ON PHYSICAL PROPERTY TENSORS INVARIANT UNDER LINE GROUPS

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Supporting information

Table 5: Rank two tensors invariant under line point groups

For each line point group family, the form of the twelve rank two physical property tensors listed in Equation (9) are tabulated. (Subsets of these tabulations can be found in the work of Milošević (1995), Dmitriev (2003), and Sirotin & Shaskolskaya (1982).)On the left-hand-side is the symbol of the line point group family followed by the form of the twelve tensors invariant under a representative line point group of the line point group types of index n = 1, 2, and n = ∞ of that family. For each line point group, the twelve tensor forms are given in an array corresponding, respectively, to the array of tensor symbols in Equation (9). Families of line groups **G1'** are not explicitly listed as for these groups, the form of tensors types not containing *a* are the same as for corresponding groups **G**, and for tensor types containing *a*, the form has only zero entries.

$\mathbf{C}_n$

n = 1

$\mathbf{C}_1 = \mathbf{1}$

$$\begin{pmatrix} xx & xy & xz \\ yx & yy & yz \\ zx & zy & zz \end{pmatrix}\begin{pmatrix} xx & xy & xz \\ yx & yy & yz \\ zx & zy & zz \end{pmatrix}\begin{pmatrix} xx & xy & xz \\ yx & yy & yz \\ zx & zy & zz \end{pmatrix}\begin{pmatrix} xx & xy & xz \\ yx & yy & yz \\ zx & zy & zz \end{pmatrix}$$

$$\begin{pmatrix} xx & xy & xz \\ xy & yy & yz \\ xz & yz & zz \end{pmatrix}\begin{pmatrix} xx & xy & xz \\ xy & yy & yz \\ xz & yz & zz \end{pmatrix}\begin{pmatrix} xx & xy & xz \\ xy & yy & yz \\ xz & yz & zz \end{pmatrix}\begin{pmatrix} xx & xy & xz \\ xy & yy & yz \\ xz & yz & zz \end{pmatrix}$$

$$\begin{pmatrix} 0 & xy & xz \\ -xy & 0 & yz \\ -xz & -yz & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & xz \\ -xy & 0 & yz \\ -xz & -yz & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & xz \\ -xy & 0 & yz \\ -xz & -yz & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & xz \\ -xy & 0 & yz \\ -xz & -yz & 0 \end{pmatrix}$$

n = 2

$\mathbf{C}_2 = \mathbf{2}_z$

$$\begin{pmatrix} xx & xy & 0 \\ yx & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & xy & 0 \\ yx & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & xy & 0 \\ yx & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & xy & 0 \\ yx & yy & 0 \\ 0 & 0 & zz \end{pmatrix}$$

$$\begin{pmatrix} xx & xy & 0 \\ xy & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & xy & 0 \\ xy & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & xy & 0 \\ xy & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & xy & 0 \\ xy & yy & 0 \\ 0 & 0 & zz \end{pmatrix}$$

$$\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

n > 2

$\mathbf{C}_\infty = \infty$

$$\begin{pmatrix} xx & xy & 0 \\ -xy & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & xy & 0 \\ -xy & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & xy & 0 \\ -xy & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & xy & 0 \\ -xy & xx & 0 \\ 0 & 0 & zz \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}$$

$$\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\mathbf{C}_{nv}$

n = 1

$\mathbf{C}_{1v} = \mathbf{m}_y$

$$\begin{pmatrix} xx & 0 & xz \\ 0 & yy & 0 \\ zx & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ yx & 0 & yz \\ 0 & zy & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & xz \\ 0 & yy & 0 \\ zx & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ yx & 0 & yz \\ 0 & zy & 0 \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & xz \\ 0 & yy & 0 \\ xz & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ xy & 0 & yz \\ 0 & yz & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & xz \\ 0 & yy & 0 \\ xz & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ xy & 0 & yz \\ 0 & yz & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & 0 \\ -xz & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & yz \\ 0 & -yz & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & 0 \\ -xz & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & yz \\ 0 & -yz & 0 \end{pmatrix}$$

n = 2

$\mathbf{C}_{2v} = \mathbf{2}_z\mathbf{m}_x\mathbf{m}_y$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ yx & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ yx & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

n > 2

$\mathbf{C}_{\infty v} = \infty\mathbf{mm}$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\mathbf{S}_{2n}$

n = 1

$\mathbf{S}_2 = \bar{\mathbf{1}}$

$$\begin{pmatrix} xx & xy & xz \\ yx & yy & yz \\ zx & zy & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & xy & xz \\ yx & yy & yz \\ zx & zy & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} xx & xy & xz \\ xy & yy & yz \\ xz & yz & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & xy & xz \\ xy & yy & yz \\ xz & yz & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & xy & xz \\ -xy & 0 & yz \\ -xz & -yz & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & xz \\ -xy & 0 & yz \\ -xz & -yz & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

n = 2

$\mathbf{S}_4 = \bar{\mathbf{4}}_z$

$$\begin{pmatrix} xx & xy & 0 \\ -xy & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & xy & 0 \\ xy & -xx & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & xy & 0 \\ -xy & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & xy & 0 \\ xy & -xx & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & xy & 0 \\ xy & -xx & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & xy & 0 \\ xy & -xx & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

n > 2

$\mathbf{C}_{\infty h} = \infty/\mathbf{m}$

$$\begin{pmatrix} xx & xy & 0 \\ -xy & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & xy & 0 \\ -xy & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

C_{nh}

n = 1

$C_{1h} = m_z$

$$\begin{pmatrix} xx & xy & 0 \\ yx & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & yz \\ zx & zy & 0 \end{pmatrix}\begin{pmatrix} xx & xy & 0 \\ yx & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & yz \\ zx & zy & 0 \end{pmatrix}$$

$$\begin{pmatrix} xx & xy & 0 \\ xy & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & yz \\ xz & yz & 0 \end{pmatrix}\begin{pmatrix} xx & xy & 0 \\ xy & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & yz \\ xz & yz & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & yz \\ -xz & -yz & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & yz \\ -xz & -yz & 0 \end{pmatrix}$$

n = 2

$C_{2h} = 2/m_z$

$$\begin{pmatrix} xx & xy & 0 \\ yx & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & xy & 0 \\ yx & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} xx & xy & 0 \\ xy & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & xy & 0 \\ xy & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

n > 2

$C_{\infty h} = \infty/m$

$$\begin{pmatrix} xx & xy & 0 \\ -xy & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & xy & 0 \\ -xy & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

D_n

n = 1

$D_1 = 2_y$

$$\begin{pmatrix} xx & 0 & xz \\ 0 & yy & 0 \\ zx & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & xz \\ 0 & yy & 0 \\ zx & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & xz \\ 0 & yy & 0 \\ zx & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & xz \\ 0 & yy & 0 \\ zx & 0 & zz \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & xz \\ 0 & yy & 0 \\ xz & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & xz \\ 0 & yy & 0 \\ xz & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & xz \\ 0 & yy & 0 \\ xz & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & xz \\ 0 & yy & 0 \\ xz & 0 & zz \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & 0 \\ -xz & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & 0 \\ -xz & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & 0 \\ -xz & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & 0 \\ -xz & 0 & 0 \end{pmatrix}$$

n = 2

$D_2 = 2_x 2_y 2_z$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

n > 2

$D_\infty = \infty 2$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

D_{nd}

n = 1

$D_{1d} = 2_y/m_y$

$$\begin{pmatrix} xx & 0 & xz \\ 0 & yy & 0 \\ zx & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & xz \\ 0 & yy & 0 \\ zx & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & xz \\ 0 & yy & 0 \\ xz & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & xz \\ 0 & yy & 0 \\ xz & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & 0 \\ -xz & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & 0 \\ -xz & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

n = 2

$D_{2d} = \bar{4}_z m_x 2_{xy}$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

n > 2

$D_{\infty h} = \infty/mmm$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\mathbf{D}_{nh}$

n = 1: $\mathbf{D}_{1h} = \mathbf{2}_x\mathbf{m}_y\mathbf{m}_z$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & yz \\ 0 & zy & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & yz \\ 0 & zy & 0 \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & yz \\ 0 & yz & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & yz \\ 0 & yz & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & yz \\ 0 & -yz & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & yz \\ 0 & -yz & 0 \end{pmatrix}$$

