 r^2 (-175 r + 96 Sqrt[(-D1 + D2) (-D1 + r)]))) - 80
    ((D2 - r)^2 (-39 b0 (33 D1^3 Sqrt[-D1 + D2] + D2^3 (-33 Sqrt[-D1 + D2] + 16 Sqrt[D2 - r]) - 4 D2^2 Sqrt[D2 - r] r -
              7 D2 Sqrt[D2 - r] r^2 - 5 Sqrt[D2 - r] r^3 - 33 D1^2 (3 D2 Sqrt[-D1 + D2] - D2 Sqrt[D2 - r] + Sqrt[D2 - r] r) +
              11 D1 (D2^2 (9 Sqrt[-D1 + D2] - 4 Sqrt[D2 - r]) + 2 D2 Sqrt[D2 - r] r + 2 Sqrt[D2 - r] r^2)) +
           b2 (D2 - r) (-429 D1^3 Sqrt[-D1 + D2] + D2^3 (429 Sqrt[-D1 + D2] - 304 Sqrt[D2 - r]) +
              28 D2^2 Sqrt[D2 - r] r + 141 D2 Sqrt[D2 - r] r^2 + 135 Sqrt[D2 - r] r^3 +
              143 D1^2 (9 D2 Sqrt[-D1 + D2] - 5 D2 Sqrt[D2 - r] + 5 Sqrt[D2 - r] r) -
              13 D1 (D2^2 (99 Sqrt[-D1 + D2] - 68 Sqrt[D2 - r]) + 26 D2 Sqrt[D2 - r] r + 42 Sqrt[D2 - r] r^2))) -
        39 a0 (-5 D2^5 Sqrt[-D1 + D2] + 22 D2^4 Sqrt[-D1 + D2] r - 33 D2^3 Sqrt[-D1 + D2] r^2 + 33 D2^2 r^3 Sqrt[-D1 + r] -
           22 D2 r^4 Sqrt[-D1 + r] + 5 r^5 Sqrt[-D1 + r] + 16 D1^5 (Sqrt[-D1 + D2] - Sqrt[-D1 + r]) +
           4 D1^4 (-9 D2 Sqrt[-D1 + D2] - 11 Sqrt[-D1 + D2] r + 11 D2 Sqrt[-D1 + r] + 9 r Sqrt[-D1 + r]) +
```

```
        D1^3 (D2^2 (17 Sqrt[-D1 + D2] - 33 Sqrt[-D1 + r]) + r^2 (33 Sqrt[-D1 + D2] - 17 Sqrt[-D1 + r]) +
            110 D2 r (Sqrt[-D1 + D2] - Sqrt[-D1 + r])) + D1 (3 D2^4 Sqrt[-D1 + D2] - 22 D2^3 Sqrt[-D1 + D2] r +
            22 D2 r^3 Sqrt[-D1 + r] - 3 r^4 Sqrt[-D1 + r] + 99 D2^2 r^2 (Sqrt[-D1 + D2] - Sqrt[-D1 + r])) +
        D1^2 (5 D2^3 Sqrt[-D1 + D2] - 5 r^3 Sqrt[-D1 + r] + 33 D2 r^2 (-3 Sqrt[-D1 + D2] + 2 Sqrt[-D1 + r]) +
            33 D2^2 r (-2 Sqrt[-D1 + D2] + 3 Sqrt[-D1 + r]))) +
     a2 (70 D2^6 Sqrt[-D1 + D2] - 195 D2^5 Sqrt[-D1 + D2] r + 429 D2^3 Sqrt[-D1 + D2] r^3 - 715 D2^2 r^4 Sqrt[-D1 + r] +
        546 D2 r^5 Sqrt[-D1 + r] - 135 r^6 Sqrt[-D1 + r] + 304 D1^6 (Sqrt[-D1 + D2] - Sqrt[-D1 + r]) +
        D1^5 (-732 D2 Sqrt[-D1 + D2] - 1092 Sqrt[-D1 + D2] r + 884 D2 Sqrt[-D1 + r] + 940 r Sqrt[-D1 + r]) +
        D1^4 (D2^2 (387 Sqrt[-D1 + D2] - 715 Sqrt[-D1 + r]) +
            26 D2 r (111 Sqrt[-D1 + D2] - 115 Sqrt[-D1 + r]) + 9 r^2 (143 Sqrt[-D1 + D2] - 95 Sqrt[-D1 + r])) +
        D1^2 (75 D2^4 Sqrt[-D1 + D2] - 663 D2^3 Sqrt[-D1 + D2] r - 10 r^4 Sqrt[-D1 + r] + 13 D2 r^3
             (99 Sqrt[-D1 + D2] - 20 Sqrt[-D1 + r]) + 429 D2^2 r^2 (9 Sqrt[-D1 + D2] - 10 Sqrt[-D1 + r])) +
        D1 (-225 D2^5 Sqrt[-D1 + D2] + 975 D2^4 Sqrt[-D1 + D2] r - 1287 D2^3 Sqrt[-D1 + D2] r^2 - 1300
              D2 r^4 Sqrt[-D1 + r] + 264 r^5 Sqrt[-D1 + r] + 143 D2^2 r^3 (-9 Sqrt[-D1 + D2] + 20 Sqrt[-D1 + r])) +
        D1^3 (121 D2^3 Sqrt[-D1 + D2] + 39 D2 r^2 (-99 Sqrt[-D1 + D2] + 80 Sqrt[-D1 + r]) + r^3 (-429
                 Sqrt[-D1 + D2] + 100 Sqrt[-D1 + r]) + 13 D2^2 r (-147 Sqrt[-D1 + D2] + 220 Sqrt[-D1 + r])))));
RGHTCFApprx11[r_, D1_, D2_] := 1/(205920 (-D1 + D2)^(7/2))
  (429 b1
      (21 D1^6 - 128 D2^6 - 35 D1^4 D2 (D2 - 10 r) + 5 D2^2 r^4 + 54 D2 r^5 -
        30 r^6 - 14 D1^5 (4 D2 + 5 r) + 64 D2^5 (2 Sqrt[(-D1 + D2) (D2 - r)] + 5 r) +
        8 D2^3 r^2 (16 Sqrt[(-D1 + D2) (D2 - r)] + 5 r) - 16 D2^4 r (16 Sqrt[(-D1 + D2) (D2 - r)] + 15 r) -
        2 D1 (-224 D2^5 + 140 D2 r^4 - 63 r^5 + 16 D2^4 (12 Sqrt[(-D1 + D2) (D2 - r)] + 35 r) - 12 D2^3 r
             (32 Sqrt[(-D1 + D2) (D2 - r)] + 35 r) + 2 D2^2 r^2 (96 Sqrt[(-D1 + D2) (D2 - r)] + 35 r)) -
        4 D1^3 (-70 D2^3 - 64 D2 Sqrt[(-D1 + D2) (D2 - r)] r + 32 Sqrt[(-D1 + D2) (D2 - r)] r^2 +
            D2^2 (32 Sqrt[(-D1 + D2) (D2 - r)] + 175 r)) +
        D1^2 (-560 D2^4 - 175 r^4 + 8 D2^3 (48 Sqrt[(-D1 + D2) (D2 - r)] + 175 r) + 4 D2 r^2
             (96 Sqrt[(-D1 + D2) (D2 - r)] + 175 r) - 6 D2^2 r (128 Sqrt[(-D1 + D2) (D2 - r)] + 175 r))) +
    80 (-39 b0 (5 D1^5 Sqrt[-D1 + D2] + 16 D2^5 (-Sqrt[-D1 + D2] + Sqrt[D2 - r]) +
            4 D2^4 (11 Sqrt[-D1 + D2] - 9 Sqrt[D2 - r]) r + D2^3 (-33 Sqrt[-D1 + D2] + 17 Sqrt[D2 - r]) r^2 +
```

```
        5 D2^2 √(D2 - r) r^3 + 3 D2 √(D2 - r) r^4 - 5 √(D2 - r) r^5 -
        D1^4 √(-D1 + D2) (3 D2 + 22 r) + D1^3 √(-D1 + D2) (-5 D2^2 + 22 D2 r + 33 r^2) +
        D1^2 (D2^3 (-17 √(-D1 + D2) + 33 √(D2 - r)) + 33 D2^2 (2 √(-D1 + D2) - 3 √(D2 - r)) r +
           99 D2 (-√(-D1 + D2) + √(D2 - r)) r^2 - 33 √(D2 - r) r^3) +
        D1 (4 D2^4 (9 √(-D1 + D2) - 11 √(D2 - r)) + 110 D2^3 (-√(-D1 + D2) + √(D2 - r)) r +
           33 D2^2 (3 √(-D1 + D2) - 2 √(D2 - r)) r^2 - 22 D2 √(D2 - r) r^3 + 22 √(D2 - r) r^4)) +
      b2 (70 D1^6 √(-D1 + D2) + 304 D2^6 (√(-D1 + D2) - √(D2 - r)) + 4 D2^5
          (-273 √(-D1 + D2) + 235 √(D2 - r)) r + 9 D2^4 (143 √(-D1 + D2) - 95 √(D2 - r)) r^2 +
        D2^3 (-429 √(-D1 + D2) + 100 √(D2 - r)) r^3 - 10 D2^2 √(D2 - r) r^4 + 264 D2 √(D2 - r) r^5 -
        135 √(D2 - r) r^6 + 75 D1^4 D2 √(-D1 + D2) (D2 + 13 r) - 15 D1^5 √(-D1 + D2) (15 D2 + 13 r) +
        D1^3 √(-D1 + D2) (121 D2^3 - 663 D2^2 r - 1287 D2 r^2 + 429 r^3) +
        D1^2 (D2^4 (387 √(-D1 + D2) - 715 √(D2 - r)) + 13 D2^3 (-147 √(-D1 + D2) + 220 √(D2 - r)) r +
           429 D2^2 (9 √(-D1 + D2) - 10 √(D2 - r)) r^2 + 143 D2 (-9 √(-D1 + D2) + 20 √(D2 - r)) r^3 -
           715 √(D2 - r) r^4) + D1 (D2^5 (-732 √(-D1 + D2) + 884 √(D2 - r)) + 26 D2^4
            (111 √(-D1 + D2) - 115 √(D2 - r)) r + 39 D2^3 (-99 √(-D1 + D2) + 80 √(D2 - r)) r^2 + 13
            D2^2 (99 √(-D1 + D2) - 20 √(D2 - r)) r^3 - 1300 D2 √(D2 - r) r^4 + 546 √(D2 - r) r^5)))) -
   1/(205 920 (-D1 + D2)^(7/2)) (D1 - r)^2 (429 a1 (128 D1^3 √((-D1 + D2) (-D1 + r)) -
      384 D1^2 D2 √((-D1 + D2) (-D1 + r)) +
      384 D1 D2^2 √((-D1 + D2) (-D1 + r)) - 128 D2^3 √((-D1 + D2) (-D1 + r)) +
      (D1 - r)^2 (79 D1^2 - 224 D1 D2 + 175 D2^2 + 66 D1 r - 126 D2 r + 30 r^2)) -
    80 (-39 a0 (-33 D2^3 √(-D1 + D2) + 33 D2^2 r √(-D1 + r) - 22 D2 r^2 √(-D1 + r) +
        5 r^3 √(-D1 + r) + D1^3 (33 √(-D1 + D2) - 16 √(-D1 + r)) +
        D1^2 (-99 D2 √(-D1 + D2) + 44 D2 √(-D1 + r) + 4 r √(-D1 + r)) +
        D1 (-22 D2 r √(-D1 + r) + 7 r^2 √(-D1 + r) + 33 D2^2 (3 √(-D1 + D2) - √(-D1 + r)))) +
      a2 (D1 - r) (-429 D2^3 √(-D1 + D2) + 715 D2^2 r √(-D1 + r) - 546 D2 r^2 √(-D1 + r) +
        135 r^3 √(-D1 + r) + D1^3 (429 √(-D1 + D2) - 304 √(-D1 + r)) +
        D1 (-338 D2 r √(-D1 + r) + 141 r^2 √(-D1 + r) + 143 D2^2 (9 √(-D1 + D2) - 5 √(-D1 + r))) +
        D1^2 (28 r √(-D1 + r) + 13 D2 (-99 √(-D1 + D2) + 68 √(-D1 + r))))));
EXTRCNTRBLft11[r_, D1_, D2_, a_, b_] :=
  1/420
```

```
    (D2 - r)^4
    (7
        a
        (5 D1^2 - 4 D1 D2 + D2^2 - 6 D1 r + 2 D2 r + 2 r^2) +
      b (-35 D1^3 + 4 D2^3 + 9 D2^2 r + 12 D2 r^2 + 10 r^3 + 21 D1^2 (2 D2 + 3 r) -
          21 D1 (D2^2 + 2 D2 r + 2 r^2)));
EXTRCNTRBRgt11[r_, D1_, D2_, a_, b_] :=
  - 1/420
    (D1 - r)^4
    (b
        (D1 - r)
        (3 D1^2 - 14 D1 D2 + 21 D2^2 + 8 D1 r - 28 D2 r + 10 r^2) -
      7 a (D1^2 + 5 D2^2 - 6 D2 r + 2 r^2 + 2 D1 (-2 D2 + r)));
```

Approximation within the interval [D0, D1]

```
OCTCFAA[r]
Simplify[D[D[OCTCFAA[r], r], r]]
```

$$\text{OcthCFApprxFnl11AA[r\_]} := 1 - \frac{3\sqrt{3}\ r}{2} + \frac{6\sqrt{2}\ r^2}{\pi} + \frac{3\ r^3}{2} - \frac{\sqrt{3}\ r^3}{2} - \frac{3\ r^3}{2\ \pi} - \frac{3\ r^2\ \text{ArcSec}[3]}{\pi};$$

$$\text{OcthCLDApprx11AA[r\_]} := \frac{12\sqrt{2}}{\pi} + 9\ r - 3\sqrt{3}\ r - \frac{9\ r}{\pi} - \frac{6\ \text{ArcSec}[3]}{\pi};$$

```
Plot[{OCTCFAA[r], OcthCFApprxFnl11AA[r]}, {r, 0, OCTDst[[1]]}]

IntervlAmpl = N[{OCTDst[[1]], OCTDst[[2]] - OCTDst[[1]],
   OCTDst[[3]] - OCTDst[[2]], OCTDst[[4]] - OCTDst[[3]]}]
```

{0.57735, 0.0350222, 0.0947343, 0.292893}

The first interval is yhe widest one, it is followd by the fourthe, the third and the second which is the smallest one.

Approximation within the interval [D1, D2]

```
Clear[a0]; Clear[a1]; Clear[a2]; Clear[a3]; Clear[a4]; Clear[a5]; Clear[a6];
Clear[b0]; Clear[b1]; Clear[b2]; Clear[b3]; Clear[b4]; Clear[b5]; Clear[b6];

cfD1Plus
cfD2Minus

CLDApprx11[r, D1, D2]

Simplify[((CLDApprx11[r, D1, D2]) /.
   {a0 → cfD1Plus[[1]], a1 → cfD1Plus[[2]], a2 → cfD1Plus[[3]], b0 → cfD2Minus[[1]],
    b1 → cfD2Minus[[2]], b2 → cfD2Minus[[3]], D1 → OCTDst[[1]], D2 → OCTDst[[2]]}),
 Assumptions → {OCTDst[[1]] < r < OCTDst[[2]]}]

FullSimplify[((CLDApprx11[r, D1, D2]) /.
     {a0 → cfD1Plus[[1]], a1 → cfD1Plus[[2]], a2 → cfD1Plus[[3]], b0 → cfD2Minus[[1]],
      b1 → cfD2Minus[[2]], b2 → cfD2Minus[[3]], D1 → OCTDst[[1]], D2 → OCTDst[[2]]}) -
   OcthCLDApprx11BB[r], Assumptions → {OCTDst[[1]] < r < OCTDst[[2]]}]
```

0

the K=1 approximation of the CLD is

```
OcthCLDApprx11BB[r_] :=
```

$$\frac{1}{9\pi}\left(\left(1-\frac{2\times 3^{3/4}\left(\sqrt{6}-4\,r\right)^{3/2}\left(44-21\sqrt{2}-42\sqrt{3}\,r+9\sqrt{6}\,r+45\,r^2\right)}{\left(-4+3\sqrt{2}\right)^{7/2}}\right)\left(\pi\left(108+256\sqrt{2}+81\,r-445\sqrt{3}\,r\right)+27\left(4\sqrt{2}-3\,r-2\,\mathrm{ArcSec}[3]\right)\right)+9\left(1+\frac{2\left(-\sqrt{3}+3\,r\right)^{3/2}\left(-379+168\sqrt{2}+12\sqrt{3}\left(-8+21\sqrt{2}\right)r-360\,r^2\right)}{3^{3/4}\left(-4+3\sqrt{2}\right)^{7/2}}\right)\left(12\sqrt{2}-9\,r+\pi\left(60+\left(9-57\sqrt{3}\right)r\right)-6\,\mathrm{ArcSec}[3]\right)\right);$$

```
Plot[{OCTDDCFBB[r], OcthCLDApprx11BB[r], OcthCLDApprx00BB[r]},
  {r, OCTDst[[1]], OCTDst[[2]]}, PlotStyle → {Directive[Blue, Thickness[0.003]],
    Directive[Red, Thickness[0.006]], Directive[Purple, Thickness[0.003]]},
  AxesLabel → {"r", "γ0''[r]"}, PlotLabel → " CLD: K=1;   D1 < r < D2"]
```

The approximation has improved.

Approximation (not yet matched) of the CLD

```
(Simplify[
   (Simplify[((LEFTCFApprx11[r, D1, D2]) /. {a0 → cfD1Plus[[1]], a1 → cfD1Plus[[2]], a2 →
          cfD1Plus[[3]], b0 → cfD2Minus[[1]], b1 → cfD2Minus[[2]],
          b2 → cfD2Minus[[3]], D1 → OCTDst[[1]], D2 → OCTDst[[2]]}),
      Assumptions → {OCTDst[[1]] < r < OCTDst[[2]]}]) /.
```

$$\left\{\sqrt{-4+3\sqrt{2}}\to\left(2^{1/4}\right)\sqrt{3-2\sqrt{2}},\ \sqrt{-8+6\sqrt{2}}\to\left(2^{3/4}\right)\sqrt{3-2\sqrt{2}},\ \sqrt{\sqrt{6}-4\,r}\to\left(2^{1/4}\right)\sqrt{\sqrt{3}-2\sqrt{2}\,r},\ \sqrt{-3\sqrt{3}+9\,r}\to 3^{3/4}\sqrt{\sqrt{3}\,r-1},\ \sqrt{-\sqrt{3}+3\,r}\to\left(3^{1/4}\right)\sqrt{\sqrt{3}\,r-1}\right\}\Big]\Big)\ /.\ \left\{\sqrt{-3+3\sqrt{3}\,r}\to\left(3^{1/2}\right)\sqrt{\sqrt{3}\,r-1}\right\}$$

ApprxOcthCF11NotMatchBB[r_] :=

$$
\begin{aligned}
&\Big(8\times 3^{1/4}\left(3+\left(-3+19\sqrt{3}\right)\pi\right)\Big(2880\sqrt{3}\left(1303+1560\sqrt{2}\right)r^{4}\sqrt{-1+\sqrt{3}\,r}\,-\\
&\quad 20\,736\left(176+273\sqrt{2}\right)r^{5}\sqrt{-1+\sqrt{3}\,r}+1\,866\,240\sqrt{3}\,r^{6}\sqrt{-1+\sqrt{3}\,r}\,+\\
&\quad \sqrt{3}\left(2^{1/4}\sqrt{3-2\sqrt{2}}\left(-239\,095+169\,083\sqrt{2}\right)+64\left(8867-5304\sqrt{2}\right)\sqrt{-1+\sqrt{3}\,r}\right)+\\
&\quad 216\sqrt{3}\,r^{2}\left(143\times 2^{1/4}\sqrt{3-2\sqrt{2}}\left(-140+99\sqrt{2}\right)+80\left(505-208\sqrt{2}\right)\sqrt{-1+\sqrt{3}\,r}\right)-\\
&\quad 72\,r^{3}\left(429\times 2^{1/4}\sqrt{3-2\sqrt{2}}\left(-140+99\sqrt{2}\right)+160\left(1327-78\sqrt{2}\right)\sqrt{-1+\sqrt{3}\,r}\right)-\\
&\quad 3\,r\left(39\times 2^{1/4}\sqrt{3-2\sqrt{2}}\left(-13\,180+9321\sqrt{2}\right)-1280\left(-1663+897\sqrt{2}\right)\sqrt{-1+\sqrt{3}\,r}\right)\Big)-\\
&\quad 936\Big(6^{1/4}\left(9964-7047\sqrt{2}\right)\sqrt{3-2\sqrt{2}}+192\times 3^{3/4}\left(257+132\sqrt{2}\right)r^{3}\sqrt{-1+\sqrt{3}\,r}\,-\\
&\quad 6912\times 3^{1/4}\left(2+11\sqrt{2}\right)r^{4}\sqrt{-1+\sqrt{3}\,r}+23\,040\times 3^{3/4}\,r^{5}\sqrt{-1+\sqrt{3}\,r}\,+\\
&\quad 64\left(264\sqrt{-2\sqrt{3}+6\,r}-425\times 3^{1/4}\sqrt{-1+\sqrt{3}\,r}\right)-\\
&\quad 24\times 3^{1/4}\,r^{2}\left(33\times 2^{1/4}\sqrt{3-2\sqrt{2}}\left(-140+99\sqrt{2}\right)+8\left(1027-396\sqrt{2}\right)\sqrt{-1+\sqrt{3}\,r}\right)+\\
&\quad 8\times 3^{3/4}\,r\left(11\times 2^{1/4}\sqrt{3-2\sqrt{2}}\left(-263+186\sqrt{2}\right)+24\left(393-220\sqrt{2}\right)\sqrt{-1+\sqrt{3}\,r}\right)\Big)\\
&\quad \left(4\sqrt{2}-\sqrt{3}+\pi+\sqrt{3}\,\pi-2\,\mathrm{ArcSec}[3]\right)+\left(\sqrt{6}-4\,r\right)^{2}\\
&\quad \Big(2^{1/4}\left(81+\left(-81+445\sqrt{3}\right)\pi\right)\left(\sqrt{6}-4\,r\right)\\
&\quad \Big(14\,157\times 3^{1/4}\sqrt{6-4\sqrt{2}}-20\,020\times 3^{1/4}\sqrt{3-2\sqrt{2}}+10\,608\sqrt{3}\sqrt{\sqrt{3}-2\sqrt{2}\,r}\,-\\
&\quad 8456\sqrt{6}\sqrt{\sqrt{3}-2\sqrt{2}\,r}+16\left(1493-507\sqrt{2}\right)r\sqrt{\sqrt{3}-2\sqrt{2}\,r}\,+\\
&\quad 24\sqrt{3}\left(-728+141\sqrt{2}\right)r^{2}\sqrt{\sqrt{3}-2\sqrt{2}\,r}+12\,960\,r^{3}\sqrt{\sqrt{3}-2\sqrt{2}\,r}\Big)+\\
&\quad 39\times 2^{1/4}\Big(1089\times 3^{1/4}\sqrt{6-4\sqrt{2}}-1540\times 3^{1/4}\sqrt{3-2\sqrt{2}}+528\sqrt{3}\sqrt{\sqrt{3}-2\sqrt{2}\,r}\,-\\
&\quad 408\sqrt{6}\sqrt{\sqrt{3}-2\sqrt{2}\,r}-48\left(-25+11\sqrt{2}\right)r\sqrt{\sqrt{3}-2\sqrt{2}\,r}+8\sqrt{3}\left(-88+21\sqrt{2}\right)\\
&\quad r^{2}\sqrt{\sqrt{3}-2\sqrt{2}\,r}+480\,r^{3}\sqrt{\sqrt{3}-2\sqrt{2}\,r}\Big)\Big(\left(432-311\sqrt{2}+81\sqrt{6}\right)\pi\,-\\
&\quad 27\left(-16\sqrt{2}+3\sqrt{6}+8\,\mathrm{ArcSec}[3]\right)\Big)\Big)\Big)\Big/\left(247\,104\times 3^{1/4}\left(-4+3\sqrt{2}\right)^{7/2}\pi\right);
\end{aligned}
$$

IDENTITIES

$$
\begin{aligned}
&\Big\{\sqrt{-4+3\sqrt{2}}\to\left(2^{1/4}\right)\sqrt{3-2\sqrt{2}},\ \sqrt{-8+6\sqrt{2}}\to\left(2^{3/4}\right)\sqrt{3-2\sqrt{2}},\\
&\ \sqrt{\sqrt{6}-4\,r}\to\left(2^{1/4}\right)\sqrt{\sqrt{3}-2\sqrt{2}\,r},\ \sqrt{-3\sqrt{3}+9\,r}\to 3^{3/4}\sqrt{\sqrt{3}\,r-1},\\
&\ \sqrt{-\sqrt{3}+3\,r}\to\left(3^{1/4}\right)\sqrt{\sqrt{3}\,r-1},\ \sqrt{-3+3\sqrt{3}\,r}\to\left(3^{1/2}\right)\sqrt{\sqrt{3}\,r-1}\Big\}
\end{aligned}
$$

```
FullSimplify[(Sqrt[-4 + 3 Sqrt[2]])^2 - ((2^(1/4)) Sqrt[3 - 2 Sqrt[2]])^2]

FullSimplify[(Sqrt[-8 + 6 Sqrt[2]])^2 - ((2^(3/4)) Sqrt[3 - 2 Sqrt[2]])^2]

FullSimplify[(Sqrt[Sqrt[6] - 4 r])^2 - ((2^(1/4)) Sqrt[Sqrt[3] - 2 Sqrt[2] r])^2]

FullSimplify[(Sqrt[-3 Sqrt[3] + 9 r])^2 - (3^(3/4) Sqrt[Sqrt[3] r - 1])^2]

FullSimplify[(Sqrt[-Sqrt[3] + 3 r])^2 - ((3^(1/4)) Sqrt[Sqrt[3] r - 1])^2]

FullSimplify[(Sqrt[-3 + 3 Sqrt[3] r])^2 - ((3^(1/2)) Sqrt[Sqrt[3] r - 1])^2]

FullSimplify[
 ((((LEFTCFApprx11[r, D1, D2]) /. {a0 → cfD1Plus[[1]], a1 → cfD1Plus[[2]], a2 → cfD1Plus[[
         3]], b0 →  cfD2Minus[[1]], b1 →  cfD2Minus[[2]], b2 →  cfD2Minus[[3]],
        D1 → OCTDst[[1]], D2 → OCTDst[[2]]}) - ApprxOcthCF11NotMatchBB[r]) /.
   {Sqrt[-4 + 3 Sqrt[2]] → (2^(1/4)) Sqrt[3 - 2 Sqrt[2]], Sqrt[-8 + 6 Sqrt[2]] → (2^(3/4)) Sqrt[3 - 2 Sqrt[2]],
    Sqrt[Sqrt[6] - 4 r] → (2^(1/4)) Sqrt[Sqrt[3] - 2 Sqrt[2] r], Sqrt[-3 Sqrt[3] + 9 r] → 3^(3/4) Sqrt[Sqrt[3] r - 1],
    Sqrt[-Sqrt[3] + 3 r] →  (3^(1/4)) Sqrt[Sqrt[3] r - 1], Sqrt[-3 + 3 Sqrt[3] r] → (3^(1/2)) Sqrt[Sqrt[3] r - 1]}),
 Assumptions → {OCTDst[[1]] < r < OCTDst[[2]]}]

Plot[{ApprxOcthCF11NotMatchBB[r], ApprxOcthCF11NotMatchBBOLD[r]},
 {r, OCTDst[[1]], OCTDst[[2]]}, PlotStyle →
  {Directive[Blue, Thickness[0.003]], Directive[Red, Thickness[0.005], Dashed]}]

Plot[{OCTCFBB[r], ApprxOcthCF11NotMatchBB[r]}, {r, OCTDst[[1]], OCTDst[[2]]},
 PlotStyle → {Directive[Blue, Thickness[0.003]], Directive[Red, Thickness[0.006]]},
 AxesLabel → {"r", "γT[r]"}]
```

The approximation is matched at r=D1

```
Simplify[
 Solve[{Limit[ApprxOcthCF11NotMatchBB[r] + a + b r, r → OCTDst[[1]], Direction → -1] - Limit[
       OcthCFApprxFnl11AA[r], r → OCTDst[[1]], Direction → 1] == 0 &&
    Limit[D[ApprxOcthCF11NotMatchBB[r] + a + b r, r], r → OCTDst[[1]], Direction → -1] -
      Limit[D[OcthCFApprxFnl11AA[r], r], r → OCTDst[[1]], Direction → 1] == 0}, {a, b}]]
```

```
Simplify[Simplify[(ApprxOcthCF11NotMatchBB[r] + a + b r) /.
  {{a → ((-42 335 271 2^(1/4) + 29 935 555 × 2^(3/4) - 69 160 392 × 2^(1/4) √3 + 48 903 777 × 2^(3/4) √3 -
        25 945 920 √(-24 + 17 √2) - 4 586 868 √(6 (-24 + 17 √2)) +
        18 347 472 √(-48 + 34 √2) + 6 486 480 √(-72 + 51 √2)) π -
      27 (990 132 × 2^(1/4) - 700 128 × 2^(3/4) - 2 561 496 × 2^(1/4) √3 + 1 811 251 × 2^(3/4) √3 -
        169 884 √(6 (-24 + 17 √2)) + 240 240 √(-72 + 51 √2) + 350 064 × 2^(1/4) ArcSec[3] -
        247 533 × 2^(3/4) ArcSec[3] + 20 592 √(-24 + 17 √2) (99 + 35 ArcSec[3]) -
        5148 √(-48 + 34 √2) (280 + 99 ArcSec[3]))) / (69 498 √(3 - 2 √2) (-4 + 3 √2)^(7/2) π),
    b → 1/(7128 √(-24 + 17 √2) π) ((10 692 √(-24 + 17 √2) - 14 256 √(-72 + 51 √2) +
        2^(1/4) (2673 - 1782 √2 - 146 267 √3 + 103 442 √6)) π -
      297 (9 × 2^(1/4) - 6 × 2^(3/4) - 544 × 2^(1/4) √3 + 384 × 2^(3/4) √3 + 36 √(-24 + 17 √2) -
        96 √(6 (-24 + 17 √2)) - 192 × 2^(1/4) √3 ArcSec[3] +
        136 × 2^(3/4) √3 ArcSec[3] + 48 √(-72 + 51 √2) ArcSec[3]))}},
  Assumptions → {OCTDst[[1]] < r < OCTDst[[2]]}]]
```

```
ApprxOcthCF11PrtlyMatchBB[r_] :=
  1/(7128 √(-24 + 17 √2) π) r ((10 692 √(-24 + 17 √2) - 14 256 √(-72 + 51 √2) +
      2^(1/4) (2673 - 1782 √2 - 146 267 √3 + 103 442 √6)) π - 297 (9 × 2^(1/4) - 6 × 2^(3/4) -
```

$$544\times 2^{1/4}\sqrt{3}+384\times 2^{3/4}\sqrt{3}+36\sqrt{-24+17\sqrt{2}}-96\sqrt{6\left(-24+17\sqrt{2}\right)}-$$

$$192\times 2^{1/4}\sqrt{3}\,\text{ArcSec}[3]+136\times 2^{3/4}\sqrt{3}\,\text{ArcSec}[3]+48\sqrt{-72+51\sqrt{2}}\,\text{ArcSec}[3]\Big)\Big)+$$

$$\Big(\Big(-42\,335\,271\;2^{1/4}+29\,935\,555\times 2^{3/4}-69\,160\,392\times 2^{1/4}\sqrt{3}+48\,903\,777\times 2^{3/4}\sqrt{3}-$$

$$25\,945\,920\sqrt{-24+17\sqrt{2}}-4\,586\,868\sqrt{6\left(-24+17\sqrt{2}\right)}+$$

$$18\,347\,472\sqrt{-48+34\sqrt{2}}+6\,486\,480\sqrt{-72+51\sqrt{2}}\Big)\pi-$$

$$27\Big(990\,132\times 2^{1/4}-700\,128\times 2^{3/4}-2\,561\,496\times 2^{1/4}\sqrt{3}+1\,811\,251\times 2^{3/4}\sqrt{3}-$$

$$169\,884\sqrt{6\left(-24+17\sqrt{2}\right)}+240\,240\sqrt{-72+51\sqrt{2}}+350\,064\times 2^{1/4}\,\text{ArcSec}[3]-$$

$$247\,533\times 2^{3/4}\,\text{ArcSec}[3]+20\,592\sqrt{-24+17\sqrt{2}}\,(99+35\,\text{ArcSec}[3])-$$

$$5148\sqrt{-48+34\sqrt{2}}\,(280+99\,\text{ArcSec}[3])\Big)\Big)\Big/\left(69\,498\sqrt{3-2\sqrt{2}}\left(-4+3\sqrt{2}\right)^{7/2}\pi\right)+$$

$$\Big(8\times 3^{1/4}\left(3+\left(-3+19\sqrt{3}\right)\pi\right)\Big(-20\,736\left(176+273\sqrt{2}\right)r^5\sqrt{-1+\sqrt{3}\,r}+$$

$$2880\left(1303+1560\sqrt{2}\right)r^4\sqrt{-3+3\sqrt{3}\,r}+1\,866\,240\,r^6\sqrt{-3+3\sqrt{3}\,r}+$$

$$\sqrt{3}\left(2^{1/4}\sqrt{3-2\sqrt{2}}\left(-239\,095+169\,083\sqrt{2}\right)+64\left(8867-5304\sqrt{2}\right)\sqrt{-1+\sqrt{3}\,r}\right)+$$

$$216\sqrt{3}\,r^2\left(143\times 2^{1/4}\sqrt{3-2\sqrt{2}}\left(-140+99\sqrt{2}\right)+80\left(505-208\sqrt{2}\right)\sqrt{-1+\sqrt{3}\,r}\right)-$$

$$72\,r^3\left(429\times 2^{1/4}\sqrt{3-2\sqrt{2}}\left(-140+99\sqrt{2}\right)+160\left(1327-78\sqrt{2}\right)\sqrt{-1+\sqrt{3}\,r}\right)-3\,r$$

$$\left(39\times 2^{1/4}\sqrt{3-2\sqrt{2}}\left(-13\,180+9321\sqrt{2}\right)-1280\left(-1663+897\sqrt{2}\right)\sqrt{-1+\sqrt{3}\,r}\right)\Big)-$$

$$936\Big(6^{1/4}\left(9964-7047\sqrt{2}\right)\sqrt{3-2\sqrt{2}}+192\times 3^{3/4}\left(257+132\sqrt{2}\right)r^3\sqrt{-1+\sqrt{3}\,r}-$$

$$6912\times 3^{1/4}\left(2+11\sqrt{2}\right)r^4\sqrt{-1+\sqrt{3}\,r}+23\,040\times 3^{3/4}\,r^5\sqrt{-1+\sqrt{3}\,r}+$$

$$64\left(264\sqrt{-2\sqrt{3}+6\,r}-425\times 3^{1/4}\sqrt{-1+\sqrt{3}\,r}\right)-$$

$$24 \times 3^{1/4} r^2 \left(33 \times 2^{1/4} \sqrt{3 - 2\sqrt{2}} \left(-140 + 99\sqrt{2}\right) + 8\left(1027 - 396\sqrt{2}\right)\sqrt{-1 + \sqrt{3}\, r}\right) +$$

$$8 \times 3^{3/4} r \left(11 \times 2^{1/4} \sqrt{3 - 2\sqrt{2}} \left(-263 + 186\sqrt{2}\right) + 24\left(393 - 220\sqrt{2}\right)\sqrt{-1 + \sqrt{3}\, r}\right)\Bigg)$$

$$\left(4\sqrt{2} - \sqrt{3} + \pi + \sqrt{3}\,\pi - 2\,\mathrm{ArcSec}[3]\right) + \left(\sqrt{6} - 4 r\right)^2$$

$$\Bigg(2^{1/4}\left(81 + \left(-81 + 445\sqrt{3}\right)\pi\right)\left(\sqrt{6} - 4 r\right)$$

$$\Bigg(14\,157 \times 3^{1/4}\sqrt{6 - 4\sqrt{2}} - 20\,020 \times 3^{1/4}\sqrt{3 - 2\sqrt{2}} + 10\,608\sqrt{3}\sqrt{\sqrt{3} - 2\sqrt{2}\, r} -$$

$$8456\sqrt{6}\sqrt{\sqrt{3} - 2\sqrt{2}\, r} + 16\left(1493 - 507\sqrt{2}\right) r \sqrt{\sqrt{3} - 2\sqrt{2}\, r} +$$

$$24\sqrt{3}\left(-728 + 141\sqrt{2}\right) r^2 \sqrt{\sqrt{3} - 2\sqrt{2}\, r} + 12\,960\, r^3 \sqrt{\sqrt{3} - 2\sqrt{2}\, r}\Bigg) + 39 \times 2^{1/4}$$

$$\Bigg(1089 \times 3^{1/4}\sqrt{6 - 4\sqrt{2}} - 1540 \times 3^{1/4}\sqrt{3 - 2\sqrt{2}} + 528\sqrt{3}\sqrt{\sqrt{3} - 2\sqrt{2}\, r} - 408$$

$$\sqrt{6}\sqrt{\sqrt{3} - 2\sqrt{2}\, r} - 48\left(-25 + 11\sqrt{2}\right) r \sqrt{\sqrt{3} - 2\sqrt{2}\, r} + 8\sqrt{3}\left(-88 + 21\sqrt{2}\right)$$

$$r^2 \sqrt{\sqrt{3} - 2\sqrt{2}\, r} + 480\, r^3 \sqrt{\sqrt{3} - 2\sqrt{2}\, r}\Bigg)\left(\left(432 - 311\sqrt{2} + 81\sqrt{6}\right)\pi -\right.$$

$$\left.\left.27\left(-16\sqrt{2} + 3\sqrt{6} + 8\,\mathrm{ArcSec}[3]\right)\right)\right)\Bigg)\Big/\left(247\,104 \times 3^{1/4}\left(-4 + 3\sqrt{2}\right)^{7/2}\pi\right);$$

```
Simplify[
 Simplify[(ApprxOcthCF11NotMatchBB[r] + a + b r) /. {{a → ((-42 335 271 2^(1/4) + 29 935 555 × 2^(3/4) -
          69 160 392 × 2^(1/4) √3 + 48 903 777 × 2^(3/4) √3 - 25 945 920 √(-24 + 17 √2) - 4 586 868
          √(6 (-24 + 17 √2)) + 18 347 472 √(-48 + 34 √2) + 6 486 480 √(-72 + 51 √2)) π -
        27 (990 132 × 2^(1/4) - 700 128 × 2^(3/4) - 2 561 496 × 2^(1/4) √3 + 1 811 251 × 2^(3/4) √3 -
          169 884 √(6 (-24 + 17 √2)) + 240 240 √(-72 + 51 √2) + 350 064 × 2^(1/4) ArcSec[3] -
          247 533 × 2^(3/4) ArcSec[3] + 20 592 √(-24 + 17 √2) (99 + 35 ArcSec[3]) - 5148
          √(-48 + 34 √2) (280 + 99 ArcSec[3]))) / (69 498 √(3 - 2 √2) (-4 + 3 √2)^(7/2) π),
      b → 1/(7128 √(-24 + 17 √2) π) ((10 692 √(-24 + 17 √2) - 14 256 √(-72 + 51 √2) +
          2^(1/4) (2673 - 1782 √2 - 146 267 √3 + 103 442 √6)) π -
        297 (9 × 2^(1/4) - 6 × 2^(3/4) - 544 × 2^(1/4) √3 + 384 × 2^(3/4) √3 + 36 √(-24 + 17 √2) -
          96 √(6 (-24 + 17 √2)) - 192 × 2^(1/4) √3 ArcSec[3] +
          136 × 2^(3/4) √3 ArcSec[3] + 48 √(-72 + 51 √2) ArcSec[3]))}},
   Assumptions → {OCTDst[[1]] < r < OCTDst[[2]]}] -
  ApprxOcthCF11PrtlyMatchBB[
   r]]
```

```
{0}
```

```
Plot[{OCTCFBB[r], ApprxOcthCF11PrtlyMatchBB[r], ApprxOcthCF11NotMatchBB[r]},
 {r, OCTDst[[1]], OCTDst[[2]]},
 PlotStyle → {Directive[Blue, Thickness[0.003]], Directive[Red, Thickness[0.004], Dashed],
   Directive[Purple, Thickness[0.004], Dotted]}, AxesLabel → {"r", "γ0[r]"}]
```

The error at r=D2 is ~0.0017, the relative error is 8%.

```
N[Limit[OCTCFBB[r] - ApprxOcthCF11PrtlyMatchBB[r], r → OCTDst[[2]], Direction → 1]]
N[Limit[OCTCFBB[r] - ApprxOcthCF11PrtlyMatchBB[r], r → OCTDst[[2]], Direction → 1] /
  (Limit[OCTCFBB[r], r → OCTDst[[2]], Direction → 1])]
```

```
0.00170296
```

```
0.0849901
```

```
ausfiga = Plot[{If[r < OCTDst[[1]], OCTCFAA[r], OCTCFBB[r] ],
   If[r < OCTDst[[1]], OcthCFApprxFnl11AA[r], ApprxOcthCF11PrtlyMatchBB[r]],
   10 * (If[r < OCTDst[[1]], OCTCFAA[r], OCTCFBB[r] ] -
      If[r < OCTDst[[1]], OcthCFApprxFnl11AA[r], ApprxOcthCF11PrtlyMatchBB[r]])},
  {r, 0, OCTDst[[2]]}, PlotRange → {{0, 1.05}, {-0.05, 1.05}}, PlotStyle →
   {Directive[Blue, Thickness[0.003]], Directive[Red, Thickness[0.004], Dashed],
    Directive[Black, Thickness[0.003], Dashed]}, AxesLabel → {"r", "γ0[r],10xΔγ"}]
```

We start now from D4 and proceed towards the left

Approximation within the interval [D3, D4]

```
cfD3Plus
cfD4Minus
```

evaluation of the CLD approximation

```
Simplify[(CLDApprx11[r, D1, D2]) /.
  {a0 → cfD3Plus[[1]], a1 → cfD3Plus[[2]], a2 → cfD3Plus[[3]], b0 → cfD4Minus[[1]],
   b1 → cfD4Minus[[2]], b2 → cfD4Minus[[3]], D1 → OCTDst[[3]], D2 → OCTDst[[4]]},
 Assumptions → {OCTDst[[3]] < r < OCTDst[[4]]}]

Simplify[Simplify[(CLDApprx11[r, D1, D2]) /.
    {a0 → cfD3Plus[[1]], a1 → cfD3Plus[[2]], a2 → cfD3Plus[[3]], b0 → cfD4Minus[[1]],
     b1 → cfD4Minus[[2]], b2 → cfD4Minus[[3]], D1 → OCTDst[[3]], D2 → OCTDst[[4]]},
   Assumptions → {OCTDst[[3]] < r < OCTDst[[4]]}] - OcthCLDApprx11DD[r]]
```

0

$$\text{OcthCLDApprx11DD[r\_]} := \left(6 + 32\sqrt{\frac{2}{3}} + \left(9 - 28\sqrt{3}\right) r + \frac{-8\sqrt{2} + 3 r}{\pi}\right) \left(1 - \frac{\left(-\frac{1}{\sqrt{2}} + r\right)^{3/2} \left(39 - 42 r + 15 r^2 + 2\sqrt{2}\,(-7 + 3 r)\right)}{8\left(1 - \frac{1}{\sqrt{2}}\right)^{7/2}}\right);$$

```
Plot[{OCTDDCFDD[r], OcthCLDApprx11DD[r], OcthCLDApprx00DD[r] },
 {r, OCTDst[[3]], OCTDst[[4]]}, AxesLabel → {"r", "γ''[r]"},
 PlotStyle → {Directive[Blue, Thickness[0.003]],
   Directive[Red, Thickness[0.006]], Directive[Purple, Thickness[0.003]]},
 PlotRange → {{OCTDst[[3]], OCTDst[[4]]}, {-2, 2}}, AxesLabel → {"r", "γ0''[r],10xΔγ"},
 PlotLabel → " CLD: K=1; D3< r < D4"]
```

The approximation has worsened!

```
Plot[{OCTDDCFDD[r] - OcthCLDApprx11DD[r]}, {r, OCTDst[[3]], OCTDst[[4]]}]
```

The largest error is ~0.1 around 0.82.The relative error 1150% !

```
N[(OCTDDCFDD[r] - OcthCLDApprx11DD[r]) /. {r → 82 / 100}]
N[((OCTDDCFDD[r] - OcthCLDApprx11DD[r]) / OCTDDCFDD[r]) /. {r → 82 / 100}]
```

1.24949

11.5111

Evaluation of the approximation of the CF

```
Simplify[
 (Simplify[Simplify[(LEFTCFApprx11[r, D1, D2]) /. {a0 → cfD3Plus[[1]], a1 → cfD3Plus[[2]],
       a2 → cfD3Plus[[3]], b0 → cfD4Minus[[1]], b1 → cfD4Minus[[2]],
       b2 → cfD4Minus[[3]], D1 → OCTDst[[3]], D2 → OCTDst[[4]]},
     Assumptions → {OCTDst[[3]] < r < OCTDst[[4]]}]]) /.
```

$$\left\{\sqrt{\frac{\sqrt{2} - 2 r}{-2 + \sqrt{2}}} \to \frac{\sqrt{r\sqrt{2} - 1}}{\sqrt{\sqrt{2} - 1}}\right\}, \text{Assumptions} \to \{\text{OCTDst[[3]]} < r < \text{OCTDst[[4]]}\}]$$

IDENTITIES

$$\left\{\sqrt{-4+3\sqrt{2}} \to \left(2^{1/4}\right)\sqrt{3-2\sqrt{2}},\right.$$
$$\sqrt{-8+6\sqrt{2}} \to \left(2^{3/4}\right)\sqrt{3-2\sqrt{2}},\ \sqrt{\sqrt{6}-4\,r} \to \left(2^{1/4}\right)\sqrt{\sqrt{3}-2\sqrt{2}\,r},$$
$$\sqrt{-3\sqrt{3}+9\,r} \to 3^{3/4}\sqrt{\sqrt{3}\,r-1},\ \sqrt{-\sqrt{3}+3\,r} \to \left(3^{1/4}\right)\sqrt{\sqrt{3}\,r-1},$$
$$\left.\sqrt{-3+3\sqrt{3}\,r} \to \left(3^{1/2}\right)\sqrt{\sqrt{3}\,r-1},\ \sqrt{\frac{\sqrt{2}-2\,r}{-2+\sqrt{2}}} \to \frac{\sqrt{r\sqrt{2}-1}}{\sqrt{\sqrt{2}-1}}\right\}$$

```
FullSimplify[Sqrt[(Sqrt[2] - 2 r)/(-2 + Sqrt[2])] - Sqrt[r Sqrt[2] - 1]/Sqrt[Sqrt[2] - 1], Assumptions → {OCTDst[[3]] < r < OCTDst[[4]]}]
```

0

```
FullSimplify[Simplify[(LEFTCFApprx11[r, D1, D2]) /.
     {a0 → cfD3Plus[[1]], a1 → cfD3Plus[[2]], a2 → cfD3Plus[[3]], b0 → cfD4Minus[[1]],
      b1 → cfD4Minus[[2]], b2 → cfD4Minus[[3]], D1 → OCTDst[[3]], D2 → OCTDst[[4]]},
   Assumptions → {OCTDst[[3]] < r < OCTDst[[4]]}] - ApprxOcthCF11NotMatchDD[r],
 Assumptions → {OCTDst[[3]] < r < OCTDst[[4]]}]
```

0

ApprxOcthCF11NotMatchDD[r_] :=

$$
\begin{aligned}
&-\frac{1}{1287\left(2-\sqrt{2}\right)^{7/2}\pi}\Bigg(3\Bigg(4582\sqrt{4-2\sqrt{2}}-6208\sqrt{2-\sqrt{2}}+7796\sqrt{-\sqrt{2}+2r}+\\
&\quad 2\left(1947+5018\sqrt{2}\right)r^4\sqrt{-\sqrt{2}+2r}-2\left(1092+1109\sqrt{2}\right)r^5\sqrt{-\sqrt{2}+2r}+\\
&\quad 540r^6\sqrt{-\sqrt{2}+2r}-4160\sqrt{-2\sqrt{2}+4r}+\\
&\quad r\left(7943\sqrt{4-2\sqrt{2}}-11934\sqrt{2-\sqrt{2}}+\left(21580-23103\sqrt{2}\right)\sqrt{-\sqrt{2}+2r}\right)+\\
&\quad 2r^2\left(17160\sqrt{4-2\sqrt{2}}-24024\sqrt{2-\sqrt{2}}+\left(22885-7137\sqrt{2}\right)\sqrt{-\sqrt{2}+2r}\right)+\\
&\quad r^3\left(3003\sqrt{4-2\sqrt{2}}-4290\sqrt{2-\sqrt{2}}-\left(6916+16129\sqrt{2}\right)\sqrt{-\sqrt{2}+2r}\right)\Bigg)+\\
&\pi\Bigg(2\left(1611+3276\sqrt{2}-39540\sqrt{3}-30680\sqrt{6}\right)r^4\sqrt{-\sqrt{2}+2r}+\\
&\quad 2\left(-7488-621\sqrt{2}+30576\sqrt{3}+6092\sqrt{6}\right)r^5\sqrt{-\sqrt{2}+2r}-\\
&\quad 540\left(-9+28\sqrt{3}\right)r^6\sqrt{-\sqrt{2}+2r}+r\Big(-2756\sqrt{12-6\sqrt{2}}+4680\sqrt{6-3\sqrt{2}}+\\
&\qquad 4563\sqrt{4-2\sqrt{2}}-4446\sqrt{2-\sqrt{2}}+43056\sqrt{-\sqrt{2}+2r}+62820\sqrt{6}\sqrt{-\sqrt{2}+2r}-\\
&\qquad 14643\sqrt{-2\sqrt{2}+4r}-55120\sqrt{-3\sqrt{2}+6r}\Big)+r^3\Big(-84084\sqrt{12-6\sqrt{2}}+\\
&\qquad 120120\sqrt{6-3\sqrt{2}}+27027\sqrt{4-2\sqrt{2}}-38610\sqrt{2-\sqrt{2}}+48672\sqrt{-\sqrt{2}+2r}+\\
&\qquad 147100\sqrt{6}\sqrt{-\sqrt{2}+2r}-20673\sqrt{-2\sqrt{2}+4r}-26000\sqrt{-3\sqrt{2}+6r}\Big)+\\
&\quad 2\Big(-6740\sqrt{12-6\sqrt{2}}+9536\sqrt{6-3\sqrt{2}}+2718\sqrt{4-2\sqrt{2}}-4302\sqrt{2-\sqrt{2}}+\\
&\qquad 1854\sqrt{-\sqrt{2}+2r}+3328\sqrt{6}\sqrt{-\sqrt{2}+2r}-2925\sqrt{-2\sqrt{2}+4r}-\\
&\qquad 6808\sqrt{-3\sqrt{2}+6r}\Big)-2r^2\Big(68640\sqrt{12-6\sqrt{2}}-96096\sqrt{6-3\sqrt{2}}-\\
&\qquad 27027\sqrt{4-2\sqrt{2}}+38610\sqrt{2-\sqrt{2}}-20169\sqrt{-\sqrt{2}+2r}-\\
&\qquad 35100\sqrt{6}\sqrt{-\sqrt{2}+2r}+27612\sqrt{-2\sqrt{2}+4r}+104140\sqrt{-3\sqrt{2}+6r}\Big)\Bigg)\Bigg);
\end{aligned}
$$

The error of the CF-approximation is -0.00223423 at D_3, i.e. ~90%

```
N[Limit[OCTCFDD[r] - ApprxOcthCF11NotMatchDD[r], r → OCTDst[[3]], Direction → -1]]
N[Limit[OCTCFDD[r] - ApprxOcthCF11NotMatchDD[r], r → OCTDst[[3]], Direction → -1] /
  Limit[OCTCFDD[r], r → OCTDst[[3]], Direction → -1]]
```

```
0.0205094

8.38517
```

```
Plot[{OCTCFDD[r], ApprxOcthCF11NotMatchDD[r], OcthCFApprxFnl00DD[r] },
 {r, OCTDst[[3]], OCTDst[[4]]}, PlotRange → {{OCTDst[[3]], OCTDst[[4]]}, {-0.02, 0.005}},
 PlotStyle → {Directive[Blue, Thickness[0.003]], Directive[Red, Thickness[0.006]],
   Directive[Magenta, Thickness[0.003]]}, AxesLabel → {"r", "γ₀[r],10xΔγ"}]
```

Approximation within the interval [D2, D3]

approximation of the CLD

```
Together[Simplify[(CLDApprx11[r, D1, D2]) /.
   {a0 → cfD2Plus[[1]], a1 → cfD2Plus[[2]], a2 → cfD2Plus[[3]], b0 → cfD3Minus[[1]],
    b1 → cfD3Minus[[2]], b2 → cfD3Minus[[3]], D1 → OCTDst[[2]], D2 → OCTDst[[3]]},
  Assumptions → {OCTDst[[2]] < r < OCTDst[[3]]}]]

Simplify[Simplify[(CLDApprx11[r, D1, D2]) /.
    {a0 → cfD2Plus[[1]], a1 → cfD2Plus[[2]], a2 → cfD2Plus[[3]], b0 → cfD3Minus[[1]],
     b1 → cfD3Minus[[2]], b2 → cfD3Minus[[3]], D1 → OCTDst[[2]], D2 → OCTDst[[3]]},
   Assumptions → {OCTDst[[2]] < r < OCTDst[[3]]}] - OcthCLDApprx11CC[r]]
```

0

```
OcthCLDApprx11CC[r_] :=
```

$$\Big(-3744\sqrt{2\left(2-\sqrt{3}\right)}+2160\sqrt{6\left(2-\sqrt{3}\right)}-16848\sqrt{2-\sqrt{3}}\,\pi-9344\sqrt{2\left(2-\sqrt{3}\right)}\,\pi+$$
$$9720\sqrt{3\left(2-\sqrt{3}\right)}\,\pi+5376\sqrt{6\left(2-\sqrt{3}\right)}\,\pi-9864\times2^{3/4}\sqrt{\sqrt{2}-2r}+$$
$$4032\times2^{3/4}\sqrt{3}\sqrt{\sqrt{2}-2r}+7398\times2^{1/4}\pi\sqrt{\sqrt{2}-2r}-16128\times2^{3/4}\pi\sqrt{\sqrt{2}-2r}-$$
$$3024\times2^{1/4}\sqrt{3}\,\pi\sqrt{\sqrt{2}-2r}+13152\times2^{3/4}\sqrt{3}\,\pi\sqrt{\sqrt{2}-2r}+5616\sqrt{2-\sqrt{3}}\,r-$$
$$3240\sqrt{3\left(2-\sqrt{3}\right)}\,r-142308\sqrt{2-\sqrt{3}}\,\pi r+82208\sqrt{3\left(2-\sqrt{3}\right)}\,\pi r+$$
$$16515\times2^{1/4}\sqrt{\sqrt{2}-2r}\,r+2520\times2^{1/4}\sqrt{3}\sqrt{\sqrt{2}-2r}\,r+37305\times2^{1/4}\pi\sqrt{\sqrt{2}-2r}\,r-$$
$$4806\times2^{3/4}\pi\sqrt{\sqrt{2}-2r}\,r-56148\times2^{1/4}\sqrt{3}\,\pi\sqrt{\sqrt{2}-2r}\,r-$$
$$1512\times2^{3/4}\sqrt{3}\,\pi\sqrt{\sqrt{2}-2r}\,r-4131\times2^{3/4}\sqrt{\sqrt{2}-2r}\,r^2-$$
$$12852\times2^{3/4}\sqrt{3}\sqrt{\sqrt{2}-2r}\,r^2+1296\times2^{1/4}\pi\sqrt{\sqrt{2}-2r}\,r^2+62343\times2^{3/4}\pi\sqrt{\sqrt{2}-2r}\,r^2+$$
$$9072\times2^{1/4}\sqrt{3}\,\pi\sqrt{\sqrt{2}-2r}\,r^2+22464\times2^{3/4}\sqrt{3}\,\pi\sqrt{\sqrt{2}-2r}\,r^2+$$
$$17928\times2^{1/4}\sqrt{\sqrt{2}-2r}\,r^3+4536\times2^{1/4}\sqrt{3}\sqrt{\sqrt{2}-2r}\,r^3-125064\times2^{1/4}\pi\sqrt{\sqrt{2}-2r}\,r^3-$$
$$6480\times2^{3/4}\pi\sqrt{\sqrt{2}-2r}\,r^3-15480\times2^{1/4}\sqrt{3}\,\pi\sqrt{\sqrt{2}-2r}\,r^3-3240\times2^{3/4}\sqrt{\sqrt{2}-2r}\,r^4-$$
$$9720\times2^{3/4}\pi\sqrt{\sqrt{2}-2r}\,r^4+30240\times2^{3/4}\sqrt{3}\,\pi\sqrt{\sqrt{2}-2r}\,r^4+9072\times2^{1/4}\sqrt{-\sqrt{6}+4r}-$$
$$8856\times2^{1/4}\sqrt{3}\sqrt{-\sqrt{6}+4r}+21504\times2^{1/4}\pi\sqrt{-\sqrt{6}+4r}+4536\times2^{3/4}\pi\sqrt{-\sqrt{6}+4r}-$$
$$20992\times2^{1/4}\sqrt{3}\,\pi\sqrt{-\sqrt{6}+4r}-4428\times2^{3/4}\sqrt{3}\,\pi\sqrt{-\sqrt{6}+4r}+$$
$$10422\times2^{3/4}r\sqrt{-\sqrt{6}+4r}+6345\times2^{3/4}\sqrt{3}\,r\sqrt{-\sqrt{6}+4r}+$$
$$13824\times2^{1/4}\pi r\sqrt{-\sqrt{6}+4r}+90905\times2^{3/4}\pi r\sqrt{-\sqrt{6}+4r}+$$
$$3024\times2^{1/4}\sqrt{3}\,\pi r\sqrt{-\sqrt{6}+4r}-14843\times2^{3/4}\sqrt{3}\,\pi r\sqrt{-\sqrt{6}+4r}-$$
$$46656\times2^{1/4}r^2\sqrt{-\sqrt{6}+4r}-3564\times2^{1/4}\sqrt{3}\,r^2\sqrt{-\sqrt{6}+4r}-$$
$$113028\times2^{1/4}\pi r^2\sqrt{-\sqrt{6}+4r}-18144\times2^{3/4}\pi r^2\sqrt{-\sqrt{6}+4r}-$$
$$57764\times2^{1/4}\sqrt{3}\,\pi r^2\sqrt{-\sqrt{6}+4r}-648\times2^{3/4}\sqrt{3}\,\pi r^2\sqrt{-\sqrt{6}+4r}+$$
$$26568\times2^{3/4}r^3\sqrt{-\sqrt{6}+4r}+486\times2^{3/4}\sqrt{3}\,r^3\sqrt{-\sqrt{6}+4r}+$$
$$12960\times2^{1/4}\pi r^3\sqrt{-\sqrt{6}+4r}+25122\times2^{3/4}\pi r^3\sqrt{-\sqrt{6}+4r}+$$
$$74274\times2^{3/4}\sqrt{3}\,\pi r^3\sqrt{-\sqrt{6}+4r}-9720\times2^{1/4}r^4\sqrt{-\sqrt{6}+4r}+$$
$$9720\times2^{1/4}\pi r^4\sqrt{-\sqrt{6}+4r}-53400\times2^{1/4}\sqrt{3}\,\pi r^4\sqrt{-\sqrt{6}+4r}+$$
$$5616\sqrt{2-\sqrt{3}}\,\mathrm{ArcSec}[3]-3240\sqrt{3\left(2-\sqrt{3}\right)}\,\mathrm{ArcSec}[3]-$$
$$2268\times2^{3/4}\sqrt{-\sqrt{6}+4r}\,\mathrm{ArcSec}[3]+2214\times2^{3/4}\sqrt{3}\sqrt{-\sqrt{6}+4r}\,\mathrm{ArcSec}[3]-$$
$$6912\times2^{1/4}r\sqrt{-\sqrt{6}+4r}\,\mathrm{ArcSec}[3]-1512\times2^{1/4}\sqrt{3}\,r\sqrt{-\sqrt{6}+4r}\,\mathrm{ArcSec}[3]+$$
$$9072\times2^{3/4}r^2\sqrt{-\sqrt{6}+4r}\,\mathrm{ArcSec}[3]+324\times2^{3/4}\sqrt{3}\,r^2\sqrt{-\sqrt{6}+4r}\,\mathrm{ArcSec}[3]-$$
$$6480\times2^{1/4}r^3\sqrt{-\sqrt{6}+4r}\,\mathrm{ArcSec}[3]\Big)\Big/\left(36\sqrt{2-\sqrt{3}}\left(-2+\sqrt{3}\right)^3\pi\right);$$

```
Plot[{OCTDDCFCC[r], OcthCLDApprx11CC[r] , OcthCLDApprx00CC[r] },
 {r, OCTDst[[2]], OCTDst[[3]]}, AxesLabel → {"r", "γ''[r]"},
 PlotStyle → {Directive[Blue, Thickness[0.003]],
   Directive[Red, Thickness[0.004]], Directive[Purple, Thickness[0.004], Dashed]},
 PlotRange → {{OCTDst[[2]], OCTDst[[3]]}, {0, 7}}, PlotLabel → " CLD: K=1; D2< r < D3"]
```

The approximation has improved. The largest error is about 58% around 0.65

```
Plot[{OCTDDCFCC[r] - OcthCLDApprx11CC[r]}, {r, OCTDst[[2]], OCTDst[[3]]}]
```

```
N[(OCTDDCFCC[65 / 100] - OcthCLDApprx11CC[65 / 100]) / OCTDDCFCC[65 / 100]]
```

```
0.582696
```

```
Plot[{If[r < OCTDst[[2]], OCTDDCFBB[r], If[r < OCTDst[[3]], OCTDDCFCC[r], OCTDDCFDD[r]]],
  If[r < OCTDst[[2]], OcthCLDApprx11BB[r],
   If[r < OCTDst[[3]], OcthCLDApprx11CC[r], OcthCLDApprx11DD[r]]]
  (*  ,If[r<OCTDst[[2]],OcthCLDApprx00BB[r],
   If[r<OCTDst[[3]],OcthCLDApprx00CC[r],OcthCLDApprx00DD[r]]] *) },
 {r, OCTDst[[1]], OCTDst[[4]]}, AxesLabel → {"r", "γ''[r]"},
 PlotStyle → {Directive[Blue, Thickness[0.003]],
   Directive[Red, Thickness[0.004], Dashed], Directive[Purple, Thickness[0.004], Dotted]},
 PlotRange → {{OCTDst[[1]], OCTDst[[4]]}, {-1.3, 10}}, AxesLabel → {"r", "γo''[r],10xΔγ''"}]
```

Overall the CLD approximation does not look very accurate.

Approximation (not matched) of the CF

```
Simplify[((LEFTCFApprx11[r, D1, D2]) /.
   {a0 → cfD2Plus[[1]], a1 → cfD2Plus[[2]], a2 → cfD2Plus[[3]], b0 → cfD3Minus[[1]],
    b1 → cfD3Minus[[2]], b2 → cfD3Minus[[3]], D1 → OCTDst[[2]], D2 → OCTDst[[3]]}),
 Assumptions → {OCTDst[[2]] < r < OCTDst[[3]]}]
```

```
FullSimplify[((LEFTCFApprx11[r, D1, D2]) /.
    {a0 → cfD2Plus[[1]], a1 → cfD2Plus[[2]], a2 → cfD2Plus[[3]], b0 → cfD3Minus[[1]],
     b1 → cfD3Minus[[2]], b2 → cfD3Minus[[3]], D1 → OCTDst[[2]], D2 → OCTDst[[3]]}) -
  ApprxOcthCF11NotMatchCC[r], Assumptions → {OCTDst[[2]] < r < OCTDst[[3]]}]
```

```
0
```

ApprxOcthCF11NotMatchCC[r_] :=

$$-\Bigg(\pi\Bigg(8640\, r^6\Big(-81\sqrt{2}\sqrt{\sqrt{2}-2r}+252\sqrt{6}\sqrt{\sqrt{2}-2r}+81\sqrt{-\sqrt{6}+4r}-445\sqrt{3}\sqrt{-\sqrt{6}+4r}\Big)+8\, r^4\Big(50\,544\sqrt{\sqrt{2}-2r}+1\,517\,643\sqrt{2}\sqrt{\sqrt{2}-2r}+185\,328\sqrt{3}\sqrt{\sqrt{2}-2r}+450\,936\sqrt{6}\sqrt{\sqrt{2}-2r}-2\,731\,665\sqrt{-\sqrt{6}+4r}-1\,178\,059\sqrt{3}\sqrt{-\sqrt{6}+4r}-25\,272\sqrt{6}\sqrt{-\sqrt{6}+4r}-370\,656\sqrt{-2\sqrt{6}+8r}\Big)-$$

$$48\, r^5 \left(267732 \sqrt{\sqrt{2} - 2 r} + 14040 \sqrt{2} \sqrt{\sqrt{2} - 2 r} + 43620 \sqrt{3} \sqrt{\sqrt{2} - 2 r} - 28080 \sqrt{-\sqrt{6} + 4 r} - 160117 \sqrt{6} \sqrt{-\sqrt{6} + 4 r} - 67781 \sqrt{-2 \sqrt{6} + 8 r}\right) +$$

$$36\, r^2 \left(429 \times 2^{1/4} \left(448 + 405 \sqrt{2}\right) \sqrt{6 - 3\sqrt{3}} - 286 \times 2^{1/4} \left(1168 + 1053 \sqrt{2}\right) \sqrt{2 - \sqrt{3}} + 170820 \sqrt{\sqrt{2} - 2 r} - 439695 \sqrt{2} \sqrt{\sqrt{2} - 2 r} - 61776 \sqrt{3} \sqrt{\sqrt{2} - 2 r} + 371048 \sqrt{6} \sqrt{\sqrt{2} - 2 r} + 576252 \sqrt{-\sqrt{6} + 4 r} - 588238 \sqrt{3} \sqrt{-\sqrt{6} + 4 r} - 104598 \sqrt{6} \sqrt{-\sqrt{6} + 4 r} + 92664 \sqrt{-2 \sqrt{6} + 8 r}\right) +$$

$$6\, r^3 \left(2938936 \times 2^{3/4} \sqrt{6 - 3\sqrt{3}} - 5087511 \times 2^{3/4} \sqrt{2 - \sqrt{3}} + 1573092 \sqrt{\sqrt{2} - 2 r} - 221832 \sqrt{2} \sqrt{\sqrt{2} - 2 r} - 3012288 \sqrt{3} \sqrt{\sqrt{2} - 2 r} - 123552 \sqrt{6} \sqrt{\sqrt{2} - 2 r} + 657072 \sqrt{-\sqrt{6} + 4 r} + 247104 \sqrt{3} \sqrt{-\sqrt{6} + 4 r} - 603071 \sqrt{6} \sqrt{-\sqrt{6} + 4 r} + 4932385 \sqrt{-2 \sqrt{6} + 8 r}\right) +$$

$$2 \left(2^{1/4} \left(612607 + 729729 \sqrt{2}\right) \sqrt{6 - 3\sqrt{3}} - 6 \times 2^{1/4} \left(177200 + 210951 \sqrt{2}\right) \sqrt{2 - \sqrt{3}} + 9 \left(76284 \sqrt{\sqrt{2} - 2 r} - 52908 \sqrt{2} \sqrt{\sqrt{2} - 2 r} - 41184 \sqrt{3} \sqrt{\sqrt{2} - 2 r} + 36752 \sqrt{6} \sqrt{\sqrt{2} - 2 r} + 64365 \sqrt{-\sqrt{6} + 4 r} - 51001 \sqrt{3} \sqrt{-\sqrt{6} + 4 r} - 31590 \sqrt{6} \sqrt{-\sqrt{6} + 4 r} + 46332 \sqrt{-2 \sqrt{6} + 8 r}\right)\right) +$$

$$3\, r \left(-13\, 2^{1/4} \left(-95040 + 194441 \sqrt{2}\right) \sqrt{2 - \sqrt{3}} + 3 \left(26 \times 2^{1/4} \left(-9108 + 18751 \sqrt{2}\right) \sqrt{6 - 3\sqrt{3}} + 1147956 \sqrt{\sqrt{2} - 2 r} - 412776 \sqrt{2} \sqrt{\sqrt{2} - 2 r} - 829840 \sqrt{3} \sqrt{\sqrt{2} - 2 r} + 205920 \sqrt{6} \sqrt{\sqrt{2} - 2 r} + 707616 \sqrt{-\sqrt{6} + 4 r} - 308880 \sqrt{3} \sqrt{-\sqrt{6} + 4 r} - 406487 \sqrt{6} \sqrt{-\sqrt{6} + 4 r} + 1024357 \sqrt{-2 \sqrt{6} + 8 r}\right)\right)\Bigg) -$$

$$27 \left(8640\, r^6 \left(\sqrt{2} \sqrt{\sqrt{2} - 2 r} + 3 \sqrt{-\sqrt{6} + 4 r}\right) - 16\, r^5 \left(4436 \sqrt{\sqrt{2} - 2 r} + 1092 \sqrt{3} \sqrt{\sqrt{2} - 2 r} + 207 \sqrt{6} \sqrt{-\sqrt{6} + 4 r} + 6396 \sqrt{-2 \sqrt{6} + 8 r} - 1560 \sqrt{-\sqrt{6} + 4 r}\, \mathrm{ArcSec}[3]\right) +\right.$$

$$
\begin{aligned}
&r\Big(-347476\sqrt{\sqrt{2}-2r}+172640\sqrt{3}\sqrt{\sqrt{2}-2r}+115101\sqrt{6}\sqrt{-\sqrt{6}+4r}-\\
&\quad 253422\sqrt{-2\sqrt{6}+8r}+78\times 2^{1/4}\sqrt{2-\sqrt{3}}\left(6077\sqrt{2}-836\,\mathrm{ArcSec}[3]\right)-\\
&\quad 9438\times 2^{1/4}\sqrt{6-3\sqrt{3}}\left(29\sqrt{2}-4\,\mathrm{ArcSec}[3]\right)+\\
&\quad 117936\sqrt{-\sqrt{6}+4r}\,\mathrm{ArcSec}[3]-51480\sqrt{3}\sqrt{-\sqrt{6}+4r}\,\mathrm{ArcSec}[3]\Big)+\\
&2r^3\Big(12870\times 2^{3/4}\sqrt{6-3\sqrt{3}}-22308\times 2^{3/4}\sqrt{2-\sqrt{3}}-89324\sqrt{\sqrt{2}-2r}-\\
&\quad 27664\sqrt{3}\sqrt{\sqrt{2}-2r}-48987\sqrt{6}\sqrt{-\sqrt{6}+4r}-52884\sqrt{-2\sqrt{6}+8r}+\\
&\quad 36504\sqrt{-\sqrt{6}+4r}\,\mathrm{ArcSec}[3]+13728\sqrt{3}\sqrt{-\sqrt{6}+4r}\,\mathrm{ArcSec}[3]\Big)+\\
&8r^4\Big(4213\sqrt{2}\sqrt{\sqrt{2}-2r}+10036\sqrt{6}\sqrt{\sqrt{2}-2r}+35073\sqrt{-\sqrt{6}+4r}+\\
&\quad 4524\sqrt{3}\sqrt{-\sqrt{6}+4r}-156\left(44+3\sqrt{3}\right)\sqrt{-2\sqrt{6}+8r}\,\mathrm{ArcSec}[3]\Big)+\\
&4r^2\Big(77975\sqrt{2}\sqrt{\sqrt{2}-2r}-28548\sqrt{6}\sqrt{\sqrt{2}-2r}-56619\sqrt{-\sqrt{6}+4r}+\\
&\quad 67275\sqrt{3}\sqrt{-\sqrt{6}+4r}-17433\sqrt{6}\sqrt{-\sqrt{6}+4r}\,\mathrm{ArcSec}[3]+\\
&\quad 15444\sqrt{-2\sqrt{6}+8r}\,\mathrm{ArcSec}[3]+6435\times 2^{1/4}\sqrt{6-3\sqrt{3}}\left(-4+3\sqrt{2}\,\mathrm{ArcSec}[3]\right)-\\
&\quad 11154\times 2^{1/4}\sqrt{2-\sqrt{3}}\left(-4+3\sqrt{2}\,\mathrm{ArcSec}[3]\right)\Big)+\\
&2\Big(30908\sqrt{2}\sqrt{\sqrt{2}-2r}-16640\sqrt{6}\sqrt{\sqrt{2}-2r}-36369\sqrt{-\sqrt{6}+4r}+\\
&\quad 23868\sqrt{3}\sqrt{-\sqrt{6}+4r}-5265\sqrt{6}\sqrt{-\sqrt{6}+4r}\,\mathrm{ArcSec}[3]+7722\sqrt{-2\sqrt{6}+8r}\\
&\quad \mathrm{ArcSec}[3]+117\times 2^{1/4}\sqrt{6-3\sqrt{3}}\left(238+33\sqrt{2}\,\mathrm{ArcSec}[3]\right)-3\times 2^{1/4}\\
&\quad \sqrt{2-\sqrt{3}}\left(16075+2236\sqrt{2}\,\mathrm{ArcSec}[3]\right)\Big)\Big)\Big)\Big)\Big/\left(46332\times 2^{3/4}\left(2-\sqrt{3}\right)^{7/2}\pi\right);
\end{aligned}
$$

```
Plot[{OCTCFCC[r], ApprxOcthCF11NotMatchCC[r]},
 {r, OCTDst[[2]], OCTDst[[3]]}, AxesLabel → {"r", "γ[r]"},
 PlotStyle → {Directive[Blue, Thickness[0.003]], Directive[Red, Thickness[0.006]]},
 AxesLabel → {"r", "γ0[r],10xΔγ"}]
```

```
Plot[{If[r < OCTDst[[2]], OCTCFBB[r], If[r < OCTDst[[3]], OCTCFCC[r], OCTCFDD[r]]],
   If[r < OCTDst[[2]], ApprxOcthCF11PrtlyMatchBB[r],
    If[r < OCTDst[[3]], ApprxOcthCF11NotMatchCC[r], ApprxOcthCF11NotMatchDD[r]]]},
 {r, OCTDst[[1]], OCTDst[[4]]}, PlotRange → {{OCTDst[[1]], OCTDst[[4]]}, {-0.020, 0.04}},
 AxesLabel → {"r", "γ[r]"},
 PlotStyle → {Directive[Blue, Thickness[0.003]], Directive[Red, Thickness[0.006]]},
 AxesLabel → {"r", "γ0[r],10xΔγ"}]
```

FINAL MATCHES

The approximation ApprxOcthCF11PrtlyMatchBB[r] becomes OcthCFApprxFnl11BB[r] ;
ApprxOcthCF11NotMatchCC[r] is matched to OcthCFApprxFnl11BB[r] at r=D2 to get OcthCFApprxFnl11CC[r];
we add EXTRCNTRBLft11[r_, D1_, D2_, a_, b_] to ApprxOcthCF11NotMatchDD[r] and match the result to OcthCFApprxFnl11CC[r] at r=D3 to get OcthCFApprxFnl11DD[r]

```
ApprxOcthCF11PrtlyMatchBB[r]
```

```
Simplify[ApprxOcthCF11PrtlyMatchBB[r] - OcthCFApprxFnl11BB[r]]
```

0

```
OcthCFApprxFnl11BB[r_] :=
```

$$\frac{1}{7128\sqrt{-24+17\sqrt{2}}\,\pi} r \Bigg(\Bigg(\Big(10692\sqrt{-24+17\sqrt{2}} - 14256\sqrt{-72+51\sqrt{2}} + 2^{1/4}\left(2673 - 1782\sqrt{2} - 146267\sqrt{3} + 103442\sqrt{6}\right)\Big)\pi - 297\Big(9\times 2^{1/4} - 6\times 2^{3/4} - 544\times 2^{1/4}\sqrt{3} + 384\times 2^{3/4}\sqrt{3} + 36\sqrt{-24+17\sqrt{2}} - 96\sqrt{6\left(-24+17\sqrt{2}\right)} - 192\times 2^{1/4}\sqrt{3}\,\text{ArcSec}[3] + 136\times 2^{3/4}\sqrt{3}\,\text{ArcSec}[3] + 48\sqrt{-72+51\sqrt{2}}\,\text{ArcSec}[3]\Big)\Bigg) + \Bigg(\Bigg(\Big(-42335271\, 2^{1/4} + 29935555\times 2^{3/4} - 69160392\times 2^{1/4}\sqrt{3} + 48903777\times 2^{3/4}\sqrt{3} - 25945920\sqrt{-24+17\sqrt{2}} - 4586868\sqrt{6\left(-24+17\sqrt{2}\right)} + 18347472\sqrt{-48+34\sqrt{2}} + 6486480\sqrt{-72+51\sqrt{2}}\Big)\pi - 27\Big(990132\times 2^{1/4} - 700128\times 2^{3/4} - 2561496\times 2^{1/4}\sqrt{3} + 1811251\times 2^{3/4}\sqrt{3} - 169884\sqrt{6\left(-24+17\sqrt{2}\right)} + 240240\sqrt{-72+51\sqrt{2}} + 350064\times 2^{1/4}\,\text{ArcSec}[3] - 247533\times 2^{3/4}\,\text{ArcSec}[3] + 20592\sqrt{-24+17\sqrt{2}}\,(99 + 35\,\text{ArcSec}[3]) - 5148\sqrt{-48+34\sqrt{2}}\,(280 + 99\,\text{ArcSec}[3])\Big)\Bigg)\Bigg/\left(69498\sqrt{3-2\sqrt{2}}\left(-4+3\sqrt{2}\right)^{7/2}\pi\right) + \left(8\times 3^{1/4}\left(3 + \left(-3+19\sqrt{3}\right)\pi\right)\right)\Big(-20736\left(176+273\sqrt{2}\right)r^5\sqrt{-1+\sqrt{3}\,r} + 2880\left(1303+1560\sqrt{2}\right)r^4\sqrt{-3+3\sqrt{3}\,r} + 1866240\,r^6\sqrt{-3+3\sqrt{3}\,r} + \sqrt{3}\left(2^{1/4}\sqrt{3-2\sqrt{2}}\left(-239095+169083\sqrt{2}\right) + 64\left(8867-5304\sqrt{2}\right)\sqrt{-1+\sqrt{3}\,r}\right) + 216\sqrt{3}\,r^2\left(143\times 2^{1/4}\sqrt{3-2\sqrt{2}}\left(-140+99\sqrt{2}\right) + 80\left(505-208\sqrt{2}\right)\sqrt{-1+\sqrt{3}\,r}\right) -$$

$$72\, r^3 \left(429 \times 2^{1/4} \sqrt{3-2\sqrt{2}} \left(-140+99\sqrt{2}\right) + 160\left(1327-78\sqrt{2}\right)\sqrt{-1+\sqrt{3}\, r}\right) - 3\, r \left(39 \times 2^{1/4}\sqrt{3-2\sqrt{2}}\left(-13\,180+9321\sqrt{2}\right) - 1280\left(-1663+897\sqrt{2}\right)\sqrt{-1+\sqrt{3}\, r}\right)\Bigg) -$$

$$936\left(6^{1/4}\left(9964-7047\sqrt{2}\right)\sqrt{3-2\sqrt{2}} + 192\times 3^{3/4}\left(257+132\sqrt{2}\right) r^3\sqrt{-1+\sqrt{3}\, r} - 6912\times 3^{1/4}\left(2+11\sqrt{2}\right) r^4\sqrt{-1+\sqrt{3}\, r} + 23\,040\times 3^{3/4} r^5\sqrt{-1+\sqrt{3}\, r} + 64\left(264\sqrt{-2\sqrt{3}+6\, r} - 425\times 3^{1/4}\sqrt{-1+\sqrt{3}\, r}\right) - 24\times 3^{1/4} r^2\left(33\times 2^{1/4}\sqrt{3-2\sqrt{2}}\left(-140+99\sqrt{2}\right) + 8\left(1027-396\sqrt{2}\right)\sqrt{-1+\sqrt{3}\, r}\right) + 8\times 3^{3/4}\, r\left(11\times 2^{1/4}\sqrt{3-2\sqrt{2}}\left(-263+186\sqrt{2}\right) + 24\left(393-220\sqrt{2}\right)\sqrt{-1+\sqrt{3}\, r}\right)\right)$$

$$\left(4\sqrt{2}-\sqrt{3}+\pi+\sqrt{3}\,\pi - 2\,\mathrm{ArcSec}[3]\right) + \left(\sqrt{6}-4\, r\right)^2$$

$$\Bigg(2^{1/4}\left(81+\left(-81+445\sqrt{3}\right)\pi\right)\left(\sqrt{6}-4\, r\right)\left(14\,157\times 3^{1/4}\sqrt{6-4\sqrt{2}} - 20\,020\times 3^{1/4}\sqrt{3-2\sqrt{2}} + 10\,608\sqrt{3}\sqrt{\sqrt{3}-2\sqrt{2}\, r} - 8456\sqrt{6}\sqrt{\sqrt{3}-2\sqrt{2}\, r} + 16\left(1493-507\sqrt{2}\right) r\sqrt{\sqrt{3}-2\sqrt{2}\, r} + 24\sqrt{3}\left(-728+141\sqrt{2}\right) r^2\sqrt{\sqrt{3}-2\sqrt{2}\, r} + 12\,960\, r^3\sqrt{\sqrt{3}-2\sqrt{2}\, r}\right) + 39\times 2^{1/4}$$

$$\left(1089\times 3^{1/4}\sqrt{6-4\sqrt{2}} - 1540\times 3^{1/4}\sqrt{3-2\sqrt{2}} + 528\sqrt{3}\sqrt{\sqrt{3}-2\sqrt{2}\, r} - 408\sqrt{6}\sqrt{\sqrt{3}-2\sqrt{2}\, r} - 48\left(-25+11\sqrt{2}\right) r\sqrt{\sqrt{3}-2\sqrt{2}\, r} + 8\sqrt{3}\left(-88+21\sqrt{2}\right) r^2\sqrt{\sqrt{3}-2\sqrt{2}\, r} + 480\, r^3\sqrt{\sqrt{3}-2\sqrt{2}\, r}\right)\left(\left(432-311\sqrt{2}+81\sqrt{6}\right)\pi - 27\left(-16\sqrt{2}+3\sqrt{6}+8\,\mathrm{ArcSec}[3]\right)\right)\Bigg)\Bigg) \Big/ \left(247\,104\times 3^{1/4}\left(-4+3\sqrt{2}\right)^{7/2}\pi\right);$$