n = 2: $\mathbf{D}_{2h} = \mathbf{m}_x\mathbf{m}_y\mathbf{m}_z$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

n > 2: $\mathbf{D}_{\infty h} = \infty/\mathbf{mmm}$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\mathbf{C}_{2n}(\mathbf{C}_n)$

n = 1: $\mathbf{C}_2(\mathbf{C}_1) = \mathbf{2}_z(\mathbf{1}) = \mathbf{2}_z'$

$$\begin{pmatrix} xx & xy & 0 \\ yx & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & xy & 0 \\ yx & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & yz \\ zx & zy & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & yz \\ zx & zy & 0 \end{pmatrix}$$

$$\begin{pmatrix} xx & xy & 0 \\ xy & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & xy & 0 \\ xy & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & yz \\ xz & yz & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & yz \\ xz & yz & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & yz \\ -xz & -yz & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & yz \\ -xz & -yz & 0 \end{pmatrix}$$

n = 2: $\mathbf{C}_4(\mathbf{C}_2) = \mathbf{4}_z(\mathbf{2}_z) = \mathbf{4}_z'$

$$\begin{pmatrix} xx & xy & 0 \\ -xy & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & xy & 0 \\ -xy & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & xy & 0 \\ xy & -xx & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & xy & 0 \\ xy & -xx & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & xy & 0 \\ xy & -xx & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & xy & 0 \\ xy & -xx & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

n > 2: $\mathbf{C}_\infty\mathbf{1}' = \infty\mathbf{1}'$

$$\begin{pmatrix} xx & xy & 0 \\ -xy & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & xy & 0 \\ -xy & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\mathbf{S}_{2n}(\mathbf{C}_n)$

n = 1: $\mathbf{S}_2(\mathbf{C}_1) = \bar{\mathbf{1}}(\mathbf{1}) = \bar{\mathbf{1}}'$

$$\begin{pmatrix} xx & xy & xz \\ yx & yy & yz \\ zx & zy & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & xy & xz \\ yx & yy & yz \\ zx & zy & zz \end{pmatrix}$$

$$\begin{pmatrix} xx & xy & xz \\ xy & yy & yz \\ xz & yz & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & xy & xz \\ xy & yy & yz \\ xz & yz & zz \end{pmatrix}$$

$$\begin{pmatrix} 0 & xy & xz \\ -xy & 0 & yz \\ -xz & -yz & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & xz \\ -xy & 0 & yz \\ -xz & -yz & 0 \end{pmatrix}$$

n = 2: $\mathbf{S}_4(\mathbf{C}_2) = \bar{\mathbf{4}}_z(\mathbf{2}_z) = \bar{\mathbf{4}}_z'$

$$\begin{pmatrix} xx & xy & 0 \\ -xy & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & xy & 0 \\ xy & -xx & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & xy & 0 \\ xy & -xx & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & xy & 0 \\ -xy & xx & 0 \\ 0 & 0 & zz \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & xy & 0 \\ xy & -xx & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & xy & 0 \\ xy & -xx & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}$$

$$\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

n > 2: $\mathbf{C}_{\infty h}(\mathbf{C}_\infty) = \infty/\mathbf{m}'$

$$\begin{pmatrix} xx & xy & 0 \\ -xy & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & xy & 0 \\ -xy & xx & 0 \\ 0 & 0 & zz \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}$$

$$\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$C_{nh}(C_n)$

n = 1

$C_{1h}(C_1) = m_z(1) = m_z'$

$$\begin{pmatrix}xx & xy & 0\\ yx & yy & 0\\ 0 & 0 & zz\end{pmatrix}\begin{pmatrix}0 & 0 & xz\\ 0 & 0 & yz\\ zx & zy & 0\end{pmatrix}\begin{pmatrix}0 & 0 & xz\\ 0 & 0 & yz\\ zx & zy & 0\end{pmatrix}\begin{pmatrix}xx & xy & 0\\ yx & yy & 0\\ 0 & 0 & zz\end{pmatrix}$$

$$\begin{pmatrix}xx & xy & 0\\ xy & yy & 0\\ 0 & 0 & zz\end{pmatrix}\begin{pmatrix}0 & 0 & xz\\ 0 & 0 & yz\\ xz & yz & 0\end{pmatrix}\begin{pmatrix}0 & 0 & xz\\ 0 & 0 & yz\\ xz & yz & 0\end{pmatrix}\begin{pmatrix}xx & xy & 0\\ xy & yy & 0\\ 0 & 0 & zz\end{pmatrix}$$

$$\begin{pmatrix}0 & xy & 0\\ -xy & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & xz\\ 0 & 0 & yz\\ -xz & -yz & 0\end{pmatrix}\begin{pmatrix}0 & 0 & xz\\ 0 & 0 & yz\\ -xz & -yz & 0\end{pmatrix}\begin{pmatrix}0 & xy & 0\\ -xy & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

n = 2

$C_{2h}(C_2) = 2_z/m_z(2_z) = 2_z/m_z'$

$$\begin{pmatrix}xx & xy & 0\\ yx & yy & 0\\ 0 & 0 & zz\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}xx & xy & 0\\ yx & yy & 0\\ 0 & 0 & zz\end{pmatrix}$$

$$\begin{pmatrix}xx & xy & 0\\ xy & yy & 0\\ 0 & 0 & zz\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}xx & xy & 0\\ xy & yy & 0\\ 0 & 0 & zz\end{pmatrix}$$

$$\begin{pmatrix}0 & xy & 0\\ -xy & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & xy & 0\\ -xy & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

n > 2

$C_{\infty h}(C_\infty) = \infty/m'$

$$\begin{pmatrix}xx & xy & 0\\ -xy & xx & 0\\ 0 & 0 & zz\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}xx & xy & 0\\ -xy & xx & 0\\ 0 & 0 & zz\end{pmatrix}$$

$$\begin{pmatrix}xx & 0 & 0\\ 0 & xx & 0\\ 0 & 0 & zz\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}xx & 0 & 0\\ 0 & xx & 0\\ 0 & 0 & zz\end{pmatrix}$$

$$\begin{pmatrix}0 & xy & 0\\ -xy & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & xy & 0\\ -xy & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$C_{2nh}(S_{2n})$

n = 1

$C_{2h}(S_2) = 2_z/m_z(\bar{1}) = 2_z'/m_z'$

$$\begin{pmatrix}xx & xy & 0\\ yx & yy & 0\\ 0 & 0 & zz\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & xz\\ 0 & 0 & yz\\ zx & zy & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$\begin{pmatrix}xx & xy & 0\\ xy & yy & 0\\ 0 & 0 & zz\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & xz\\ 0 & 0 & yz\\ xz & yz & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$\begin{pmatrix}0 & xy & 0\\ -xy & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & xz\\ 0 & 0 & yz\\ -xz & -yz & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

n = 2

$C_{4h}(S_4) = 4_z/m_z(\bar{4}_z) = 4_z'/m_z'$

$$\begin{pmatrix}xx & xy & 0\\ -xy & xx & 0\\ 0 & 0 & zz\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}xx & xy & 0\\ xy & -xx & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$\begin{pmatrix}xx & 0 & 0\\ 0 & xx & 0\\ 0 & 0 & zz\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}xx & xy & 0\\ xy & -xx & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$\begin{pmatrix}0 & xy & 0\\ -xy & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

n > 2

$C_{\infty h}\,1' = \infty/m1'$

$$\begin{pmatrix}xx & xy & 0\\ -xy & xx & 0\\ 0 & 0 & zz\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$\begin{pmatrix}xx & 0 & 0\\ 0 & xx & 0\\ 0 & 0 & zz\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$\begin{pmatrix}0 & xy & 0\\ -xy & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$C_{2nh}(C_{nh})$

n = 1

$C_{2h}(C_{1h}) = 2_z/m_z(m_z) = 2_z'/m_z$

$$\begin{pmatrix}xx & xy & 0\\ yx & yy & 0\\ 0 & 0 & zz\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & xz\\ 0 & 0 & yz\\ zx & zy & 0\end{pmatrix}$$

$$\begin{pmatrix}xx & xy & 0\\ xy & yy & 0\\ 0 & 0 & zz\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & xz\\ 0 & 0 & yz\\ xz & yz & 0\end{pmatrix}$$

$$\begin{pmatrix}0 & xy & 0\\ -xy & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & xz\\ 0 & 0 & yz\\ -xz & -yz & 0\end{pmatrix}$$

n = 2

$C_{4h}(C_{2h}) = 4_z/m_z(2_z/m_z) = 4_z'/m_z$

$$\begin{pmatrix}xx & xy & 0\\ -xy & xx & 0\\ 0 & 0 & zz\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}xx & xy & 0\\ xy & -xx & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$\begin{pmatrix}xx & 0 & 0\\ 0 & xx & 0\\ 0 & 0 & zz\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}xx & xy & 0\\ xy & -xx & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$\begin{pmatrix}0 & xy & 0\\ -xy & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

n > 2

$C_{\infty h}1' = \infty/m1'$

$$\begin{pmatrix}xx & xy & 0\\ -xy & xx & 0\\ 0 & 0 & zz\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$\begin{pmatrix}xx & 0 & 0\\ 0 & xx & 0\\ 0 & 0 & zz\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$\begin{pmatrix}0 & xy & 0\\ -xy & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$D_n(C_n)$

n = 1

$D_1(C_1) = 2_y(1) = 2_y'$

$$\begin{pmatrix} xx & 0 & xz \\ 0 & yy & 0 \\ zx & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & xz \\ 0 & yy & 0 \\ zx & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ yx & 0 & yz \\ 0 & zy & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ yx & 0 & yz \\ 0 & zy & 0 \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & xz \\ 0 & yy & 0 \\ xz & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & xz \\ 0 & yy & 0 \\ xz & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ xy & 0 & yz \\ 0 & yz & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ xy & 0 & yz \\ 0 & yz & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & 0 \\ -xz & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & 0 \\ -xz & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & yz \\ 0 & -yz & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & yz \\ 0 & -yz & 0 \end{pmatrix}$$

n = 2

$D_2(C_2) = 2_x2_y2_z(2_z) = 2_x'2_y'2_z$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ yx & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ yx & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

n > 2

$D_\infty(C_\infty) = \infty 2'$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$D_{2n}(D_n)$

n = 1

$D_2(D_1) = 2_x2_y2(2_y) = 2_x'2_y2_z'$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & 0 \\ zx & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & 0 \\ zx & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & 0 \\ xz & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & 0 \\ xz & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & 0 \\ -xz & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & 0 \\ -xz & 0 & 0 \end{pmatrix}$$

n = 2

$D_4(D_2) = 4_z2_x2_{xy}(2_x2_y2_z) = 4_z'2_x2_{xy}'$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & -xx & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & -xx & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & -xx & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & -xx & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

n > 2

$D_\infty 1' = \infty 21'$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$C_{nv}(C_n)$

n = 1

$C_{1v}(C_1) = m_y(1) = m_y'$

$$\begin{pmatrix} xx & 0 & xz \\ 0 & yy & 0 \\ zx & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ yx & 0 & yz \\ 0 & zy & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ yx & 0 & yz \\ 0 & zy & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & xz \\ 0 & yy & 0 \\ zx & 0 & zz \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & xz \\ 0 & yy & 0 \\ xz & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ xy & 0 & yz \\ 0 & yz & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ xy & 0 & yz \\ 0 & yz & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & xz \\ 0 & yy & 0 \\ xz & 0 & zz \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & 0 \\ -xz & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & yz \\ 0 & -yz & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & yz \\ 0 & -yz & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & 0 \\ -xz & 0 & 0 \end{pmatrix}$$

n = 2

$C_{2v}(C_2) = 2_zm_xm_y(2_z) = 2_zm_x'm_y'$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ yx & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ yx & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

n > 2

$C_{\infty v}(C_\infty) = \infty m'm'$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\mathbf{C_{2nv}(C_{nv})}$

n = 1

$\mathbf{C_{2v}(C_{1v}) = 2_z m_x m_y(m_y) = 2_z' m_x' m_y}$

$$\begin{pmatrix}xx&0&0\\0&yy&0\\0&0&zz\end{pmatrix}\begin{pmatrix}0&xy&0\\yx&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&xz\\0&0&0\\zx&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&yz\\0&zy&0\end{pmatrix}$$

$$\begin{pmatrix}xx&0&0\\0&yy&0\\0&0&zz\end{pmatrix}\begin{pmatrix}0&xy&0\\xy&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&xz\\0&0&0\\xz&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&yz\\0&yz&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&xy&0\\-xy&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&xz\\0&0&0\\-xz&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&yz\\0&-yz&0\end{pmatrix}$$

n = 2

$\mathbf{C_{2nv}(C_{nv}) = 4_z m_x m_{xy}(2_z m_x m_y) = 4_z' m_x m_{xy}'}$

$$\begin{pmatrix}xx&0&0\\0&xx&0\\0&0&zz\end{pmatrix}\begin{pmatrix}0&xy&0\\-xy&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}xx&0&0\\0&-xx&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&xy&0\\xy&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}xx&0&0\\0&xx&0\\0&0&zz\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}xx&0&0\\0&-xx&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&xy&0\\xy&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&xy&0\\-xy&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

n > 2

$\mathbf{C_{\infty v}1' = \infty mm1'}$

$$\begin{pmatrix}xx&0&0\\0&xx&0\\0&0&zz\end{pmatrix}\begin{pmatrix}0&xy&0\\-xy&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}xx&0&0\\0&xx&0\\0&0&zz\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&xy&0\\-xy&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$\mathbf{D_{nd}(S_{2n})}$

n = 1

$\mathbf{D_{1d}(S_2) = 2_y/m_y(\bar{1}) = 2_y'/m_y'}$

$$\begin{pmatrix}xx&0&xz\\0&yy&0\\zx&0&zz\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&xy&0\\yx&0&yz\\0&zy&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}xx&0&xz\\0&yy&0\\xz&0&zz\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&xy&0\\xy&0&yz\\0&yz&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&xz\\0&0&0\\-xz&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&xy&0\\-xy&0&yz\\0&-yz&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

n = 2

$\mathbf{D_{2d}(S_4) = \bar{4}_z m_x 2_{xy}(\bar{4}_z) = \bar{4}_z m_x' 2_{xy}'}$

$$\begin{pmatrix}xx&0&0\\0&xx&0\\0&0&zz\end{pmatrix}\begin{pmatrix}0&xy&0\\xy&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&xy&0\\-xy&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}xx&0&0\\0&-xx&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}xx&0&0\\0&xx&0\\0&0&zz\end{pmatrix}\begin{pmatrix}0&xy&0\\xy&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}xx&0&0\\0&-xx&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&xy&0\\-xy&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

n > 2

$\mathbf{D_{\infty h}(C_{\infty h}) = \infty/mm'm'}$

$$\begin{pmatrix}xx&0&0\\0&xx&0\\0&0&zz\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&xy&0\\-xy&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}xx&0&0\\0&xx&0\\0&0&zz\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&xy&0\\-xy&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$\mathbf{D_{nd}(D_n)}$

n = 1

$\mathbf{D_{1d}(D_1) = 2_y/m_y(2_y) = 2_y/m_y'}$

$$\begin{pmatrix}xx&0&xz\\0&yy&0\\zx&0&zz\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}xx&0&xz\\0&yy&0\\zx&0&zz\end{pmatrix}$$

$$\begin{pmatrix}xx&0&xz\\0&yy&0\\xz&0&zz\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}xx&0&xz\\0&yy&0\\xz&0&zz\end{pmatrix}$$

$$\begin{pmatrix}0&0&xz\\0&0&0\\-xz&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&xz\\0&0&0\\-xz&0&0\end{pmatrix}$$

n = 2

$\mathbf{D_{2d}(D_2) = \bar{4}_z m_x 2_{xy}(2_{xy} 2_{\bar{x}y} 2_z) = \bar{4}_z' m_x' 2_{xy}}$

$$\begin{pmatrix}xx&0&0\\0&xx&0\\0&0&zz\end{pmatrix}\begin{pmatrix}0&xy&0\\xy&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&xy&0\\xy&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}xx&0&0\\0&xx&0\\0&0&zz\end{pmatrix}$$