```
Simplify[Solve[
  {Simplify[Limit[ApprxOcthCF11NotMatchCC[r] + a + b r, r → OCTDst[[2]], Direction → -1] -
       Limit[OcthCFApprxFnl11BB[r], r → OCTDst[[2]], Direction → 1]] == 0 &&
    Simplify[Limit[D[ApprxOcthCF11NotMatchCC[r] + a + b r, r],
        r → OCTDst[[2]], Direction → -11] -
       Limit[D[OcthCFApprxFnl11BB[r], r], r → OCTDst[[2]], Direction → 1]] == 0}, {a, b}]]
```

```
Simplify[
 Simplify[(ApprxOcthCF11NotMatchCC[r] + a + b r) /. {a -> ((-164 216 241 Sqrt[-24 + 17 Sqrt[2]] + 4
        (29 030 139 Sqrt[-48 + 34 Sqrt[2]] + 21 488 868 Sqrt[-72 + 51 Sqrt[2]] + 2^(1/4) (-42 335 271 + 29 935 555
              Sqrt[2] - 69 160 392 Sqrt[3] + 48 903 777 Sqrt[6] - 15 195 000 × 2^(1/4) Sqrt[-72 + 51 Sqrt[2]]))) π -
      54 (66 690 Sqrt[-24 + 17 Sqrt[2]] (11 + 14 ArcSec[3]) - 741 Sqrt[-48 + 34 Sqrt[2]]
          (700 + 891 ArcSec[3]) + 2 (15 Sqrt[-72 + 51 Sqrt[2]] (81 059 + 13 104 ArcSec[3]) +
           2^(1/4) (990 132 - 700 128 Sqrt[2] - 2 561 496 Sqrt[3] + 1 811 251 Sqrt[6] - 429 (-816 + 577 Sqrt[2])
              ArcSec[3] - 486 × 2^(1/4) Sqrt[-72 + 51 Sqrt[2]] (1769 + 286 ArcSec[3]))))) /
     (277 992 Sqrt[3 - 2 Sqrt[2]] (-4 + 3 Sqrt[2])^(7/2) π), b -> 1/(14 256 Sqrt[-24 + 17 Sqrt[2]] π)
     ((5346 × 2^(1/4) - 3564 × 2^(3/4) - 292 534 × 2^(1/4) Sqrt[3] + 206 884 × 2^(3/4) Sqrt[3] + 44 178 Sqrt[-24 + 17 Sqrt[2]] -
          21 384 Sqrt[6 (-24 + 17 Sqrt[2])] + 42 768 Sqrt[-48 + 34 Sqrt[2]] - 44 091 Sqrt[-72 + 51 Sqrt[2]]) π -
      54 (-175 Sqrt[-24 + 17 Sqrt[2]] + 16 Sqrt[-72 + 51 Sqrt[2]] (20 + 33 ArcSec[3]) +
          11 × 2^(1/4) (9 - 6 Sqrt[2] - 544 Sqrt[3] + 384 Sqrt[6] + 8 (-24 Sqrt[3] + 17 Sqrt[6] + 3 × 2^(1/4) Sqrt[-24 + 17 Sqrt[2]])
               ArcSec[3] - 12 × 2^(1/4) Sqrt[-72 + 51 Sqrt[2]] (8 + ArcSec[3]))))},
  Assumptions -> {OCTDst[[2]] < r < OCTDst[[3]]}]]
```

```
OcthCFApprxFnl11CC[r_] := 1/(14 256 Sqrt[-24 + 17 Sqrt[2]] π) r
   ((5346 × 2^(1/4) - 3564 × 2^(3/4) - 292 534 × 2^(1/4) Sqrt[3] + 206 884 × 2^(3/4) Sqrt[3] + 44 178 Sqrt[-24 + 17 Sqrt[2]] -
```

$$21384\sqrt{6\left(-24+17\sqrt{2}\right)}+42768\sqrt{-48+34\sqrt{2}}-44091\sqrt{-72+51\sqrt{2}}\Bigg)\pi-$$

$$54\left(-175\sqrt{-24+17\sqrt{2}}+16\sqrt{-72+51\sqrt{2}}\ (20+33\,\mathrm{ArcSec}[3])+\right.$$

$$11\times2^{1/4}\left(9-6\sqrt{2}-544\sqrt{3}+384\sqrt{6}+8\left(-24\sqrt{3}+17\sqrt{6}+3\times2^{1/4}\sqrt{-24+17\sqrt{2}}\right)\right.$$

$$\left.\left.\mathrm{ArcSec}[3]-12\times2^{1/4}\sqrt{-72+51\sqrt{2}}\ (8+\mathrm{ArcSec}[3])\Big)\right)\right)+$$

$$\Bigg(-\pi\Bigg(8640\,r^6\left(-81\sqrt{2}\sqrt{\sqrt{2}-2r}+252\sqrt{6}\sqrt{\sqrt{2}-2r}+81\sqrt{-\sqrt{6}+4r}-\right.$$

$$\left.445\sqrt{3}\sqrt{-\sqrt{6}+4r}\right)+8\,r^4\left(50544\sqrt{\sqrt{2}-2r}+1517643\sqrt{2}\sqrt{\sqrt{2}-2r}+\right.$$

$$185328\sqrt{3}\sqrt{\sqrt{2}-2r}+450936\sqrt{6}\sqrt{\sqrt{2}-2r}-2731665\sqrt{-\sqrt{6}+4r}-$$

$$\left.1178059\sqrt{3}\sqrt{-\sqrt{6}+4r}-25272\sqrt{6}\sqrt{-\sqrt{6}+4r}-370656\sqrt{-2\sqrt{6}+8r}\right)-$$

$$48\,r^5\left(267732\sqrt{\sqrt{2}-2r}+14040\sqrt{2}\sqrt{\sqrt{2}-2r}+43620\sqrt{3}\sqrt{\sqrt{2}-2r}-\right.$$

$$\left.28080\sqrt{-\sqrt{6}+4r}-160117\sqrt{6}\sqrt{-\sqrt{6}+4r}-67781\sqrt{-2\sqrt{6}+8r}\right)+$$

$$36\,r^2\left(429\times2^{1/4}\left(448+405\sqrt{2}\right)\sqrt{6-3\sqrt{3}}-286\times2^{1/4}\left(1168+1053\sqrt{2}\right)\sqrt{2-\sqrt{3}}+\right.$$

$$170820\sqrt{\sqrt{2}-2r}-439695\sqrt{2}\sqrt{\sqrt{2}-2r}-61776\sqrt{3}\sqrt{\sqrt{2}-2r}+$$

$$371048\sqrt{6}\sqrt{\sqrt{2}-2r}+576252\sqrt{-\sqrt{6}+4r}-588238\sqrt{3}\sqrt{-\sqrt{6}+4r}-$$

$$\left.104598\sqrt{6}\sqrt{-\sqrt{6}+4r}+92664\sqrt{-2\sqrt{6}+8r}\right)+$$

$$6\,r^3\left(2938936\times2^{3/4}\sqrt{6-3\sqrt{3}}-5087511\times2^{3/4}\sqrt{2-\sqrt{3}}+\right.$$

$$1573092\sqrt{\sqrt{2}-2r}-221832\sqrt{2}\sqrt{\sqrt{2}-2r}-3012288\sqrt{3}\sqrt{\sqrt{2}-2r}-$$

$$123552\sqrt{6}\sqrt{\sqrt{2}-2r}+657072\sqrt{-\sqrt{6}+4r}+247104\sqrt{3}\sqrt{-\sqrt{6}+4r}-$$

$$\left.603071\sqrt{6}\sqrt{-\sqrt{6}+4r}+4932385\sqrt{-2\sqrt{6}+8r}\right)+$$

$$2\left(2^{1/4}\left(612607+729729\sqrt{2}\right)\sqrt{6-3\sqrt{3}}-6\times2^{1/4}\left(177200+210951\sqrt{2}\right)\sqrt{2-\sqrt{3}}+\right.$$

$$9\left(76284\sqrt{\sqrt{2}-2r}-52908\sqrt{2}\sqrt{\sqrt{2}-2r}-41184\sqrt{3}\sqrt{\sqrt{2}-2r}+\right.$$

$$36752 \sqrt{6} \sqrt{\sqrt{2} - 2 r} + 64365 \sqrt{-\sqrt{6} + 4 r} - 51001 \sqrt{3}$$

$$\sqrt{-\sqrt{6} + 4 r} - 31590 \sqrt{6} \sqrt{-\sqrt{6} + 4 r} + 46332 \sqrt{-2 \sqrt{6} + 8 r})) +$$

$$3 r (-13\ 2^{1/4} (-95040 + 194441 \sqrt{2}) \sqrt{2 - \sqrt{3}} + 3 (26 \times 2^{1/4} (-9108 + 18751 \sqrt{2})$$

$$\sqrt{6 - 3 \sqrt{3}} + 1147956 \sqrt{\sqrt{2} - 2 r} - 412776 \sqrt{2} \sqrt{\sqrt{2} - 2 r} - 829840 \sqrt{3}$$

$$\sqrt{\sqrt{2} - 2 r} + 205920 \sqrt{6} \sqrt{\sqrt{2} - 2 r} + 707616 \sqrt{-\sqrt{6} + 4 r} - 308880 \sqrt{3}$$

$$\sqrt{-\sqrt{6} + 4 r} - 406487 \sqrt{6} \sqrt{-\sqrt{6} + 4 r} + 1024357 \sqrt{-2 \sqrt{6} + 8 r}))) +$$

$$27 (8640 r^6 (\sqrt{2} \sqrt{\sqrt{2} - 2 r} + 3 \sqrt{-\sqrt{6} + 4 r}) - 16 r^5 (4436 \sqrt{\sqrt{2} - 2 r} +$$

$$1092 \sqrt{3} \sqrt{\sqrt{2} - 2 r} + 207 \sqrt{6} \sqrt{-\sqrt{6} + 4 r} +$$

$$6396 \sqrt{-2 \sqrt{6} + 8 r} - 1560 \sqrt{-\sqrt{6} + 4 r}\ \text{ArcSec}[3]) +$$

$$r (-347476 \sqrt{\sqrt{2} - 2 r} + 172640 \sqrt{3} \sqrt{\sqrt{2} - 2 r} + 115101 \sqrt{6} \sqrt{-\sqrt{6} + 4 r} -$$

$$253422 \sqrt{-2 \sqrt{6} + 8 r} + 78 \times 2^{1/4} \sqrt{2 - \sqrt{3}} (6077 \sqrt{2} - 836\ \text{ArcSec}[3]) -$$

$$9438 \times 2^{1/4} \sqrt{6 - 3 \sqrt{3}} (29 \sqrt{2} - 4\ \text{ArcSec}[3]) +$$

$$117936 \sqrt{-\sqrt{6} + 4 r}\ \text{ArcSec}[3] - 51480 \sqrt{3} \sqrt{-\sqrt{6} + 4 r}\ \text{ArcSec}[3]) +$$

$$2 r^3 (12870 \times 2^{3/4} \sqrt{6 - 3 \sqrt{3}} - 22308 \times 2^{3/4} \sqrt{2 - \sqrt{3}} - 89324 \sqrt{\sqrt{2} - 2 r} -$$

$$27664 \sqrt{3} \sqrt{\sqrt{2} - 2 r} - 48987 \sqrt{6} \sqrt{-\sqrt{6} + 4 r} - 52884 \sqrt{-2 \sqrt{6} + 8 r} +$$

$$36504 \sqrt{-\sqrt{6} + 4 r}\ \text{ArcSec}[3] + 13728 \sqrt{3} \sqrt{-\sqrt{6} + 4 r}\ \text{ArcSec}[3]) +$$

$$8 r^4 (4213 \sqrt{2} \sqrt{\sqrt{2} - 2 r} + 10036 \sqrt{6} \sqrt{\sqrt{2} - 2 r} + 35073 \sqrt{-\sqrt{6} + 4 r} +$$

$$4524 \sqrt{3} \sqrt{-\sqrt{6} + 4 r} - 156 (44 + 3 \sqrt{3}) \sqrt{-2 \sqrt{6} + 8 r}\ \text{ArcSec}[3]) +$$

$$4 r^2 (77975 \sqrt{2} \sqrt{\sqrt{2} - 2 r} - 28548 \sqrt{6} \sqrt{\sqrt{2} - 2 r} - 56619 \sqrt{-\sqrt{6} + 4 r} +$$

$$67275 \sqrt{3} \sqrt{-\sqrt{6} + 4 r} - 17433 \sqrt{6} \sqrt{-\sqrt{6} + 4 r}\ \text{ArcSec}[3] +$$

$$15444 \sqrt{-2 \sqrt{6} + 8 r}\ \text{ArcSec}[3] + 6435 \times 2^{1/4} \sqrt{6 - 3 \sqrt{3}} (-4 + 3 \sqrt{2}\ \text{ArcSec}[3]) -$$

```
            11 154 × 2^(1/4) Sqrt[2 - Sqrt[3]] (-4 + 3 Sqrt[2] ArcSec[3])) +
          2 (30 908 Sqrt[2] Sqrt[Sqrt[2] - 2 r] - 16 640 Sqrt[6] Sqrt[Sqrt[2] - 2 r] - 36 369 Sqrt[-Sqrt[6] + 4 r] +
            23 868 Sqrt[3] Sqrt[-Sqrt[6] + 4 r] - 5265 Sqrt[6] Sqrt[-Sqrt[6] + 4 r] ArcSec[3] +
            7722 Sqrt[-2 Sqrt[6] + 8 r] ArcSec[3] + 117 × 2^(1/4) Sqrt[6 - 3 Sqrt[3]] (238 + 33 Sqrt[2] ArcSec[3]) -
            3 × 2^(1/4) Sqrt[2 - Sqrt[3]] (16 075 + 2236 Sqrt[2] ArcSec[3]))))) /
      (46 332 × 2^(3/4) (2 - Sqrt[3])^(7/2) π) + (((-164 216 241 Sqrt[-24 + 17 Sqrt[2]] +
          4 (29 030 139 Sqrt[-48 + 34 Sqrt[2]] + 21 488 868 Sqrt[-72 + 51 Sqrt[2]] + 2^(1/4) (-42 335 271 + 29 935 555
                Sqrt[2] - 69 160 392 Sqrt[3] + 48 903 777 Sqrt[6] - 15 195 000 × 2^(1/4) Sqrt[-72 + 51 Sqrt[2]]))) π -
        54 (66 690 Sqrt[-24 + 17 Sqrt[2]] (11 + 14 ArcSec[3]) - 741 Sqrt[-48 + 34 Sqrt[2]]
            (700 + 891 ArcSec[3]) + 2 (15 Sqrt[-72 + 51 Sqrt[2]] (81 059 + 13 104 ArcSec[3]) +
            2^(1/4) (990 132 - 700 128 Sqrt[2] - 2 561 496 Sqrt[3] + 1 811 251 Sqrt[6] - 429
                (-816 + 577 Sqrt[2]) ArcSec[3] - 486 × 2^(1/4) Sqrt[-72 + 51 Sqrt[2]]
                (1769 + 286 ArcSec[3]))))) / (277 992 Sqrt[3 - 2 Sqrt[2]] (-4 + 3 Sqrt[2])^(7/2) π);
```

```
Plot[{If[r < OCTDst[[2]], OCTCFBB[r], If[r < OCTDst[[3]], OCTCFCC[r], OCTCFDD[r]]],
  If[r < OCTDst[[2]], OcthCFApprxFnl11BB[r],
   If[r < OCTDst[[3]], OcthCFApprxFnl11CC[r], ApprxOcthCF11NotMatchDD[r]]]},
 {r, OCTDst[[1]], OCTDst[[4]]}, AxesLabel → {"r", "γ[r]"},
 PlotRange → {{OCTDst[[2]], OCTDst[[4]]}, {-0.02, 0.025}},
 PlotStyle → {Directive[Blue, Thickness[0.003]],
   Directive[Red, Thickness[0.004]], Directive[Black, Thickness[0.003], Dotted]}]
```

```
(EXTRCNTRBLft11[r, D1, D2, a, b]) /. {D1 → OCTDst[[3]], D2 → OCTDst[[4]]}
```

```
Simplify[
 Solve[{Simplify[Limit[(ApprxOcthCF11NotMatchDD[r] + (EXTRCNTRBLft11[r, D1, D2, a, b]) /.
          {D1 → OCTDst[[3]], D2 → OCTDst[[4]]}), r → OCTDst[[3]], Direction → -1] -
       Limit[OcthCFApprxFnl11CC[r], r → OCTDst[[3]], Direction → 1]] == 0 &&
    Simplify[Limit[D[(ApprxOcthCF11NotMatchDD[r] + (EXTRCNTRBLft11[r, D1, D2, a, b]) /.
           {D1 → OCTDst[[3]], D2 → OCTDst[[4]]}), r], r → OCTDst[[3]], Direction → -1] -
       Limit[D[OcthCFApprxFnl11CC[r], r], r → OCTDst[[3]], Direction → 1]] == 0}, {a, b}]]
```

```
Simplify[((ApprxOcthCF11NotMatchDD[r] + (EXTRCNTRBLft11[r, D1, D2, a, b]) /.
    {D1 → OCTDst[[3]], D2 → OCTDst[[4]]})) /.
```

$$\Big\{\Big\{a \to \Big(20\,\Big(\Big(-279\,008\,604 + 197\,165\,493\sqrt{2} + 146\,006\,784\sqrt{3} - 103\,224\,912\sqrt{6} - 17\,208\,990\sqrt{1154 - 816\sqrt{2}} - 26\,610\,048\sqrt{577 - 408\sqrt{2}} + 21\,576\,888\sqrt{3\left(577 - 408\sqrt{2}\right)} + 10\,156\,455\sqrt{6\left(577 - 408\sqrt{2}\right)} + 6\,482\,098\times 2^{1/4}\sqrt{-24 + 17\sqrt{2}} - 4\,644\,288\times 2^{3/4}\sqrt{-24 + 17\sqrt{2}} - 20\,520\,546\times 2^{1/4}\sqrt{-72 + 51\sqrt{2}} + 14\,528\,352\times 2^{3/4}\sqrt{-72 + 51\sqrt{2}}\Big)\pi - 54\,\Big(1\,244\,880 - 881\,790\sqrt{2} + 4\,128\,096\sqrt{3} - 2\,922\,066\sqrt{6} + 147\,264\sqrt{3\left(577 - 408\sqrt{2}\right)} + 292\,032\sqrt{6\left(577 - 408\sqrt{2}\right)} - 63\,063\times 2^{1/4}\sqrt{-24 + 17\sqrt{2}} + 40\,326\times 2^{3/4}\sqrt{-24 + 17\sqrt{2}} - 436\,264\times 2^{1/4}\sqrt{-72 + 51\sqrt{2}} + 311\,490\times 2^{3/4}\sqrt{-72 + 51\sqrt{2}} + 1\,587\,222\,\mathrm{ArcSec}[3] - 1\,120\,392\sqrt{2}\,\mathrm{ArcSec}[3] + 668\,304\sqrt{3}\,\mathrm{ArcSec}[3] - 471\,744\sqrt{6}\,\mathrm{ArcSec}[3] - 133\,848\sqrt{3\left(577 - 408\sqrt{2}\right)}\,\mathrm{ArcSec}[3] - 20\,592\sqrt{6\left(577 - 408\sqrt{2}\right)}\,\mathrm{ArcSec}[3] - 24\,024\times 2^{1/4}\sqrt{-24 + 17\sqrt{2}}\,\mathrm{ArcSec}[3] + 18\,018\times 2^{3/4}\sqrt{-24 + 17\sqrt{2}}\,\mathrm{ArcSec}[3] - 219\,648\times 2^{1/4}\sqrt{-72 + 51\sqrt{2}}\,\mathrm{ArcSec}[3] + 154\,440\times 2^{3/4}\sqrt{-72 + 51\sqrt{2}}\,\mathrm{ArcSec}[3] - 264\sqrt{577 - 408\sqrt{2}}\,(-1087 + 234\,\mathrm{ArcSec}[3]) - 3\sqrt{1154 - 816\sqrt{2}}\,(50\,977 + 27\,456\,\mathrm{ArcSec}[3])\Big)\Big)\Big) \Big/ \left(11\,583\left(-4 + 3\sqrt{2}\right)^{7/2}\sqrt{-140 + 99\sqrt{2}}\,\pi\right),$$

$$b \to -\Big(280\,\Big(\Big(-23\,383\,746 + 16\,474\,626\sqrt{2} + 12\,223\,392\sqrt{3} - 8\,634\,720\sqrt{6} - 8\,494\,452\sqrt{1154 - 816\sqrt{2}} - 12\,152\,646\sqrt{577 - 408\sqrt{2}} + 8\,656\,551\sqrt{3\left(577 - 408\sqrt{2}\right)} + 5\,945\,346\sqrt{6\left(577 - 408\sqrt{2}\right)} + 505\,232\times 2^{1/4}\sqrt{-24 + 17\sqrt{2}} - 386\,033\times 2^{3/4}\sqrt{-24 + 17\sqrt{2}} - 559\,416\times 2^{1/4}\sqrt{-72 + 51\sqrt{2}} + 404\,535\times 2^{3/4}\sqrt{-72 + 51\sqrt{2}}\Big)\pi -$$

$$27\left(207480-148200\sqrt{2}+689160\sqrt{3}-490296\sqrt{6}+254592\sqrt{3\left(577-408\sqrt{2}\right)}+\right.$$
$$197184\sqrt{6\left(577-408\sqrt{2}\right)}-13728\times 2^{1/4}\sqrt{-24+17\sqrt{2}}+$$
$$5577\times 2^{3/4}\sqrt{-24+17\sqrt{2}}-24232\times 2^{1/4}\sqrt{-72+51\sqrt{2}}+$$
$$20008\times 2^{3/4}\sqrt{-72+51\sqrt{2}}+\sqrt{577-408\sqrt{2}}\ (67266-82368\,\text{ArcSec}[3])+$$
$$\sqrt{1154-816\sqrt{2}}\ (1644-61776\,\text{ArcSec}[3])+266760\,\text{ArcSec}[3]-$$
$$186732\sqrt{2}\,\text{ArcSec}[3]+112320\sqrt{3}\,\text{ArcSec}[3]-78624\sqrt{6}\,\text{ArcSec}[3]-82368$$
$$\sqrt{3\left(577-408\sqrt{2}\right)}\,\text{ArcSec}[3]-51480\sqrt{6\left(577-408\sqrt{2}\right)}\,\text{ArcSec}[3]-3432\times$$
$$2^{1/4}\sqrt{-24+17\sqrt{2}}\,\text{ArcSec}[3]+3432\times 2^{3/4}\sqrt{-24+17\sqrt{2}}\,\text{ArcSec}[3]-20592\times$$
$$\left.2^{1/4}\sqrt{-72+51\sqrt{2}}\,\text{ArcSec}[3]+13728\times 2^{3/4}\sqrt{-72+51\sqrt{2}}\,\text{ArcSec}[3]\right)\Bigg)\Bigg)\Bigg)\Big/$$
$$\left(11583\left(-4+3\sqrt{2}\right)^{7/2}\sqrt{-140+99\sqrt{2}}\ \pi\right)\}\}\text{, Assumptions} \to \{\text{OCTDst}[[$$
$$3]] < r < \text{OCTDst}[[$$
$$4]]\}\Big]$$

```
OcthCFApprxFnl11DD[r_] :=
```

$$-\frac{1}{1287\left(2-\sqrt{2}\right)^{7/2}\pi}\left(3\left(4582\sqrt{4-2\sqrt{2}}-6208\sqrt{2-\sqrt{2}}+7796\sqrt{-\sqrt{2}+2r}+\right.\right.$$
$$2\left(1947+5018\sqrt{2}\right)r^4\sqrt{-\sqrt{2}+2r}-2\left(1092+1109\sqrt{2}\right)r^5\sqrt{-\sqrt{2}+2r}+$$
$$540r^6\sqrt{-\sqrt{2}+2r}-4160\sqrt{-2\sqrt{2}+4r}+$$
$$r\left(7943\sqrt{4-2\sqrt{2}}-11934\sqrt{2-\sqrt{2}}+\left(21580-23103\sqrt{2}\right)\sqrt{-\sqrt{2}+2r}\right)+$$
$$2r^2\left(17160\sqrt{4-2\sqrt{2}}-24024\sqrt{2-\sqrt{2}}+\left(22885-7137\sqrt{2}\right)\sqrt{-\sqrt{2}+2r}\right)+$$
$$\left.r^3\left(3003\sqrt{4-2\sqrt{2}}-4290\sqrt{2-\sqrt{2}}-\left(6916+16129\sqrt{2}\right)\sqrt{-\sqrt{2}+2r}\right)\right)+$$
$$\pi\left(2\left(1611+3276\sqrt{2}-39540\sqrt{3}-30680\sqrt{6}\right)r^4\sqrt{-\sqrt{2}+2r}+\right.$$
$$2\left(-7488-621\sqrt{2}+30576\sqrt{3}+6092\sqrt{6}\right)r^5\sqrt{-\sqrt{2}+2r}-$$

$$540\left(-9+28\sqrt{3}\right)r^6\sqrt{-\sqrt{2}+2r}+r\left(-2756\sqrt{12-6\sqrt{2}}+4680\sqrt{6-3\sqrt{2}}+\right.$$
$$4563\sqrt{4-2\sqrt{2}}-4446\sqrt{2-\sqrt{2}}+43056\sqrt{-\sqrt{2}+2r}+62820\sqrt{6}\sqrt{-\sqrt{2}+2r}-$$
$$\left.14643\sqrt{-2\sqrt{2}+4r}-55120\sqrt{-3\sqrt{2}+6r}\right)+r^3\left(-84084\sqrt{12-6\sqrt{2}}+\right.$$
$$120120\sqrt{6-3\sqrt{2}}+27027\sqrt{4-2\sqrt{2}}-38610\sqrt{2-\sqrt{2}}+48672\sqrt{-\sqrt{2}+2r}+$$
$$\left.147100\sqrt{6}\sqrt{-\sqrt{2}+2r}-20673\sqrt{-2\sqrt{2}+4r}-26000\sqrt{-3\sqrt{2}+6r}\right)+$$
$$2\left(-6740\sqrt{12-6\sqrt{2}}+9536\sqrt{6-3\sqrt{2}}+2718\sqrt{4-2\sqrt{2}}-4302\sqrt{2-\sqrt{2}}+\right.$$
$$1854\sqrt{-\sqrt{2}+2r}+3328\sqrt{6}\sqrt{-\sqrt{2}+2r}-2925\sqrt{-2\sqrt{2}+4r}-$$
$$\left.6808\sqrt{-3\sqrt{2}+6r}\right)-2r^2\left(68640\sqrt{12-6\sqrt{2}}-96096\sqrt{6-3\sqrt{2}}-\right.$$
$$27027\sqrt{4-2\sqrt{2}}+38610\sqrt{2-\sqrt{2}}-20169\sqrt{-\sqrt{2}+2r}-$$
$$\left.\left.\left.35100\sqrt{6}\sqrt{-\sqrt{2}+2r}+27612\sqrt{-2\sqrt{2}+4r}+104140\sqrt{-3\sqrt{2}+6r}\right)\right)\right)+$$
$$\left((-1+r)^4\left(\frac{1}{2}\left(-100+77\sqrt{2}+6\left(-27+14\sqrt{2}\right)r+12\left(-4+7\sqrt{2}\right)r^2-40r^3\right)\right.\right.$$
$$\left(\left(\left(-23383746+16474626\sqrt{2}+12223392\sqrt{3}-8634720\sqrt{6}-\right.\right.\right.$$
$$8494452\sqrt{1154-816\sqrt{2}}-12152646\sqrt{577-408\sqrt{2}}+$$
$$8656551\sqrt{3\left(577-408\sqrt{2}\right)}+5945346\sqrt{6\left(577-408\sqrt{2}\right)}+$$
$$505232\times 2^{1/4}\sqrt{-24+17\sqrt{2}}-386033\times 2^{3/4}\sqrt{-24+17\sqrt{2}}-$$
$$\left.559416\times 2^{1/4}\sqrt{-72+51\sqrt{2}}+404535\times 2^{3/4}\sqrt{-72+51\sqrt{2}}\right)\pi-$$
$$27\left(207480-148200\sqrt{2}+689160\sqrt{3}-490296\sqrt{6}+254592\sqrt{3\left(577-408\sqrt{2}\right)}+\right.$$
$$197184\sqrt{6\left(577-408\sqrt{2}\right)}-13728\times 2^{1/4}\sqrt{-24+17\sqrt{2}}+5577\times 2^{3/4}$$
$$\sqrt{-24+17\sqrt{2}}-24232\times 2^{1/4}\sqrt{-72+51\sqrt{2}}+20008\times 2^{3/4}\sqrt{-72+51\sqrt{2}}+$$
$$\sqrt{577-408\sqrt{2}}\left(67266-82368\,\mathrm{ArcSec}[3]\right)+\sqrt{1154-816\sqrt{2}}\left(1644-\right.$$
$$\left.61776\,\mathrm{ArcSec}[3]\right)+266760\,\mathrm{ArcSec}[3]-186732\sqrt{2}\,\mathrm{ArcSec}[3]+112320\sqrt{3}$$

$$\text{ArcSec}[3] - 78624\sqrt{6}\,\text{ArcSec}[3] - 82368\sqrt{3\left(577-408\sqrt{2}\right)}\,\text{ArcSec}[3] - 51480\sqrt{6\left(577-408\sqrt{2}\right)}\,\text{ArcSec}[3] - 3432\times 2^{1/4}\sqrt{-24+17\sqrt{2}}\,\text{ArcSec}[3] + 3432\times 2^{3/4}\sqrt{-24+17\sqrt{2}}\,\text{ArcSec}[3] - 20592\times 2^{1/4}\sqrt{-72+51\sqrt{2}}\,\text{ArcSec}[3] + 13728\times 2^{3/4}\sqrt{-72+51\sqrt{2}}\,\text{ArcSec}[3]\Big)\Big) - \frac{1}{2}\left(-7+4\sqrt{2}+\left(-4+6\sqrt{2}\right)r - 4r^2\right)$$

$$\Big(\Big(-279008604 + 197165493\sqrt{2} + 146006784\sqrt{3} - 103224912\sqrt{6} - 17208990\sqrt{1154-816\sqrt{2}} - 26610048\sqrt{577-408\sqrt{2}} + 21576888\sqrt{3\left(577-408\sqrt{2}\right)} + 10156455\sqrt{6\left(577-408\sqrt{2}\right)} + 6482098\times 2^{1/4}\sqrt{-24+17\sqrt{2}} - 4644288\times 2^{3/4}\sqrt{-24+17\sqrt{2}} - 20520546\times 2^{1/4}\sqrt{-72+51\sqrt{2}} + 14528352\times 2^{3/4}\sqrt{-72+51\sqrt{2}}\Big)\pi -$$

$$54\Big(1244880 - 881790\sqrt{2} + 4128096\sqrt{3} - 2922066\sqrt{6} + 147264\sqrt{3\left(577-408\sqrt{2}\right)} + 292032\sqrt{6\left(577-408\sqrt{2}\right)} - 63063\times 2^{1/4}\sqrt{-24+17\sqrt{2}} + 40326\times 2^{3/4}\sqrt{-24+17\sqrt{2}} - 436264\times 2^{1/4}\sqrt{-72+51\sqrt{2}} + 311490\times 2^{3/4}\sqrt{-72+51\sqrt{2}} + 1587222\,\text{ArcSec}[3] - 1120392\sqrt{2}\,\text{ArcSec}[3] + 668304\sqrt{3}\,\text{ArcSec}[3] - 471744\sqrt{6}\,\text{ArcSec}[3] - 133848\sqrt{3\left(577-408\sqrt{2}\right)}\,\text{ArcSec}[3] - 20592\sqrt{6\left(577-408\sqrt{2}\right)}\,\text{ArcSec}[3] - 24024\times 2^{1/4}\sqrt{-24+17\sqrt{2}}\,\text{ArcSec}[3] + 18018\times 2^{3/4}\sqrt{-24+17\sqrt{2}}\,\text{ArcSec}[3] - 219648\times 2^{1/4}\sqrt{-72+51\sqrt{2}}\,\text{ArcSec}[3] + 154440\times 2^{3/4}\sqrt{-72+51\sqrt{2}}\,\text{ArcSec}[3] - 264\sqrt{577-408\sqrt{2}}\,(-1087+234\,\text{ArcSec}[3]) - 3\sqrt{1154-816\sqrt{2}}\,(50977+27456\,\text{ArcSec}[3])\Big)\Big)\Big)\Big)\Big/\left(34749\left(-4+3\sqrt{2}\right)^{7/2}\sqrt{-140+99\sqrt{2}}\,\pi\right);$$

In[24]:=
```
fig11 = Plot[{If[r < OCTDst[[1]], OCTCFAA[r],
    If[r < OCTDst[[2]], OCTCFBB[r], If[r < OCTDst[[3]], OCTCFCC[r], OCTCFDD[r]]]],
   If[r < OCTDst[[1]], OcthCFApprxFnl11AA[r], If[r < OCTDst[[2]], OcthCFApprxFnl11BB[r],
     If[r < OCTDst[[3]], OcthCFApprxFnl11CC[r], OcthCFApprxFnl11DD[r]]]],
   10 * If[r < OCTDst[[1]], (OCTCFAA[r] - OcthCFApprxFnl11AA[r]),
     If[r < OCTDst[[2]], (OCTCFBB[r] - OcthCFApprxFnl11BB[r]), If[r < OCTDst[[3]],
       (OCTCFCC[r] - OcthCFApprxFnl11CC[r]), (OCTCFDD[r] - OcthCFApprxFnl11DD[r])]]]},
  {r, 0, 1}, PlotStyle → {Directive[Blue, Thickness[0.005]],
    Directive[Red, Thickness[0.005], Dashed], Directive[Black, Thickness[0.003], Dotted]},
  AxesLabel → {"r", " γ0(r), 10xΔγ"}]
```

In[22]:=
```
Plot[{If[r < OCTDst[[1]], OCTDDCFAA[r],
    If[r < OCTDst[[2]], OCTDDCFBB[r], If[r < OCTDst[[3]], OCTDDCFCC[r], OCTDDCFDD[r]]]],
   If[r < OCTDst[[1]], OcthCLDApprx11AA[r], If[r < OCTDst[[2]], OcthCLDApprx11BB[r],
     If[r < OCTDst[[3]], OcthCLDApprx11CC[r], OcthCLDApprx11DD[r]]]]
    (* ,If[r<OCTDst[[1]],(OCTDDCFAA[r]-OcthCLDApprx11AA[r]),
    If[r<OCTDst[[2]],(OCTDDCFBB[r]-OcthCLDApprx11BB[r]),
     If[r<OCTDst[[3]],(OCTDDCFCC[r]-OcthCLDApprx11CC[r]),
      (OCTDDCFDD[r]-OcthCLDApprx11DD[r])]]] *) }, {r, -0, 1},
  PlotRange → {{0, 1}, {-1.2, 10}}, PlotStyle → {Directive[Blue, Thickness[0.003]],
    Directive[Red, Thickness[0.005], Dashed], Directive[Black, Thickness[0.003], Dotted]},
  AxesLabel → {"r", " γ0"(r), Δγ""}]
```

The final formulae of the tetrahedron-CF approximation for the case K=1 and with the BB procedure B are

In[23]:=
```
Plot[{If[r < OCTDst[[1]], OCTDDCFAA[r],
    If[r < OCTDst[[2]], OCTDDCFBB[r], If[r < OCTDst[[3]], OCTDDCFCC[r], OCTDDCFDD[r]]]],
   If[r < OCTDst[[1]], OcthCLDApprx11AA[r], If[r < OCTDst[[2]], OcthCLDApprx11BB[r],
     If[r < OCTDst[[3]], OcthCLDApprx11CC[r], OcthCLDApprx11DD[r]]]]
    (* ,If[r<OCTDst[[1]],(OCTDDCFAA[r]-OcthCLDApprx11AA[r]),
    If[r<OCTDst[[2]],(OCTDDCFBB[r]-OcthCLDApprx11BB[r]),
     If[r<OCTDst[[3]],(OCTDDCFCC[r]-OcthCLDApprx11CC[r]),
      (OCTDDCFDD[r]-OcthCLDApprx11DD[r])]]] *) }, {r, -0, 1},
  PlotRange → {{0.5, 1}, {-1.2, 10}}, PlotStyle → {Directive[Blue, Thickness[0.003]],
    Directive[Red, Thickness[0.005], Dashed], Directive[Black, Thickness[0.003], Dotted]},
  AxesLabel → {"r", " γ0"(r), Δγ""}]
```

FINAL FORMULAE OF THE CF AND CLD APPROXIMATION IN THE CASE K = 1

In[20]:=

$$\text{OcthCFApprxFnl11AA}[r\_] := 1 - \frac{3\sqrt{3}\,r}{2} + \frac{6\sqrt{2}\,r^2}{\pi} + \frac{3\,r^3}{2} - \frac{\sqrt{3}\,r^3}{2} - \frac{3\,r^3}{2\,\pi} - \frac{3\,r^2\,\text{ArcSec}[3]}{\pi};$$

$$\text{OcthCFApprxFnl11BB}[r\_] :=$$

$$\frac{1}{7128\sqrt{-24+17\sqrt{2}}\,\pi}\, r \Bigg(\Bigg(10\,692\sqrt{-24+17\sqrt{2}} - 14\,256\sqrt{-72+51\sqrt{2}} + 2^{1/4}\left(2673 - 1782\sqrt{2} - 146\,267\sqrt{3} + 103\,442\sqrt{6}\right)\Bigg)\pi - 297\Bigg(9\times 2^{1/4} - 6\times 2^{3/4} - 544\times 2^{1/4}\sqrt{3} + 384\times 2^{3/4}\sqrt{3} + 36\sqrt{-24+17\sqrt{2}} - 96\sqrt{6\left(-24+17\sqrt{2}\right)} - 192\times 2^{1/4}\sqrt{3}\,\text{ArcSec}[3] + 136\times 2^{3/4}\sqrt{3}\,\text{ArcSec}[3] + 48\sqrt{-72+51\sqrt{2}}\,\text{ArcSec}[3]\Bigg)\Bigg) +$$

$$\Big(\Big(-42\,335\,271\,2^{1/4} + 29\,935\,555\times 2^{3/4} - 69\,160\,392\times 2^{1/4}\sqrt{3} + 48\,903\,777\times 2^{3/4}\sqrt{3} -$$

$$25\,945\,920\sqrt{-24+17\sqrt{2}}-4\,586\,868\sqrt{6\left(-24+17\sqrt{2}\right)}+$$
$$18\,347\,472\sqrt{-48+34\sqrt{2}}+6\,486\,480\sqrt{-72+51\sqrt{2}}\Bigg)\pi-$$
$$27\Big(990\,132\times 2^{1/4}-700\,128\times 2^{3/4}-2\,561\,496\times 2^{1/4}\sqrt{3}+1\,811\,251\times 2^{3/4}\sqrt{3}-$$
$$169\,884\sqrt{6\left(-24+17\sqrt{2}\right)}+240\,240\sqrt{-72+51\sqrt{2}}+350\,064\times 2^{1/4}\,\mathrm{ArcSec}[3]-$$
$$247\,533\times 2^{3/4}\,\mathrm{ArcSec}[3]+20\,592\sqrt{-24+17\sqrt{2}}\,(99+35\,\mathrm{ArcSec}[3])-$$
$$5148\sqrt{-48+34\sqrt{2}}\,(280+99\,\mathrm{ArcSec}[3])\Big)\Bigg)\Bigg/\left(69\,498\sqrt{3-2\sqrt{2}}\left(-4+3\sqrt{2}\right)^{7/2}\pi\right)+$$
$$\Bigg(8\times 3^{1/4}\left(3+\left(-3+19\sqrt{3}\right)\pi\right)\Big(-20\,736\left(176+273\sqrt{2}\right)r^5\sqrt{-1+\sqrt{3}\,r}+$$
$$2880\left(1303+1560\sqrt{2}\right)r^4\sqrt{-3+3\sqrt{3}\,r}+1\,866\,240\,r^6\sqrt{-3+3\sqrt{3}\,r}+$$
$$\sqrt{3}\left(2^{1/4}\sqrt{3-2\sqrt{2}}\left(-239\,095+169\,083\sqrt{2}\right)+64\left(8867-5304\sqrt{2}\right)\sqrt{-1+\sqrt{3}\,r}\right)+$$
$$216\sqrt{3}\,r^2\left(143\times 2^{1/4}\sqrt{3-2\sqrt{2}}\left(-140+99\sqrt{2}\right)+80\left(505-208\sqrt{2}\right)\sqrt{-1+\sqrt{3}\,r}\right)-$$
$$72\,r^3\left(429\times 2^{1/4}\sqrt{3-2\sqrt{2}}\left(-140+99\sqrt{2}\right)+160\left(1327-78\sqrt{2}\right)\sqrt{-1+\sqrt{3}\,r}\right)-$$
$$3\,r\left(39\times 2^{1/4}\sqrt{3-2\sqrt{2}}\left(-13\,180+9321\sqrt{2}\right)-1280\left(-1663+897\sqrt{2}\right)\sqrt{-1+\sqrt{3}\,r}\right)\Big)-$$
$$936\Big(6^{1/4}\left(9964-7047\sqrt{2}\right)\sqrt{3-2\sqrt{2}}+192\times 3^{3/4}\left(257+132\sqrt{2}\right)r^3\sqrt{-1+\sqrt{3}\,r}-$$
$$6912\times 3^{1/4}\left(2+11\sqrt{2}\right)r^4\sqrt{-1+\sqrt{3}\,r}+23\,040\times 3^{3/4}\,r^5\sqrt{-1+\sqrt{3}\,r}+$$
$$64\left(264\sqrt{-2\sqrt{3}+6\,r}-425\times 3^{1/4}\sqrt{-1+\sqrt{3}\,r}\right)-$$
$$24\times 3^{1/4}\,r^2\left(33\times 2^{1/4}\sqrt{3-2\sqrt{2}}\left(-140+99\sqrt{2}\right)+8\left(1027-396\sqrt{2}\right)\sqrt{-1+\sqrt{3}\,r}\right)+$$
$$8\times 3^{3/4}\,r\left(11\times 2^{1/4}\sqrt{3-2\sqrt{2}}\left(-263+186\sqrt{2}\right)+24\left(393-220\sqrt{2}\right)\sqrt{-1+\sqrt{3}\,r}\right)\Big)$$
$$\left(4\sqrt{2}-\sqrt{3}+\pi+\sqrt{3}\,\pi-2\,\mathrm{ArcSec}[3]\right)+\left(\sqrt{6}-4\,r\right)^2$$
$$\Big(2^{1/4}\left(81+\left(-81+445\sqrt{3}\right)\pi\right)\left(\sqrt{6}-4\,r\right)$$

$$\left(14157\times 3^{1/4}\sqrt{6-4\sqrt{2}}-20020\times 3^{1/4}\sqrt{3-2\sqrt{2}}+10608\sqrt{3}\sqrt{\sqrt{3}-2\sqrt{2}\,r}-\right.$$

$$8456\sqrt{6}\sqrt{\sqrt{3}-2\sqrt{2}\,r}+16\left(1493-507\sqrt{2}\right)r\sqrt{\sqrt{3}-2\sqrt{2}\,r}+$$

$$\left.24\sqrt{3}\left(-728+141\sqrt{2}\right)r^2\sqrt{\sqrt{3}-2\sqrt{2}\,r}+12960\,r^3\sqrt{\sqrt{3}-2\sqrt{2}\,r}\right)+$$

$$39\times 2^{1/4}\left(1089\times 3^{1/4}\sqrt{6-4\sqrt{2}}-1540\times 3^{1/4}\sqrt{3-2\sqrt{2}}+528\sqrt{3}\sqrt{\sqrt{3}-2\sqrt{2}\,r}-\right.$$

$$408\sqrt{6}\sqrt{\sqrt{3}-2\sqrt{2}\,r}-48\left(-25+11\sqrt{2}\right)r\sqrt{\sqrt{3}-2\sqrt{2}\,r}+$$

$$\left.8\sqrt{3}\left(-88+21\sqrt{2}\right)r^2\sqrt{\sqrt{3}-2\sqrt{2}\,r}+480\,r^3\sqrt{\sqrt{3}-2\sqrt{2}\,r}\right)$$

$$\left.\left.\left.\left(\left(432-311\sqrt{2}+81\sqrt{6}\right)\pi-27\left(-16\sqrt{2}+3\sqrt{6}+8\,\text{ArcSec}[3]\right)\right)\right)\right)\right)\Big/$$

$$\left(247104\times 3^{1/4}\left(-4+3\sqrt{2}\right)^{7/2}\pi\right);\ \text{OcthCFApprxFnl11CC}[$$

$$\text{r\_}]:=\frac{1}{14256\sqrt{-24+17\sqrt{2}}\,\pi}$$

$$r$$

$$\left(\left(\left(5346\times 2^{1/4}-3564\times 2^{3/4}-292534\times 2^{1/4}\sqrt{3}+206884\times 2^{3/4}\sqrt{3}+44178\sqrt{-24+17\sqrt{2}}-\right.\right.\right.$$

$$\left.21384\sqrt{6\left(-24+17\sqrt{2}\right)}+42768\sqrt{-48+34\sqrt{2}}-44091\sqrt{-72+51\sqrt{2}}\right)\pi-$$

$$54\left(-175\sqrt{-24+17\sqrt{2}}+16\sqrt{-72+51\sqrt{2}}\,(20+33\,\text{ArcSec}[3])+\right.$$

$$11\times 2^{1/4}\left(9-6\sqrt{2}-544\sqrt{3}+384\sqrt{6}+8\left(-24\sqrt{3}+17\sqrt{6}+3\times 2^{1/4}\sqrt{-24+17\sqrt{2}}\right)\right.$$

$$\left.\left.\left.\text{ArcSec}[3]-12\times 2^{1/4}\sqrt{-72+51\sqrt{2}}\,(8+\text{ArcSec}[3])\right)\right)\right)+$$

$$\left(-\pi\left(8640\,r^6\left(-81\sqrt{2}\sqrt{\sqrt{2}-2r}+252\sqrt{6}\sqrt{\sqrt{2}-2r}+81\sqrt{-\sqrt{6}+4r}-\right.\right.\right.$$

$$\left.445\sqrt{3}\sqrt{-\sqrt{6}+4r}\right)+8\,r^4\left(50544\sqrt{\sqrt{2}-2r}+1517643\sqrt{2}\sqrt{\sqrt{2}-2r}+\right.$$

$$185328\sqrt{3}\sqrt{\sqrt{2}-2r}+450936\sqrt{6}\sqrt{\sqrt{2}-2r}-2731665\sqrt{-\sqrt{6}+4r}-$$

$$\left.1178059\sqrt{3}\sqrt{-\sqrt{6}+4r}-25272\sqrt{6}\sqrt{-\sqrt{6}+4r}-370656\sqrt{-2\sqrt{6}+8r}\right)-$$

$$48 r^5 \left(267732 \sqrt{\sqrt{2}-2 r}+14040 \sqrt{2} \sqrt{\sqrt{2}-2 r}+43620 \sqrt{3} \sqrt{\sqrt{2}-2 r}-28080 \sqrt{-\sqrt{6}+4 r}-160117 \sqrt{6} \sqrt{-\sqrt{6}+4 r}-67781 \sqrt{-2 \sqrt{6}+8 r}\right)+$$

$$36 r^2 \left(429\times 2^{1/4} \left(448+405 \sqrt{2}\right) \sqrt{6-3 \sqrt{3}}-286\times 2^{1/4} \left(1168+1053 \sqrt{2}\right) \sqrt{2-\sqrt{3}}+170820 \sqrt{\sqrt{2}-2 r}-439695 \sqrt{2} \sqrt{\sqrt{2}-2 r}-61776 \sqrt{3} \sqrt{\sqrt{2}-2 r}+371048 \sqrt{6} \sqrt{\sqrt{2}-2 r}+576252 \sqrt{-\sqrt{6}+4 r}-588238 \sqrt{3} \sqrt{-\sqrt{6}+4 r}-104598 \sqrt{6} \sqrt{-\sqrt{6}+4 r}+92664 \sqrt{-2 \sqrt{6}+8 r}\right)+$$

$$6 r^3 \left(2938936\times 2^{3/4} \sqrt{6-3 \sqrt{3}}-5087511\times 2^{3/4} \sqrt{2-\sqrt{3}}+1573092 \sqrt{\sqrt{2}-2 r}-221832 \sqrt{2} \sqrt{\sqrt{2}-2 r}-3012288 \sqrt{3} \sqrt{\sqrt{2}-2 r}-123552 \sqrt{6} \sqrt{\sqrt{2}-2 r}+657072 \sqrt{-\sqrt{6}+4 r}+247104 \sqrt{3} \sqrt{-\sqrt{6}+4 r}-603071 \sqrt{6} \sqrt{-\sqrt{6}+4 r}+4932385 \sqrt{-2 \sqrt{6}+8 r}\right)+2 \left(2^{1/4} \left(612607+729729 \sqrt{2}\right) \sqrt{6-3 \sqrt{3}}-6\times 2^{1/4} \left(177200+210951 \sqrt{2}\right) \sqrt{2-\sqrt{3}}+9 \left(76284 \sqrt{\sqrt{2}-2 r}-52908 \sqrt{2} \sqrt{\sqrt{2}-2 r}-41184 \sqrt{3} \sqrt{\sqrt{2}-2 r}+36752 \sqrt{6} \sqrt{\sqrt{2}-2 r}+64365 \sqrt{-\sqrt{6}+4 r}-51001 \sqrt{3} \sqrt{-\sqrt{6}+4 r}-31590 \sqrt{6} \sqrt{-\sqrt{6}+4 r}+46332 \sqrt{-2 \sqrt{6}+8 r}\right)\right)+$$

$$3 r \left(-13\, 2^{1/4} \left(-95040+194441 \sqrt{2}\right) \sqrt{2-\sqrt{3}}+3 \left(26\times 2^{1/4} \left(-9108+18751 \sqrt{2}\right) \sqrt{6-3 \sqrt{3}}+1147956 \sqrt{\sqrt{2}-2 r}-412776 \sqrt{2} \sqrt{\sqrt{2}-2 r}-829840 \sqrt{3} \sqrt{\sqrt{2}-2 r}+205920 \sqrt{6} \sqrt{\sqrt{2}-2 r}+707616 \sqrt{-\sqrt{6}+4 r}-308880 \sqrt{3} \sqrt{-\sqrt{6}+4 r}-406487 \sqrt{6} \sqrt{-\sqrt{6}+4 r}+1024357 \sqrt{-2 \sqrt{6}+8 r}\right)\right)\Bigg)+$$

$$27 \Bigg(8640 r^6 \left(\sqrt{2} \sqrt{\sqrt{2}-2 r}+3 \sqrt{-\sqrt{6}+4 r}\right)-16 r^5 \Big(4436 \sqrt{\sqrt{2}-2 r}+1092 \sqrt{3} \sqrt{\sqrt{2}-2 r}+207 \sqrt{6} \sqrt{-\sqrt{6}+4 r}+6396 \sqrt{-2 \sqrt{6}+8 r}-1560 \sqrt{-\sqrt{6}+4 r}\ \mathrm{ArcSec}[3]\Big)+r \Big(-347476 \sqrt{\sqrt{2}-2 r}+172640 \sqrt{3} \sqrt{\sqrt{2}-2 r}+115101 \sqrt{6} \sqrt{-\sqrt{6}+4 r}-253422 \sqrt{-2 \sqrt{6}+8 r}+78\times 2^{1/4} \sqrt{2-\sqrt{3}} \left(6077 \sqrt{2}-836\, \mathrm{ArcSec}[3]\right)-9438\times 2^{1/4} \sqrt{6-3 \sqrt{3}} \left(29 \sqrt{2}-4\, \mathrm{ArcSec}[3]\right)+$$

$$117936\sqrt{-\sqrt{6}+4r}\,\text{ArcSec}[3]-51480\sqrt{3}\sqrt{-\sqrt{6}+4r}\,\text{ArcSec}[3]\Big)+$$

$$2r^3\Big(12870\times 2^{3/4}\sqrt{6-3\sqrt{3}}-22308\times 2^{3/4}\sqrt{2-\sqrt{3}}-89324\sqrt{\sqrt{2}-2r}-$$
$$27664\sqrt{3}\sqrt{\sqrt{2}-2r}-48987\sqrt{6}\sqrt{-\sqrt{6}+4r}-52884\sqrt{-2\sqrt{6}+8r}+$$
$$36504\sqrt{-\sqrt{6}+4r}\,\text{ArcSec}[3]+13728\sqrt{3}\sqrt{-\sqrt{6}+4r}\,\text{ArcSec}[3]\Big)+$$

$$8r^4\Big(4213\sqrt{2}\sqrt{\sqrt{2}-2r}+10036\sqrt{6}\sqrt{\sqrt{2}-2r}+35073\sqrt{-\sqrt{6}+4r}+$$
$$4524\sqrt{3}\sqrt{-\sqrt{6}+4r}-156\left(44+3\sqrt{3}\right)\sqrt{-2\sqrt{6}+8r}\,\text{ArcSec}[3]\Big)+$$

$$4r^2\Big(77975\sqrt{2}\sqrt{\sqrt{2}-2r}-28548\sqrt{6}\sqrt{\sqrt{2}-2r}-56619\sqrt{-\sqrt{6}+4r}+$$
$$67275\sqrt{3}\sqrt{-\sqrt{6}+4r}-17433\sqrt{6}\sqrt{-\sqrt{6}+4r}\,\text{ArcSec}[3]+$$
$$15444\sqrt{-2\sqrt{6}+8r}\,\text{ArcSec}[3]+6435\times 2^{1/4}\sqrt{6-3\sqrt{3}}\left(-4+3\sqrt{2}\,\text{ArcSec}[3]\right)-$$
$$11154\times 2^{1/4}\sqrt{2-\sqrt{3}}\left(-4+3\sqrt{2}\,\text{ArcSec}[3]\right)\Big)+$$

$$2\Big(30908\sqrt{2}\sqrt{\sqrt{2}-2r}-16640\sqrt{6}\sqrt{\sqrt{2}-2r}-36369\sqrt{-\sqrt{6}+4r}+$$
$$23868\sqrt{3}\sqrt{-\sqrt{6}+4r}-5265\sqrt{6}\sqrt{-\sqrt{6}+4r}\,\text{ArcSec}[3]+$$
$$7722\sqrt{-2\sqrt{6}+8r}\,\text{ArcSec}[3]+117\times 2^{1/4}\sqrt{6-3\sqrt{3}}\left(238+33\sqrt{2}\,\text{ArcSec}[3]\right)-$$
$$3\times 2^{1/4}\sqrt{2-\sqrt{3}}\left(16075+2236\sqrt{2}\,\text{ArcSec}[3]\right)\Big)\Big)\Big)\Big)\Big/$$

$$\left(46332\times 2^{3/4}\left(2-\sqrt{3}\right)^{7/2}\pi\right)+\Big(\Big(-164216241\sqrt{-24+17\sqrt{2}}+$$
$$4\Big(29030139\sqrt{-48+34\sqrt{2}}+21488868\sqrt{-72+51\sqrt{2}}+2^{1/4}\Big(-42335271+29935555$$
$$\sqrt{2}-69160392\sqrt{3}+48903777\sqrt{6}-15195000\times 2^{1/4}\sqrt{-72+51\sqrt{2}}\Big)\Big)\Big)\pi-$$
$$54\Big(66690\sqrt{-24+17\sqrt{2}}\,(11+14\,\text{ArcSec}[3])-741\sqrt{-48+34\sqrt{2}}\,(700+891\,\text{ArcSec}[3])+$$
$$2\Big(15\sqrt{-72+51\sqrt{2}}\,(81059+13104\,\text{ArcSec}[3])+$$
$$2^{1/4}\Big(990132-700128\sqrt{2}-2561496\sqrt{3}+1811251\sqrt{6}-429\left(-816+577\sqrt{2}\right)$$

```
ArcSec[3] - 486 × 2^(1/4) Sqrt[-72 + 51 Sqrt[2]] (1769 + 286 ArcSec[3])))))/
(277 992 Sqrt[3 - 2 Sqrt[2]] (-4 + 3 Sqrt[2])^(7/2) π); OcthCFApprxFnl11DD[
r_] := -1/(1287 (2 - Sqrt[2])^(7/2) π)
(3 (4582 Sqrt[4 - 2 Sqrt[2]] - 6208 Sqrt[2 - Sqrt[2]] + 7796 Sqrt[-Sqrt[2] + 2 r] +
    2 (1947 + 5018 Sqrt[2]) r^4 Sqrt[-Sqrt[2] + 2 r] - 2 (1092 + 1109 Sqrt[2]) r^5 Sqrt[-Sqrt[2] + 2 r] +
    540 r^6 Sqrt[-Sqrt[2] + 2 r] - 4160 Sqrt[-2 Sqrt[2] + 4 r] +
    r (7943 Sqrt[4 - 2 Sqrt[2]] - 11 934 Sqrt[2 - Sqrt[2]] + (21 580 - 23 103 Sqrt[2]) Sqrt[-Sqrt[2] + 2 r]) +
    2 r^2 (17 160 Sqrt[4 - 2 Sqrt[2]] - 24 024 Sqrt[2 - Sqrt[2]] + (22 885 - 7137 Sqrt[2]) Sqrt[-Sqrt[2] + 2 r]) +
    r^3 (3003 Sqrt[4 - 2 Sqrt[2]] - 4290 Sqrt[2 - Sqrt[2]] - (6916 + 16 129 Sqrt[2]) Sqrt[-Sqrt[2] + 2 r])) +
  π (2 (1611 + 3276 Sqrt[2] - 39 540 Sqrt[3] - 30 680 Sqrt[6]) r^4 Sqrt[-Sqrt[2] + 2 r] +
    2 (-7488 - 621 Sqrt[2] + 30 576 Sqrt[3] + 6092 Sqrt[6]) r^5 Sqrt[-Sqrt[2] + 2 r] -
    540 (-9 + 28 Sqrt[3]) r^6 Sqrt[-Sqrt[2] + 2 r] + r (-2756 Sqrt[12 - 6 Sqrt[2]] + 4680 Sqrt[6 - 3 Sqrt[2]] +
      4563 Sqrt[4 - 2 Sqrt[2]] - 4446 Sqrt[2 - Sqrt[2]] + 43 056 Sqrt[-Sqrt[2] + 2 r] + 62 820 Sqrt[6] Sqrt[-Sqrt[2] + 2 r] -
      14 643 Sqrt[-2 Sqrt[2] + 4 r] - 55 120 Sqrt[-3 Sqrt[2] + 6 r]) + r^3 (-84 084 Sqrt[12 - 6 Sqrt[2]] +
      120 120 Sqrt[6 - 3 Sqrt[2]] + 27 027 Sqrt[4 - 2 Sqrt[2]] - 38 610 Sqrt[2 - Sqrt[2]] + 48 672 Sqrt[-Sqrt[2] + 2 r] +
      147 100 Sqrt[6] Sqrt[-Sqrt[2] + 2 r] - 20 673 Sqrt[-2 Sqrt[2] + 4 r] - 26 000 Sqrt[-3 Sqrt[2] + 6 r]) +
    2 (-6740 Sqrt[12 - 6 Sqrt[2]] + 9536 Sqrt[6 - 3 Sqrt[2]] + 2718 Sqrt[4 - 2 Sqrt[2]] - 4302 Sqrt[2 - Sqrt[2]] +
      1854 Sqrt[-Sqrt[2] + 2 r] + 3328 Sqrt[6] Sqrt[-Sqrt[2] + 2 r] - 2925 Sqrt[-2 Sqrt[2] + 4 r] -
      6808 Sqrt[-3 Sqrt[2] + 6 r]) - 2 r^2 (68 640 Sqrt[12 - 6 Sqrt[2]] - 96 096 Sqrt[6 - 3 Sqrt[2]] -
      27 027 Sqrt[4 - 2 Sqrt[2]] + 38 610 Sqrt[2 - Sqrt[2]] - 20 169 Sqrt[-Sqrt[2] + 2 r] -
```

$$35100\sqrt{6}\sqrt{-\sqrt{2}+2r}+27612\sqrt{-2\sqrt{2}+4r}+104140\sqrt{-3\sqrt{2}+6r}\Big)\Big)\Big)+$$

$$\Big((-1+r)^4\Big(\frac{1}{2}\left(-100+77\sqrt{2}+6\left(-27+14\sqrt{2}\right)r+12\left(-4+7\sqrt{2}\right)r^2-40r^3\right)$$

$$\Big(\Big(-23383746+16474626\sqrt{2}+12223392\sqrt{3}-8634720\sqrt{6}-8494452\sqrt{1154-816\sqrt{2}}-12152646\sqrt{577-408\sqrt{2}}+8656551\sqrt{3\left(577-408\sqrt{2}\right)}+5945346\sqrt{6\left(577-408\sqrt{2}\right)}+505232\times 2^{1/4}\sqrt{-24+17\sqrt{2}}-386033\times 2^{3/4}\sqrt{-24+17\sqrt{2}}-559416\times 2^{1/4}\sqrt{-72+51\sqrt{2}}+404535\times 2^{3/4}\sqrt{-72+51\sqrt{2}}\Big)$$

$$\pi-27\Big(207480-148200\sqrt{2}+689160\sqrt{3}-490296\sqrt{6}+254592\sqrt{3\left(577-408\sqrt{2}\right)}+197184\sqrt{6\left(577-408\sqrt{2}\right)}-13728\times 2^{1/4}\sqrt{-24+17\sqrt{2}}+5577\times 2^{3/4}\sqrt{-24+17\sqrt{2}}-24232\times 2^{1/4}\sqrt{-72+51\sqrt{2}}+20008\times 2^{3/4}\sqrt{-72+51\sqrt{2}}+\sqrt{577-408\sqrt{2}}\,(67266-82368\,\mathrm{ArcSec}[3])+\sqrt{1154-816\sqrt{2}}\,(1644-61776\,\mathrm{ArcSec}[3])+266760\,\mathrm{ArcSec}[3]-186732\sqrt{2}\,\mathrm{ArcSec}[3]+112320\sqrt{3}\,\mathrm{ArcSec}[3]-78624\sqrt{6}\,\mathrm{ArcSec}[3]-82368\sqrt{3\left(577-408\sqrt{2}\right)}\,\mathrm{ArcSec}[3]-51480\sqrt{6\left(577-408\sqrt{2}\right)}\,\mathrm{ArcSec}[3]-3432\times 2^{1/4}\sqrt{-24+17\sqrt{2}}\,\mathrm{ArcSec}[3]+3432\times 2^{3/4}\sqrt{-24+17\sqrt{2}}\,\mathrm{ArcSec}[3]-20592\times 2^{1/4}\sqrt{-72+51\sqrt{2}}\,\mathrm{ArcSec}[3]+13728\times 2^{3/4}\sqrt{-72+51\sqrt{2}}\,\mathrm{ArcSec}[3]\Big)\Big)-$$

$$\frac{1}{2}\left(-7+4\sqrt{2}+\left(-4+6\sqrt{2}\right)r-4r^2\right)\Big(\Big(-279008604+197165493\sqrt{2}+146006784\sqrt{3}-103224912\sqrt{6}-17208990\sqrt{1154-816\sqrt{2}}-26610048\sqrt{577-408\sqrt{2}}+21576888\sqrt{3\left(577-408\sqrt{2}\right)}+10156455\sqrt{6\left(577-408\sqrt{2}\right)}+6482098\times 2^{1/4}\sqrt{-24+17\sqrt{2}}-4644288\times 2^{3/4}\sqrt{-24+17\sqrt{2}}-$$

$$20520546\times 2^{1/4}\sqrt{-72+51\sqrt{2}}+14528352\times 2^{3/4}\sqrt{-72+51\sqrt{2}}\Big)\pi-$$
$$54\Big(1244880-881790\sqrt{2}+4128096\sqrt{3}-2922066\sqrt{6}+$$
$$147264\sqrt{3\left(577-408\sqrt{2}\right)}+292032\sqrt{6\left(577-408\sqrt{2}\right)}-$$
$$63063\times 2^{1/4}\sqrt{-24+17\sqrt{2}}+40326\times 2^{3/4}\sqrt{-24+17\sqrt{2}}-436264\times 2^{1/4}$$
$$\sqrt{-72+51\sqrt{2}}+311490\times 2^{3/4}\sqrt{-72+51\sqrt{2}}+1587222\,\text{ArcSec}[3]-$$
$$1120392\sqrt{2}\,\text{ArcSec}[3]+668304\sqrt{3}\,\text{ArcSec}[3]-471744\sqrt{6}\,\text{ArcSec}[3]-$$
$$133848\sqrt{3\left(577-408\sqrt{2}\right)}\,\text{ArcSec}[3]-20592\sqrt{6\left(577-408\sqrt{2}\right)}\,\text{ArcSec}[3]-$$
$$24024\times 2^{1/4}\sqrt{-24+17\sqrt{2}}\,\text{ArcSec}[3]+18018\times 2^{3/4}\sqrt{-24+17\sqrt{2}}\,\text{ArcSec}[3]-$$
$$219648\times 2^{1/4}\sqrt{-72+51\sqrt{2}}\,\text{ArcSec}[3]+154440\times 2^{3/4}\sqrt{-72+51\sqrt{2}}$$
$$\text{ArcSec}[3]-264\sqrt{577-408\sqrt{2}}\,(-1087+234\,\text{ArcSec}[3])-3\sqrt{1154-816\sqrt{2}}$$
$$(50977+27456\,\text{ArcSec}[3])\Big)\Big)\Big)\Big)\Big)\Big/\left(34749\left(-4+3\sqrt{2}\right)^{7/2}\sqrt{-140+99\sqrt{2}}\,\pi\right);$$

In[21]:=

$$\text{OcthCLDApprx11AA}[r\_]:=\frac{12\sqrt{2}}{\pi}+9r-3\sqrt{3}\,r-\frac{9r}{\pi}-\frac{6\,\text{ArcSec}[3]}{\pi};\ \text{OcthCLDApprx11BB}[r\_]:=$$
$$\frac{1}{9\pi}\left(\left(1-\frac{2\times 3^{3/4}\left(\sqrt{6}-4r\right)^{3/2}\left(44-21\sqrt{2}-42\sqrt{3}\,r+9\sqrt{6}\,r+45r^2\right)}{\left(-4+3\sqrt{2}\right)^{7/2}}\right)\right.$$
$$\left(\pi\left(108+256\sqrt{2}+81r-445\sqrt{3}\,r\right)+27\left(4\sqrt{2}-3r-2\,\text{ArcSec}[3]\right)\right)+$$
$$9\left(1+\frac{2\left(-\sqrt{3}+3r\right)^{3/2}\left(-379+168\sqrt{2}+12\sqrt{3}\left(-8+21\sqrt{2}\right)r-360r^2\right)}{3^{3/4}\left(-4+3\sqrt{2}\right)^{7/2}}\right)$$
$$\left.\left(12\sqrt{2}-9r+\pi\left(60+\left(9-57\sqrt{3}\right)r\right)-6\,\text{ArcSec}[3]\right)\right);$$

$$\text{OcthCLDApprx11CC}[r\_]:=\Big(-3744\sqrt{2\left(2-\sqrt{3}\right)}+2160\sqrt{6\left(2-\sqrt{3}\right)}-16848\sqrt{2-\sqrt{3}}\,\pi-$$

$$9344 \sqrt{2\left(2-\sqrt{3}\right)}\,\pi + 9720 \sqrt{3\left(2-\sqrt{3}\right)}\,\pi + 5376 \sqrt{6\left(2-\sqrt{3}\right)}\,\pi - 9864 \times 2^{3/4} \sqrt{\sqrt{2}-2r} +$$

$$4032 \times 2^{3/4} \sqrt{3} \sqrt{\sqrt{2}-2r} + 7398 \times 2^{1/4} \pi \sqrt{\sqrt{2}-2r} - 16128 \times 2^{3/4} \pi \sqrt{\sqrt{2}-2r} -$$

$$3024 \times 2^{1/4} \sqrt{3}\, \pi \sqrt{\sqrt{2}-2r} + 13152 \times 2^{3/4} \sqrt{3}\, \pi \sqrt{\sqrt{2}-2r} + 5616 \sqrt{2-\sqrt{3}}\, r -$$

$$3240 \sqrt{3\left(2-\sqrt{3}\right)}\, r - 142308 \sqrt{2-\sqrt{3}}\, \pi r + 82208 \sqrt{3\left(2-\sqrt{3}\right)}\, \pi r +$$

$$16515 \times 2^{1/4} \sqrt{\sqrt{2}-2r}\, r + 2520 \times 2^{1/4} \sqrt{3} \sqrt{\sqrt{2}-2r}\, r + 37305 \times 2^{1/4} \pi \sqrt{\sqrt{2}-2r}\, r -$$

$$4806 \times 2^{3/4} \pi \sqrt{\sqrt{2}-2r}\, r - 56148 \times 2^{1/4} \sqrt{3}\, \pi \sqrt{\sqrt{2}-2r}\, r - 1512 \times 2^{3/4} \sqrt{3}\, \pi \sqrt{\sqrt{2}-2r}\, r -$$

$$4131 \times 2^{3/4} \sqrt{\sqrt{2}-2r}\, r^2 - 12852 \times 2^{3/4} \sqrt{3} \sqrt{\sqrt{2}-2r}\, r^2 + 1296 \times 2^{1/4} \pi \sqrt{\sqrt{2}-2r}\, r^2 +$$

$$62343 \times 2^{3/4} \pi \sqrt{\sqrt{2}-2r}\, r^2 + 9072 \times 2^{1/4} \sqrt{3}\, \pi \sqrt{\sqrt{2}-2r}\, r^2 +$$

$$22464 \times 2^{3/4} \sqrt{3}\, \pi \sqrt{\sqrt{2}-2r}\, r^2 + 17928 \times 2^{1/4} \sqrt{\sqrt{2}-2r}\, r^3 +$$

$$4536 \times 2^{1/4} \sqrt{3} \sqrt{\sqrt{2}-2r}\, r^3 - 125064 \times 2^{1/4} \pi \sqrt{\sqrt{2}-2r}\, r^3 - 6480 \times 2^{3/4} \pi \sqrt{\sqrt{2}-2r}\, r^3 -$$

$$15480 \times 2^{1/4} \sqrt{3}\, \pi \sqrt{\sqrt{2}-2r}\, r^3 - 3240 \times 2^{3/4} \sqrt{\sqrt{2}-2r}\, r^4 - 9720 \times 2^{3/4} \pi \sqrt{\sqrt{2}-2r}\, r^4 +$$

$$30240 \times 2^{3/4} \sqrt{3}\, \pi \sqrt{\sqrt{2}-2r}\, r^4 + 9072 \times 2^{1/4} \sqrt{-\sqrt{6}+4r} - 8856 \times 2^{1/4} \sqrt{3} \sqrt{-\sqrt{6}+4r} +$$

$$21504 \times 2^{1/4} \pi \sqrt{-\sqrt{6}+4r} + 4536 \times 2^{3/4} \pi \sqrt{-\sqrt{6}+4r} - 20992 \times 2^{1/4} \sqrt{3}\, \pi \sqrt{-\sqrt{6}+4r} -$$

$$4428 \times 2^{3/4} \sqrt{3}\, \pi \sqrt{-\sqrt{6}+4r} + 10422 \times 2^{3/4} r \sqrt{-\sqrt{6}+4r} + 6345 \times 2^{3/4} \sqrt{3}\, r \sqrt{-\sqrt{6}+4r} +$$

$$13824 \times 2^{1/4} \pi r \sqrt{-\sqrt{6}+4r} + 90905 \times 2^{3/4} \pi r \sqrt{-\sqrt{6}+4r} +$$

$$3024 \times 2^{1/4} \sqrt{3}\, \pi r \sqrt{-\sqrt{6}+4r} - 14843 \times 2^{3/4} \sqrt{3}\, \pi r \sqrt{-\sqrt{6}+4r} -$$

$$46656 \times 2^{1/4} r^2 \sqrt{-\sqrt{6}+4r} - 3564 \times 2^{1/4} \sqrt{3}\, r^2 \sqrt{-\sqrt{6}+4r} -$$

$$113028 \times 2^{1/4} \pi r^2 \sqrt{-\sqrt{6}+4r} - 18144 \times 2^{3/4} \pi r^2 \sqrt{-\sqrt{6}+4r} -$$

$$57764 \times 2^{1/4} \sqrt{3}\, \pi r^2 \sqrt{-\sqrt{6}+4r} - 648 \times 2^{3/4} \sqrt{3}\, \pi r^2 \sqrt{-\sqrt{6}+4r} +$$

$$26568 \times 2^{3/4} r^3 \sqrt{-\sqrt{6}+4r} + 486 \times 2^{3/4} \sqrt{3}\, r^3 \sqrt{-\sqrt{6}+4r} + 12960 \times 2^{1/4} \pi r^3 \sqrt{-\sqrt{6}+4r} +$$

$$25122 \times 2^{3/4} \pi r^3 \sqrt{-\sqrt{6}+4r} + 74274 \times 2^{3/4} \sqrt{3}\, \pi r^3 \sqrt{-\sqrt{6}+4r} -$$

$$9720 \times 2^{1/4} r^4 \sqrt{-\sqrt{6}+4r} + 9720 \times 2^{1/4} \pi r^4 \sqrt{-\sqrt{6}+4r} -$$

$$53400 \times 2^{1/4} \sqrt{3}\, \pi r^4 \sqrt{-\sqrt{6}+4r} + 5616 \sqrt{2-\sqrt{3}}\, \text{ArcSec}[3] -$$

$$3240 \sqrt{3\left(2-\sqrt{3}\right)}\, \text{ArcSec}[3] - 2268 \times 2^{3/4} \sqrt{-\sqrt{6}+4r}\, \text{ArcSec}[3] +$$

$$2214 \times 2^{3/4} \sqrt{3} \sqrt{-\sqrt{6}+4r}\, \text{ArcSec}[3] - 6912 \times 2^{1/4} r \sqrt{-\sqrt{6}+4r}\, \text{ArcSec}[3] -$$

$$1512 \times 2^{1/4} \sqrt{3}\, r \sqrt{-\sqrt{6}+4r}\, \text{ArcSec}[3] + 9072 \times 2^{3/4} r^2 \sqrt{-\sqrt{6}+4r}\, \text{ArcSec}[3] +$$

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324 × 2^(3/4) √3 r² √(-√6 + 4 r) ArcSec[3] - 6480 × 2^(1/4) r³ √(-√6 + 4 r) ArcSec[3]) /
(36 √(2 - √3) (-2 + √3)³ π); OcthCLDApprx11DD[r_] :=
(6 + 32 √(2/3) + (9 - 28 √3) r + (-8 √2 + 3 r)/π)
(1 - ((-1/√2 + r)^(3/2) (39 - 42 r + 15 r² + 2 √2 (-7 + 3 r)))/(8 (1 - 1/√2)^(7/2)));
```

Check of the sum rule $4\pi \int_0^\infty r^2 \gamma(r)\, dr = V_p$

In our case $V_p =$ VOcth = 1 / 6.

We find 0.11777 with an error ~0.041395 (~ 24%) that will appear in the FT, i.e. ΔI(0) ~+0.041.

```
N[1/6]

0.166667

4 π (NIntegrate[r² * OcthCFApprxFnl11AA[r], {r, 0, OCTDst[[1]]},
    WorkingPrecision → 30, PrecisionGoal → 15] + NIntegrate[r² * OcthCFApprxFnl11BB[r],
    {r, OCTDst[[1]], OCTDst[[2]]}, WorkingPrecision → 30, PrecisionGoal → 15] +
   NIntegrate[r² * OcthCFApprxFnl11CC[r], {r, OCTDst[[2]], OCTDst[[3]]},
    WorkingPrecision → 30, PrecisionGoal → 15 ] + NIntegrate[r² * OcthCFApprxFnl11DD[r],
    {r, OCTDst[[3]], OCTDst[[4]]}, WorkingPrecision → 30, PrecisionGoal → 15 ])

0.125271435567193889268872365297

N[VOcth - 0.125271435567193889]
N[(VOcth - 0.125271435567193889) / VOcth]

0.0413952

0.248371
```

The approximation K = 0 is much better than the K = 1 one!

The case K = 2

$$\text{LeftCLD22}[r\_, D1\_, D2\_] := \left(1 + \frac{1}{16\,(-D1 + D2)^{11/2}}\right.$$

$$\left.(16\,(D1 - D2)^{3} - 40\,(D1 - D2)^{2}\,(D2 - r) + 70\,(D1 - D2)\,(D2 - r)^{2} - 105\,(D2 - r)^{3})\,(-D1 + r)^{5/2}\right)$$

$$\left(a0 + a4\,(D1 - r)^{2} + a2\,(-D1 + r) + \sqrt{-D1 + r}\,(a1 + a3\,(-D1 + r))\right);$$

$$\text{RgtCLD22}[r\_, D1\_, D2\_] := \left(b0 + (b1 + b3\,(D2 - r))\,\sqrt{D2 - r} + b2\,(D2 - r) + b4\,(D2 - r)^{2}\right)$$

$$\left(1 - \frac{1}{16\,(-D1 + D2)^{11/2}}\,(D2 - r)^{5/2}\,\left(-231\,D1^{3} + 16\,D2^{3} + 40\,D2^{2}\,r +\right.\right.$$

$$\left.\left.70\,D2\,r^{2} + 105\,r^{3} + 99\,D1^{2}\,(2\,D2 + 5\,r) - 11\,D1\,(8\,D2^{2} + 20\,D2\,r + 35\,r^{2})\right)\right);$$

$$\text{CLDApprx22}[r\_, D1\_, D2\_] := \left(1 + \frac{1}{16\,(-D1 + D2)^{11/2}}\right.$$

$$\left.(16\,(D1 - D2)^{3} - 40\,(D1 - D2)^{2}\,(D2 - r) + 70\,(D1 - D2)\,(D2 - r)^{2} - 105\,(D2 - r)^{3})\,(-D1 + r)^{5/2}\right)$$

$$\left(a0 + a4\,(D1 - r)^{2} + a2\,(-D1 + r) + \sqrt{-D1 + r}\,(a1 + a3\,(-D1 + r))\right) +$$

$$\left(b0 + (b1 + b3\,(D2 - r))\,\sqrt{D2 - r} + b2\,(D2 - r) + b4\,(D2 - r)^{2}\right)$$

$$\left(1 - \frac{1}{16\,(-D1 + D2)^{11/2}}\,(D2 - r)^{5/2}\,\left(-231\,D1^{3} + 16\,D2^{3} + 40\,D2^{2}\,r + 70\,D2\,r^{2} + 105\,r^{3} +\right.\right.$$

$$\left.\left.99\,D1^{2}\,(2\,D2 + 5\,r) - 11\,D1\,(8\,D2^{2} + 20\,D2\,r + 35\,r^{2})\right)\right);\quad \text{LEFTCFApprx22}[r\_, D1\_, D2\_] :=$$

$$(29\,393\,a1\,(-512\,D1^{8} + 2816\,D1^{7}\,D2 - 6336\,D1^{6}\,D2^{2} + 7392\,D1^{5}\,D2^{3} - 4620\,D1^{4}\,D2^{4} + 1386\,D1^{3}\,D2^{5} +$$

$$462\,D1^{2}\,D2^{6} - 462\,D1\,D2^{7} + 99\,D2^{8} + 1280\,D1^{7}\,r - 7040\,D1^{6}\,D2\,r + 15\,840\,D1^{5}\,D2^{2}\,r -$$

$$18\,480\,D1^{4}\,D2^{3}\,r + 11\,550\,D1^{3}\,D2^{4}\,r - 6930\,D1^{2}\,D2^{5}\,r + 2310\,D1\,D2^{6}\,r - 330\,D2^{7}\,r - 960\,D1^{6}\,r^{2} +$$

$$5280\,D1^{5}\,D2\,r^{2} - 11\,880\,D1^{4}\,D2^{2}\,r^{2} + 13\,860\,D1^{3}\,D2^{3}\,r^{2} + 160\,D1^{5}\,r^{3} - 880\,D1^{4}\,D2\,r^{3} + 1980\,D1^{3}$$

$$D2^{2}\,r^{3} - 13\,860\,D1^{2}\,D2^{3}\,r^{3} + 20\,D1^{4}\,r^{4} - 110\,D1^{3}\,D2\,r^{4} + 8910\,D1^{2}\,D2^{2}\,r^{4} + 6930\,D1\,D2^{3}\,r^{4} +$$

$$6\,D1^{3}\,r^{5} - 3498\,D1^{2}\,D2\,r^{5} - 7722\,D1\,D2^{2}\,r^{5} - 1386\,D2^{3}\,r^{5} + 580\,D1^{2}\,r^{6} + 3740\,D1\,D2\,r^{6} +$$

$$1980\,D2^{2}\,r^{6} - 700\,D1\,r^{7} - 1100\,D2\,r^{7} + 225\,r^{8} + 512\,(-D1 + D2)^{11/2}\,(-D1 + r)^{5/2}) -$$

$$4199\,a3\,(1225\,D1^{9} + 275\,(D1 - D2)^{9} + 693\,(D1 - D2)^{8}\,r + 6468\,(-D1 + D2)^{3}\,r^{6} -$$

$$9900\,(D1 - D2)^{2}\,r^{7} + 5775\,(-D1 + D2)\,r^{8} - 1225\,r^{9} - 1536\,(-D1 + D2)^{11/2}\,(-D1 + r)^{7/2} -$$

$$525\,D1^{8}\,(11\,D1 - 11\,D2 + 21\,r) + 300\,D1^{7}\,(33\,(D1 - D2)^{2} + 154\,(D1 - D2)\,r + 147\,r^{2}) -$$

$$420\,D1^{2}\,r^{4}\,(231\,(D1 - D2)^{3} + 495\,(D1 - D2)^{2}\,r + 385\,(D1 - D2)\,r^{2} + 105\,r^{3}) +$$

$$420\,D1^{3}\,r^{3}\,(308\,(D1 - D2)^{3} + 825\,(D1 - D2)^{2}\,r + 770\,(D1 - D2)\,r^{2} + 245\,r^{3}) -$$

$$210\,D1^{4}\,r^{2}\,(462\,(D1 - D2)^{3} + 1650\,(D1 - D2)^{2}\,r + 1925\,(D1 - D2)\,r^{2} + 735\,r^{3}) -$$

$$84\,D1^{6}\,(77\,(D1 - D2)^{3} + 825\,(D1 - D2)^{2}\,r + 1925\,(D1 - D2)\,r^{2} + 1225\,r^{3}) + 42\,D1^{5}\,r$$

$$(924\,(D1 - D2)^{3} + 4950\,(D1 - D2)^{2}\,r + 7700\,(D1 - D2)\,r^{2} + 3675\,r^{3}) + 21\,D1\,(-33\,(D1 - D2)^{8} +$$

$$1848\,(D1 - D2)^{3}\,r^{5} + 3300\,(D1 - D2)^{2}\,r^{6} + 2200\,(D1 - D2)\,r^{7} + 525\,r^{8})) - 1120$$