$$\begin{pmatrix}xx&0&0\\0&xx&0\\0&0&zz\end{pmatrix}\begin{pmatrix}0&xy&0\\xy&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&xy&0\\xy&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}xx&0&0\\0&xx&0\\0&0&zz\end{pmatrix}$$

$$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

n > 2

$\mathbf{D_{\infty h}(D_\infty) = \infty/m'm'm'}$

$$\begin{pmatrix}xx&0&0\\0&xx&0\\0&0&zz\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}xx&0&0\\0&xx&0\\0&0&zz\end{pmatrix}$$

$$\begin{pmatrix}xx&0&0\\0&xx&0\\0&0&zz\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}xx&0&0\\0&xx&0\\0&0&zz\end{pmatrix}$$

$$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$D_{nd}(C_{nv})$

n = 1

$D_{1d}(C_{1v}) = 2_y/m_y(m_y) = 2_y'/m_y$

$$\begin{pmatrix} xx & 0 & xz \\ 0 & yy & 0 \\ zx & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ yx & 0 & yz \\ 0 & zy & 0 \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & xz \\ 0 & yy & 0 \\ xz & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ xy & 0 & yz \\ 0 & yz & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & 0 \\ -xz & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & yz \\ 0 & -yz & 0 \end{pmatrix}$$

n = 2

$D_{2d}(C_{2v}) = \bar{4}_z m_x 2_{xy}(2_z m_x m_y) = \bar{4}_z' m_x 2_{xy}'$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & -xx & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & -xx & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

n > 2

$D_{\infty h}(C_{\infty v}) = \infty/m'mm$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$D_{nh}(C_{nh})$

n = 1

$D_{1h}(C_{1h}) = 2_x m_y m_z(m_z) = 2_x' m_y' m_z$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & yz \\ 0 & zy & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ yx & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & 0 \\ zx & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & yz \\ 0 & yz & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & 0 \\ xz & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & yz \\ 0 & -yz & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & 0 \\ -xz & 0 & 0 \end{pmatrix}$$

n = 2

$D_{2h}(C_{2h}) = m_x m_y m_z(2_z/m_z) = m_x' m_y' m_z$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ yx & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

n > 2

$D_{\infty h}(C_{\infty h}) = \infty/mm'm'$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & xy & 0 \\ -xy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$D_{nh}(D_n)$

n = 1

$D_{1h}(D_1) = 2_x m_y m_z(2_x) = 2_x m_y' m_z'$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & yz \\ 0 & zy & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & yz \\ 0 & zy & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & yz \\ 0 & yz & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & yz \\ 0 & yz & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & yz \\ 0 & -yz & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & yz \\ 0 & -yz & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

n = 2

$D_{2h}(D_2) = m_x m_y m_z(2_x 2_y 2_z) = m_x' m_y' m_z'$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & yy & 0 \\ 0 & 0 & zz \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

n > 2

$D_{\infty h}(D_\infty) = \infty/m'm'm'$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}$$

$$\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} xx & 0 & 0 \\ 0 & xx & 0 \\ 0 & 0 & zz \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

n = 1

$\mathbf{D}_{nh}(\mathbf{C}_{nv})$ $\mathbf{D}_{1h}(\mathbf{C}_{1v}) = 2_x m_y\, m_z(m_y) = 2_x' m_y\, m_z'$

$$\begin{pmatrix}xx&0&0\\0&yy&0\\0&0&zz\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&yz\\0&zy&0\end{pmatrix}\begin{pmatrix}0&0&xz\\0&0&0\\zx&0&0\end{pmatrix}\begin{pmatrix}0&xy&0\\yx&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}xx&0&0\\0&yy&0\\0&0&zz\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&yz\\0&yz&0\end{pmatrix}\begin{pmatrix}0&0&xz\\0&0&0\\xz&0&0\end{pmatrix}\begin{pmatrix}0&xy&0\\xy&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&yz\\0&-yz&0\end{pmatrix}\begin{pmatrix}0&0&xz\\0&0&0\\-xz&0&0\end{pmatrix}\begin{pmatrix}0&xy&0\\-xy&0&0\\0&0&0\end{pmatrix}$$

n = 2

$\mathbf{D}_{2h}(\mathbf{C}_{2v}) = m_x m_y\, m_z(2_z m_x m_y) = m_x m_y\, m_z'$

$$\begin{pmatrix}xx&0&0\\0&yy&0\\0&0&zz\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&xy&0\\yx&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}xx&0&0\\0&yy&0\\0&0&zz\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&xy&0\\xy&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&xy&0\\-xy&0&0\\0&0&0\end{pmatrix}$$

n > 2

$\mathbf{D}_{\infty h}(\mathbf{C}_{\infty v}) = \infty/m'mm$

$$\begin{pmatrix}xx&0&0\\0&xx&0\\0&0&zz\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&xy&0\\-xy&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}xx&0&0\\0&xx&0\\0&0&zz\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&xy&0\\-xy&0&0\\0&0&0\end{pmatrix}$$

n = 1

$\mathbf{D}_{2nh}(\mathbf{D}_{nd})$ $\mathbf{D}_{2h}(\mathbf{D}_{1d}) = m_x m_y m_z(2_y/m_y) = m_x' m_y\, m_z'$

$$\begin{pmatrix}xx&0&0\\0&yy&0\\0&0&zz\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&xz\\0&0&0\\zx&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}xx&0&0\\0&yy&0\\0&0&zz\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&xz\\0&0&0\\xz&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&xz\\0&0&0\\-xz&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

n = 2

$\mathbf{D}_{4h}(\mathbf{D}_{2d}) = 4_z/m_z m_x m_{xy}(\bar{4}_z\, m_x 2_{xy}) = 4_z'/m_z' m_x m_{xy}'$

$$\begin{pmatrix}xx&0&0\\0&xx&0\\0&0&zz\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&xy&0\\xy&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}xx&0&0\\0&xx&0\\0&0&zz\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&xy&0\\xy&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

n > 2

$\mathbf{D}_{\infty h}1' = \infty/mmm1'$

$$\begin{pmatrix}xx&0&0\\0&xx&0\\0&0&zz\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}xx&0&0\\0&xx&0\\0&0&zz\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

n = 1

$\mathbf{D}_{2nh}(\mathbf{D}_{nh})$ $\mathbf{D}_{2h}(\mathbf{D}_{1h}) = m_x m_y m_z(2_x m_y m_z) = m_x' m_y m_z$

$$\begin{pmatrix}xx&0&0\\0&yy&0\\0&0&zz\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&yz\\0&zy&0\end{pmatrix}$$

$$\begin{pmatrix}xx&0&0\\0&yy&0\\0&0&zz\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&yz\\0&yz&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&yz\\0&-yz&0\end{pmatrix}$$

n = 2

$\mathbf{D}_{4h}(\mathbf{D}_{2h}) = 4_z/m_z m_x m_{xy}(m_x m_y m_z) = 4_z'/m_z m_x m_{xy}'$

$$\begin{pmatrix}xx&0&0\\0&xx&0\\0&0&zz\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}xx&0&0\\0&-xx&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}xx&0&0\\0&xx&0\\0&0&zz\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}xx&0&0\\0&-xx&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

n > 2

$\mathbf{D}_{\infty h}1' = \infty/mmm1'$

$$\begin{pmatrix}xx&0&0\\0&xx&0\\0&0&zz\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}xx&0&0\\0&xx&0\\0&0&zz\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

Appendix A:

We first show that for a tensor V^m and a group $\mathbf{C}_n$ that the components which satisfy the 3^m conditions

$$m\text{-product}_s = C_n\,(m\text{-product}_s) \qquad \text{(A1)}$$

$s = 1, 2, \ldots, 3^m$, also satisfy, the $3^m\,(n - 1)$ conditions

$$m\text{-product}_s = (C_n)^j(m\text{-product}_s) \qquad \text{(A2)}$$

$s = 1, 2, \ldots, 3^m$ and $j = 2, 3, \ldots, n$:

Substitute equation (A1) into the right-hand-side of equation (A1):

$$m\text{-product}_s = C_n\,(C_n\,(m\text{-product}_s))$$

$$=(C_n)^2(m\text{-product}_s)$$

Repeat an additional j-2 times to obtain equation (A2).