$$(323\,(a0\,(143\,(-D1 + D2)^{3} + 195\,(D1 - D2)^{2}\,(D1 - r) + 105\,(-D1 + D2)\,(D1 - r)^{2} + 21\,(D1 - r)^{3})$$

```
            (-D1 + r)^(9/2) - 2 a0 (-D1 + D2)^(11/2) (16 D1^2 + 7 D2^2 + 16 D1 (D2 - 3 r) - 30 D2 r + 39 r^2)) +
        (-D1 + D2)^(11/2) (-693 a4 (D1 - D2)^4 + 1596 (D1 - D2)^3 (a2 + a4 (D1 - r)) -
           4522 a2 (D1 - D2)^2 (D1 - r) + 4199 (D1 - r)^3 (2 a2 + a4 (-D1 + r))) -
        (-D1 + r)^(11/2) (2907 (D1 - D2)^2 (-15 a2 + 11 a4 (D1 - r)) (D1 - r) +
           1463 (-D1 + D2) (-17 a2 + 13 a4 (D1 - r)) (D1 - r)^2 +
           273 (-19 a2 + 15 a4 (D1 - r)) (D1 - r)^3 - 2261 (-D1 + D2)^3 (13 a2 + 9 a4 (-D1 + r)))) +
     (D2 - r)^2 (29 393 b1 (-281 (D1 - D2)^6 + 388 (D1 - D2)^5 (D1 - r) + 97 (D1 - D2)^4 (D1 - r)^2 +
           34 (-D1 + D2)^3 (D1 - r)^3 - 145 (D1 - D2)^2 (D1 - r)^4 +
           250 (-D1 + D2) (D1 - r)^5 + 225 (D1 - r)^6 + 512 (-D1 + D2)^(11/2) Sqrt[D2 - r]) +
        4199 b3 (-1118 (D1 - D2)^6 + 1329 (D1 - D2)^5 (D1 - r) + 621 (D1 - D2)^4 (D1 - r)^2 +
           118 (D1 - D2)^3 (D1 - r)^3 - 600 (D1 - D2)^2 (D1 - r)^4 + 1575 (-D1 + D2) (D1 - r)^5 +
           1225 (D1 - r)^6 + 1536 (-D1 + D2)^(11/2) Sqrt[D2 - r]) (D2 - r) + 1120
         (323 b0 (78 (-D1 + D2)^(11/2) + (32 (D1 - D2)^3 + 48 (D1 - D2)^2 (D1 - r) + 42 (D1 - D2) (D1 - r)^2 +
                 21 (D1 - r)^3) (D2 - r)^(5/2)) + 4199 (-D1 + D2)^(11/2) (2 b2 + b4 (D2 - r)) (D2 - r) +
           (D2 - r)^(7/2) (-3296 b4 (D1 - D2)^4 - 7 (-D1 + D2) (1330 b2 + 377 b4 (D1 - r)) (D1 - r)^2 +
              273 (19 b2 + 15 b4 (D1 - r)) (D1 - r)^3 - 48 (-D1 + D2)^3 (114 b2 + 61 b4 (-D1 + r)) +
              2 (D1 - D2)^2 (D1 - r) (4712 b2 + 255 b4 (-D1 + r)))))) /
  (56 434 560 (-D1 + D2)^(11/2)); RGHTCFApprx22[r_, D1_, D2_] :=
 (29 393 b1 (99 D1^8 - 462 D1^7 D2 + 462 D1^6 D2^2 + 1386 D1^5 D2^3 - 4620 D1^4 D2^4 + 7392 D1^3 D2^5 -
        6336 D1^2 D2^6 + 2816 D1 D2^7 - 512 D2^8 + 512 (-D1 + D2)^(11/2) (D2 - r)^(5/2) - 330 D1^7 r +
        2310 D1^6 D2 r - 6930 D1^5 D2^2 r + 11 550 D1^4 D2^3 r - 18 480 D1^3 D2^4 r + 15 840 D1^2 D2^5 r -
        7040 D1 D2^6 r + 1280 D2^7 r + 13 860 D1^3 D2^3 r^2 - 11 880 D1^2 D2^4 r^2 + 5280 D1 D2^5 r^2 -
        960 D2^6 r^2 - 13 860 D1^3 D2^2 r^3 + 1980 D1^2 D2^3 r^3 - 880 D1 D2^4 r^3 + 160 D2^5 r^3 + 6930 D1^3 D2 r^4 +
        8910 D1^2 D2^2 r^4 - 110 D1 D2^3 r^4 + 20 D2^4 r^4 - 1386 D1^3 r^5 - 7722 D1^2 D2 r^5 - 3498 D1 D2^2 r^5 +
        6 D2^3 r^5 + 1980 D1^2 r^6 + 3740 D1 D2 r^6 + 580 D2^2 r^6 - 1100 D1 r^7 - 700 D2 r^7 + 225 r^8) +
     4199 b3 (-275 D1^9 + 1782 D1^8 D2 - 4356 D1^7 D2^2 + 3696 D1^6 D2^3 + 4158 D1^5 D2^4 -
        13 860 D1^4 D2^5 + 22 176 D1^3 D2^6 - 19 008 D1^2 D2^7 + 8448 D1 D2^8 - 1536 D2^9 +
        1536 (-D1 + D2)^(11/2) (D2 - r)^(7/2) + 693 D1^8 r - 5544 D1^7 D2 r + 19 404 D1^6 D2^2 r -
        38 808 D1^5 D2^3 r + 48 510 D1^4 D2^4 r - 77 616 D1^3 D2^5 r + 66 528 D1^2 D2^6 r - 29 568 D1 D2^7 r +
        5376 D2^8 r + 97 020 D1^3 D2^4 r^2 - 83 160 D1^2 D2^5 r^2 + 36 960 D1 D2^6 r^2 - 6720 D2^7 r^2 -
        129 360 D1^3 D2^3 r^3 + 41 580 D1^2 D2^4 r^3 - 18 480 D1 D2^5 r^3 + 3360 D2^6 r^3 + 97 020 D1^3 D2^2 r^4 +
        55 440 D1^2 D2^3 r^4 + 2310 D1 D2^4 r^4 - 420 D2^5 r^4 - 38 808 D1^3 D2 r^5 - 91 476 D1^2 D2^2 r^5 -
        24 024 D1 D2^3 r^5 - 42 D2^4 r^5 + 6468 D1^3 r^6 + 49 896 D1^2 D2 r^6 + 42 504 D1 D2^2 r^6 + 4032 D2^3 r^6 -
        9900 D1^2 r^7 - 26 400 D1 D2 r^7 - 7800 D2^2 r^7 + 5775 D1 r^8 + 5250 D2 r^8 - 1225 r^9) -
     1120 (b4 (-D1 + D2)^(11/2) (-3296 (D1 - D2)^4 + 15 200 (D1 - D2)^3 (D1 - r) -
           25 194 (D1 - D2)^2 (D1 - r)^2 + 16 796 (D1 - D2) (D1 - r)^3 - 4199 (D1 - r)^4) -
        19 b2 (288 (D1 - D2)^3 + 496 (D1 - D2)^2 (D1 - r) + 490 (D1 - D2) (D1 - r)^2 + 273 (D1 - r)^3)
         (D2 - r)^(11/2) - b4 (3296 (D1 - D2)^3 + 6224 (D1 - D2)^2 (D1 - r) +
           6734 (D1 - D2) (D1 - r)^2 + 4095 (D1 - r)^3) (D2 - r)^(13/2) + 38 b2 (-D1 + D2)^(11/2)
         (144 (D1 - D2)^3 + 663 (D1 - D2) (D1 - r)^2 - 221 (D1 - r)^3 + 544 (D1 - D2)^2 (-D1 + r)) +
```

```
323 (-b0 (32 (D1 - D2)^3 + 48 (D1 - D2)^2 (D1 - r) + 42 (D1 - D2) (D1 - r)^2 + 21 (D1 - r)^3)
      (D2 - r)^(9/2) - 2 b0 (-D1 + D2)^(11/2) (7 D1^2 + 16 D1 D2 + 16 D2^2 - 30 D1 r - 48 D2 r + 39 r^2))) -
  (D1 - r)^2 (-29 393 a1 ((1386 (-D1 + D2)^3 + 1980 (D1 - D2)^2 (D1 - r) + 1100 (-D1 + D2) (D1 - r)^2 +
        225 (D1 - r)^3) (D1 - r)^3 + 512 (-D1 + D2)^(11/2) Sqrt[-D1 + r]) + 4199 a3 (D1 - r)
     ((6468 (-D1 + D2)^3 + 9900 (D1 - D2)^2 (D1 - r) + 5775 (-D1 + D2) (D1 - r)^2 + 1225 (D1 - r)^3)
      (D1 - r)^3 + 1536 (-D1 + D2)^(11/2) Sqrt[-D1 + r]) +
    1120 (-4199 (-D1 + D2)^(11/2) (6 a0 + (-2 a2 + a4 (D1 - r)) (D1 - r)) +
      (-D1 + r)^(5/2) (969 (D1 - D2)^2 (65 a0 + 3 (-15 a2 + 11 a4 (D1 - r)) (D1 - r)) (D1 - r) +
        133 (-D1 + D2) (255 a0 + 11 (-17 a2 + 13 a4 (D1 - r)) (D1 - r)) (D1 - r)^2 +
        21 (323 a0 + 13 (-19 a2 + 15 a4 (D1 - r)) (D1 - r)) (D1 - r)^3 + 323 (-D1 + D2)^3
         (143 a0 - 7 (D1 - r) (13 a2 + 9 a4 (-D1 + r))))))) / (56 434 560 (-D1 + D2)^(11/2));

EXTRCNTRBLft22[r_, D1_, D2_, a_, b_] := 1/2520 (D2 - r)^5
  (-9 a (14 D1^3 - D2^3 - 3 D2^2 r - 5 D2 r^2 - 5 r^3 - 14 D1^2 (D2 + 2 r) + 2 D1 (3 D2^2 + 8 D2 r + 10 r^2)) +
   b (126 D1^4 + 5 D2^4 + 16 D2^3 r + 30 D2^2 r^2 + 40 D2 r^3 + 35 r^4 - 168 D1^3 (D2 + 2 r) +
     36 D1^2 (3 D2^2 + 8 D2 r + 10 r^2) - 36 D1 (D2^3 + 3 D2^2 r + 5 D2 r^2 + 5 r^3)));

EXTRCNTRBRgt22[r_, D1_, D2_, a_, b_] := 1/2520 (D1 - r)^5
  (9 a (D1^3 - 6 D1^2 D2 + 14 D1 D2^2 - 14 D2^3 + (3 D1^2 - 16 D1 D2 + 28 D2^2) r + 5 (D1 - 4 D2) r^2 + 5 r^3) -
   b (D1 - r) (4 D1^3 - 27 D1^2 D2 + 72 D1 D2^2 - 84 D2^3 +
     15 (D1^2 - 6 D1 D2 + 12 D2^2) r + 15 (2 D1 - 9 D2) r^2 + 35 r^3));
```

Approximation within the interval [D0, D1]

```
OCTCFAA[r]
Simplify[D[D[OCTCFAA[r], r], r]]
```

```
OcthCFApprxFnl22AA[r_] := 1 - (3 Sqrt[3] r)/2 + (6 Sqrt[2] r^2)/π + (3 r^3)/2 - (Sqrt[3] r^3)/2 - (3 r^3)/(2 π) - (3 r^2 ArcSec[3])/π;

OcthCLDApprx22AA[r_] := (12 Sqrt[2])/π + 9 r - 3 Sqrt[3] r - (9 r)/π - (6 ArcSec[3])/π;
```

```
Plot[{OCTCFAA[r], OcthCFApprxFnl22AA[r]}, {r, 0, OCTDst[[1]]}]

IntervlAmpl = N[{OCTDst[[1]], OCTDst[[2]] - OCTDst[[1]],
   OCTDst[[3]] - OCTDst[[2]], OCTDst[[4]] - OCTDst[[3]]}]
{0.57735, 0.0350222, 0.0947343, 0.292893}
```

The first interval is yhe widest one, it is followd by the fourthe, the third and the second which is the smallest one.

Approximation within the interval [D1, D2]

```
Clear[a0]; Clear[a1]; Clear[a2]; Clear[a3]; Clear[a4]; Clear[a5]; Clear[a6];
Clear[b0]; Clear[b1]; Clear[b2]; Clear[b3]; Clear[b4]; Clear[b5]; Clear[b6];

cfD1Plus
cfD2Minus

CLDApprx22[r, D1, D2]
```

```
Simplify[((CLDApprx22[r, D1, D2]) /.
   {a0 → cfD1Plus[[1]], a1 → cfD1Plus[[2]], a2 → cfD1Plus[[3]], a3 → cfD1Plus[[4]],
    a4 → cfD1Plus[[5]], b0 → cfD2Minus[[1]], b1 → cfD2Minus[[2]], b2 → cfD2Minus[[3]],
    b3 → cfD2Minus[[4]], b4 → cfD2Minus[[5]], D1 → OCTDst[[1]], D2 → OCTDst[[2]]}),
 Assumptions → {OCTDst[[1]] < r < OCTDst[[2]]}]
```

```
FullSimplify[
 ((CLDApprx22[r, D1, D2]) /. {a0 → cfD1Plus[[1]], a1 → cfD1Plus[[2]], a2 → cfD1Plus[[3]],
     a3 → cfD1Plus[[4]], a4 → cfD1Plus[[5]], b0 → cfD2Minus[[1]],
     b1 → cfD2Minus[[2]], b2 → cfD2Minus[[3]], b3 → cfD2Minus[[4]],
     b4 → cfD2Minus[[5]], D1 → OCTDst[[1]], D2 → OCTDst[[2]]}) -
  OcthCLDApprx22BB[r], Assumptions → {OCTDst[[1]] < r < OCTDst[[2]]}]
```

0

the K=0 approximation of the CLD

$$\text{OcthCLDApprx22BB}[r\_] := \frac{1}{9\pi}\left(\left(1 - \frac{1}{\left(-4+3\sqrt{2}\right)^{11/2}}\, 6\times 3^{3/4}\left(\sqrt{6}-4r\right)^{5/2}\left(4\sqrt{3}\left(-55+27\sqrt{2}\right) - 30\left(-36+11\sqrt{2}\right)r + 35\sqrt{3}\left(-22+3\sqrt{2}\right)r^2 + 630 r^3\right)\right)\left(\pi\left(108+640\sqrt{2}+\left(81-1469\sqrt{3}\right)r + 1024\sqrt{2}\,r^2\right) + 27\left(4\sqrt{2}-3r-2\,\text{ArcSec}[3]\right)\right) + 27\left(1 - \frac{1}{3\times 3^{3/4}\left(-4+3\sqrt{2}\right)^{11/2}}\, 4\left(-\sqrt{3}+3r\right)^{5/2}\left(\sqrt{3}\left(-7640+8349\sqrt{2}\right) + 60\left(-955+264\sqrt{2}\right)r + 840\sqrt{3}\left(-8+33\sqrt{2}\right)r^2 - 30240 r^3\right)\right)\left(4\sqrt{2}-3r+\pi\left(44+\left(3-67\sqrt{3}\right)r+72r^2\right) - 2\,\text{ArcSec}[3]\right)\right);$$

```
Plot[{OCTDDCFBB[r], OcthCLDApprx22BB[r], OcthCLDApprx11BB[r]},
 {r, OCTDst[[1]], OCTDst[[2]]}, PlotStyle → {Directive[Blue, Thickness[0.003]],
   Directive[Red, Thickness[0.006]], Directive[Purple, Thickness[0.003]]},
 AxesLabel → {"r", "γ0''[r]"}, PlotLabel → "CLD:  K=2; D1 < r < D2 "]
```

The approximation has improved.

Approximation (not yet matched) of the CLD

```
Simplify[((LEFTCFApprx22[r, D1, D2]) /.
   {a0 → cfD1Plus[[1]], a1 → cfD1Plus[[2]], a2 → cfD1Plus[[3]], a3 → cfD1Plus[[4]],
    a4 → cfD1Plus[[5]], b0 → cfD2Minus[[1]], b1 → cfD2Minus[[2]], b2 → cfD2Minus[[3]],
    b3 → cfD2Minus[[4]], b4 → cfD2Minus[[5]], D1 → OCTDst[[1]], D2 → OCTDst[[2]]}),
 Assumptions → {OCTDst[[1]] < r < OCTDst[[2]]}]
```

```
Simplify[
 ((LEFTCFApprx22[r, D1, D2]) /. {a0 → cfD1Plus[[1]], a1 → cfD1Plus[[2]], a2 → cfD1Plus[[3]],
     a3 → cfD1Plus[[4]], a4 → cfD1Plus[[5]], b0 → cfD2Minus[[1]],
     b1 → cfD2Minus[[2]], b2 → cfD2Minus[[3]], b3 → cfD2Minus[[4]],
     b4 → cfD2Minus[[5]], D1 → OCTDst[[1]], D2 → OCTDst[[2]]}) -
  ApprxOcthCF22NotMatchBB[r], Assumptions → {OCTDst[[1]] < r < OCTDst[[2]]}]
```

ApprxOcthCF22NotMatchBB[r_] :=

$$\frac{1}{146965\left(-4+3\sqrt{2}\right)^{11/2}}48\times 3^{3/4}\Bigg(-1120\Bigg(\frac{1}{18432\times 3^{3/4}}\left(-4+3\sqrt{2}\right)^{11/2}\Bigg(\frac{2079}{8}\left(-577+408\sqrt{2}\right)-\frac{133\sqrt{3}\left(-4+3\sqrt{2}\right)^{3}\left(-3+\pi\left(3+5\sqrt{3}-72r\right)\right)}{16\pi}+\frac{2261\left(34-24\sqrt{2}\right)\left(3+\left(-3+19\sqrt{3}\right)\pi\right)\left(\sqrt{3}-3r\right)}{24\pi}-\frac{8398\left(3+\pi\left(-3+31\sqrt{3}-36r\right)\right)\left(\sqrt{3}-3r\right)^{3}}{9\pi}\Bigg)-\frac{1}{576\pi}\Big(6783\sqrt{3}\left(-4+3\sqrt{2}\right)^{3}\left(39+\pi\left(-39+463\sqrt{3}-648r\right)\right)+313956\left(34-24\sqrt{2}\right)\left(5+\pi\left(-5+61\sqrt{3}-88r\right)\right)\left(\sqrt{3}-3r\right)+23408\sqrt{3}\left(-4+3\sqrt{2}\right)\left(51+\pi\left(-51+635\sqrt{3}-936r\right)\right)\left(\sqrt{3}-3r\right)^{2}+17472\left(57+\pi\left(-57+721\sqrt{3}-1080r\right)\right)\left(\sqrt{3}-3r\right)^{3}\Big)\left(-\frac{1}{\sqrt{3}}+r\right)^{11/2}+\frac{1}{82944\sqrt{3}\pi}323\Big(\frac{3}{8}3^{3/4}\left(-4+3\sqrt{2}\right)^{11/2}\left(-191-96\sqrt{2}+12\sqrt{3}\left(32+15\sqrt{2}\right)r-936r^{2}\right)+\frac{32}{3}\left(-\sqrt{3}+3r\right)^{9/2}\Big(\sqrt{3}\left(-9448+7317\sqrt{2}\right)+36\left(-713+360\sqrt{2}\right)r+504\sqrt{3}\left(-8+15\sqrt{2}\right)r^{2}-6048r^{3}\Big)\Big)\left(4\sqrt{2}-\sqrt{3}+\pi+\sqrt{3}\pi-2\,\mathrm{ArcSec}[3]\right)\Bigg)+\frac{1}{497664}35\left(\sqrt{6}-4r\right)^{2}\Bigg(\frac{4199\times 3^{1/4}\left(-4+3\sqrt{2}\right)^{11/2}\left(\sqrt{6}-4r\right)\left(81+\pi\left(-81+701\sqrt{3}-512\sqrt{2}r\right)\right)}{\pi}-\frac{1}{\pi}144\left(\sqrt{6}-4r\right)^{7/2}\Big(513\Big(\sqrt{3}\left(-6154+4005\sqrt{2}\right)-18\left(-951+493\sqrt{2}\right)r+21\sqrt{3}\left(-374+105\sqrt{2}\right)r^{2}+4914r^{3}\Big)+\pi\Big(-91430470+59035779\sqrt{2}+3157002\sqrt{3}-2054565\sqrt{6}-6\left(1463589-758727\sqrt{2}-9099009\sqrt{3}+3107963\sqrt{6}\right)r-3\left(15072662+15223527\sqrt{2}-1343034\sqrt{3}+377055\sqrt{6}\right)r^{2}+182\left(-13851+31551\sqrt{3}+214016\sqrt{6}\right)r^{3}-25159680\sqrt{2}r^{4}\Big)\Big)+969\Big(13\times 3^{1/4}\left(-4+3\sqrt{2}\right)^{11/2}-16\left(\sqrt{6}-4r\right)^{5/2}\Big(5\sqrt{3}\left(-122+81\sqrt{2}\right)-18\left(-83+45\sqrt{2}\right)r+63\sqrt{3}\left(-10+3\sqrt{2}\right)r^{2}+378r^{3}\Big)\Big)\left(432-311\sqrt{2}+81\sqrt{6}-\frac{27\left(-16\sqrt{2}+3\sqrt{6}+8\,\mathrm{ArcSec}[3]\right)}{\pi}\right)\Bigg)\Bigg);$$

```
Plot[{OCTCFBB[r], ApprxOcthCF22NotMatchBB[r]}, {r, OCTDst[[1]], OCTDst[[2]]},
 PlotStyle → {Directive[Blue, Thickness[0.003]], Directive[Red, Thickness[0.006]]},
 AxesLabel → {"r", "γT[r]"}]
```

The approximation is matched at r=D1

```
Simplify[
 Solve[{Limit[ApprxOcthCF22NotMatchBB[r] + a + b r, r → OCTDst[[1]], Direction → -1] - Limit[
       OcthCFApprxFnl22AA[r], r → OCTDst[[1]], Direction → 1] == 0 &&
    Limit[D[ApprxOcthCF22NotMatchBB[r] + a + b r, r], r → OCTDst[[1]], Direction → -1] -
      Limit[D[OcthCFApprxFnl22AA[r], r], r → OCTDst[[1]], Direction → 1] == 0}, {a, b}]]
```

Simplify[Simplify[(ApprxOcthCF22NotMatchBB[r] + a + b r) /.

$$\left\{a \to \frac{1}{10\,883\,808\,\pi}\left(-\left(-26\,868\,843 + 8\,902\,834\sqrt{2} + 262\,656\sqrt{3} + 2\,760\,966\sqrt{6}\right)\pi + 1026\left(-23\,256\sqrt{2} + 256\sqrt{3} + 2691\sqrt{6} + 11\,628\,\mathrm{ArcSec}[3]\right)\right),\right.$$

$$\left. b \to \frac{1}{143\,208\,\pi}\left(\left(235\,467 - 989\,977\sqrt{3} + 499\,630\sqrt{6}\right)\pi - 4131\left(57 - 160\sqrt{3} - 32\sqrt{6} + 8\sqrt{3}\left(2 + 5\sqrt{2}\right)\mathrm{ArcSec}[3]\right)\right)\right\},$$

Assumptions → {OCTDst[[1]] < r < OCTDst[[2]]}]]

ApprxOcthCF22PrtlyMatchBB[r_] :=

$$\frac{1}{10\,883\,808\,\pi}\left(-\left(-26\,868\,843 + 8\,902\,834\sqrt{2} + 262\,656\sqrt{3} + 2\,760\,966\sqrt{6}\right)\pi + 1026\left(-23\,256\sqrt{2} + 256\sqrt{3} + 2691\sqrt{6} + 11\,628\,\mathrm{ArcSec}[3]\right)\right) +$$

$$\frac{1}{143\,208\,\pi}\, r\left(\left(235\,467 - 989\,977\sqrt{3} + 499\,630\sqrt{6}\right)\pi - 4131\left(57 - 160\sqrt{3} - 32\sqrt{6} + 8\sqrt{3}\left(2 + 5\sqrt{2}\right)\mathrm{ArcSec}[3]\right)\right) +$$

$$\frac{1}{146\,965\left(-4 + 3\sqrt{2}\right)^{11/2}}\, 48 \times 3^{3/4}\left(-1120\left(\frac{1}{18\,432 \times 3^{3/4}}\left(-4 + 3\sqrt{2}\right)^{11/2}\right.\right.$$

$$\left(\frac{2079}{8}\left(-577 + 408\sqrt{2}\right) - \frac{133\sqrt{3}\left(-4 + 3\sqrt{2}\right)^{3}\left(-3 + \pi\left(3 + 5\sqrt{3} - 72\,r\right)\right)}{16\,\pi} +\right.$$

$$\frac{2261\left(34 - 24\sqrt{2}\right)\left(3 + \left(-3 + 19\sqrt{3}\right)\pi\right)\left(\sqrt{3} - 3\,r\right)}{24\,\pi} -$$

$$\left.\frac{8398\left(3 + \pi\left(-3 + 31\sqrt{3} - 36\,r\right)\right)\left(\sqrt{3} - 3\,r\right)^{3}}{9\,\pi}\right) -$$

$$\frac{1}{576\,\pi}\Big(6783\sqrt{3}\,(-4+3\sqrt{2})^3\,(39+\pi(-39+463\sqrt{3}-648\,r))+$$
$$313\,956\,(34-24\sqrt{2})\,(5+\pi(-5+61\sqrt{3}-88\,r))\,(\sqrt{3}-3\,r)+$$
$$23\,408\sqrt{3}\,(-4+3\sqrt{2})\,(51+\pi(-51+635\sqrt{3}-936\,r))\,(\sqrt{3}-3\,r)^2+$$
$$17\,472\,(57+\pi(-57+721\sqrt{3}-1080\,r))\,(\sqrt{3}-3\,r)^3\Big)\Big(-\frac{1}{\sqrt{3}}+r\Big)^{11/2}+$$
$$\frac{1}{82\,944\sqrt{3}\,\pi}\,323\,\Big(\frac{3}{8}\,3^{3/4}\,(-4+3\sqrt{2})^{11/2}\,\big(-191-96\sqrt{2}+12\sqrt{3}\,(32+15\sqrt{2})\,r-$$
$$936\,r^2\big)+\frac{32}{3}\,(-\sqrt{3}+3\,r)^{9/2}\,\big(\sqrt{3}\,(-9448+7317\sqrt{2})+$$
$$36\,(-713+360\sqrt{2})\,r+504\sqrt{3}\,(-8+15\sqrt{2})\,r^2-6048\,r^3\big)\Big)$$
$$\big(4\sqrt{2}-\sqrt{3}+\pi+\sqrt{3}\,\pi-2\,\mathrm{ArcSec}[3]\big)\Big)+\frac{1}{497\,664}\,35\,(\sqrt{6}-4\,r)^2$$
$$\Bigg(\frac{4199\times 3^{1/4}\,(-4+3\sqrt{2})^{11/2}\,(\sqrt{6}-4\,r)\,(81+\pi(-81+701\sqrt{3}-512\sqrt{2}\,r))}{\pi}-$$
$$\frac{1}{\pi}\,144\,(\sqrt{6}-4\,r)^{7/2}\,\Big(513\,\big(\sqrt{3}\,(-6154+4005\sqrt{2})-18\,(-951+493\sqrt{2})\,r+21\sqrt{3}$$
$$(-374+105\sqrt{2})\,r^2+4914\,r^3\big)+\pi\,\big(-91\,430\,470+59\,035\,779\sqrt{2}+3\,157\,002$$
$$\sqrt{3}-2\,054\,565\sqrt{6}-6\,(1\,463\,589-758\,727\sqrt{2}-9\,099\,009\sqrt{3}+3\,107\,963$$
$$\sqrt{6})\,r-3\,(15\,072\,662+15\,223\,527\sqrt{2}-1\,343\,034\sqrt{3}+377\,055\sqrt{6})\,r^2+$$
$$182\,(-13\,851+31\,551\sqrt{3}+214\,016\sqrt{6})\,r^3-25\,159\,680\sqrt{2}\,r^4\big)\Big)+$$
$$969\,\Big(13\times 3^{1/4}\,(-4+3\sqrt{2})^{11/2}-16\,(\sqrt{6}-4\,r)^{5/2}\,\big(5\sqrt{3}\,(-122+81\sqrt{2})-$$
$$18\,(-83+45\sqrt{2})\,r+63\sqrt{3}\,(-10+3\sqrt{2})\,r^2+378\,r^3\big)\Big)$$
$$\Big(432-311\sqrt{2}+81\sqrt{6}-\frac{27\,(-16\sqrt{2}+3\sqrt{6}+8\,\mathrm{ArcSec}[3])}{\pi}\Big)\Bigg)\Bigg)\Bigg);$$

```
Simplify[Simplify[(ApprxOcthCF22NotMatchBB[r] + a + b r) /.
   {a → 1/(10 883 808 π) (-(-26 868 843 + 8 902 834 √2 + 262 656 √3 + 2 760 966 √6) π +
        1026 (-23 256 √2 + 256 √3 + 2691 √6 + 11 628 ArcSec[3])),
    b → 1/(143 208 π) ((235 467 - 989 977 √3 + 499 630 √6) π -
        4131 (57 - 160 √3 - 32 √6 + 8 √3 (2 + 5 √2) ArcSec[3]))},
   Assumptions → {OCTDst[[1]] < r < OCTDst[[2]]}] - ApprxOcthCF22PrtlyMatchBB[r]]

Plot[{OCTCFBB[r], ApprxOcthCF22PrtlyMatchBB[r], ApprxOcthCF22NotMatchBB[r]},
 {r, OCTDst[[1]], OCTDst[[2]]},
 PlotStyle → {Directive[Blue, Thickness[0.003]], Directive[Red, Thickness[0.004], Dashed],
   Directive[Purple, Thickness[0.004], Dotted]}, AxesLabel → {"r", "γ0[r]"}]
```

The error at r=D2 is ~0.0011, the relative error is 6%.

```
N[Limit[OCTCFBB[r] - ApprxOcthCF22PrtlyMatchBB[r], r → OCTDst[[2]], Direction → 1]]
N[Limit[OCTCFBB[r] - ApprxOcthCF22PrtlyMatchBB[r], r → OCTDst[[2]], Direction → 1] /
  (Limit[OCTCFBB[r], r → OCTDst[[2]], Direction → 1])]
```

```
0.00116442

0.058113
```

```
ausfiga = Plot[{If[r < OCTDst[[1]], OCTCFAA[r], OCTCFBB[r] ],
   If[r < OCTDst[[1]], OcthCFApprxFnl22AA[r], ApprxOcthCF22PrtlyMatchBB[r]],
   10 * (If[r < OCTDst[[1]], OCTCFAA[r], OCTCFBB[r] ] -
      If[r < OCTDst[[1]], OcthCFApprxFnl22AA[r], ApprxOcthCF22PrtlyMatchBB[r]])},
  {r, 0, OCTDst[[2]]}, PlotRange → {{0, 1.05}, {-0.05, 1.05}}, PlotStyle →
   {Directive[Blue, Thickness[0.003]], Directive[Red, Thickness[0.004], Dashed],
    Directive[Black, Thickness[0.003], Dashed]}, AxesLabel → {"r", "γ0[r],10xΔγ"}]
```

We start now from D4 and proceed towards the left

Approximation within the interval [D3, D4]

```
cfD3Plus
cfD4Minus
```

evaluation of the CLD approximation

```
Simplify[(CLDApprx22[r, D1, D2]) /.
  {a0 → cfD3Plus[[1]], a1 → cfD3Plus[[2]], a2 → cfD3Plus[[3]], a3 → cfD3Plus[[4]],
   a4 → cfD3Plus[[5]], b0 → cfD4Minus[[1]], b1 → cfD4Minus[[2]], b2 → cfD4Minus[[3]],
   b3 → cfD4Minus[[4]], b4 → cfD4Minus[[5]], D1 → OCTDst[[3]], D2 → OCTDst[[4]]},
 Assumptions → {OCTDst[[3]] < r < OCTDst[[4]]}]

Simplify[Simplify[(CLDApprx22[r, D1, D2]) /.
    {a0 → cfD3Plus[[1]], a1 → cfD3Plus[[2]], a2 → cfD3Plus[[3]], a3 → cfD3Plus[[4]],
     a4 → cfD3Plus[[5]], b0 → cfD4Minus[[1]], b1 → cfD4Minus[[2]], b2 → cfD4Minus[[3]],
     b3 → cfD4Minus[[4]], b4 → cfD4Minus[[5]], D1 → OCTDst[[3]], D2 → OCTDst[[4]]},
   Assumptions → {OCTDst[[3]] < r < OCTDst[[4]]}] - OcthCLDApprx22DD[r]]
```

0

OcthCLDApprx22DD[r_] :=

$$\frac{1}{3\pi}\left(-60\sqrt{2}+18\pi+80\sqrt{6}\,\pi+153\,r+27\,\pi\,r-276\sqrt{3}\,\pi\,r-72\sqrt{2}\,r^2+96\sqrt{6}\,\pi\,r^2+288\times2^{1/4}\left(-\sqrt{2}+2\,r\right)^{3/2}\right)\left(1+\frac{\left(-\sqrt{2}+2\,r\right)^{5/2}\left(-275+103\sqrt{2}+\left(515-110\sqrt{2}\right)r+35\left(-11+\sqrt{2}\right)r^2+105\,r^3\right)}{2\left(2-\sqrt{2}\right)^{11/2}}\right);$$

```
Plot[{OCTDDCFDD[r], OcthCLDApprx22DD[r], OcthCLDApprx11DD[r]},
 {r, OCTDst[[3]], OCTDst[[4]]}, AxesLabel → {"r", "γ"[r]"},
 PlotStyle → {Directive[Blue, Thickness[0.003]],
   Directive[Red, Thickness[0.006]], Directive[Purple, Thickness[0.003], Dashed]},
 PlotRange → {{OCTDst[[3]], OCTDst[[4]]}, {-2, 2}}, AxesLabel → {"r", "γ0"[r],10xΔγ"},
 PlotLabel → "CLD: K=2; D3 < r < D4"]
```

The approximation has improved

```
Plot[{OCTDDCFDD[r] - OcthCLDApprx22DD[r]}, {r, OCTDst[[3]], OCTDst[[4]]}]
```

The largest error is ~ -0.7 around 0.84.The relative error 1079% !

```
N[(OCTDDCFDD[r] - OcthCLDApprx22DD[r]) /. {r → 84 / 100}]
N[((OCTDDCFDD[r] - OcthCLDApprx22DD[r]) / OCTDDCFDD[r]) /. {r → 84 / 100}]
```

-0.728566

-10.7963

Evaluation of the CF approximation withint [D3,D4]

```
Simplify[
 (Simplify[Simplify[(LEFTCFApprx22[r, D1, D2]) /. {a0 → cfD3Plus[[1]], a1 → cfD3Plus[[2]],
       a2 → cfD3Plus[[3]], a3 → cfD3Plus[[4]], a4 → cfD3Plus[[5]],
       b0 → cfD4Minus[[1]], b1 → cfD4Minus[[2]], b2 → cfD4Minus[[3]],
       b3 → cfD4Minus[[4]], b4 → cfD4Minus[[5]], D1 → OCTDst[[3]], D2 → OCTDst[[4]]},
     Assumptions → {OCTDst[[3]] < r < OCTDst[[4]]}]]) /.
```

$$\left\{\sqrt{\frac{\sqrt{2}-2r}{-2+\sqrt{2}}} \to \frac{\sqrt{r\sqrt{2}-1}}{\sqrt{\sqrt{2}-1}}\right\}\text{, Assumptions} \to \{\text{OCTDst[[3]]} < r < \text{OCTDst[[4]]}\}\Big]$$

$$\text{FullSimplify}\left[\sqrt{\frac{\sqrt{2}-2r}{-2+\sqrt{2}}} - \frac{\sqrt{r\sqrt{2}-1}}{\sqrt{\sqrt{2}-1}}\text{, Assumptions} \to \{\text{OCTDst[[3]]} < r < \text{OCTDst[[4]]}\}\right]$$

```
FullSimplify[Simplify[(LEFTCFApprx22[r, D1, D2]) /.
    {a0 → cfD3Plus[[1]], a1 → cfD3Plus[[2]], a2 → cfD3Plus[[3]], a3 → cfD3Plus[[4]],
     a4 → cfD3Plus[[5]], b0 → cfD4Minus[[1]], b1 → cfD4Minus[[2]], b2 → cfD4Minus[[3]],
     b3 → cfD4Minus[[4]], b4 → cfD4Minus[[5]], D1 → OCTDst[[3]], D2 → OCTDst[[4]]},
   Assumptions → {OCTDst[[3]] < r < OCTDst[[4]]}] - ApprxOcthCF22NotMatchDD[r],
 Assumptions → {OCTDst[[3]] < r < OCTDst[[4]]}]
```

0

$$
\begin{aligned}
&\text{ApprxOcthCF22NotMatchDD[r\_] := } \frac{1}{56\,434\,560\left(1-\frac{1}{\sqrt{2}}\right)^{11/2}} \\
&\Bigg(\frac{1}{\pi}\,403\,104\times 2^{3/4}\Bigg(-3773+2307\sqrt{2}-105\left(935+263\sqrt{2}\right)r^4+21\left(4357+2420\sqrt{2}\right)r^5- \\
&\quad 504\left(110+103\sqrt{2}\right)r^6+1200\left(23+22\sqrt{2}\right)r^7-1050\left(11+5\sqrt{2}\right)r^8+2450\,r^9- \\
&\quad 5568\sqrt{1-\sqrt{2}-\left(-2+\sqrt{2}\right)r}+3936\sqrt{2}\sqrt{1-\sqrt{2}-\left(-2+\sqrt{2}\right)r}+6\,r^3\Bigg(-3605+ \\
&\qquad 11\,550\sqrt{2}-2624\sqrt{1-\sqrt{2}-\left(-2+\sqrt{2}\right)r}+1856\sqrt{2}\sqrt{1-\sqrt{2}-\left(-2+\sqrt{2}\right)r}\Bigg)+ \\
&\quad 6\,r^2\Bigg(-9625+3605\sqrt{2}-5568\sqrt{1-\sqrt{2}-\left(-2+\sqrt{2}\right)r}+ \\
&\qquad 3936\sqrt{2}\sqrt{1-\sqrt{2}-\left(-2+\sqrt{2}\right)r}\Bigg)+3\,r\Bigg(-20\,783+16\,016\sqrt{2}- \\
&\qquad 7872\sqrt{1-\sqrt{2}-\left(-2+\sqrt{2}\right)r}+5568\sqrt{2}\sqrt{1-\sqrt{2}-\left(-2+\sqrt{2}\right)r}\Bigg)\Bigg)-\frac{1}{\sqrt{2}}\,35 \\
&\quad\Bigg(646\left(6-10\sqrt{\frac{2}{3}}+\frac{9}{\sqrt{2}}-\frac{13}{\sqrt{2}\,\pi}\right)\Bigg(-\left(2-\sqrt{2}\right)^{11/2}\left(15+8\sqrt{2}\,(1-3\,r)-30\,r+39\,r^2\right)- \\
&\qquad \left(-\sqrt{2}+2\,r\right)^{9/2}\left(5\left(-43+25\sqrt{2}\right)+\left(219-90\sqrt{2}\right)r+21\left(-5+\sqrt{2}\right)r^2+21\,r^3\right)\Bigg)- \\
&\quad \frac{1}{\pi}\left(-\sqrt{2}+2\,r\right)^{11/2}\Bigg(-808\,659+439\,581\sqrt{2}+3\left(83\,285+104\,842\sqrt{2}\right)r+ \\
&\qquad \left(90\,915-747\,363\sqrt{2}\right)r^2+273\left(-175+1672\sqrt{2}\right)r^3-98\,280\sqrt{2}\,r^4+ \\
&\qquad \pi\Bigg(-380\,817+212\,895\sqrt{2}+2\,093\,724\sqrt{3}-1\,153\,828\sqrt{6}+\Big(434\,853-168\,606\sqrt{2}- \\
&\qquad\quad 1\,492\,748\sqrt{3}+30\,248\sqrt{6}\Big)r+\Big(-223\,839+41\,895\sqrt{2}+475\,684\sqrt{3}+ \\
&\qquad\quad 884\,764\sqrt{6}\Big)r^2-91\left(-513+668\sqrt{3}+6688\sqrt{6}\right)r^3+131\,040\sqrt{6}\,r^4\Bigg)\Bigg)+ \\
&\quad \left(2-\sqrt{2}\right)^{11/2}\Bigg(-\frac{693\left(-2+\sqrt{2}\right)^4\left(-3+4\sqrt{3}\,\pi\right)}{\sqrt{2}\,\pi}-4522\left(-1+\frac{1}{\sqrt{2}}\right)^2\left(9-28\sqrt{3}+\frac{3}{\pi}\right) \\
&\qquad \left(\frac{1}{\sqrt{2}}-r\right)+\frac{4199\left(\sqrt{2}-2\,r\right)^3\left(15-12\sqrt{2}\,r+\pi\left(9-44\sqrt{3}+16\sqrt{6}\,r\right)\right)}{4\,\pi}- \\
&\qquad \frac{399\left(-2+\sqrt{2}\right)^3\left(21-24\sqrt{2}\,r+\pi\left(-9-4\sqrt{3}+32\sqrt{6}\,r\right)\right)}{2\,\pi}\Bigg)\Bigg)\Bigg)\Bigg);
\end{aligned}
$$

The error of the CF-approximation is -0.0121185 at D_3, i.e. ~495%

```
N[Limit[OCTCFDD[r] - ApprxOcthCF22NotMatchDD[r], r → OCTDst[[3]], Direction → -1]]
N[Limit[OCTCFDD[r] - ApprxOcthCF22NotMatchDD[r], r → OCTDst[[3]], Direction → -1] /
  Limit[OCTCFDD[r], r → OCTDst[[3]], Direction → -1]]
```

```
OcthCFApprxFnl11DD[r]

ApprxOcthCF11NotMatchDD[r]

Plot[{OCTCFDD[r], ApprxOcthCF22NotMatchDD[r](*, ApprxOcthCF11NotMatchDD[r] *)},
 {r, OCTDst[[3]], OCTDst[[4]]}, PlotRange → {{OCTDst[[3]], OCTDst[[4]]}, {-0.02, 0.02}},
 PlotStyle → {Directive[Blue, Thickness[0.003]], Directive[Red, Thickness[0.006]],
   Directive[Magenta, Thickness[0.008], Dotted]}, AxesLabel → {"r", "γ0[r],10xΔγ"}]
```

Approximation within the interval [D2, D3]

approximation of the CLD

```
Simplify[(CLDApprx22[r, D1, D2]) /.
  {a0 → cfD2Plus[[1]], a1 → cfD2Plus[[2]], a2 → cfD2Plus[[3]], a3 → cfD2Plus[[4]],
   a4 → cfD2Plus[[5]], b0 → cfD3Minus[[1]], b1 → cfD3Minus[[2]], b2 → cfD3Minus[[3]],
   b3 → cfD3Minus[[4]], b4 → cfD3Minus[[5]], D1 → OCTDst[[2]], D2 → OCTDst[[3]]},
 Assumptions → {OCTDst[[2]] < r < OCTDst[[3]]}]

Simplify[Simplify[(CLDApprx22[r, D1, D2]) /.
    {a0 → cfD2Plus[[1]], a1 → cfD2Plus[[2]], a2 → cfD2Plus[[3]], a3 → cfD2Plus[[4]],
     a4 → cfD2Plus[[5]], b0 → cfD3Minus[[1]], b1 → cfD3Minus[[2]], b2 → cfD3Minus[[3]],
     b3 → cfD3Minus[[4]], b4 → cfD3Minus[[5]], D1 → OCTDst[[2]], D2 → OCTDst[[3]]},
   Assumptions → {OCTDst[[2]] < r < OCTDst[[3]]}] - OcthCLDApprx22CC[r]]
```

0

$$\text{OcthCLDApprx22CC[r\_]} := \left(6 + 80\sqrt{\frac{2}{3}} + \left(9 - 92\sqrt{3}\right) r + 32\sqrt{6}\, r^2 + \frac{52\sqrt{2} - 237 r + 120\sqrt{2}\, r^2}{\pi}\right)$$
$$\left(1 + \frac{1}{8 \times 2^{1/4}\left(2-\sqrt{3}\right)^{11/2}}\left(\sqrt{2} - 2r\right)^{5/2}\right.$$
$$\left.\left(\sqrt{2}\left(-1316 + 1045\sqrt{3}\right) + 20\left(-329 + 88\sqrt{3}\right) r + 280\sqrt{2}\left(-4 + 11\sqrt{3}\right) r^2 - 3360 r^3\right)\right) +$$
$$\frac{1}{9\pi}\left(1 + \frac{1}{4 \times 2^{3/4}\left(2-\sqrt{3}\right)^{11/2}}\left(-\sqrt{6} + 4r\right)^{5/2}\left(3\sqrt{2}\left(-99 + 35\sqrt{3}\right) + \right.\right.$$
$$\left.\left.\left(1050 - 220\sqrt{3}\right) r + 70\sqrt{2}\left(-11 + \sqrt{3}\right) r^2 + 420 r^3\right)\right)$$
$$\left(\pi\left(108 + 640\sqrt{2} + \left(81 - 1469\sqrt{3}\right) r + 1024\sqrt{2}\, r^2\right) + 27\left(4\sqrt{2} - 3r - 2\,\text{ArcSec}[3]\right)\right);$$

```
Plot[{OCTDDCFCC[r], OcthCLDApprx22CC[r], OcthCLDApprx11CC[r]},
 {r, OCTDst[[2]], OCTDst[[3]]}, AxesLabel → {"r", "γ''[r]"},
 PlotStyle → {Directive[Blue, Thickness[0.003]],
   Directive[Red, Thickness[0.004]], Directive[Purple, Thickness[0.004], Dashed]},
 PlotRange → {{OCTDst[[2]], OCTDst[[3]]}, {0, 7}}, AxesLabel → {"r", "γ0''[r],10xΔγ"},
 PlotLabel → "CLD: K=1; D2 <r<D3"]
```

The approximation has improved. The largest error is about 44% around 0.65

```
Plot[{OCTDDCFCC[r] - OcthCLDApprx22CC[r]}, {r, OCTDst[[2]], OCTDst[[3]]}]
```

```
N[(OCTDDCFCC[65 / 100] - OcthCLDApprx22CC[65 / 100]) / OCTDDCFCC[65 / 100]]
```

```
0.445361
```

```
Plot[{If[r < OCTDst[[2]], OCTDDCFBB[r], If[r < OCTDst[[3]], OCTDDCFCC[r], OCTDDCFDD[r]]],
   If[r < OCTDst[[2]], OcthCLDApprx22BB[r],
    If[r < OCTDst[[3]], OcthCLDApprx22CC[r], OcthCLDApprx22DD[r]]], If[r < OCTDst[[2]],
    OcthCLDApprx00BB[r], If[r < OCTDst[[3]], OcthCLDApprx00CC[r], OcthCLDApprx00DD[r]]]},
  {r, OCTDst[[1]], OCTDst[[4]]}, AxesLabel → {"r", "γ"[r]"},
  PlotStyle → {Directive[Blue, Thickness[0.003]],
    Directive[Red, Thickness[0.004], Dashed], Directive[Purple, Thickness[0.004], Dotted]},
  PlotRange →  {{OCTDst[[1]], OCTDst[[4]]}, {-1.3, 10}}, AxesLabel → {"r", "γo"[r],10xΔγ"}]
```

Overall the CLD approximation does not look very accurate.