To show that for a tensor V^m and a group $\mathbf{C}_n$ that the components which satisfy the 3^m conditions

$$n(m\text{-product}_s) = \sum_{j=1}^{n}(C_n)^j(m-\text{product}_s) \qquad \text{(A3)}$$

satisfy each of the equations (A1) and (A2), we operate on both sides of (A3) with $(C_n)^t$ where "t" is an integer:

$$(C_n)^t\,n(m\text{-product}_s) = (C_n)^t \sum_{j=1}^{n}(C_n)^j(m-\text{product}_s)$$

$$n(C_n)^t\,(m\text{-product}_s) = \sum_{j=1}^{n}(C_n)^{j+t}(m-\text{product}_s)$$

$(C_n)^{j+t}$ can be replaced with $(C_n)^{(j+t)\mathrm{mod}(n)}$, and as j runs from 1 to n, (j+t)mod(n) also runs from 1 to n. Therefore:

$$n(C_n)^t\,(m\text{-product}_s) = \sum_{j=1}^{n}(C_n)^j(m-\text{product}_s) = n(m\text{-product}_s)$$

and finally:

$$(m\text{-product}_s) = (C_n)^t\,(m\text{-product}_s)$$

Taking $t = 1, 2, \ldots, n$ we have equations (A1) and (A2).

Appendix B: Form of V^2 invariant under $\mathbf{C}_n$

For n = 1 ,we have invariant under $\mathbf{C}_1$, the general form of the tensor is

$V^2 = \begin{pmatrix} xx & xy & xz \\ yx & yy & yz \\ zx & zy & zz \end{pmatrix}$. For $n \geq 2$, applying equation (5) to each of the components, one has using equations (6a,b):

$$nxz = \sum_{j=1}^{n} (x\cos(\frac{2\pi}{n})j - y\sin(\frac{2\pi}{n})j)z; \quad (B1)$$

$$nzx = \sum_{j=1}^{n} z(x\cos(\frac{2\pi}{n})j - y\sin(\frac{2\pi}{n})j); \quad (B2)$$

$$nyz = \sum_{j=1}^{n} (x\sin(\frac{2\pi}{n})j + y\cos(\frac{2\pi}{n})j)z; \quad (B3)$$

$$nzy = \sum_{j=1}^{n} z(x\sin(\frac{2\pi}{n})j + y\cos(\frac{2\pi}{n})j); \quad (B4)$$

$$nzz = nzz; \quad (B5)$$

$$nxx = \sum_{j=1}^{n} (x\cos(\frac{2\pi}{n})j - y\sin(\frac{2\pi}{n})j)^2; \quad (B6)$$

$$nyy = \sum_{j=1}^{n} (x\sin(\frac{2\pi}{n})j + y\cos(\frac{2\pi}{n})j)^2; \quad (B7)$$

$$nxy = \sum_{j=1}^{n} (x\cos(\frac{2\pi}{n})j - y\sin(\frac{2\pi}{n})j)(x\sin(\frac{2\pi}{n})j + y\cos(\frac{2\pi}{n})j); \quad (B8)$$

$$nyx = \sum_{j=1}^{n} (x\sin(\frac{2\pi}{n})j + y\cos(\frac{2\pi}{n})j)(x\cos(\frac{2\pi}{n})j - y\sin(\frac{2\pi}{n})j); \quad (B9)$$

Since $\sum_{j=1}^{n}\sin aj=\frac{\sin\frac{n+1}{2}a\sin\frac{na}{2}}{\sin\frac{a}{2}}$ and $\sum_{j=1}^{n}\cos aj=\frac{\cos\frac{n+1}{2}a\sin\frac{na}{2}}{\sin\frac{a}{2}}$ (Gradshteyn and Ryshik, 2007), we have for n≥2 that $\sum_{j=1}^{n}\cos\frac{2\pi}{n}j=0$ and $\sum_{j=1}^{n}\sin\frac{2\pi}{n}j=0$ and from equations (B1-B4) we have:

$$xz = zx = yz = zy = 0. \qquad \text{(B10)}$$

From equation (B5) we have the trivial zz = zz, i.e. no condition on component zz.

The remaining four conditions can be rewritten as:

$$\begin{aligned}
nxx &= xx\sum_{j=1}^{n}\cos^2\frac{2\pi}{n}j + yy\sum_{j=1}^{n}\sin^2\frac{2\pi}{n}j - (xy+yx)\sum_{j=1}^{n}\cos\frac{2\pi}{n}j\sin\frac{2\pi}{n}j \\
nyy &= yy\sum_{j=1}^{n}\cos^2\frac{2\pi}{n}j + xx\sum_{j=1}^{n}\sin^2\frac{2\pi}{n}j + (xy+yx)\sum_{j=1}^{n}\cos\frac{2\pi}{n}j\sin\frac{2\pi}{n}j \\
nxy &= xy\sum_{j=1}^{n}\cos^2\frac{2\pi}{n}j - yx\sum_{j=1}^{n}\sin^2\frac{2\pi}{n}j + (xx-yy)\sum_{j=1}^{n}\cos\frac{2\pi}{n}j\sin\frac{2\pi}{n}j \\
nyx &= yx\sum_{j=1}^{n}\cos^2\frac{2\pi}{n}j - xy\sum_{j=1}^{n}\sin^2\frac{2\pi}{n}j + (xx-yy)\sum_{j=1}^{n}\cos\frac{2\pi}{n}j\sin\frac{2\pi}{n}j
\end{aligned} \qquad \text{(B11)}$$

The values of the summations in these conditions are for n=2

$$\sum_{j=1}^{2}\cos^2\pi j = 2,\quad \sum_{j=1}^{2}\sin^2\pi j = 0,\quad \text{and}\quad \sum_{j=1}^{2}\cos\pi j\sin\pi j = 0$$

and consequently equations (B11) give no conditions on the components xx, yy, xy, and yx. For n > 2, using (Gradshteyn and Ryshik, 2007)

$$\sum_{k=1}^{n}\cos^2 kx=\frac{n}{2}+\frac{\cos(n+1)x\sin nx}{2\sin x},\quad \sum_{k=1}^{n}\sin^2 kx=\frac{n}{2}-\frac{\cos(n+1)x\sin nx}{2\sin x},\quad \text{and}$$

$\sum_{k=1}^{n}\cos kx\sin kx=\frac{1}{2}\sum_{k=1}^{n}\sin 2kx=\frac{1}{2}\frac{\sin(n+1)x\sin nx}{\sin x}$, we have:

$\sum_{j=1}^{n}\cos^2\frac{2\pi}{n}j = \sum_{j=1}^{n}\sin^2\frac{2\pi}{n}j = \frac{n}{2}$ and $\sum_{j=1}^{n}\cos\frac{2\pi}{n}j\sin\frac{2\pi}{n}j = 0$. Using these summations and equations (B11) we have no conditions on components xx and xy and the condition yx = -xy.

Using these summations and conditions we obtain that the form of the physical property tensor V^2 invariant under $\mathbf{C}_2$ is $V^2 = \begin{pmatrix} xx & xy & 0 \\ yx & yy & 0 \\ 0 & 0 & zz \end{pmatrix}$ and that the form invariant under $\mathbf{C}_n$, for n>2, is $V^2 = \begin{pmatrix} xx & xy & 0 \\ -xy & xx & 0 \\ 0 & 0 & zz \end{pmatrix}$. These results agree with form of the physical property tensor V^2 invariant under $\mathbf{C}_n$ for n=2,3,4, and 6 given by Sirotin & Shaskolskaya (1982).

Appendix C: Form of V^2 invariant under $\mathbf{C}_\theta$

The tensor V^2 in terms of the m-products of its components is $\begin{pmatrix} xx & xy & xz \\ yx & yy & yz \\ zx & zy & zz \end{pmatrix}$.

Applying equation (9) to each of the m-products:

$$2\pi xx = \int_{\theta=0}^{\theta=2\pi} (x\cos\theta - y\sin\theta)^2 d\theta$$

$$= xx\int_{\theta=0}^{\theta=2\pi} \cos^2\theta d\theta + yy\int_{\theta=0}^{\theta=2\pi} \sin^2\theta d\theta - (xy + yx)\int_{\theta=0}^{\theta=2\pi} \cos\theta\sin\theta d\theta \quad \text{(C1)}$$

$$2\pi yy = \int_{\theta=0}^{\theta=2\pi} (x\sin\theta + y\cos\theta)^2 d\theta$$

$$= yy\int_{\theta=0}^{\theta=2\pi} \cos^2\theta d\theta + xx\int_{\theta=0}^{\theta=2\pi} \sin^2\theta d\theta + (xy + yx)\int_{\theta=0}^{\theta=2\pi} \cos\theta\sin\theta d\theta \quad \text{(C2)}$$

$$2\pi zz = zz\int_{\theta=0}^{\theta=2\pi} d\theta \quad \text{(C3)}$$

$$2\pi xy = \int_{\theta=0}^{\theta=2\pi} (x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta) d\theta$$

$$= xy\int_{\theta=0}^{\theta=2\pi} \cos^2\theta d\theta - yx\int_{\theta=0}^{\theta=2\pi} \sin^2\theta d\theta + (xx - yy)\int_{\theta=0}^{\theta=2\pi} \cos\theta\sin\theta d\theta \quad \text{(C4)}$$

$$2\pi yx = \int_{\theta=0}^{\theta=2\pi} (x\sin\theta + y\cos\theta)(x\cos\theta - y\sin\theta) d\theta$$

$$= yx\int_{\theta=0}^{\theta=2\pi} \cos^2\theta d\theta - xy\int_{\theta=0}^{\theta=2\pi} \sin^2\theta d\theta + (xx - yy)\int_{\theta=0}^{\theta=2\pi} \cos\theta\sin\theta d\theta \quad \text{(C5)}$$

$$2\pi xz = \int_{\theta=0}^{\theta=2\pi} (x\cos\theta - y\sin\theta)z d\theta = xz\int_{\theta=0}^{\theta=2\pi} \cos\theta d\theta - yz\int_{\theta=0}^{\theta=2\pi} \sin\theta d\theta$$

(C6)

$$2\pi zx = \int_{\theta=0}^{\theta=2\pi} z(x\cos\theta - y\sin\theta) d\theta = zx\int_{\theta=0}^{\theta=2\pi} \cos\theta d\theta - zy\int_{\theta=0}^{\theta=2\pi} \sin\theta d\theta \quad \text{(C7)}$$

$$2\pi yz = \int_{\theta=0}^{\theta=2\pi} (x\sin\theta + y\cos\theta)z d\theta = yz\int_{\theta=0}^{\theta=2\pi} \cos\theta d\theta + xz\int_{\theta=0}^{\theta=2\pi} \sin\theta d\theta$$

(C8)

$$2\pi zy = \int_{\theta=0}^{\theta=2\pi} z(x\sin\theta + y\cos\theta)d\theta = zy\int_{\theta=0}^{\theta=2\pi}\cos\theta d\theta + zx\int_{\theta=0}^{\theta=2\pi}\sin\theta d\theta \qquad \text{(C9)}$$

Using $\int_{\theta=0}^{\theta=2\pi}\cos^2\theta d\theta = \pi$, $\int_{\theta=0}^{\theta=2\pi}\sin\theta^2 d\theta = \pi$, and $\int_{\theta=0}^{\theta=2\pi}\cos\theta\sin\theta d\theta = 0$, we have xx = yy from Equations (C1,C2), no condition on zz from Equation (C3), yx = -xy from Equations (C4, C5), and xz = zx = yz =zy =0 from equations (C6 – C9). Therefore, the form of tensor V^2 invariant under **C∞** is $\begin{pmatrix} xx & xy & 0 \\ -xy & xx & 0 \\ 0 & 0 & zz \end{pmatrix}$.

Appendix D: Proof of the Theorem: *The form of the physical property tensor V^m invariant under a group* $\boldsymbol{C}_n$ *, with n > m, is independent of n.*

The proof is by showing that when n > m all conditions on the components of the tensor, which determine the form of the tensor, are independent of n. This is done , as explained in the text, by showing that for $s \leq m$, $0 \leq b \leq s$, and $n>m$ $\sum_{j=1}^{n} \cos^{s-b}\frac{2\pi}{n}j \sin^{b}\frac{2\pi}{n}j$ = 0 or $\propto n$. The proof is divided into eight parts depending on the value and parity of the exponents of the trigonometric functions:

We use the following formulae each specialized from general trigonometric summations given in Gradshteyn & Ryshik (2007). The A, B, C, and D's are real numbers, and a, b, and N are integers.

$$\text{b even: } \sin^{b}\frac{2\pi}{n}j = A_{oo} + \sum_{k=0}^{b/2-1} A_k \cos(b-2k)\frac{2\pi}{n}j$$

$$\cos^{s-b}\frac{2\pi}{n}j = B_{oo} + \sum_{k=0}^{(s-b)/2-1} B_k \cos(s-b-2k)\frac{2\pi}{n}j$$

$$\text{b odd: } \sin^{b}\frac{2\pi}{n}j = \sum_{k=0}^{(b-1)/2} C_k \sin(b-2k)\frac{2\pi}{n}j$$

$$\cos^{s-b}\frac{2\pi}{n}j = \sum_{k=0}^{(s-b-1)/2} D_k \cos(s-b-2k)\frac{2\pi}{n}j$$

$$\sum_{j=1}^{n} \sin N\frac{2\pi}{n}j = \frac{\sin[N\pi + (\frac{N}{n})\pi]\sin N\pi}{\sin(\frac{N}{n})\pi}$$

$$\sum_{j=1}^{n} \cos N \frac{2\pi}{n} j = \frac{\cos[N\pi + (\frac{N}{n})\pi] \sin N\pi}{\sin(\frac{N}{n})\pi}$$

Proof:

1) <u>b=0 s even</u>

$$\sum_{j=1}^{n} \cos^s \frac{2\pi}{n} j = nB_{oo} + \sum_{j=1}^{n} \sum_{k=0}^{s/2-1} B_k \cos(s-2k)\frac{2\pi}{n} j$$

$$= nB_{oo} + \sum_{k=0}^{s/2-1} B_k (\sum_{j=1}^{n} \cos(s-2k)\frac{2\pi}{n} j)$$

$$= nB_{oo} + \sum_{k=0}^{s/2-1} B_k \frac{\cos[(s-2k)\pi + (\frac{s-2k}{n})\pi] \sin(s-2k)\pi}{\sin(\frac{s-2k}{n})\pi}$$

and since the cosine function is periodic with cos(θ+2π) = cosθ, we have:

$$= nB_{oo} + \sum_{k=0}^{s/2-1} B_k \frac{\cos[(\frac{s-2k}{n})\pi] \sin(s-2k)\pi}{\sin(\frac{s-2k}{n})\pi}$$

Since n>m>s>s-2k, $\frac{s-2k}{n}\pi < \pi$ and the denominator in the second term is not zero. The numerator is zero since s-2k is even and then sin(s-2k)π=0. Therefore:

$$\sum_{j=1}^{n} \cos^s \frac{2\pi}{n} j \propto n \,.$$

2) <u>b=0 s odd</u>

$$\sum_{j=1}^{n} \cos^s \frac{2\pi}{n} j = \sum_{j=1}^{n} \sum_{k=0}^{(s-1)/2} D_k \cos(s-2k)\frac{2\pi}{n} j$$

$$= \sum_{k=0}^{(s-1)/2} D_k (\sum_{j=1}^{n} \cos(s-2k)\frac{2\pi}{n} j)$$

$$= \sum_{k=0}^{(s-1)/2} D_k \frac{\cos[(s-2k)\pi + (\frac{s-2k}{n})\pi] \sin(s-2k)\pi}{\sin(\frac{s-2k}{n})\pi}$$

s-2k= s, s-2,s-4,….,1 are all odd, (s-2k)π is π+a multiple of 2π. since cos is periodic in

2π and cos(θ+π) = -cosθ we have:

$$= - \sum_{k=0}^{(s-1)/2} D_k \frac{\cos(\frac{s-2k}{n})\pi \sin(s-2k)\pi}{\sin(\frac{s-2k}{n})\pi}$$