Approximation (not matched) of the CF within [D2,D3]

```
Simplify[((LEFTCFApprx22[r, D1, D2]) /.
    {a0 → cfD2Plus[[1]], a1 → cfD2Plus[[2]], a2 → cfD2Plus[[3]], a3 → cfD2Plus[[4]],
     a4 → cfD2Plus[[5]], b0 →  cfD3Minus[[1]], b1 →  cfD3Minus[[2]], b2 →  cfD3Minus[[3]],
     b3 →  cfD3Minus[[4]], b4 →  cfD3Minus[[5]], D1 → OCTDst[[2]], D2 → OCTDst[[3]]}),
  Assumptions → {OCTDst[[2]] < r < OCTDst[[3]]}]
```

```
FullSimplify[
  ((LEFTCFApprx22[r, D1, D2]) /. {a0 → cfD2Plus[[1]], a1 → cfD2Plus[[2]], a2 → cfD2Plus[[3]],
      a3 → cfD2Plus[[4]], a4 → cfD2Plus[[5]], b0 →  cfD3Minus[[1]],
      b1 →  cfD3Minus[[2]], b2 →  cfD3Minus[[3]], b3 →  cfD3Minus[[4]],
      b4 →  cfD3Minus[[5]], D1 → OCTDst[[2]], D2 → OCTDst[[3]]}) -
   ApprxOcthCF22NotMatchCC[r], Assumptions → {OCTDst[[2]] < r < OCTDst[[3]]}]
```

```
0
```

```
ApprxOcthCF22NotMatchCC[r_] :=
```

$$\Bigg(-\frac{1}{\pi}4\times 2^{3/4}\left(2-\sqrt{3}\right)^{11/2}\Big(-1539\left(\sqrt{2}\left(84+115\sqrt{3}\right)-68\left(17+7\sqrt{3}\right)r+1326\sqrt{6}\,r^2-1768\,r^3\right)+\pi\Big(-3\sqrt{2}\left(1\,637\,329+235\,089\sqrt{3}\right)+152\left(137\,616+104\,027\sqrt{3}\right)r+226\,746\sqrt{2}\left(-319+9\sqrt{3}\right)r^2+33\,592\left(-81+1469\sqrt{3}\right)r^3-17\,199\,104\sqrt{2}\,r^4\Big)\Big)+\frac{1}{\pi}\left(-\sqrt{6}+4\,r\right)^{11/2}\Big(-1539\left(\sqrt{2}\left(-2567+1281\sqrt{3}\right)+\left(5334-1972\sqrt{3}\right)r+14\sqrt{2}\left(-187+35\sqrt{3}\right)r^2+1092\,r^3\right)+2\pi\Big(\sqrt{2}\left(-30\,333\,717+20\,140\,570\sqrt{3}\right)-3\left(-5\,268\,833+8\,893\,225\sqrt{3}\right)r+\left(24\,123\,792\sqrt{2}+7\,913\,386\sqrt{6}\right)r^2-182\left(209\,399+10\,517\sqrt{3}\right)r^3+8\,386\,560\sqrt{2}\,r^4\Big)\Big)+72\left(\frac{1}{\sqrt{2}}-r\right)^2\Bigg(-\frac{8398\times 2^{3/4}\left(2-\sqrt{3}\right)^{11/2}\left(\sqrt{2}-2\,r\right)\left(-57+60\sqrt{2}\,r+\pi\left(9-44\sqrt{3}+16\sqrt{6}\,r\right)\right)}{\pi}+323\left(6-10\sqrt{\frac{2}{3}}+\frac{9}{\sqrt{2}}-\frac{13}{\sqrt{2}\,\pi}\right)\Big(156\times 2^{3/4}\left(2-\sqrt{3}\right)^{11/2}+4\sqrt{2}\left(\sqrt{2}-2\,r\right)^{5/2}\left(5\sqrt{2}\left(-332+201\sqrt{3}\right)+12\left(-259+120\sqrt{3}\right)r+168\sqrt{2}\left(-4+5\sqrt{3}\right)r^2-672\,r^3\right)+\frac{1}{\pi}2\sqrt{2}\left(\sqrt{2}-2\,r\right)^{7/2}\Big(-3\Big(7\,720\,300\sqrt{2}-4\,828\,261\sqrt{6}+4\left(648\,809+511\,556\sqrt{3}\right)r+8\sqrt{2}\left(-1\,896\,440+300\,751\sqrt{3}\right)r^2+2912\left(-1217+4180\sqrt{3}\right)r^3-5\,241\,600\sqrt{2}\,r^4\Big)+\pi\Big(30\,937\,776\sqrt{2}-16\,863\,227\sqrt{6}-4\left(-6\,763\,335+6\,046\,588\sqrt{3}\right)r+8\sqrt{2}\left(1\,594\,632+757\,057\sqrt{3}\right)r^2-486\,304\left(57+4\sqrt{3}\right)r^3+4\,193\,280\sqrt{6}\,r^4\Big)\Big)\Big)-646\Big(-4\,2^{3/4}\left(2-\sqrt{3}\right)^{11/2}\left(\frac{19}{2}+4\sqrt{3}-3\sqrt{2}\left(5+4\sqrt{3}\right)r+39\,r^2\right)-\frac{1}{2}\left(-\sqrt{6}+4\,r\right)^{9/2}\left(\sqrt{2}\left(-251+129\sqrt{3}\right)-6\left(-77+30\sqrt{3}\right)r+42\sqrt{2}\left(-5+\sqrt{3}\right)r^2+84\,r^3\right)\Big)\left(432-311\sqrt{2}+81\sqrt{6}-\frac{27\left(-16\sqrt{2}+3\sqrt{6}+8\,\mathrm{ArcSec}[3]\right)}{\pi}\right)\Bigg)\Bigg)\Big/\left(7\,255\,872\times 2^{3/4}\left(2-\sqrt{3}\right)^{11/2}\right);$$

```
Plot[{OCTCFCC[r], ApprxOcthCF22NotMatchCC[r]},
 {r, OCTDst[[2]], OCTDst[[3]]}, AxesLabel → {"r", "γ[r]"},
 PlotStyle → {Directive[Blue, Thickness[0.003]], Directive[Red, Thickness[0.006]]},
 AxesLabel → {"r", "γ0[r],10xΔγ"}]
```

```
Plot[{If[r < OCTDst[[2]], OCTCFBB[r], If[r < OCTDst[[3]], OCTCFCC[r], OCTCFDD[r]]],
   If[r < OCTDst[[2]], ApprxOcthCF22PrtlyMatchBB[r],
    If[r < OCTDst[[3]], ApprxOcthCF22NotMatchCC[r], ApprxOcthCF22NotMatchDD[r]]]},
  {r, OCTDst[[1]], OCTDst[[4]]}, PlotRange → {{OCTDst[[1]], OCTDst[[4]]}, {-0.020, 0.04}},
  AxesLabel → {"r", "γ[r]"},
  PlotStyle → {Directive[Blue, Thickness[0.003]], Directive[Red, Thickness[0.006]]},
  AxesLabel → {"r", "γ0[r]"}]
```

FINAL MATCHES

The approximation ApprxOcthCF22PrtlyMatchBB[r] becomes OcthCFApprxFnl22BB[r] ;
the approximation ApprxOcthCF22NotMatchDD[r] becomes OcthCFApprxFnl22DD[r] because we want to preserve the positivity of the CF. Finally, macthing ApprxOcthCF22NotMatchCC[r] to OcthCFApprxFnl22BB[r] at r=D2 it becomes ApprxOcthCF22PrtlyMatchCC[r]. Finally, adding to it EXTRCNTRBRgt22[r_, D1_, D2_, a_, b_] and matching the result to OcthCFApprxFnl22DD[r] at r=D3 we obtain OcthCFApprxFnl22DD[r]

```
ApprxOcthCF22PrtlyMatchBB[r]
```

```
OcthCFApprxFnl22BB[r_] :=
```

$$\frac{1}{10\,883\,808\,\pi}\Big(\big(26\,868\,843 - 8\,902\,834\sqrt{2} - 262\,656\sqrt{3} - 2\,760\,966\sqrt{6}\big)\pi + 1026\big(-23\,256\sqrt{2} + 256\sqrt{3} + 2691\sqrt{6} + 11\,628\,\text{ArcSec}[3]\big)\Big) +$$

$$\frac{1}{143\,208\,\pi}\, r\,\Big(\big(235\,467 - 989\,977\sqrt{3} + 499\,630\sqrt{6}\big)\pi - 4131\big(57 - 160\sqrt{3} - 32\sqrt{6} + 8\sqrt{3}\big(2 + 5\sqrt{2}\big)\text{ArcSec}[3]\big)\Big) +$$

$$\frac{1}{146\,965\left(-4 + 3\sqrt{2}\right)^{11/2}}\, 48 \times 3^{3/4}\Bigg(-1120\Bigg(\frac{1}{18\,432 \times 3^{3/4}}\left(-4 + 3\sqrt{2}\right)^{11/2}$$

$$\Bigg(\frac{2079}{8}\left(-577 + 408\sqrt{2}\right) - \frac{133\sqrt{3}\left(-4 + 3\sqrt{2}\right)^{3}\left(-3 + \pi\left(3 + 5\sqrt{3} - 72\,r\right)\right)}{16\,\pi} +$$

$$\frac{2261\left(34 - 24\sqrt{2}\right)\left(3 + \left(-3 + 19\sqrt{3}\right)\pi\right)\left(\sqrt{3} - 3\,r\right)}{24\,\pi} -$$

$$\frac{8398\left(3 + \pi\left(-3 + 31\sqrt{3} - 36\,r\right)\right)\left(\sqrt{3} - 3\,r\right)^{3}}{9\,\pi}\Bigg) - \frac{1}{576\,\pi}\Big(6783\sqrt{3}\left(-4 + 3\sqrt{2}\right)^{3}$$

$$\left(39 + \pi\left(-39 + 463\sqrt{3} - 648\,r\right)\right) + 313\,956\left(34 - 24\sqrt{2}\right)\left(5 + \pi\left(-5 + 61\sqrt{3} - 88\,r\right)\right)$$

$$\left(\sqrt{3} - 3\,r\right) + 23\,408\sqrt{3}\left(-4 + 3\sqrt{2}\right)\left(51 + \pi\left(-51 + 635\sqrt{3} - 936\,r\right)\right)\left(\sqrt{3} - 3\,r\right)^{2} +$$

$$17472\left(57+\pi\left(-57+721\sqrt{3}-1080\,r\right)\right)\left(\sqrt{3}-3\,r\right)^{3}\Big)\left(-\frac{1}{\sqrt{3}}+r\right)^{11/2}+\frac{1}{82944\sqrt{3}\,\pi}$$

$$323\left(\frac{3}{8}\,3^{3/4}\left(-4+3\sqrt{2}\right)^{11/2}\left(-191-96\sqrt{2}+12\sqrt{3}\left(32+15\sqrt{2}\right)r-936\,r^{2}\right)+\right.$$

$$\frac{32}{3}\left(-\sqrt{3}+3\,r\right)^{9/2}\left(\sqrt{3}\left(-9448+7317\sqrt{2}\right)+\right.$$

$$\left.\left.36\left(-713+360\sqrt{2}\right)r+504\sqrt{3}\left(-8+15\sqrt{2}\right)r^{2}-6048\,r^{3}\right)\right)$$

$$\left.\left(4\sqrt{2}-\sqrt{3}+\pi+\sqrt{3}\,\pi-2\,\mathrm{ArcSec}[3]\right)\right)+\frac{1}{497664}\,35\left(\sqrt{6}-4\,r\right)^{2}$$

$$\left(\frac{4199\times 3^{1/4}\left(-4+3\sqrt{2}\right)^{11/2}\left(\sqrt{6}-4\,r\right)\left(81+\pi\left(-81+701\sqrt{3}-512\sqrt{2}\,r\right)\right)}{\pi}-\right.$$

$$\frac{1}{\pi}\,144\left(\sqrt{6}-4\,r\right)^{7/2}\left(513\left(\sqrt{3}\left(-6154+4005\sqrt{2}\right)-18\left(-951+493\sqrt{2}\right)r+21\sqrt{3}\right.\right.$$

$$\left.\left(-374+105\sqrt{2}\right)r^{2}+4914\,r^{3}\right)+\pi\left(-91430470+59035779\sqrt{2}+3157002\sqrt{3}-\right.$$

$$2054565\sqrt{6}-6\left(1463589-758727\sqrt{2}-9099009\sqrt{3}+3107963\sqrt{6}\right)r-$$

$$3\left(15072662+15223527\sqrt{2}-1343034\sqrt{3}+377055\sqrt{6}\right)r^{2}+$$

$$\left.\left.182\left(-13851+31551\sqrt{3}+214016\sqrt{6}\right)r^{3}-25159680\sqrt{2}\,r^{4}\right)\right)+$$

$$969\left(13\times 3^{1/4}\left(-4+3\sqrt{2}\right)^{11/2}-16\left(\sqrt{6}-4\,r\right)^{5/2}\left(5\sqrt{3}\left(-122+81\sqrt{2}\right)-\right.\right.$$

$$\left.\left.18\left(-83+45\sqrt{2}\right)r+63\sqrt{3}\left(-10+3\sqrt{2}\right)r^{2}+378\,r^{3}\right)\right)$$

$$\left.\left.\left.\left(432-311\sqrt{2}+81\sqrt{6}-\frac{27\left(-16\sqrt{2}+3\sqrt{6}+8\,\mathrm{ArcSec}[3]\right)}{\pi}\right)\right)\right)\right)$$

ApprxOcthCF22NotMatchDD[r]

$$
\begin{aligned}
&\texttt{OcthCFApprxFnl22DD[r\_] :=} \frac{1}{56\,434\,560 \left(1-\frac{1}{\sqrt{2}}\right)^{11/2}} \\
&\Bigg(\frac{1}{\pi} 403\,104 \times 2^{3/4} \Big(-3773+2307\sqrt{2}-105\left(935+263\sqrt{2}\right) r^4+21\left(4357+2420\sqrt{2}\right) r^5- \\
&\quad 504\left(110+103\sqrt{2}\right) r^6+1200\left(23+22\sqrt{2}\right) r^7-1050\left(11+5\sqrt{2}\right) r^8+2450 r^9- \\
&\quad 5568\sqrt{1-\sqrt{2}-\left(-2+\sqrt{2}\right) r}+3936\sqrt{2}\sqrt{1-\sqrt{2}-\left(-2+\sqrt{2}\right) r}+6 r^3 \Big(-3605+ \\
&\qquad 11\,550\sqrt{2}-2624\sqrt{1-\sqrt{2}-\left(-2+\sqrt{2}\right) r}+1856\sqrt{2}\sqrt{1-\sqrt{2}-\left(-2+\sqrt{2}\right) r}\Big)+ \\
&\quad 6 r^2\Big(-9625+3605\sqrt{2}-5568\sqrt{1-\sqrt{2}-\left(-2+\sqrt{2}\right) r}+ \\
&\qquad 3936\sqrt{2}\sqrt{1-\sqrt{2}-\left(-2+\sqrt{2}\right) r}\Big)+3 r\Big(-20\,783+16\,016\sqrt{2}- \\
&\qquad 7872\sqrt{1-\sqrt{2}-\left(-2+\sqrt{2}\right) r}+5568\sqrt{2}\sqrt{1-\sqrt{2}-\left(-2+\sqrt{2}\right) r}\Big)\Big)-\frac{1}{\sqrt{2}} 35 \\
&\;\Bigg(646\left(6-10\sqrt{\frac{2}{3}}+\frac{9}{\sqrt{2}}-\frac{13}{\sqrt{2}\,\pi}\right)\Big(-\left(2-\sqrt{2}\right)^{11/2}\left(15+8\sqrt{2}\,(1-3 r)-30 r+39 r^2\right)- \\
&\qquad \left(-\sqrt{2}+2 r\right)^{9/2}\left(5\left(-43+25\sqrt{2}\right)+\left(219-90\sqrt{2}\right) r+21\left(-5+\sqrt{2}\right) r^2+21 r^3\right)\Big)- \\
&\quad \frac{1}{\pi}\left(-\sqrt{2}+2 r\right)^{11/2}\Big(-808\,659+439\,581\sqrt{2}+3\left(83\,285+104\,842\sqrt{2}\right) r+ \\
&\qquad \left(90\,915-747\,363\sqrt{2}\right) r^2+273\left(-175+1672\sqrt{2}\right) r^3-98\,280\sqrt{2}\, r^4+ \\
&\qquad \pi\Big(-380\,817+212\,895\sqrt{2}+2\,093\,724\sqrt{3}-1\,153\,828\sqrt{6}+\big(434\,853-168\,606\sqrt{2}- \\
&\qquad\quad 1\,492\,748\sqrt{3}+30\,248\sqrt{6}\big) r+\big(-223\,839+41\,895\sqrt{2}+475\,684\sqrt{3}+ \\
&\qquad\quad 884\,764\sqrt{6}\big) r^2-91\left(-513+668\sqrt{3}+6688\sqrt{6}\right) r^3+131\,040\sqrt{6}\, r^4\Big)\Big)+ \\
&\quad \left(2-\sqrt{2}\right)^{11/2}\Bigg(-\frac{693\left(-2+\sqrt{2}\right)^4\left(-3+4\sqrt{3}\,\pi\right)}{\sqrt{2}\,\pi}-4522\left(-1+\frac{1}{\sqrt{2}}\right)^2\left(9-28\sqrt{3}+\frac{3}{\pi}\right) \\
&\qquad \left(\frac{1}{\sqrt{2}}-r\right)+\frac{4199\left(\sqrt{2}-2 r\right)^3\left(15-12\sqrt{2}\, r+\pi\left(9-44\sqrt{3}+16\sqrt{6}\, r\right)\right)}{4\pi}- \\
&\qquad \frac{399\left(-2+\sqrt{2}\right)^3\left(21-24\sqrt{2}\, r+\pi\left(-9-4\sqrt{3}+32\sqrt{6}\, r\right)\right)}{2\pi}\Bigg)\Bigg)\Bigg);
\end{aligned}
$$

```
Plot[{If[r < OCTDst[[2]], OCTCFBB[r], If[r < OCTDst[[3]], OCTCFCC[r], OCTCFDD[r]]],
   If[r < OCTDst[[2]], OcthCFApprxFnl22BB[r],
    If[r < OCTDst[[3]], ApprxOcthCF22NotMatchCC[r], OcthCFApprxFnl22DD[r]]]},
  {r, OCTDst[[1]], OCTDst[[4]]}, PlotRange → {{OCTDst[[1]], OCTDst[[4]]}, {-0.020, 0.04}},
  AxesLabel → {"r", "γ[r]"},
  PlotStyle → {Directive[Blue, Thickness[0.003]], Directive[Red, Thickness[0.006]]},
  AxesLabel → {"r", "γ0[r]"}]

Simplify[Limit[ApprxOcthCF22NotMatchCC[r] + a + b r, r → OCTDst[[2]], Direction → -1] -
   Limit[OcthCFApprxFnl22BB[r], r → OCTDst[[2]], Direction → 1]]

Simplify[Limit[D[ApprxOcthCF22NotMatchCC[r] + a + b r, r], r → OCTDst[[2]], Direction → -1] -
   Limit[D[OcthCFApprxFnl22BB[r], r], r → OCTDst[[2]], Direction → 1]]

FullSimplify[Solve[
   {Simplify[Limit[ApprxOcthCF22NotMatchCC[r] + a + b r, r → OCTDst[[2]], Direction → -1] -
        Limit[OcthCFApprxFnl22BB[r], r → OCTDst[[2]], Direction → 1]] == 0 && Simplify[
       Limit[D[ApprxOcthCF22NotMatchCC[r] + a + b r, r], r → OCTDst[[2]], Direction → -1] -
        Limit[D[OcthCFApprxFnl22BB[r], r], r → OCTDst[[2]], Direction → 1]] == 0}, {a, b}]]
```

Simplify[(ApprxOcthCF22NotMatchCC[r] + a + b r) /.

$$\left\{a \to -\frac{1}{10\,883\,808\,\pi}\left(\left(-28\,020\,015 + 8\,534\,791\sqrt{2} + 60\sqrt{3}\left(60\,192 + 33\,697\sqrt{2}\right)\right)\pi + 27\left(377\,165\sqrt{2} - 9728\sqrt{3} + 195\,508\sqrt{6} - 3876\left(73 + 32\sqrt{3}\right)\text{ArcSec}[3]\right)\right),\right.$$
$$\left. b \to \frac{1}{286\,416\,\pi}\left(\left(1\,881\,648 + 991\,440\sqrt{2} - 2\,849\,567\sqrt{3} + 503\,540\sqrt{6}\right)\pi + 324\left(6191 - 294\sqrt{3} + 816\sqrt{6} - 102\left(10\sqrt{2} + 4\sqrt{3} + 5\sqrt{6}\right)\text{ArcSec}[3]\right)\right)\right\},$$

Assumptions → {OCTDst[[2]] < r < OCTDst[[3]]}]

Simplify[((ApprxOcthCF22NotMatchCC[r] + a + b r) /.

$$\left\{a \to -\frac{1}{10\,883\,808\,\pi}\left(\left(-28\,020\,015 + 8\,534\,791\sqrt{2} + 60\sqrt{3}\left(60\,192 + 33\,697\sqrt{2}\right)\right)\pi + 27\left(377\,165\sqrt{2} - 9728\sqrt{3} + 195\,508\sqrt{6} - 3876\left(73 + 32\sqrt{3}\right)\text{ArcSec}[3]\right)\right),\right.$$
$$\left. b \to \frac{1}{286\,416\,\pi}\left(\left(1\,881\,648 + 991\,440\sqrt{2} - 2\,849\,567\sqrt{3} + 503\,540\sqrt{6}\right)\pi + 324\left(6191 - 294\sqrt{3} + 816\sqrt{6} - 102\left(10\sqrt{2} + 4\sqrt{3} + 5\sqrt{6}\right)\text{ArcSec}[3]\right)\right)\right\}) -$$

ApprxOcthCF22PrtlyMatchCC[r], Assumptions → {OCTDst[[2]] < r < OCTDst[[3]]}]

ApprxOcthCF22PrtlyMatchCC[r_] :=

$$\frac{1}{43\,535\,232}\left(-\frac{1}{\pi}4\left(\left(-28\,020\,015 + 8\,534\,791\sqrt{2} + 60\sqrt{3}\left(60\,192 + 33\,697\sqrt{2}\right)\right)\pi + \right.\right.$$

$$27\left(377165\sqrt{2}-9728\sqrt{3}+195508\sqrt{6}-3876\left(73+32\sqrt{3}\right)\text{ArcSec}[3]\right)\Big)+$$
$$\frac{1}{\pi}152\,r\left(\left(1881648+991440\sqrt{2}-2849567\sqrt{3}+503540\sqrt{6}\right)\pi+\right.$$
$$\left.324\left(6191-294\sqrt{3}+816\sqrt{6}-102\left(10\sqrt{2}+4\sqrt{3}+5\sqrt{6}\right)\text{ArcSec}[3]\right)\right)+$$
$$\frac{1}{\left(2-\sqrt{3}\right)^{11/2}}3\times 2^{1/4}\left(-\frac{1}{\pi}4\times 2^{3/4}\left(2-\sqrt{3}\right)^{11/2}\right.$$
$$\left(-1539\left(\sqrt{2}\left(84+115\sqrt{3}\right)-68\left(17+7\sqrt{3}\right)r+1326\sqrt{6}\,r^2-1768\,r^3\right)+\right.$$
$$\pi\left(-3\sqrt{2}\left(1637329+235089\sqrt{3}\right)+152\left(137616+104027\sqrt{3}\right)r+\right.$$
$$\left.\left.226746\sqrt{2}\left(-319+9\sqrt{3}\right)r^2+33592\left(-81+1469\sqrt{3}\right)r^3-17199104\sqrt{2}\,r^4\right)\right)+$$
$$\frac{1}{\pi}\left(-\sqrt{6}+4r\right)^{11/2}\left(-1539\left(\sqrt{2}\left(-2567+1281\sqrt{3}\right)+\left(5334-1972\sqrt{3}\right)r+\right.\right.$$
$$\left.14\sqrt{2}\left(-187+35\sqrt{3}\right)r^2+1092\,r^3\right)+2\pi\left(\sqrt{2}\left(-30333717+20140570\sqrt{3}\right)-\right.$$
$$3\left(-5268833+8893225\sqrt{3}\right)r+\left(24123792\sqrt{2}+7913386\sqrt{6}\right)r^2-$$
$$\left.\left.182\left(209399+10517\sqrt{3}\right)r^3+8386560\sqrt{2}\,r^4\right)\right)+72\left(\frac{1}{\sqrt{2}}-r\right)^2$$
$$\left(-\frac{8398\times 2^{3/4}\left(2-\sqrt{3}\right)^{11/2}\left(\sqrt{2}-2r\right)\left(-57+60\sqrt{2}\,r+\pi\left(9-44\sqrt{3}+16\sqrt{6}\,r\right)\right)}{\pi}+\right.$$
$$323\left(6-10\sqrt{\frac{2}{3}}+\frac{9}{\sqrt{2}}-\frac{13}{\sqrt{2}\,\pi}\right)\left(156\times 2^{3/4}\left(2-\sqrt{3}\right)^{11/2}+\right.$$
$$4\sqrt{2}\left(\sqrt{2}-2r\right)^{5/2}\left(5\sqrt{2}\left(-332+201\sqrt{3}\right)+12\left(-259+120\sqrt{3}\right)r+\right.$$
$$\left.168\sqrt{2}\left(-4+5\sqrt{3}\right)r^2-672\,r^3\right)+\frac{1}{\pi}2\sqrt{2}\left(\sqrt{2}-2r\right)^{7/2}$$
$$\left(-3\left(7720300\sqrt{2}-4828261\sqrt{6}+4\left(648809+511556\sqrt{3}\right)r+8\sqrt{2}\left(-1896440+\right.\right.\right.$$
$$\left.\left.300751\sqrt{3}\right)r^2+2912\left(-1217+4180\sqrt{3}\right)r^3-5241600\sqrt{2}\,r^4\right)+$$
$$\pi\left(30937776\sqrt{2}-16863227\sqrt{6}-4\left(-6763335+6046588\sqrt{3}\right)r+8\sqrt{2}\right.$$
$$\left.\left.\left.\left(1594632+757057\sqrt{3}\right)r^2-486304\left(57+4\sqrt{3}\right)r^3+4193280\sqrt{6}\,r^4\right)\right)\right)-$$

$$646\left(-4\,2^{3/4}\left(2-\sqrt{3}\right)^{11/2}\left(\frac{19}{2}+4\sqrt{3}-3\sqrt{2}\left(5+4\sqrt{3}\right)r+39r^2\right)-\frac{1}{2}\left(-\sqrt{6}+4r\right)^{9/2}\right.$$
$$\left(\sqrt{2}\left(-251+129\sqrt{3}\right)-6\left(-77+30\sqrt{3}\right)r+42\sqrt{2}\left(-5+\sqrt{3}\right)r^2+84r^3\right)\Bigg)$$
$$\left.\left(432-311\sqrt{2}+81\sqrt{6}-\frac{27\left(-16\sqrt{2}+3\sqrt{6}+8\,\mathrm{ArcSec}[3]\right)}{\pi}\right)\right)\Bigg);$$

```
Plot[{If[r < OCTDst[[2]], OCTCFBB[r], OCTCFCC[r]],
   If[r < OCTDst[[2]], OcthCFApprxFnl22BB[r], ApprxOcthCF22PrtlyMatchCC[r]]
   (*  ,If[r<OCTDst[[2]],OcthCFApprxFnl22BB[r],OcthCFApprxFnl22CC[r]] *) },
  {r, OCTDst[[1]], OCTDst[[3]] }, AxesLabel → {"r", "γ[r]"},
  PlotStyle → {Directive[Blue, Thickness[0.003]],
    Directive[Red, Thickness[0.004]], Directive[Black, Thickness[0.003], Dotted]}]
```

```
N[OCTDst]
```

```
{0.57735, 0.612372, 0.707107, 1.}
```

```
OCTCFCC[r]
```

```
Plot[{If[r < OCTDst[[2]], OCTCFBB[r], If[r < OCTDst[[3]], OCTCFCC[r], OCTCFDD[r] ]] ,
   If[r < OCTDst[[2]], OcthCFApprxFnl22BB[r],
    If[r < OCTDst[[3]], ApprxOcthCF22PrtlyMatchCC[r], OcthCFApprxFnl22DD[r]]] },
  {r, OCTDst[[1]], OCTDst[[4]]}, AxesLabel → {"r", "γ[r]"},
  PlotRange → {{OCTDst[[1]], OCTDst[[4]]}, {-0.03, 0.05}},
  PlotStyle → {Directive[Blue, Thickness[0.003]],
    Directive[Red, Thickness[0.004]], Directive[Black, Thickness[0.003], Dotted]}]
```

ApprxOcthCF22NotMatchCc[r] is now matched to OcthCFApprxFnl22DD[r]

```
Simplify[(EXTRCNTRBRgt22[r, D1, D2, a, b]) /. {D1 → OCTDst[[2]], D2 → OCTDst[[3]]}]
Simplify[(EXTRCNTRBLft22[r, D1, D2, a, b]) /. {D1 → OCTDst[[2]], D2 → OCTDst[[3]]}]
```

EXTRCNTRBLft22act[r_] := $-\frac{1}{2580480}\left(\sqrt{2}-2r\right)^5$
$\left(18a\left(\sqrt{2}\left(-46+33\sqrt{3}\right)+64\left(-3+\sqrt{3}\right)r+40\sqrt{2}\left(-1+2\sqrt{3}\right)r^2-80r^3\right)+b\left(-1255+648\sqrt{3}+32\sqrt{2}\left(-58+45\sqrt{3}\right)r+480\left(-10+3\sqrt{3}\right)r^2+160\sqrt{2}\left(-4+9\sqrt{3}\right)r^3-1120r^4\right)\right);$

EXTRCNTRBRgt22act[r_] := $-\frac{1}{82575360}\left(\sqrt{6}-4r\right)^5$
$\left(-9a\left(\sqrt{2}\left(-148+59\sqrt{3}\right)+\left(484-128\sqrt{3}\right)r+40\sqrt{2}\left(-8+\sqrt{3}\right)r^2+160r^3\right)+b\left(450-417\sqrt{3}+3\sqrt{2}\left(98+155\sqrt{3}\right)r-180\left(15+2\sqrt{3}\right)r^2+40\sqrt{2}\left(54+\sqrt{3}\right)r^3-1120r^4\right)\right);$

```
Series[EXTRCNTRBLft22act[r], {r, OCTDst[[2]], 4}]
```

```
Series[EXTRCNTRBLft22act[r], {r, OCTDst[[3]], 4}]
```

```
Series[EXTRCNTRBRgt22act[r], {r, OCTDst[[2]], 3}]
```

```
Series[EXTRCNTRBRgt22act[r], {r, OCTDst[[3]], 6}]
```

```
Limit[ApprxOcthCF22NotMatchCC[r], r → OCTDst[[3]], Direction → 1]
```

```
0
```

```
Limit[EXTRCNTRBRgt22act[r], r → OCTDst[[3]], Direction → 1]
```

```
Limit[OcthCFApprxFnl22DD[r], r → OCTDst[[3]], Direction → -1]
```

```
Limit[D[ApprxOcthCF22NotMatchCC[r], r], r → OCTDst[[3]], Direction → 1]
Limit[D[EXTRCNTRBRgt22act[r], r], r → OCTDst[[3]], Direction → 1]
```

```
Limit[D[OcthCFApprxFnl22DD[r], r], r → OCTDst[[3]], Direction → -1]

Simplify[
 Solve[{Simplify[Limit[ApprxOcthCF22PrtlyMatchCC[r], r → OCTDst[[3]], Direction → 1] +
       Limit[EXTRCNTRBRgt22act[r], r → OCTDst[[3]], Direction → 1] -
       Limit[OcthCFApprxFnl22DD[r], r → OCTDst[[3]], Direction → -1]] == 0 &&
    Simplify[Limit[D[ApprxOcthCF22PrtlyMatchCC[r], r], r → OCTDst[[3]], Direction → 1] +
       Limit[D[EXTRCNTRBRgt22act[r], r], r → OCTDst[[3]], Direction → 1] -
       Limit[D[OcthCFApprxFnl22DD[r], r], r → OCTDst[[3]], Direction → -1]] == 0}, {a, b}]]
```

Simplify[(ApprxOcthCF22PrtlyMatchCC[r] + EXTRCNTRBRgt22act[r]) /.

$$\Big\{a \to -\frac{1}{340119\left(2-\sqrt{2}\right)^{11/2}\pi}\,4096\Big(-35\sqrt{2-\sqrt{2}}\,\big(-3139696019251800 + 2220112804936521\sqrt{2} - 1812704342080746\sqrt{3} + 1281782725697354\sqrt{6}\big)\pi + 162\Big(35\sqrt{4-2\sqrt{2}}\,(3054536155141 + 120481733226\,\text{ArcSec}[3]) - 35\sqrt{2-\sqrt{2}}\,(4319094679585 + 170646206244\,\text{ArcSec}[3]) + 4\Big(-665\sqrt{6-3\sqrt{2}}\,(32810927309 + 1296350436\,\text{ArcSec}[3]) + 2^{1/4}\Big(415701\big(-869137442 + 614572749\sqrt{2} - 501796736\sqrt{3} + 354823742\sqrt{6}\big) + 35\times 2^{1/4}\sqrt{6-3\sqrt{2}}\,(440884317779 + 17390040282\,\text{ArcSec}[3])\Big)\Big)\Big)\Big),$$

$$b \to -\frac{1}{37791\left(2-\sqrt{2}\right)^{11/2}\pi}\,8192\left(708158977 + 408855776\sqrt{3}\right)\Big(35\sqrt{2-\sqrt{2}}\,\big(9841852687064 - 6962416670200\sqrt{2} - 5682193316070\sqrt{3} + 4019751030077\sqrt{6}\big)\pi - 108\Big(-4760\sqrt{2-\sqrt{2}}\,(498787061 + 73987710\,\text{ArcSec}[3]) + 3\Big(140\sqrt{4-2\sqrt{2}}\,(3998433803 + 593215340\,\text{ArcSec}[3]) + 70\sqrt{6-3\sqrt{2}}\,(6527430815 + 968248056\,\text{ArcSec}[3]) + 19\times 2^{1/4}\Big(4862\big(18333833 - 12964832\sqrt{2} - 10585036\sqrt{3} + 7485244\sqrt{6}\big) - 105\times 2^{1/4}\sqrt{6-3\sqrt{2}}\,(161999803 + 24034600\,\text{ArcSec}[3])\Big)\Big)\Big)\Big)\Big\}\Big]$$

```
OcthCFApprxFnl22CC[r_] :=
```

$$\frac{1}{43\,535\,232}\left(-\frac{1}{\pi}4\left(\left(-28\,020\,015+8\,534\,791\sqrt{2}+60\sqrt{3}\left(60\,192+33\,697\sqrt{2}\right)\right)\pi+27\left(377\,165\sqrt{2}-9728\sqrt{3}+195\,508\sqrt{6}-3876\left(73+32\sqrt{3}\right)\text{ArcSec}[3]\right)\right)+\frac{1}{\pi}152\,r\left(\left(1\,881\,648+991\,440\sqrt{2}-2\,849\,567\sqrt{3}+503\,540\sqrt{6}\right)\pi+324\left(6191-294\sqrt{3}+816\sqrt{6}-102\left(10\sqrt{2}+4\sqrt{3}+5\sqrt{6}\right)\text{ArcSec}[3]\right)\right)+\frac{1}{\left(2-\sqrt{3}\right)^{11/2}}3\times 2^{1/4}\left(-\frac{1}{\pi}4\times 2^{3/4}\left(2-\sqrt{3}\right)^{11/2}\left(-1539\left(\sqrt{2}\left(84+115\sqrt{3}\right)-68\left(17+7\sqrt{3}\right)r+1326\sqrt{6}\,r^2-1768\,r^3\right)+\pi\left(-3\sqrt{2}\left(1\,637\,329+235\,089\sqrt{3}\right)+152\left(137\,616+104\,027\sqrt{3}\right)r+226\,746\sqrt{2}\left(-319+9\sqrt{3}\right)r^2+33\,592\left(-81+1469\sqrt{3}\right)r^3-17\,199\,104\sqrt{2}\,r^4\right)\right)+\frac{1}{\pi}\left(-\sqrt{6}+4\,r\right)^{11/2}\left(-1539\left(\sqrt{2}\left(-2567+1281\sqrt{3}\right)+\left(5334-1972\sqrt{3}\right)r+14\sqrt{2}\left(-187+35\sqrt{3}\right)r^2+1092\,r^3\right)+2\pi\left(\sqrt{2}\left(-30\,333\,717+20\,140\,570\sqrt{3}\right)-3\left(-5\,268\,833+8\,893\,225\sqrt{3}\right)r+\left(24\,123\,792\sqrt{2}+7\,913\,386\sqrt{6}\right)r^2-182\left(209\,399+10\,517\sqrt{3}\right)r^3+8\,386\,560\sqrt{2}\,r^4\right)\right)+72\left(\frac{1}{\sqrt{2}}-r\right)^2\left(-\frac{8398\times 2^{3/4}\left(2-\sqrt{3}\right)^{11/2}\left(\sqrt{2}-2\,r\right)\left(-57+60\sqrt{2}\,r+\pi\left(9-44\sqrt{3}+16\sqrt{6}\,r\right)\right)}{\pi}+323\left(6-10\sqrt{\frac{2}{3}}+\frac{9}{\sqrt{2}}-\frac{13}{\sqrt{2}\,\pi}\right)\left(156\times 2^{3/4}\left(2-\sqrt{3}\right)^{11/2}+4\sqrt{2}\left(\sqrt{2}-2\,r\right)^{5/2}\left(5\sqrt{2}\left(-332+201\sqrt{3}\right)+12\left(-259+120\sqrt{3}\right)r+168\sqrt{2}\left(-4+5\sqrt{3}\right)r^2-672\,r^3\right)\right)+\frac{1}{\pi}2\sqrt{2}\left(\sqrt{2}-2\,r\right)^{7/2}\left(-3\left(7\,720\,300\sqrt{2}-4\,828\,261\sqrt{6}+4\left(648\,809+511\,556\sqrt{3}\right)r+8\sqrt{2}\left(-1\,896\,440+300\,751\sqrt{3}\right)r^2+2912\left(-1217+4180\sqrt{3}\right)r^3-5\,241\,600\sqrt{2}\,r^4\right)+\pi\left(30\,937\,776\sqrt{2}-16\,863\,227\sqrt{6}-4\left(-6\,763\,335+6\,046\,588\sqrt{3}\right)r+8\sqrt{2}\right.\right.$$

$$\left(1\,594\,632+757\,057\sqrt{3}\right) r^2-486\,304\left(57+4\sqrt{3}\right) r^3+4\,193\,280\sqrt{6}\; r^4\right)\Big)\Bigg) -$$

$$646\left(-4\; 2^{3/4}\left(2-\sqrt{3}\right)^{11/2}\left(\frac{19}{2}+4\sqrt{3}-3\sqrt{2}\left(5+4\sqrt{3}\right) r+39\, r^2\right)-\frac{1}{2}\left(-\sqrt{6}+4\, r\right)^{9/2}\right.$$

$$\left(\sqrt{2}\left(-251+129\sqrt{3}\right)-6\left(-77+30\sqrt{3}\right) r+42\sqrt{2}\left(-5+\sqrt{3}\right) r^2+84\, r^3\right)\Big)$$

$$\left(432-311\sqrt{2}+81\sqrt{6}-\frac{27\left(-16\sqrt{2}+3\sqrt{6}+8\,\mathrm{ArcSec}[3]\right)}{\pi}\right)\Bigg)\Bigg)\Bigg) -$$

$$\Bigg(\left(\sqrt{6}-4\, r\right)^5\Bigg(-2\left(708\,158\,977+408\,855\,776\sqrt{3}\right)\left(450-417\sqrt{3}+3\sqrt{2}\left(98+155\sqrt{3}\right) r-\right.$$

$$\left.180\left(15+2\sqrt{3}\right) r^2+40\sqrt{2}\left(54+\sqrt{3}\right) r^3-1120\, r^4\right)$$

$$\Bigg(35\sqrt{2-\sqrt{2}}\,\Big(9\,841\,852\,687\,064-6\,962\,416\,670\,200\sqrt{2}-$$

$$5\,682\,193\,316\,070\sqrt{3}+4\,019\,751\,030\,077\sqrt{6}\Big)\pi-$$

$$108\Bigg(-4760\sqrt{2-\sqrt{2}}\,(498\,787\,061+73\,987\,710\,\mathrm{ArcSec}[3])+$$

$$3\Bigg(140\sqrt{4-2\sqrt{2}}\,(3\,998\,433\,803+593\,215\,340\,\mathrm{ArcSec}[3])+$$

$$70\sqrt{6-3\sqrt{2}}\,(6\,527\,430\,815+968\,248\,056\,\mathrm{ArcSec}[3])+$$

$$19\times 2^{1/4}\Bigg(4862\left(18\,333\,833-12\,964\,832\sqrt{2}-10\,585\,036\sqrt{3}+7\,485\,244\sqrt{6}\right)-$$

$$105\times 2^{1/4}\sqrt{6-3\sqrt{2}}\,(161\,999\,803+24\,034\,600\,\mathrm{ArcSec}[3])\Bigg)\Bigg)\Bigg)\Bigg)+$$

$$\left(\sqrt{2}\left(-148+59\sqrt{3}\right)+\left(484-128\sqrt{3}\right) r+40\sqrt{2}\left(-8+\sqrt{3}\right) r^2+160\, r^3\right)$$

$$\Bigg(-35\sqrt{2-\sqrt{2}}\,\Big(-3\,139\,696\,019\,251\,800+2\,220\,112\,804\,936\,521\sqrt{2}-$$

$$1\,812\,704\,342\,080\,746\sqrt{3}+1\,281\,782\,725\,697\,354\sqrt{6}\Big)\pi+$$

$$162\Bigg(35\sqrt{4-2\sqrt{2}}\,(3\,054\,536\,155\,141+120\,481\,733\,226\,\mathrm{ArcSec}[3])-$$

$$35\sqrt{2-\sqrt{2}}\,(4\,319\,094\,679\,585+170\,646\,206\,244\,\mathrm{ArcSec}[3])+$$

$$4\Bigg(-665\sqrt{6-3\sqrt{2}}\,(32\,810\,927\,309+1\,296\,350\,436\,\mathrm{ArcSec}[3])+$$

$$2^{1/4}\left(415701\left(-869137442+614572749\sqrt{2}-501796736\sqrt{3}+354823742\sqrt{6}\right)+35\times 2^{1/4}\sqrt{6-3\sqrt{2}}\,(440884317779+17390040282\,\text{ArcSec}[3])\right)\Big)\Big)\Big)\Big)\Big/\left(761866560\left(2-\sqrt{2}\right)^{11/2}\pi\right);$$

In[27]:=
```
fig22 = Plot[{If[r < OCTDst[[1]], OCTCFAA[r],
    If[r < OCTDst[[2]], OCTCFBB[r], If[r < OCTDst[[3]], OCTCFCC[r], OCTCFDD[r]]]],
   If[r < OCTDst[[1]], OcthCFApprxFnl22AA[r], If[r < OCTDst[[2]], OcthCFApprxFnl22BB[r],
     If[r < OCTDst[[3]], OcthCFApprxFnl22CC[r], OcthCFApprxFnl22DD[r]]]],
   10 * If[r < OCTDst[[1]], (OCTCFAA[r] - OcthCFApprxFnl22AA[r]),
     If[r < OCTDst[[2]], (OCTCFBB[r] - OcthCFApprxFnl22BB[r]), If[r < OCTDst[[3]],
       (OCTCFCC[r] - OcthCFApprxFnl22CC[r]), (OCTCFDD[r] - OcthCFApprxFnl22DD[r])]]]},
  {r, 0.35, 1}, PlotStyle → {Directive[Blue, Thickness[0.005]],
    Directive[Red, Thickness[0.005], Dashed], Directive[Black, Thickness[0.003], Dotted]},
  AxesLabel → {"r", " γ0(r), 10xΔγ"}]

Show[{fig22, fig11}]

Plot[{If[r < OCTDst[[1]], OCTDDCFAA[r],
   If[r < OCTDst[[2]], OCTDDCFBB[r], If[r < OCTDst[[3]], OCTDDCFCC[r], OCTDDCFDD[r]]]],
  If[r < OCTDst[[1]], OcthCLDApprx22AA[r], If[r < OCTDst[[2]], OcthCLDApprx22BB[r],
    If[r < OCTDst[[3]], OcthCLDApprx22CC[r], OcthCLDApprx22DD[r]]]]}, {r, -0, 1},
 PlotRange → {{0, 1}, {-1.2, 10}}, PlotStyle → {Directive[Blue, Thickness[0.003]],
   Directive[Red, Thickness[0.005], Dashed], Directive[Black, Thickness[0.003], Dotted]},
 AxesLabel → {"r", " γ0''(r), Δγ''"}]
```

The final formulae of the tetrahedron-CF approximation for the case K=1 and with the BB procedure B are

In[25]:=

$$\text{OcthCFApprxFnl22AA}[r\_] := 1-\frac{3\sqrt{3}\,r}{2}+\frac{6\sqrt{2}\,r^2}{\pi}+\frac{3r^3}{2}-\frac{\sqrt{3}\,r^3}{2}-\frac{3r^3}{2\pi}-\frac{3r^2\,\text{ArcSec}[3]}{\pi};$$

$$\text{OcthCFApprxFnl22BB}[r\_] := \frac{1}{10883808\pi}\left(\left(26868843-8902834\sqrt{2}-262656\sqrt{3}-2760966\sqrt{6}\right)\pi+1026\left(-23256\sqrt{2}+256\sqrt{3}+2691\sqrt{6}+11628\,\text{ArcSec}[3]\right)\right)+\frac{1}{143208\pi}r\left(\left(235467-989977\sqrt{3}+499630\sqrt{6}\right)\pi-4131\left(57-160\sqrt{3}-32\sqrt{6}+8\sqrt{3}\left(2+5\sqrt{2}\right)\text{ArcSec}[3]\right)\right)+\frac{1}{146965\left(-4+3\sqrt{2}\right)^{11/2}}48\times 3^{3/4}\left(-1120\left(\frac{1}{18432\times 3^{3/4}}\left(-4+3\sqrt{2}\right)^{11/2}\right.\right.$$

$$\left(\frac{2079}{8}\left(-577+408\sqrt{2}\right)-\frac{133\sqrt{3}\left(-4+3\sqrt{2}\right)^3\left(-3+\pi\left(3+5\sqrt{3}-72\,r\right)\right)}{16\,\pi}+\right.$$

$$\frac{2261\left(34-24\sqrt{2}\right)\left(3+\left(-3+19\sqrt{3}\right)\pi\right)\left(\sqrt{3}-3\,r\right)}{24\,\pi}-$$

$$\left.\frac{8398\left(3+\pi\left(-3+31\sqrt{3}-36\,r\right)\right)\left(\sqrt{3}-3\,r\right)^3}{9\,\pi}\right)-\frac{1}{576\,\pi}\left(6783\sqrt{3}\left(-4+3\sqrt{2}\right)^3\right.$$

$$\left(39+\pi\left(-39+463\sqrt{3}-648\,r\right)\right)+313\,956\left(34-24\sqrt{2}\right)\left(5+\pi\left(-5+61\sqrt{3}-88\,r\right)\right)$$

$$\left(\sqrt{3}-3\,r\right)+23\,408\sqrt{3}\left(-4+3\sqrt{2}\right)\left(51+\pi\left(-51+635\sqrt{3}-936\,r\right)\right)\left(\sqrt{3}-3\,r\right)^2+$$

$$\left.17\,472\left(57+\pi\left(-57+721\sqrt{3}-1080\,r\right)\right)\left(\sqrt{3}-3\,r\right)^3\right)\left(-\frac{1}{\sqrt{3}}+r\right)^{11/2}+\frac{1}{82\,944\sqrt{3}\,\pi}$$

$$323\left(\frac{3}{8}3^{3/4}\left(-4+3\sqrt{2}\right)^{11/2}\left(-191-96\sqrt{2}+12\sqrt{3}\left(32+15\sqrt{2}\right)r-936\,r^2\right)+\right.$$

$$\frac{32}{3}\left(-\sqrt{3}+3\,r\right)^{9/2}\left(\sqrt{3}\left(-9448+7317\sqrt{2}\right)+\right.$$

$$\left.\left.36\left(-713+360\sqrt{2}\right)r+504\sqrt{3}\left(-8+15\sqrt{2}\right)r^2-6048\,r^3\right)\right)$$

$$\left.\left(4\sqrt{2}-\sqrt{3}+\pi+\sqrt{3}\,\pi-2\,\mathrm{ArcSec}[3]\right)\right)+\frac{1}{497\,664}35\left(\sqrt{6}-4\,r\right)^2$$

$$\left(\frac{4199\times3^{1/4}\left(-4+3\sqrt{2}\right)^{11/2}\left(\sqrt{6}-4\,r\right)\left(81+\pi\left(-81+701\sqrt{3}-512\sqrt{2}\,r\right)\right)}{\pi}-\right.$$

$$\frac{1}{\pi}144\left(\sqrt{6}-4\,r\right)^{7/2}\left(513\left(\sqrt{3}\left(-6154+4005\sqrt{2}\right)-18\left(-951+493\sqrt{2}\right)r+21\sqrt{3}\right.\right.$$

$$\left.\left(-374+105\sqrt{2}\right)r^2+4914\,r^3\right)+\pi\left(-91\,430\,470+59\,035\,779\sqrt{2}+3\,157\,002\sqrt{3}-\right.$$

$$2\,054\,565\sqrt{6}-6\left(1\,463\,589-758\,727\sqrt{2}-9\,099\,009\sqrt{3}+3\,107\,963\sqrt{6}\right)r-$$

$$3\left(15\,072\,662+15\,223\,527\sqrt{2}-1\,343\,034\sqrt{3}+377\,055\sqrt{6}\right)r^2+$$

$$\left.\left.182\left(-13\,851+31\,551\sqrt{3}+214\,016\sqrt{6}\right)r^3-25\,159\,680\sqrt{2}\,r^4\right)\right)+$$

$$969\left(13\times3^{1/4}\left(-4+3\sqrt{2}\right)^{11/2}-16\left(\sqrt{6}-4\,r\right)^{5/2}\left(5\sqrt{3}\left(-122+81\sqrt{2}\right)-\right.\right.$$

$$18\left(-83+45\sqrt{2}\right)r+63\sqrt{3}\left(-10+3\sqrt{2}\right)r^{2}+378r^{3}\Big)\Big)\Big(432-311\sqrt{2}+81\sqrt{6}-$$

$$\frac{27\left(-16\sqrt{2}+3\sqrt{6}+8\,\mathrm{ArcSec}[3]\right)}{\pi}\Big)\Big)\Big)\Big);\ \mathrm{OcthCFApprxFnl22CC}[r\_]:=$$

$$\frac{1}{43\,535\,232}\Bigg(-\frac{1}{\pi}4\Big(\Big(-28\,020\,015+8\,534\,791\sqrt{2}+60\sqrt{3}\left(60\,192+33\,697\sqrt{2}\right)\Big)\pi+$$

$$27\left(377\,165\sqrt{2}-9728\sqrt{3}+195\,508\sqrt{6}-3876\left(73+32\sqrt{3}\right)\mathrm{ArcSec}[3]\right)\Big)+$$

$$\frac{1}{\pi}152\,r\Big(\Big(1\,881\,648+991\,440\sqrt{2}-2\,849\,567\sqrt{3}+503\,540\sqrt{6}\Big)\pi+$$

$$324\left(6191-294\sqrt{3}+816\sqrt{6}-102\left(10\sqrt{2}+4\sqrt{3}+5\sqrt{6}\right)\mathrm{ArcSec}[3]\right)\Big)+$$

$$\frac{1}{\left(2-\sqrt{3}\right)^{11/2}}3\times 2^{1/4}\Bigg(-\frac{1}{\pi}4\times 2^{3/4}\left(2-\sqrt{3}\right)^{11/2}$$

$$\Big(-1539\left(\sqrt{2}\left(84+115\sqrt{3}\right)-68\left(17+7\sqrt{3}\right)r+1326\sqrt{6}\,r^{2}-1768r^{3}\right)+$$

$$\pi\Big(-3\sqrt{2}\left(1\,637\,329+235\,089\sqrt{3}\right)+152\left(137\,616+104\,027\sqrt{3}\right)r+$$

$$226\,746\sqrt{2}\left(-319+9\sqrt{3}\right)r^{2}+33\,592\left(-81+1469\sqrt{3}\right)r^{3}-17\,199\,104\sqrt{2}\,r^{4}\Big)\Big)+$$

$$\frac{1}{\pi}\left(-\sqrt{6}+4r\right)^{11/2}\Big(-1539\Big(\sqrt{2}\left(-2567+1281\sqrt{3}\right)+\left(5334-1972\sqrt{3}\right)r+$$

$$14\sqrt{2}\left(-187+35\sqrt{3}\right)r^{2}+1092r^{3}\Big)+2\pi\Big(\sqrt{2}\left(-30\,333\,717+20\,140\,570\sqrt{3}\right)-$$

$$3\left(-5\,268\,833+8\,893\,225\sqrt{3}\right)r+\left(24\,123\,792\sqrt{2}+7\,913\,386\sqrt{6}\right)r^{2}-$$

$$182\left(209\,399+10\,517\sqrt{3}\right)r^{3}+8\,386\,560\sqrt{2}\,r^{4}\Big)\Big)+72\left(\frac{1}{\sqrt{2}}-r\right)^{2}$$

$$\Bigg(-\frac{8398\times 2^{3/4}\left(2-\sqrt{3}\right)^{11/2}\left(\sqrt{2}-2r\right)\left(-57+60\sqrt{2}\,r+\pi\left(9-44\sqrt{3}+16\sqrt{6}\,r\right)\right)}{\pi}+$$

$$323\left(6-10\sqrt{\frac{2}{3}}+\frac{9}{\sqrt{2}}-\frac{13}{\sqrt{2}\,\pi}\right)\Big(156\times 2^{3/4}\left(2-\sqrt{3}\right)^{11/2}+$$

$$4\sqrt{2}\left(\sqrt{2}-2r\right)^{5/2}\Big(5\sqrt{2}\left(-332+201\sqrt{3}\right)+12\left(-259+120\sqrt{3}\right)r+$$

$$168\sqrt{2}\ (-4+5\sqrt{3})\ r^2-672\,r^3))+\frac{1}{\pi}2\sqrt{2}\ (\sqrt{2}-2\,r)^{7/2}$$

$$(-3\ (7\,720\,300\sqrt{2}-4\,828\,261\sqrt{6}+4\ (648\,809+511\,556\sqrt{3})\ r+8\sqrt{2}\ (-1\,896\,440+$$

$$300\,751\sqrt{3})\ r^2+2912\ (-1217+4180\sqrt{3})\ r^3-5\,241\,600\sqrt{2}\ r^4)+$$

$$\pi\ (30\,937\,776\sqrt{2}-16\,863\,227\sqrt{6}-4\ (-6\,763\,335+6\,046\,588\sqrt{3})\ r+8\sqrt{2}$$

$$(1\,594\,632+757\,057\sqrt{3})\ r^2-486\,304\ (57+4\sqrt{3})\ r^3+4\,193\,280\sqrt{6}\ r^4)))-$$

$$646\ (-4\ 2^{3/4}\ (2-\sqrt{3})^{11/2}\ (\frac{19}{2}+4\sqrt{3}-3\sqrt{2}\ (5+4\sqrt{3})\ r+39\,r^2)-\frac{1}{2}\ (-\sqrt{6}+4\,r)^{9/2}$$

$$(\sqrt{2}\ (-251+129\sqrt{3})-6\ (-77+30\sqrt{3})\ r+42\sqrt{2}\ (-5+\sqrt{3})\ r^2+84\,r^3))$$

$$(432-311\sqrt{2}+81\sqrt{6}-\frac{27\ (-16\sqrt{2}+3\sqrt{6}+8\,\mathrm{ArcSec}[3])}{\pi})))-$$

$$((\sqrt{6}-4\,r)^5\ (-2\ (708\,158\,977+408\,855\,776\sqrt{3})\ (450-417\sqrt{3}+3\sqrt{2}\ (98+155\sqrt{3})\ r-$$

$$180\ (15+2\sqrt{3})\ r^2+40\sqrt{2}\ (54+\sqrt{3})\ r^3-1120\,r^4)$$

$$(35\sqrt{2-\sqrt{2}}\ (9\,841\,852\,687\,064-6\,962\,416\,670\,200\sqrt{2}-5\,682\,193\,316\,070\sqrt{3}+$$

$$4\,019\,751\,030\,077\sqrt{6})\ \pi-108\ (-4760\sqrt{2-\sqrt{2}}\ (498\,787\,061+73\,987\,710$$

$$\mathrm{ArcSec}[3])+3\ (140\sqrt{4-2\sqrt{2}}\ (3\,998\,433\,803+593\,215\,340\,\mathrm{ArcSec}[3])+$$

$$70\sqrt{6-3\sqrt{2}}\ (6\,527\,430\,815+968\,248\,056\,\mathrm{ArcSec}[3])+$$

$$19\times 2^{1/4}\ (4862\ (18\,333\,833-12\,964\,832\sqrt{2}-10\,585\,036\sqrt{3}+7\,485\,244\sqrt{6})-$$

$$105\times 2^{1/4}\sqrt{6-3\sqrt{2}}\ (161\,999\,803+24\,034\,600\,\mathrm{ArcSec}[3])))))+$$

$$(\sqrt{2}\ (-148+59\sqrt{3})+(484-128\sqrt{3})\ r+40\sqrt{2}\ (-8+\sqrt{3})\ r^2+160\,r^3)$$

$$(-35\sqrt{2-\sqrt{2}}\ (-3\,139\,696\,019\,251\,800+2\,220\,112\,804\,936\,521\sqrt{2}-$$

$$1\,812\,704\,342\,080\,746\sqrt{3}+1\,281\,782\,725\,697\,354\sqrt{6})\ \pi+$$

$$162\ (35\sqrt{4-2\sqrt{2}}\ (3\,054\,536\,155\,141+120\,481\,733\,226\,\mathrm{ArcSec}[3])-$$

$$35\sqrt{2-\sqrt{2}}\ (4\,319\,094\,679\,585 + 170\,646\,206\,244\ \mathrm{ArcSec}[3]) +$$
$$4\left(-665\sqrt{6-3\sqrt{2}}\ (32\,810\,927\,309 + 1\,296\,350\,436\ \mathrm{ArcSec}[3]) + 2^{1/4}\right.$$
$$\left(415\,701\left(-869\,137\,442 + 614\,572\,749\sqrt{2} - 501\,796\,736\sqrt{3} + 354\,823\,742\sqrt{6}\right) +\right.$$
$$\left.\left.35\times 2^{1/4}\sqrt{6-3\sqrt{2}}\ (440\,884\,317\,779 + 17\,390\,040\,282\ \mathrm{ArcSec}[3])\right)\right)\Big)\Big)\Big)\Big)\Big/$$
$$\left(761\,866\,560\left(2-\sqrt{2}\right)^{11/2}\pi\right);\ \mathrm{OcthCFApprxFnl22DD}[r\_] :=$$

$$\frac{1}{56\,434\,560\left(1-\frac{1}{\sqrt{2}}\right)^{11/2}}$$
$$\left(\frac{1}{\pi}\right.$$
$$403\,104\times$$
$$2^{3/4}$$
$$\left(-3773 + 2307\sqrt{2} - 105\left(935 + 263\sqrt{2}\right)r^4 +\right.$$
$$21\left(4357 + 2420\sqrt{2}\right)r^5 - 504\left(110 + 103\sqrt{2}\right)r^6 +$$
$$1200\left(23 + 22\sqrt{2}\right)r^7 - 1050\left(11 + 5\sqrt{2}\right)r^8 +$$
$$2450\,r^9 - 5568\sqrt{1-\sqrt{2}-\left(-2+\sqrt{2}\right)r} +$$
$$3936\sqrt{2}\sqrt{1-\sqrt{2}-\left(-2+\sqrt{2}\right)r} +$$
$$6\,r^3\left(-3605 + 11\,550\sqrt{2} - 2624\sqrt{1-\sqrt{2}-\left(-2+\sqrt{2}\right)r} +\right.$$
$$\left.1856\sqrt{2}\sqrt{1-\sqrt{2}-\left(-2+\sqrt{2}\right)r}\right) + 6\,r^2\left(-9625 + 3605\sqrt{2} -\right.$$
$$\left.5568\sqrt{1-\sqrt{2}-\left(-2+\sqrt{2}\right)r} + 3936\sqrt{2}\sqrt{1-\sqrt{2}-\left(-2+\sqrt{2}\right)r}\right) +$$
$$3\,r\left(-20\,783 + 16\,016\sqrt{2} - 7872\sqrt{1-\sqrt{2}-\left(-2+\sqrt{2}\right)r} +\right.$$

$$5568\sqrt{2}\sqrt{1-\sqrt{2}-\left(-2+\sqrt{2}\right)r}\Bigg)\Bigg) -$$

$$\frac{1}{\sqrt{2}}35\left(646\left(6-10\sqrt{\frac{2}{3}}+\frac{9}{\sqrt{2}}-\frac{13}{\sqrt{2}\,\pi}\right)\left(-\left(2-\sqrt{2}\right)^{11/2}\left(15+8\sqrt{2}\,(1-3\,r)-30\,r+39\,r^2\right)-\right.\right.$$

$$\left(-\sqrt{2}+2\,r\right)^{9/2}\left(5\left(-43+25\sqrt{2}\right)+\left(219-90\sqrt{2}\right)r+21\left(-5+\sqrt{2}\right)r^2+21\,r^3\right)\Big)-$$

$$\frac{1}{\pi}\left(-\sqrt{2}+2\,r\right)^{11/2}\Big(-808\,659+439\,581\sqrt{2}+3\left(83\,285+104\,842\sqrt{2}\right)r+$$

$$\left(90\,915-747\,363\sqrt{2}\right)r^2+273\left(-175+1672\sqrt{2}\right)r^3-98\,280\sqrt{2}\,r^4+$$

$$\pi\Big(-380\,817+212\,895\sqrt{2}+2\,093\,724\sqrt{3}-1\,153\,828\sqrt{6}+\Big(434\,853-168\,606\sqrt{2}-$$

$$1\,492\,748\sqrt{3}+30\,248\sqrt{6}\Big)r+\Big(-223\,839+41\,895\sqrt{2}+475\,684\sqrt{3}+$$

$$884\,764\sqrt{6}\Big)r^2-91\left(-513+668\sqrt{3}+6688\sqrt{6}\right)r^3+131\,040\sqrt{6}\,r^4\Big)\Big)+$$

$$\left(2-\sqrt{2}\right)^{11/2}\left(-\frac{693\left(-2+\sqrt{2}\right)^4\left(-3+4\sqrt{3}\,\pi\right)}{\sqrt{2}\,\pi}-4522\left(-1+\frac{1}{\sqrt{2}}\right)^2\left(9-28\sqrt{3}+\frac{3}{\pi}\right)\right.$$

$$\left(\frac{1}{\sqrt{2}}-r\right)+\frac{4199\left(\sqrt{2}-2\,r\right)^3\left(15-12\sqrt{2}\,r+\pi\left(9-44\sqrt{3}+16\sqrt{6}\,r\right)\right)}{4\,\pi}-$$

$$\left.\left.\left.\frac{399\left(-2+\sqrt{2}\right)^3\left(21-24\sqrt{2}\,r+\pi\left(-9-4\sqrt{3}+32\sqrt{6}\,r\right)\right)}{2\,\pi}\right)\right)\right);$$

In[26]:=

$$\text{OcthCLDApprx22AA}[r\_] := \frac{12\sqrt{2}}{\pi} + 9r - 3\sqrt{3}\,r - \frac{9r}{\pi} - \frac{6\,\text{ArcSec}[3]}{\pi};$$

$$\text{OcthCLDApprx22BB}[r\_] := \frac{1}{9\pi}\Bigg(\Bigg(1 - \frac{1}{\left(-4+3\sqrt{2}\right)^{11/2}}\,6\times 3^{3/4}\left(\sqrt{6}-4r\right)^{5/2}\left(4\sqrt{3}\left(-55+27\sqrt{2}\right) - 30\left(-36+11\sqrt{2}\right)r + 35\sqrt{3}\left(-22+3\sqrt{2}\right)r^2 + 630r^3\right)\Bigg)\left(\pi\left(108+640\sqrt{2}+\left(81-1469\sqrt{3}\right)r+1024\sqrt{2}\,r^2\right)+27\left(4\sqrt{2}-3r-2\,\text{ArcSec}[3]\right)\right) + 27\Bigg(1 - \frac{1}{3\times 3^{3/4}\left(-4+3\sqrt{2}\right)^{11/2}}\,4\left(-\sqrt{3}+3r\right)^{5/2}\left(\sqrt{3}\left(-7640+8349\sqrt{2}\right)+60\left(-955+264\sqrt{2}\right)r+840\sqrt{3}\left(-8+33\sqrt{2}\right)r^2-30\,240\,r^3\right)\Bigg)\left(4\sqrt{2}-3r+\pi\left(44+\left(3-67\sqrt{3}\right)r+72r^2\right)-2\,\text{ArcSec}[3]\right)\Bigg);$$

$$\text{OcthCLDApprx22CC}[r\_] := \left(6+80\sqrt{\frac{2}{3}}+\left(9-92\sqrt{3}\right)r+32\sqrt{6}\,r^2+\frac{52\sqrt{2}-237r+120\sqrt{2}\,r^2}{\pi}\right)\left(1+\frac{1}{8\times 2^{1/4}\left(2-\sqrt{3}\right)^{11/2}}\left(\sqrt{2}-2r\right)^{5/2}\left(\sqrt{2}\left(-1316+1045\sqrt{3}\right)+20\left(-329+88\sqrt{3}\right)r+280\sqrt{2}\left(-4+11\sqrt{3}\right)r^2-3360r^3\right)\right)+\frac{1}{9\pi}\left(1+\frac{\left(-\sqrt{6}+4r\right)^{5/2}\left(3\sqrt{2}\left(-99+35\sqrt{3}\right)+\left(1050-220\sqrt{3}\right)r+70\sqrt{2}\left(-11+\sqrt{3}\right)r^2+420r^3\right)}{4\times 2^{3/4}\left(2-\sqrt{3}\right)^{11/2}}\right)\left(\pi\left(108+640\sqrt{2}+\left(81-1469\sqrt{3}\right)r+1024\sqrt{2}\,r^2\right)+27\left(4\sqrt{2}-3r-2\,\text{ArcSec}[3]\right)\right);$$

$$\text{OcthCLDApprx22DD}[r\_] := \frac{1}{3\pi}\left(-60\sqrt{2}+18\pi+80\sqrt{6}\,\pi+153r+27\pi r-276\sqrt{3}\,\pi r-72\sqrt{2}\,r^2+96\sqrt{6}\,\pi r^2+288\times 2^{1/4}\left(-\sqrt{2}+2r\right)^{3/2}\right)\left(1+\frac{\left(-\sqrt{2}+2r\right)^{5/2}\left(-275+103\sqrt{2}+\left(515-110\sqrt{2}\right)r+35\left(-11+\sqrt{2}\right)r^2+105r^3\right)}{2\left(2-\sqrt{2}\right)^{11/2}}\right);$$

Check of the sum rule $4\pi\int_0^\infty r^2\,\gamma(r)\,dr = V_p$

In our case V_p = `VOcth = 1 / 6.`

We find 0.11777 with an error ~ -0.0098 (~ 6%) that will appear in the FT, i.e. ΔI(0) ~ -0.0098.

```
N[1/6]
```

0.166667

```
4 π (NIntegrate[r^2 * OcthCFApprxFnl22AA[r], {r, 0, OCTDst[[1]]},
     WorkingPrecision → 30, PrecisionGoal → 15] + NIntegrate[r^2 * OcthCFApprxFnl22BB[r],
     {r, OCTDst[[1]], OCTDst[[2]]}, WorkingPrecision → 30, PrecisionGoal → 15] +
    NIntegrate[r^2 * OcthCFApprxFnl22CC[r], {r, OCTDst[[2]], OCTDst[[3]]},
     WorkingPrecision → 30, PrecisionGoal → 15 ] + NIntegrate[r^2 * OcthCFApprxFnl22DD[r],
     {r, OCTDst[[3]], OCTDst[[4]]}, WorkingPrecision → 30, PrecisionGoal → 15 ])
```

0.176480811450121089451773832601

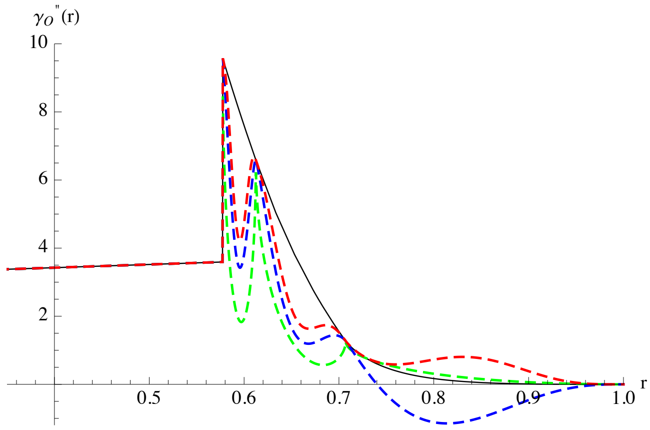
```
N[VOcth - 0.176480811450121089451773]
N[(VOcth - 0.176480811450121089451773) / VOcth]
```

-0.00981414

-0.0588849

```
In[28]:= FigOCTAllCLDAprx = Plot[{If[r < OCTDst[[1]], OCTDDCFAA[r],
      If[r < OCTDst[[2]], OCTDDCFBB[r], If[r < OCTDst[[3]], OCTDDCFCC[r], OCTDDCFDD[r]]]],
    If[r < OCTDst[[1]], OcthCLDApprx00AA[r], If[r < OCTDst[[2]], OcthCLDApprx00BB[r],
      If[r < OCTDst[[3]], OcthCLDApprx00CC[r], OcthCLDApprx00DD[r]]]],
    If[r < OCTDst[[1]], OcthCLDApprx11AA[r], If[r < OCTDst[[2]], OcthCLDApprx11BB[r],
      If[r < OCTDst[[3]], OcthCLDApprx11CC[r], OcthCLDApprx11DD[r]]]],
    If[r < OCTDst[[1]], OcthCLDApprx22AA[r], If[r < OCTDst[[2]], OcthCLDApprx22BB[r],
      If[r < OCTDst[[3]], OcthCLDApprx22CC[r], OcthCLDApprx22DD[r]]]] }, {r, -0, 1},
   PlotRange → {{0.35, 1}, {-1.2, 10}}, PlotStyle → {Directive[Black, Thickness[0.002]],
     Directive[Green, Thickness[0.004], Dashed], Directive[Blue, Thickness[0.004], Dashed],
     Directive[Red, Thickness[0.004], Dashed]}, AxesLabel → {"r", " γO"(r)"}]
```

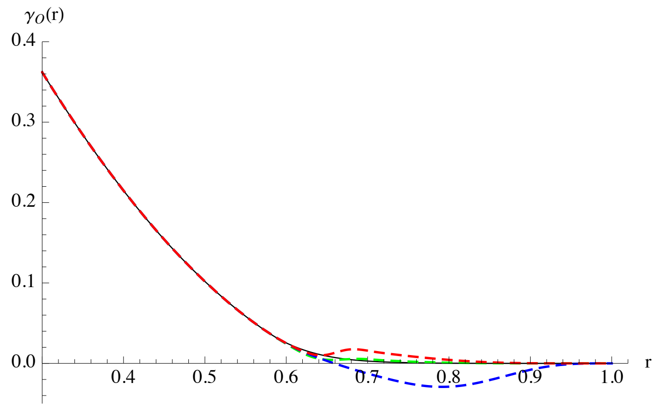
Out[28]=




```
Export["FigOCTAllCLDAprx.eps", FigOCTAllCLDAprx];
```

In[29]:=
```
FigOCTAllCFAprx = Plot[{If[r < OCTDst[[1]], OCTCFAA[r],
    If[r < OCTDst[[2]], OCTCFBB[r], If[r < OCTDst[[3]], OCTCFCC[r], OCTCFDD[r]]]],
   If[r < OCTDst[[1]], OcthCFApprxFnl00AA[r], If[r < OCTDst[[2]], OcthCFApprxFnl00BB[r],
     If[r < OCTDst[[3]], OcthCFApprxFnl00CC[r], OcthCFApprxFnl00DD[r]]]],
   If[r < OCTDst[[1]], OcthCFApprxFnl11AA[r], If[r < OCTDst[[2]], OcthCFApprxFnl11BB[r],
     If[r < OCTDst[[3]], OcthCFApprxFnl11CC[r], OcthCFApprxFnl11DD[r]]]],
   If[r < OCTDst[[1]], OcthCFApprxFnl22AA[r], If[r < OCTDst[[2]], OcthCFApprxFnl22BB[r],
     If[r < OCTDst[[3]], OcthCFApprxFnl22CC[r], OcthCFApprxFnl22DD[r]]]] }, {r, 0.3, 1},
  PlotRange → {{0.3, 1.02}, {-0.05, 0.4}}, PlotStyle → {Directive[Black, Thickness[0.002]],
    Directive[Green, Thickness[0.004], Dashed], Directive[Blue, Thickness[0.004], Dashed],
    Directive[Red, Thickness[0.004], Dashed]}, AxesLabel → {"r", " γO(r)"}]
```

Out[29]=




```
Export["FigOCTAllCFAprx.eps", FigOCTAllCFAprx];
```

EVALUATION OF THE FOURIER TRANSFORM

We have $I(q) = \frac{4\pi}{q}\int_0^\infty r\ \mathrm{Sin}[q\,r]\,\gamma[r]\,dr = \frac{2\pi S}{V\,q^4} - \frac{4\pi}{q^4}\int_0^\infty (2\,\mathrm{Cos}[q\,r] + r\,q\ \mathrm{Sin}[q\,r])\,\gamma''[r]\,dr$

The last expression is obtained integrating twice by part the first integral considering the successive primitives of r Sin[q r] [See Ciccariello J. Appl. Cryst. 38, 97, (2005)].
The use of the last expression is "dangerous" when q < 1 as we already found in the tetrahedron case.

Consequently we shall confine ourselves to the first formula.