Since the denominator is never zero and sin(s-2k)π=0 :

$$\sum_{j=1}^{n} \cos^s \frac{2\pi}{n} j = 0$$

3) <u>b=s s even</u>

$$\sum_{j=1}^{n} \sin^s \frac{2\pi}{n} j \quad = nA_{oo} + \sum_{j=1}^{n} \sum_{k=0}^{s/2-1} A_k \cos(s-2k)\frac{2\pi}{n} j = nA_{oo}$$ using the analogous argument as in the case of b=0 and s even, we have:

$$\sum_{j=1}^{n} \sin^s \frac{2\pi}{n} j \propto n$$

4) <u>b=s s odd</u>

$$\sum_{j=1}^{n} \sin^s \frac{2\pi}{n} j \quad = \sum_{j=1}^{n} \sum_{k=0}^{(s-1)/2} C_k \sin(s-2k)\frac{2\pi}{n} j$$

$$= \sum_{k=0}^{(s-1)/2} C_k (\sum_{j=1}^{n} \sin(s-2k)\frac{2\pi}{n} j)$$

$$= \sum_{k=0}^{(s-1)/2} C_k \frac{\sin[(s-2k)\pi + (\frac{s-2k}{n})\pi] \sin(s-2k)\pi}{\sin(\frac{s-2k}{n})\pi}$$

s-2k = s,s-2,s-4,….,1 all odd, (s-2k)π is π+a multiple of 2π, since sin is periodic in 2π and sin(θ+π) = -sinθ we have:

$$= - \sum_{k=0}^{(s-1)/2} C_k \frac{\sin(\frac{s-2k}{n})\pi \sin(s-2k)\pi}{\sin(\frac{s-2k}{n})\pi}$$

Since denominator is never zero and sin(s-2k)π=0 we have:

$$\sum_{j=1}^{n} \sin^s \frac{2\pi}{n} j = 0$$

5) b≠0 or s s-b even, b even

$$\sum_{j=1}^{n} \cos^{s-b} \frac{2\pi}{n} j \sin^b \frac{2\pi}{n} j$$

$$= \sum_{j=1}^{n} \left(B_{oo} + \sum_{k=0}^{(s-b)/2-1} B_k \cos(s-b-2k)\frac{2\pi}{n} j\right)\left(A_{oo} + \sum_{t=0}^{b/2-1} A_k \cos(b-2t)\frac{2\pi}{n} j\right)$$

$$= nB_{oo}A_{oo}$$

$$+ \sum_{j=1}^{n} B_{oo} \sum_{t=0}^{b/2-1} A_k \cos(b-2t)\frac{2\pi}{n} j$$

$$+ \sum_{j=1}^{n} A_{oo} \sum_{k=0}^{(s-b)/2-1} B_k \cos(s-b-2k)\frac{2\pi}{n} j$$

$$+ \sum_{j=1}^{n} \sum_{k=0}^{(s-b)/2-1} B_k \cos(s-b-2k)\frac{2\pi}{n} j \sum_{t=0}^{b/2-1} A_k \cos(b-2t)\frac{2\pi}{n} j$$

The second term of the four terms:

$$\sum_{j=1}^{n} B_{oo} \sum_{t=0}^{b/2-1} A_k \cos(b-2t)\frac{2\pi}{n} j$$

$$= B_{oo} \sum_{t=0}^{b/2-1} A_k \left(\sum_{j=1}^{n} \cos(b-2t)\frac{2\pi}{n} j\right)$$

$$= B_{oo} \sum_{t=0}^{b/2-1} A_k \frac{\cos[(b-2t)\pi + (\frac{b-2t}{n})\pi] \sin(b-2t)\pi}{\sin(\frac{b-2t}{n})\pi}$$

b-2t is even , (b-2t)π is then a multiple of 2π and cosine is periodic in 2π, so

$$= B_{oo} \sum_{t=0}^{b/2-1} A_k \frac{\cos(\frac{b-2t}{n})\pi \sin(b-2t)\pi}{\sin(\frac{b-2t}{n})\pi}$$

the denominator is never zero and sin(b-2t)π=0 , so:

$$\sum_{j=1}^{n} B_{oo} \sum_{t=0}^{b/2-1} A_k \cos(b-2t)\frac{2\pi}{n} j = 0$$

The third term is:

$$\sum_{j=1}^{n} A_{oo} \sum_{k=0}^{(s-b)/2-1} B_k \cos(s-b-2k)\frac{2\pi}{n} j$$

$$= A_{oo}\sum_{k=0}^{(s-b)/2-1} B_k\left(\sum_{j=1}^{n} \cos(s-b-2k)\frac{2\pi}{n}j\right)$$

$$= A_{oo}\sum_{k=0}^{(s-b)/2-1} B_k \frac{\cos[(s-b-2k)\pi + (\frac{s-b-2k}{n})\pi]\sin(s-b-2k)\pi}{\sin(\frac{s-b-2k}{n})\pi}$$

s-b-2k is even, (s-b-2k)π is then a multiple of 2π and cosine is periodic in 2π, so

$$= A_{oo}\sum_{k=0}^{(s-b)/2-1} B_k \frac{\cos(\frac{s-b-2k}{n})\pi \sin(s-b-2k)\pi}{\sin(\frac{s-b-2k}{n})\pi}$$

The denominator is never zero, sin(b-2k)π=0 so

$$\sum_{j=1}^{n} A_{oo}\sum_{k=0}^{(s-b)/2-1} B_k \cos(s-b-2k)\frac{2\pi}{n}j = 0$$

The fourth term:

$$\sum_{j=1}^{n} \sum_{k=0}^{(s-b)/2-1} B_k \cos(s-b-2k)\frac{2\pi}{n}j \sum_{t=0}^{b/2-1} A_k \cos(b-2t)\frac{2\pi}{n}j$$

$$= \sum_{k=0}^{(s-b)/2-1} B_k \sum_{t=0}^{b/2-1} A_k \sum_{j=1}^{n} \cos(s-b-2k)\frac{2\pi}{n}j\cos(b-2t)\frac{2\pi}{n}j$$

$$= ½ \sum_{k=0}^{(s-b)/2-1} B_k \sum_{t=0}^{b/2-1} A_k \sum_{j=1}^{n} [\cos(s-2b-2k+2t)\frac{2\pi}{n}j + \cos(s-2k-2t)\frac{2\pi}{n}j]$$

$$=½ \sum_{k=0}^{(s-b)/2-1} B_k \sum_{t=0}^{b/2-1} A_k \left(\sum_{j=1}^{n}\cos(s-2b-2k+2t)\frac{2\pi}{n}j\right)$$

$$+½ \sum_{k=0}^{(s-b)/2-1} B_k \sum_{t=0}^{b/2-1} A_k \left(\sum_{j=1}^{n}\cos(s-2k-2t)\frac{2\pi}{n}j\right)$$

In the first half of this:

$\sum_{j=1}^{n}\cos(s-2b-2k+2t)\frac{2\pi}{n}j = n$ if s=2b-2k+2t=0. If s-2b-2k+2t ≠ 0 then

$$\sum_{j=1}^{n}\cos(s-2b-2k+2t)\frac{2\pi}{n}j =$$

$$= \frac{\cos[(s-2b-2k+2t)\pi + (\frac{s-2b-2k+2t}{n})\pi]\sin(s-2b-2k+2t)\pi}{\sin(\frac{s-2b-2k+2t}{n})\pi}$$

s-2b-2k+2t is even so

$$=\frac{\cos(\frac{s-2b-2k+2t}{n})\pi\sin(s-2b-2k+2t)\pi}{\sin(\frac{s-2b-2k+2t}{n})\pi}$$

and since the denominator not zero, sin(s-2b-2k+2t)π=0 we have:

$$\sum_{j=1}^{n}\cos(s-2b-2k+2t)\frac{2\pi}{n}j=0 .$$

In the second half of the fourth term:

$\sum_{j=1}^{n}\cos(s-2k-2t)\frac{2\pi}{n}j$ =n if s-2k-2t =0. If s-2k-2t ≠ 0 then:

$$\sum_{j=1}^{n}\cos(s-2k-2t)\frac{2\pi}{n}j=\frac{\cos[(s-2k-2t)\pi+(\frac{s-2k-2t}{n})\pi]\sin(s-2k-2t)\pi}{\sin(\frac{s-2k-2t}{n})\pi}$$

s-2k-2t is even so

$$=\frac{\cos(\frac{s-2k-2t}{n})\pi\sin(s-2k-2t)\pi}{\sin(\frac{s-2k-2t}{n})\pi}$$

The denominator not zero, sin(s-2k-2t)π=0 and we have:

$$\sum_{j=1}^{n}\cos(s-2k-2t)\frac{2\pi}{n}j=0$$

Therefore, for b∓0 or s, s-b even and b even:

$\sum_{j=1}^{n}\cos^{s-b}\frac{2\pi}{n}j\sin^{b}\frac{2\pi}{n}j$ =0 or is proportional to n.