In[51]:=
```
Npnt = 832; Qgrid = Table[0, {i, 1, Npnt}];
Solve[Log[10, 10^(-5) f^20] == Log[10, 10^(-2)], f];
Solve[{Log[10, 10^(-2) f^30] == Log[10, 99 / 100]}, f];
Qstep = 8 / 10; Q0 = 1;
Do[ Qgrid[[i]] = 10^(-5) ((10^(3/20))^(i - 1)), {i, 1, 21}];
Do[J = i + 21; Qgrid[[J]] = ((3^(1/15) 11^(1/30))^(J - 21) / 100), {i, 1, 30}];
Q0 = 1; Do[J = i + 51; Qgrid[[J]] = Q0 + Qstep * (i - 1), {i, 1, 25}]; Kont = 76;
Do[Q0 = 500 * (K - 1); If[Q0 < 1, Q0 = 100];
  Do[Kont = Kont + 1; Qgrid[[Kont]] = Q0 + Qstep * (i - 1);, {i, 1, 36}];, {K, 1, 21}];
Kont
Npnt = Kont;
N[Qgrid]
N[Qgrid[[Npnt]]]

N[Qgrid];
```

In[64]:=
```
OctExctFormFact = Table[{N[Qgrid[[i]], 15], 0}, {i, 1, Npnt}] ;
OctApprxtd00FormFact = Table[{N[Qgrid[[i]], 15], 0}, {i, 1, Npnt}] ;
OctApprxtd11FormFact = Table[{N[Qgrid[[i]], 15], 0}, {i, 1, Npnt}] ;
OctApprxtd22FormFact = Table[{N[Qgrid[[i]], 15], 0}, {i, 1, Npnt}] ;
```

In[65]:=
```
Do[qact = Qgrid[[i]]; OctExctFormFact[[i, 2]] =
   (4 π / qact) (NIntegrate[OCTCFAA[r] * r * Sin[qact * r], {r, 0, OCTDst[[1]]},
       WorkingPrecision → 50, PrecisionGoal → 15] + NIntegrate[OCTCFBB[r] * r * Sin[qact * r],
       {r, OCTDst[[1]], OCTDst[[2]]}, WorkingPrecision → 100, PrecisionGoal → 15] +
      NIntegrate[OCTCFCC[r] * r * Sin[qact * r], {r, OCTDst[[2]], OCTDst[[3]]},
       WorkingPrecision → 100, PrecisionGoal → 15] +
      NIntegrate[OCTCFDD[r] * r * Sin[qact * r], {r, OCTDst[[3]], OCTDst[[4]]},
       WorkingPrecision → 100, PrecisionGoal → 15]), {i, 1, Npnt, 1}];
```

In[66]:=
```
Do[qact = Qgrid[[i]]; OctApprxtd00FormFact[[i, 2]] =
   (4 π / qact) N[(NIntegrate[OcthCFApprxFnl00AA[r] * r * Sin[qact * r],
        {r, 0, OCTDst[[1]]}, WorkingPrecision → 100, PrecisionGoal → 15] +
       NIntegrate[OcthCFApprxFnl00BB[r] * r * Sin[qact * r],
        {r, OCTDst[[1]], OCTDst[[2]]}, WorkingPrecision → 100, PrecisionGoal → 15] +
       NIntegrate[OcthCFApprxFnl00CC[r] * r * Sin[qact * r], {r, OCTDst[[2]], OCTDst[[3]]},
        WorkingPrecision → 100, PrecisionGoal → 15] +
       NIntegrate[OcthCFApprxFnl00DD[r] * r * Sin[qact * r], {r, OCTDst[[3]], OCTDst[[4]]},
        WorkingPrecision → 100, PrecisionGoal → 15])];, {i, 1, Npnt, 1}];
```

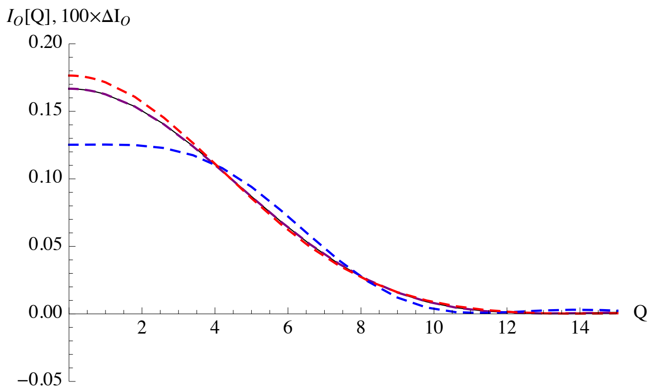
In[67]:=
```
Do[qact = Qgrid[[i]]; OctApprxtd11FormFact[[i, 2]] =
   (4 π / qact) N[(NIntegrate[OcthCFApprxFnl11AA[r] * r * Sin[qact * r],
        {r, 0, OCTDst[[1]]}, WorkingPrecision → 100, PrecisionGoal → 15] +
       NIntegrate[OcthCFApprxFnl11BB[r] * r * Sin[qact * r],
        {r, OCTDst[[1]], OCTDst[[2]]}, WorkingPrecision → 100, PrecisionGoal → 15] +
       NIntegrate[OcthCFApprxFnl11CC[r] * r * Sin[qact * r], {r, OCTDst[[2]], OCTDst[[3]]},
        WorkingPrecision → 100, PrecisionGoal → 15] +
       NIntegrate[OcthCFApprxFnl11DD[r] * r * Sin[qact * r], {r, OCTDst[[3]], OCTDst[[4]]},
        WorkingPrecision → 100, PrecisionGoal → 15]), 15], {i, 1, Npnt, 1}];
```

In[68]:=
```
Do[qact = Qgrid[[i]]; OctApprxtd22FormFact[[i, 2]] =
   (4 π / qact) N[(NIntegrate[OcthCFApprxFnl22AA[r] * r * Sin[qact * r],
        {r, 0, OCTDst[[1]]}, WorkingPrecision → 100, PrecisionGoal → 15] +
       NIntegrate[OcthCFApprxFnl22BB[r] * r * Sin[qact * r],
        {r, OCTDst[[1]], OCTDst[[2]]}, WorkingPrecision → 100, PrecisionGoal → 15] +
       NIntegrate[OcthCFApprxFnl22CC[r] * r * Sin[qact * r], {r, OCTDst[[2]], OCTDst[[3]]},
        WorkingPrecision → 100, PrecisionGoal → 15] +
       NIntegrate[OcthCFApprxFnl22DD[r] * r * Sin[qact * r], {r, OCTDst[[3]], OCTDst[[4]]},
        WorkingPrecision → 100, PrecisionGoal → 15]), 15], {i, 1, Npnt, 1}];

OctApprxtd22FormFact
```

In[107]:=
```
FigOCTFormFactAllApprx =
 ListPlot[{N[OctExctFormFact], N[ OctApprxtd00FormFact], N[ OctApprxtd11FormFact],
   N[ OctApprxtd22FormFact]}, Joined → True, PlotRange → {{0, 15}, {-0.05, 0.20}},
  PlotStyle → {Directive[Black, Thickness[0.002]], Directive[Purple,
     Thickness[0.004], Dashed], Directive[Blue, Thickness[0.004], Dashed],
    Directive[Red, Thickness[0.004], Dashed]}, AxesLabel → {"Q", "I₀[Q], 100×ΔI₀"}]
```

Out[107]=



In[108]:=
```
Export["FigOCTFormFactAllApprx.eps", FigOCTFormFactAllApprx];
```

In[70]:=
```
prdpltOctExctInt = Table[{N[OctExctFormFact[[i, 1]]],
    N[OctExctFormFact[[i, 1]]^4 * OctExctFormFact[[i, 2]]]}, {i, 1, Kont}];
prdpltOctAp00Int = Table[{N[OctApprxtd00FormFact[[i, 1]]],
    N[ OctApprxtd00FormFact[[i, 1]]^4 * OctApprxtd00FormFact[[i, 2]]]}, {i, 1, Kont}];
prdpltOctAp11Int = Table[{N[OctApprxtd11FormFact[[i, 1]]],
    N[ OctApprxtd11FormFact[[i, 1]]^4 * OctApprxtd11FormFact[[i, 2]]]}, {i, 1, Kont}];
prdpltOctAp22Int = Table[{N[OctApprxtd22FormFact[[i, 1]]],
    N[ OctApprxtd22FormFact[[i, 1]]^4 * OctApprxtd22FormFact[[i, 2]]]}, {i, 1, Kont}];

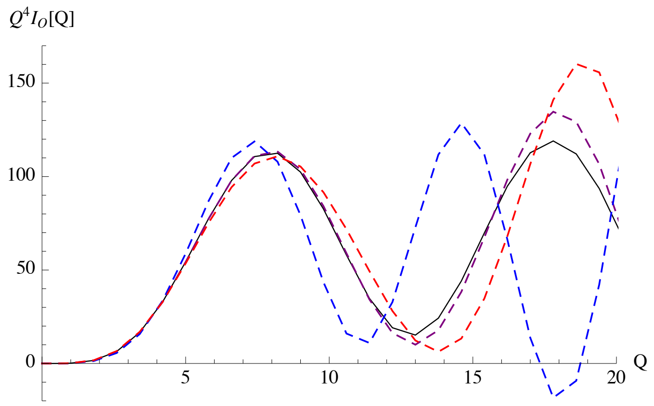
prdpltOctExctInt
prdpltOctAp00Int
prdpltOctAp11Int
prdpltOctAp22Int

OCTFigFormFact00 = ListPlot[{N[OctExctFormFact], N[ OctApprxtd00FormFact],
    Table[{N[OctExctFormFact[[i, 1]]],
      100 * N[OctExctFormFact[[i, 2]] - OctApprxtd00FormFact[[i, 2]]]}, {i, 1, Npnt}]},
   Joined → True, PlotRange → {{0, 15}, {-0.05, 0.20}}, PlotStyle →
    {Directive[Black, Thickness[0.003]], Directive[Green, Thickness[0.005], Dashed],
     Directive[Blue, Thickness[0.003], Dotted]}, AxesLabel → {"Q", "Io[Q]", 100×ΔIo"}]

Export["OCTFigFormFactAll.eps", OCTFigFormFactAll];
Export["OCTFigFormFactAll.PDF", OCTFigFormFactAll];
```

In[109]:=
```
OCTFigPorPltALL020 =
 ListPlot[{prdpltOctExctInt , prdpltOctAp00Int , prdpltOctAp11Int , prdpltOctAp22Int },
  Joined → True, PlotRange → {{0, 20}, {-20, 170}},
  PlotStyle → {Directive[Black, Thickness[0.002]], Directive[Purple,
     Thickness[0.003], Dashed], Directive[Blue, Thickness[0.003], Dashed],
    Directive[Red, Thickness[0.003], Dashed]}, AxesLabel → {"Q", "Q^4Io[Q]"}]
```

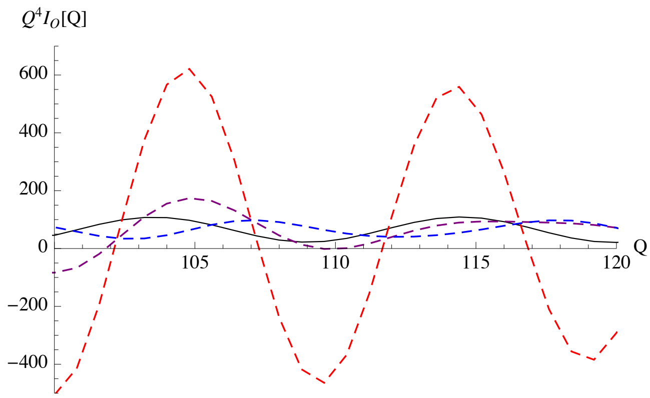
Out[109]=



In[110]:=
```
OCTFigPorPltALL0120 =
 ListPlot[{prdpltOctExctInt , prdpltOctAp00Int , prdpltOctAp11Int , prdpltOctAp22Int },
  Joined → True, PlotRange → {{0, 120}, {-500, 700}},
  PlotStyle → {Directive[Black, Thickness[0.002]], Directive[Purple,
     Thickness[0.003], Dashed], Directive[Blue, Thickness[0.003], Dashed],
    Directive[Red, Thickness[0.003], Dashed]}, AxesLabel → {"Q", "Q^4Io[Q]"}]
```

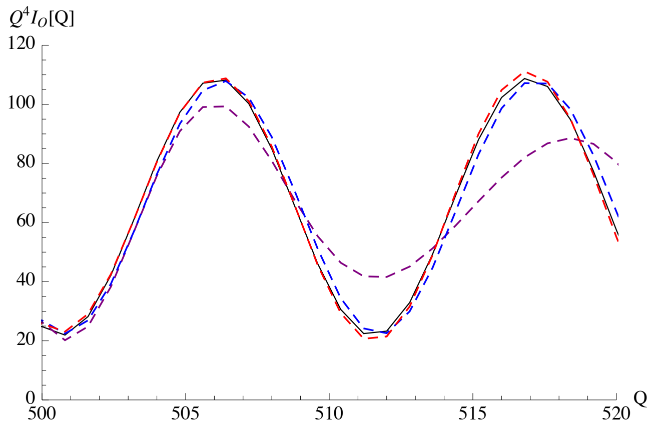
In[111]:=
```
OCTFigPorPltALL100120 =
 ListPlot[{prdpltOctExctInt , prdpltOctAp00Int , prdpltOctAp11Int , prdpltOctAp22Int },
  Joined → True, PlotRange → {{100, 120}, {-500, 700}},
  PlotStyle → {Directive[Black, Thickness[0.002]], Directive[Purple,
     Thickness[0.003], Dashed], Directive[Blue, Thickness[0.003], Dashed],
    Directive[Red, Thickness[0.003], Dashed]}, AxesLabel → {"Q", "Q^4 I_O[Q]"}]
```

Out[111]=



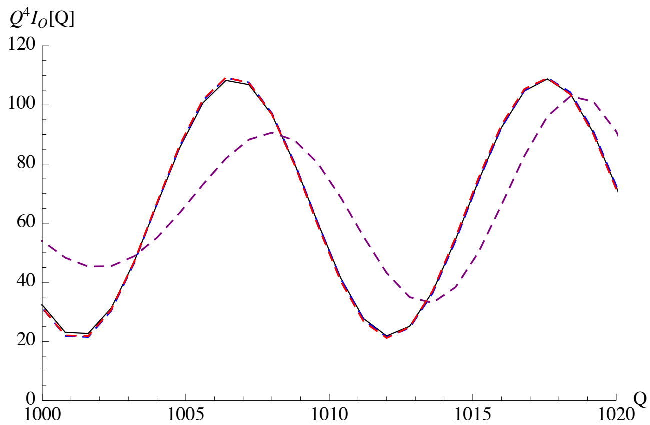
In[112]:=
```
OCTFigPorPltALL500520 =
 ListPlot[{prdpltOctExctInt , prdpltOctAp00Int , prdpltOctAp11Int , prdpltOctAp22Int },
  Joined → True, PlotRange → {{500, 520}, {-0, 120}},
  PlotStyle → {Directive[Black, Thickness[0.002]], Directive[Purple,
     Thickness[0.003], Dashed], Directive[Blue, Thickness[0.003], Dashed],
    Directive[Red, Thickness[0.003], Dashed]}, AxesLabel → {"Q", "Q^4 I_O[Q]"}]
```

Out[112]=



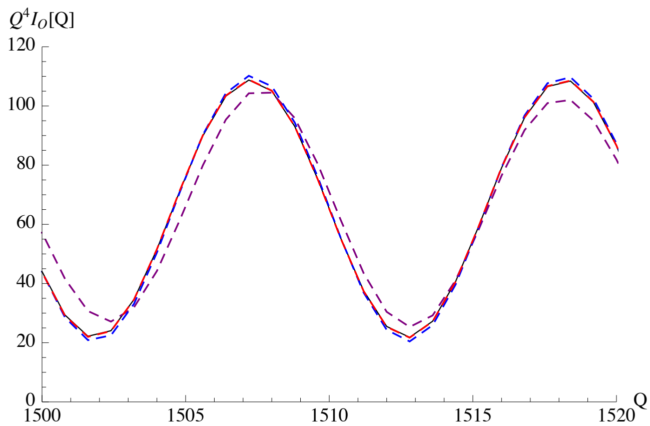
In[114]:=
```
OCTFigPorPltALL10001020 =
 ListPlot[{prdpltOctExctInt , prdpltOctAp00Int , prdpltOctAp11Int , prdpltOctAp22Int },
  Joined → True, PlotRange → {{1000, 1020}, {0, 120}},
  PlotStyle → {Directive[Black, Thickness[0.002]], Directive[Purple,
     Thickness[0.003], Dashed], Directive[Blue, Thickness[0.003], Dashed],
    Directive[Red, Thickness[0.003], Dashed]}, AxesLabel → {"Q", "Q⁴Iₒ[Q]"} ]
```

Out[114]=



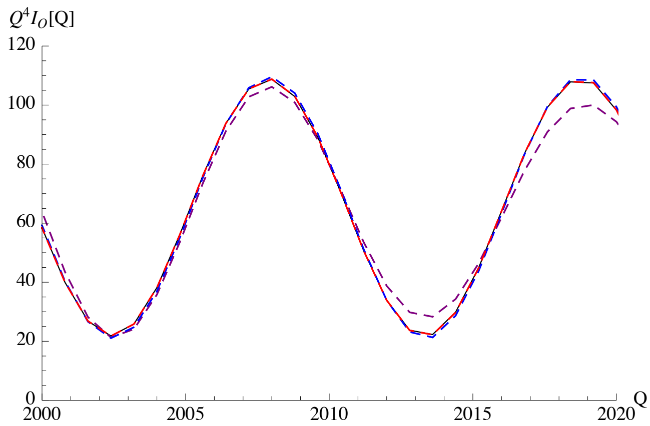
In[115]:=
```
OCTFigPorPltALL15001520 =
 ListPlot[{prdpltOctExctInt , prdpltOctAp00Int , prdpltOctAp11Int , prdpltOctAp22Int },
  Joined → True, PlotRange → {{1500, 1520}, {0, 120}},
  PlotStyle → {Directive[Black, Thickness[0.002]], Directive[Purple,
     Thickness[0.003], Dashed], Directive[Blue, Thickness[0.003], Dashed],
    Directive[Red, Thickness[0.003], Dashed]}, AxesLabel → {"Q", "Q⁴Iₒ[Q]"} ]
```

Out[115]=



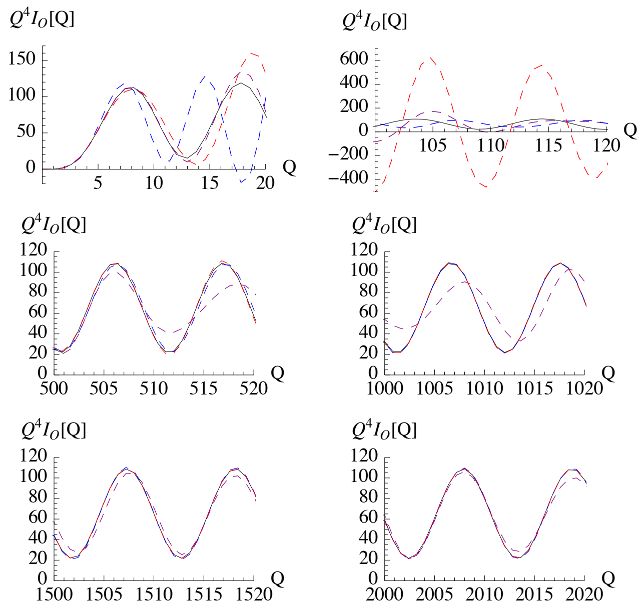
In[116]:=
```
OCTFigPorPltALL20002020 =
 ListPlot[{prdpltOctExctInt , prdpltOctAp00Int , prdpltOctAp11Int , prdpltOctAp22Int },
  Joined → True, PlotRange → {{2000, 2020}, {0, 120}},
  PlotStyle → {Directive[Black, Thickness[0.002]], Directive[Purple,
     Thickness[0.003], Dashed], Directive[Blue, Thickness[0.003], Dashed],
    Directive[Red, Thickness[0.003], Dashed]}, AxesLabel → {"Q", "Q^4 I_O[Q]"}]
```

Out[116]=



In[117]:=
```
FigOCTPrdPltALLAprxLargQ =
 Show[GraphicsArray[{{OCTFigPorPltALL020, OCTFigPorPltALL100120}, {OCTFigPorPltALL500520,
     OCTFigPorPltALL1001020}, {OCTFigPorPltALL15001520, OCTFigPorPltALL20002020}}]]
```

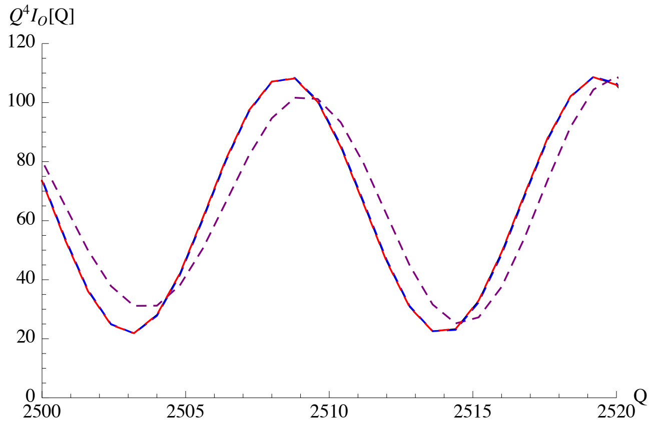
Out[117]=



In[118]:=
```
Export["FigOCTPrdPltALLAprxLargQ.eps", FigOCTPrdPltALLAprxLargQ];
```

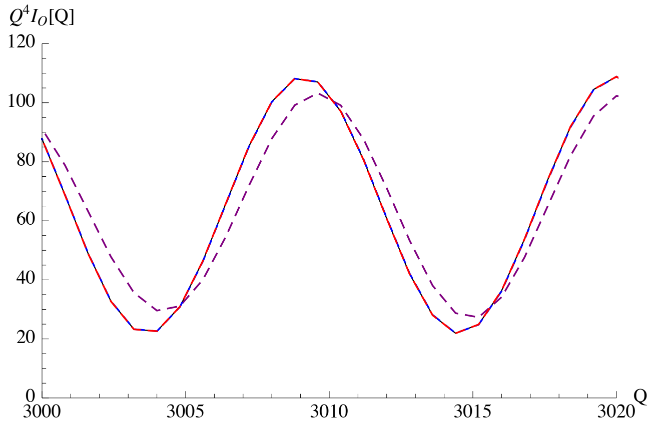
```
OCTFigPorPltALL25002520 =
 ListPlot[{prdpltOctExctInt , prdpltOctAp00Int , prdpltOctAp11Int , prdpltOctAp22Int },
  Joined → True, PlotRange → {{2500, 2520}, {0, 120}},
  PlotStyle → {Directive[Black, Thickness[0.002]], Directive[Purple,
      Thickness[0.003], Dashed], Directive[Blue, Thickness[0.003], Dashed],
    Directive[Red, Thickness[0.003], Dashed]}, AxesLabel → {"Q", "Q⁴I₀[Q]"}]
```

Out[89]=



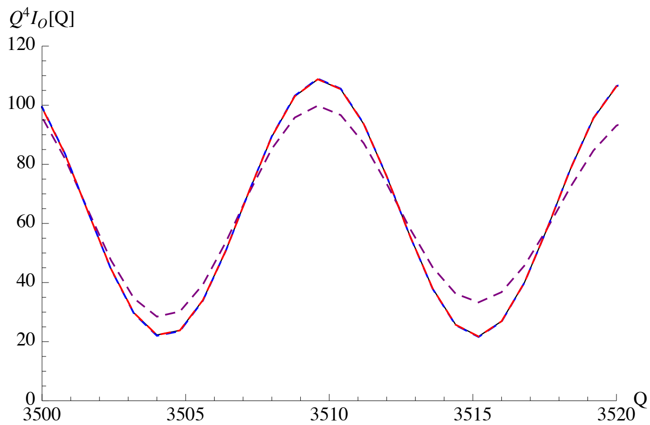

```
OCTFigPorPltALL30003020 =
 ListPlot[{prdpltOctExctInt , prdpltOctAp00Int , prdpltOctAp11Int , prdpltOctAp22Int },
  Joined → True, PlotRange → {{3000, 3020}, {0, 120}},
  PlotStyle → {Directive[Black, Thickness[0.002]], Directive[Purple,
      Thickness[0.003], Dashed], Directive[Blue, Thickness[0.003], Dashed],
    Directive[Red, Thickness[0.003], Dashed]}, AxesLabel → {"Q", "Q⁴I₀[Q]"}]
```

Out[91]=



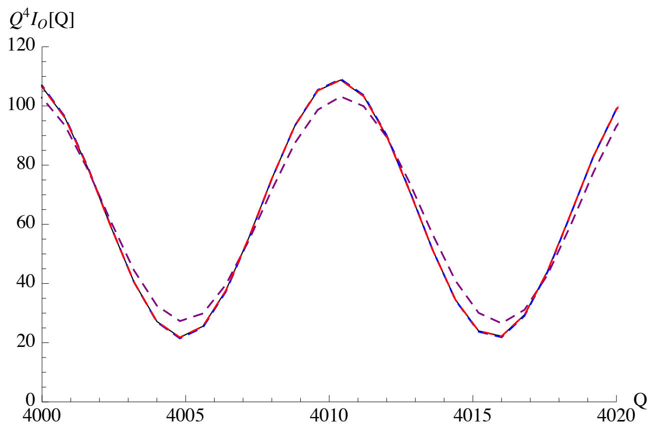
```
OCTFigPorPltALL35003520 =
 ListPlot[{prdpltOctExctInt , prdpltOctAp00Int , prdpltOctAp11Int , prdpltOctAp22Int },
  Joined → True, PlotRange → {{3500, 3520}, {0, 120}},
  PlotStyle → {Directive[Black, Thickness[0.002]], Directive[Purple,
      Thickness[0.003], Dashed], Directive[Blue, Thickness[0.003], Dashed],
    Directive[Red, Thickness[0.003], Dashed]}, AxesLabel → {"Q", "Q^4 I_o[Q]"}]
```

Out[92]=



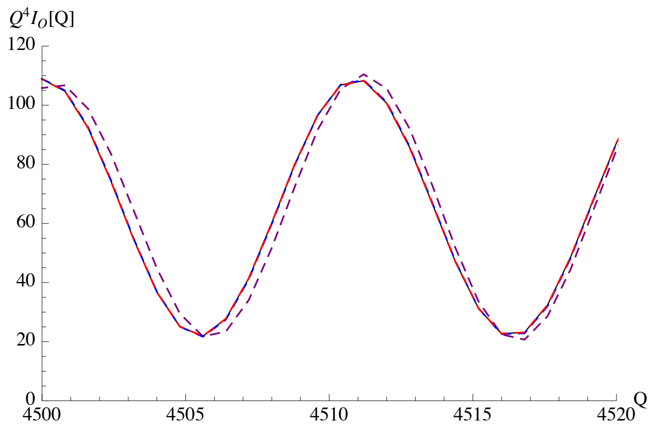

```
OCTFigPorPltALL40004020 =
 ListPlot[{prdpltOctExctInt , prdpltOctAp00Int , prdpltOctAp11Int , prdpltOctAp22Int },
  Joined → True, PlotRange → {{4000, 4020}, {0, 120}},
  PlotStyle → {Directive[Black, Thickness[0.002]], Directive[Purple,
      Thickness[0.003], Dashed], Directive[Blue, Thickness[0.003], Dashed],
    Directive[Red, Thickness[0.003], Dashed]}, AxesLabel → {"Q", "Q^4 I_o[Q]"}]
```

Out[93]=



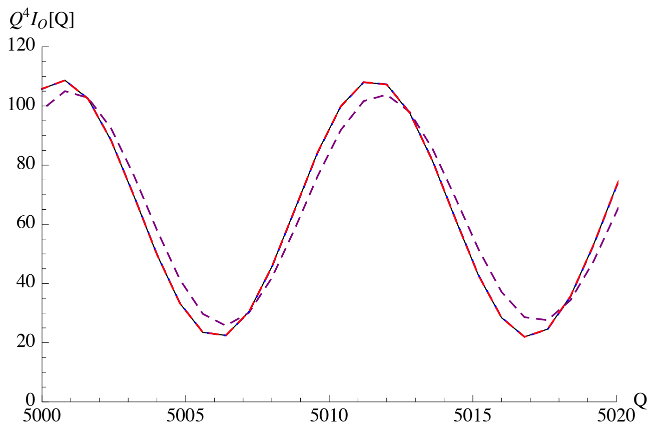
```
OCTFigPorPltALL45004520 =
 ListPlot[{prdpltOctExctInt , prdpltOctAp00Int , prdpltOctAp11Int , prdpltOctAp22Int },
  Joined → True, PlotRange → {{4500, 4520}, {0, 120}},
  PlotStyle → {Directive[Black, Thickness[0.002]], Directive[Purple,
      Thickness[0.003], Dashed], Directive[Blue, Thickness[0.003], Dashed],
    Directive[Red, Thickness[0.003], Dashed]}, AxesLabel → {"Q", "Q^4I_O[Q]"}]
```

Out[105]=



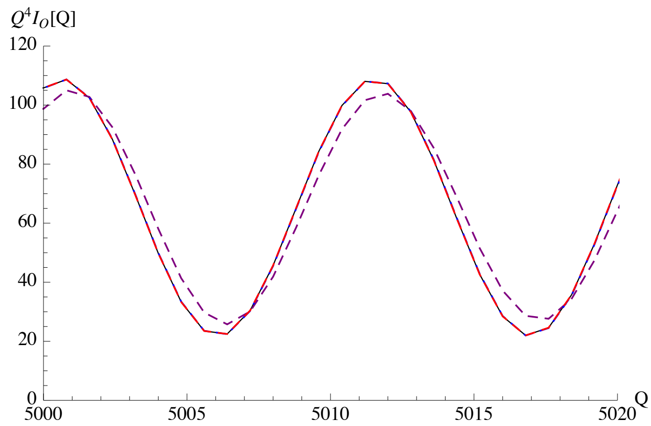

```
OCTFigPorPltALL50005020 =
 ListPlot[{prdpltOctExctInt , prdpltOctAp00Int , prdpltOctAp11Int , prdpltOctAp22Int },
  Joined → True, PlotRange → {{5000, 5020}, {0, 120}},
  PlotStyle → {Directive[Black, Thickness[0.002]], Directive[Purple,
      Thickness[0.003], Dashed], Directive[Blue, Thickness[0.003], Dashed],
    Directive[Red, Thickness[0.003], Dashed]}, AxesLabel → {"Q", "Q^4I_O[Q]"}]
```

Out[94]=



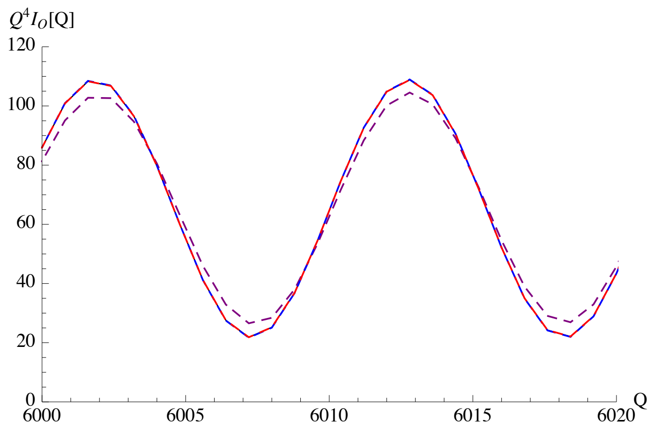
```
OCTFigPorPltALL55005520 =
 ListPlot[{prdpltOctExctInt , prdpltOctAp00Int , prdpltOctAp11Int , prdpltOctAp22Int },
  Joined → True, PlotRange → {{5000, 5020}, {0, 120}},
  PlotStyle → {Directive[Black, Thickness[0.002]], Directive[Purple,
     Thickness[0.003], Dashed], Directive[Blue, Thickness[0.003], Dashed],
    Directive[Red, Thickness[0.003], Dashed]}, AxesLabel → {"Q", "Q⁴Iₒ[Q]"}]
```

Out[104]=



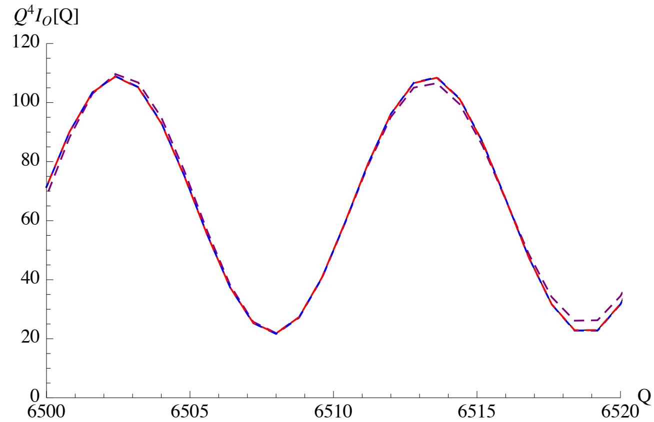

```
OCTFigPorPltALL60006020 =
 ListPlot[{prdpltOctExctInt , prdpltOctAp00Int , prdpltOctAp11Int , prdpltOctAp22Int },
  Joined → True, PlotRange → {{6000, 6020}, {0, 120}},
  PlotStyle → {Directive[Black, Thickness[0.002]], Directive[Purple,
     Thickness[0.003], Dashed], Directive[Blue, Thickness[0.003], Dashed],
    Directive[Red, Thickness[0.003], Dashed]}, AxesLabel → {"Q", "Q⁴Iₒ[Q]"}]
```

Out[95]=



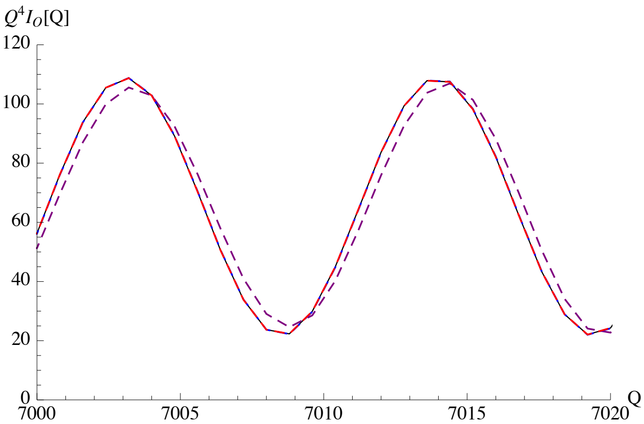
```
OCTFigPorPltALL65006520 =
 ListPlot[{prdpltOctExctInt , prdpltOctAp00Int , prdpltOctAp11Int , prdpltOctAp22Int },
  Joined → True, PlotRange → {{6500, 6520}, {0, 120}},
  PlotStyle → {Directive[Black, Thickness[0.002]], Directive[Purple,
      Thickness[0.003], Dashed], Directive[Blue, Thickness[0.003], Dashed],
    Directive[Red, Thickness[0.003], Dashed]}, AxesLabel → {"Q", "Q^4 Io[Q]"}]
```

Out[103]=



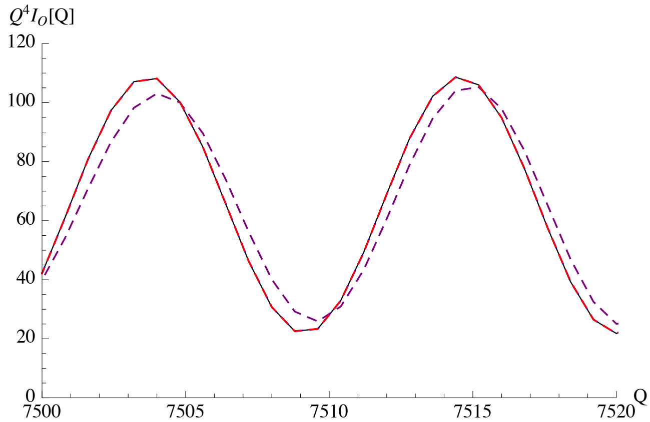

```
OCTFigPorPltALL70007020 =
 ListPlot[{prdpltOctExctInt , prdpltOctAp00Int , prdpltOctAp11Int , prdpltOctAp22Int },
  Joined → True, PlotRange → {{7000, 7020}, {0, 120}},
  PlotStyle → {Directive[Black, Thickness[0.002]], Directive[Purple,
      Thickness[0.003], Dashed], Directive[Blue, Thickness[0.003], Dashed],
    Directive[Red, Thickness[0.003], Dashed]}, AxesLabel → {"Q", "Q^4 Io[Q]"}]
```

Out[96]=



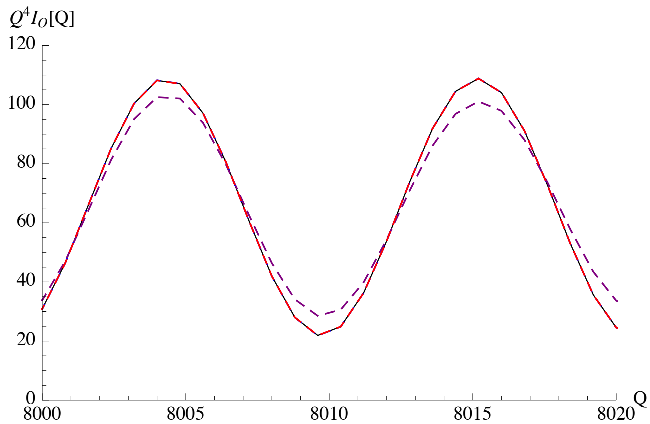
```
OCTFigPorPltALL75007520 =
 ListPlot[{prdpltOctExctInt , prdpltOctAp00Int , prdpltOctAp11Int , prdpltOctAp22Int },
  Joined → True, PlotRange → {{7500, 7520}, {0, 120}},
  PlotStyle → {Directive[Black, Thickness[0.002]], Directive[Purple,
     Thickness[0.003], Dashed], Directive[Blue, Thickness[0.003], Dashed],
    Directive[Red, Thickness[0.003], Dashed]}, AxesLabel → {"Q", "Q⁴I₀[Q]"}]
```

Out[102]=



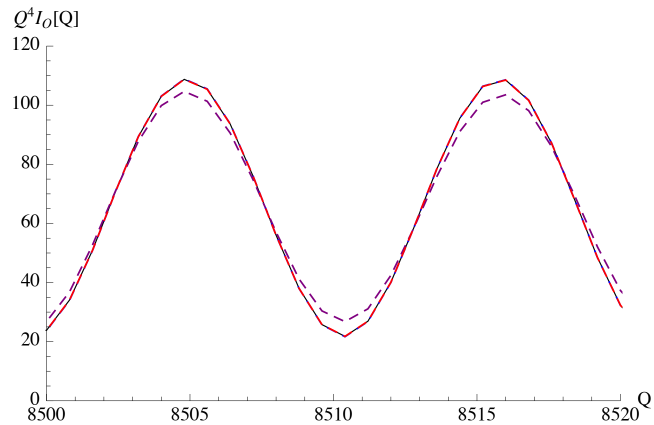

```
OCTFigPorPltALL8000020 =
 ListPlot[{prdpltOctExctInt , prdpltOctAp00Int , prdpltOctAp11Int , prdpltOctAp22Int },
  Joined → True, PlotRange → {{8000, 8020}, {0, 120}},
  PlotStyle → {Directive[Black, Thickness[0.002]], Directive[Purple,
     Thickness[0.003], Dashed], Directive[Blue, Thickness[0.003], Dashed],
    Directive[Red, Thickness[0.003], Dashed]}, AxesLabel → {"Q", "Q⁴I₀[Q]"}]
```

Out[97]=



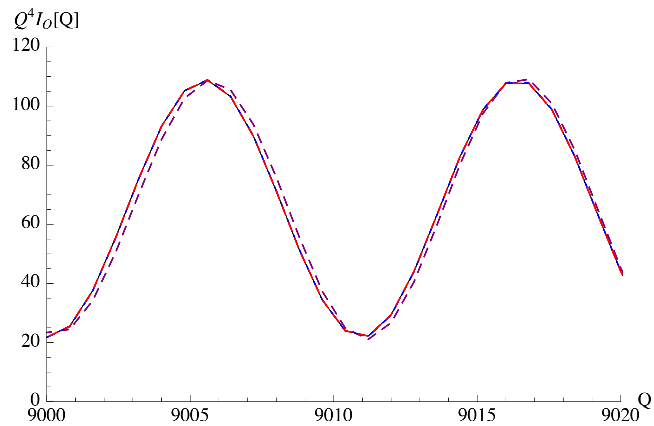
```
OCTFigPorPltALL85008520 =
 ListPlot[{prdpltOctExctInt , prdpltOctAp00Int , prdpltOctAp11Int , prdpltOctAp22Int },
  Joined → True, PlotRange → {{8500, 8520}, {0, 120}},
  PlotStyle → {Directive[Black, Thickness[0.002]], Directive[Purple,
     Thickness[0.003], Dashed], Directive[Blue, Thickness[0.003], Dashed],
    Directive[Red, Thickness[0.003], Dashed]}, AxesLabel → {"Q", "Q^4I_O[Q]"}]
```

Out[101]=



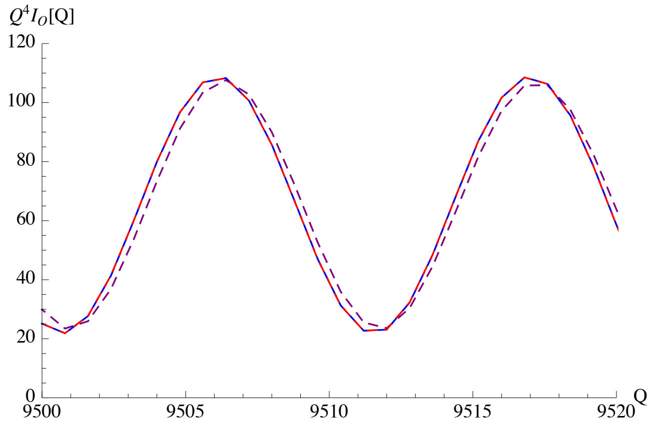

```
OCTFigPorPltALL90009020 =
 ListPlot[{prdpltOctExctInt , prdpltOctAp00Int , prdpltOctAp11Int , prdpltOctAp22Int },
  Joined → True, PlotRange → {{9000, 9020}, {0, 120}},
  PlotStyle → {Directive[Black, Thickness[0.002]], Directive[Purple,
     Thickness[0.003], Dashed], Directive[Blue, Thickness[0.003], Dashed],
    Directive[Red, Thickness[0.003], Dashed]}, AxesLabel → {"Q", "Q^4I_O[Q]"}]
```

Out[98]=



```
OCTFigPorPltALL95009520 =
 ListPlot[{prdpltOctExctInt , prdpltOctAp00Int , prdpltOctAp11Int , prdpltOctAp22Int },
  Joined → True, PlotRange → {{9500, 9520}, {0, 120}},
  PlotStyle → {Directive[Black, Thickness[0.002]], Directive[Purple,
      Thickness[0.003], Dashed], Directive[Blue, Thickness[0.003], Dashed],
    Directive[Red, Thickness[0.003], Dashed]}, AxesLabel → {"Q", "Q⁴I₀[Q]"}]
```

Out[100]=




```
OCTFigPorPltALL1000010020 =
 ListPlot[{prdpltOctExctInt , prdpltOctAp00Int , prdpltOctAp11Int , prdpltOctAp22Int },
  Joined → True, PlotRange → {{10 000, 10 020}, {0, 120}},
  PlotStyle → {Directive[Black, Thickness[0.002]], Directive[Purple,
      Thickness[0.003], Dashed], Directive[Blue, Thickness[0.003], Dashed],
    Directive[Red, Thickness[0.003], Dashed]}, AxesLabel → {"Q", "Q⁴I₀[Q]"}]
```

Out[99]=