6) <u>b∓0 or s s-b even, b odd</u>

$$\sum_{j=1}^{n}\cos^{s-b}\frac{2\pi}{n}j\sin^{b}\frac{2\pi}{n}j=\sum_{j=1}^{n}(B_{oo}+\sum_{t=0}^{(s-b)/2-1}B_t\cos(s-b-2t)\frac{2\pi}{n}j)(\sum_{k=0}^{(b-1)/2}C_k\sin(b-2k)\frac{2\pi}{n}j)$$

$$=\sum_{j=1}^{n}B_{oo}\sum_{k=0}^{(b-1)/2}C_k\sin(b-2k)\frac{2\pi}{n}j$$

$$+\sum_{j=1}^{n}\sum_{t=0}^{(s-b)/2-1}B_t\cos(s-b-2t)\frac{2\pi}{n}j\sum_{k=0}^{(b-1)/2}C_k\sin(b-2k)\frac{2\pi}{n}j$$

In the first term

$$\sum_{j=1}^{n} B_{oo} \sum_{k=0}^{(b-1)/2} C_k \sin(b-2k)\frac{2\pi}{n} j = B_{oo} \sum_{k=0}^{(b-1)/2} C_k (\sum_{j=1}^{n} \sin(b-2k)\frac{2\pi}{n} j)$$

we have:

$$\sum_{j=1}^{n} \sin(b-2k)\frac{2\pi}{n} j = \frac{\sin[(b-2k)\pi + (\frac{b-2k}{n})\pi]\sin(b-2k)\pi}{\sin(\frac{b-2k}{n})\pi}$$

Since b-2k is odd, (b-2k)π = π+ a multiple of 2π, and sin(θ+π) = -sinθ, we have:

$$= - \frac{\sin(\frac{b-2k}{n})\pi \sin(b-2k)\pi}{\sin(\frac{b-2k}{n})\pi}$$

The denominator is not zero and sin(b-2k)π=0 , so:

$$\sum_{j=1}^{n} B_{oo} \sum_{k=0}^{(b-1)/2} C_k \sin(b-2k)\frac{2\pi}{n} j = 0$$

In the second term

$$\sum_{j=1}^{n} \sum_{t=0}^{(s-b)/2-1} B_t \cos(s-b-2t)\frac{2\pi}{n} j \sum_{k=0}^{(b-1)/2} C_k \sin(b-2k)\frac{2\pi}{n} j$$

$$= \sum_{t=0}^{(s-b)/2-1} B_t \sum_{k=0}^{(b-1)/2} C_k (\sum_{j=1}^{n} \cos(s-b-2t)\frac{2\pi}{n} j \sin(b-2k)\frac{2\pi}{n} j)$$

we have:

$$\sum_{j=1}^{n} \cos(s-b-2t)\frac{2\pi}{n} j \sin(b-2k)\frac{2\pi}{n} j$$

$$=½ \sum_{j=1}^{n} \sin(s-2k-2t)\frac{2\pi}{n} j - ½ \sum_{j=1}^{n} \sin(s-2b-2t+2k)\frac{2\pi}{n} j$$

In the first part

$\sum_{j=1}^{n} \sin(s-2k-2t)\frac{2\pi}{n} j$ = 0 if s-2k-2t=0, if s-2k-2t≠0 then:

$$= \frac{\sin[(s-2k-2t)\pi + (\frac{s-2k-2t}{n})\pi]\sin(s-2k-2t)\pi}{\sin(\frac{s-2k-2t}{n})\pi}$$

Since s-2k-2t is odd, we have

$$= - \frac{\sin(\frac{s-2k-2t}{n})\pi \sin(s-2k-2t)\pi}{\sin(\frac{s-2k-2t}{n})\pi}$$

and since s-2k-2t≠0 :

$$\sum_{j=1}^{n} \sin(s-2k-2t)\frac{2\pi}{n} j = 0 \ .$$

In the second part

$$\sum_{j=1}^{n} \sin(s-2b-2t+2k)\frac{2\pi}{n} j \ = 0 \text{ if } s\text{-}2b\text{-}2t\text{+}2k = 0$$

if s-2b-2t+2k≠0 then:

$$= \frac{\sin[(s-2b+2k-2t)\pi + (\frac{s-2b+2k-2t}{n})\pi]\sin(s-2b+2k-2t)\pi}{\sin(\frac{s-2b+2k-2t}{n})\pi}$$

Since s-2b+2k-2t is odd

$$= - \frac{\sin(\frac{s-2b+2k-2t}{n})\pi \sin(s-2b+2k-2t)\pi}{\sin(\frac{s-2b+2k-2t}{n})\pi} = 0 \ .$$

Therefore

$$\sum_{j=1}^{n} \sum_{t=0}^{(s-b)/2-1} B_t \cos(s-b-2t)\frac{2\pi}{n} j \sum_{k=0}^{(b-1)/2} C_k \sin(b-2k)\frac{2\pi}{n} j \ = 0$$

and

$$\sum_{j=1}^{n} \cos^{s-b}\frac{2\pi}{n} j \sin^{b}\frac{2\pi}{n} j = 0 \text{ for } b \neq 0 \text{ or } s, \ \ s-b \text{ even and } b \text{ odd.}$$

7) <u>b≠0 or s   s-b odd, b odd</u>

$$\sum_{j=1}^{n}\cos^{s-b}\frac{2\pi}{n}j\sin^{b}\frac{2\pi}{n}j = \sum_{j=1}^{n}\ \left(\sum_{k=0}^{(s-b-1)/2} D_k\cos(s-b-2k)\frac{2\pi}{n}j\right)\left(\sum_{t=0}^{(b-1)/2} C_t\sin(b-2t)\frac{2\pi}{n}j\right)$$

$$= \sum_{k=0}^{(s-b-1)/2} D_k \sum_{t=0}^{(b-1)/2} C_t\left(\sum_{j=1}^{n}\ \cos(s-b-2k)\frac{2\pi}{n}j\sin(b-2t)\frac{2\pi}{n}j\right)$$

Here we have

$$\sum_{j=1}^{n}\ \cos(s-b-2k)\frac{2\pi}{n}j\sin(b-2t)\frac{2\pi}{n}j=$$

$$= ½\sum_{j=1}^{n}\ \sin(s-2k-2t)\frac{2\pi}{n}j\ -\ ½\sum_{j=1}^{n}\ \sin(s-2b-2k+2t)\frac{2\pi}{n}j$$

In the first term

$\sum_{j=1}^{n}\ \sin(s-2k-2t)\frac{2\pi}{n}j$ =0 if s-2k-2t = 0 and if s-2k-2t ≠0 :

$$= \frac{\sin[(s-2k-2t)\pi+(\frac{s-2k-2t}{n})\pi]\sin(s-2k-2t)\pi}{\sin(\frac{s-2k-2t}{n})\pi}$$

Since s-2k-2t is even

$$= \frac{\sin(\frac{s-2k-2t}{n})\pi\sin(s-2k-2t)\pi}{\sin(\frac{s-2k-2t}{n})\pi}$$

and since s-2k-2t≠0:

$$\sum_{j=1}^{n}\ \sin(s-2k-2t)\frac{2\pi}{n}j = 0\ .$$

In the second term

$\sum_{j=1}^{n}\ \sin(s-2b-2k+2t)\frac{2\pi}{n}j$ = 0 if s-2b-2k+2t =0, and if s-2b-2k+2t≠0

$$= \frac{\sin[(s-2b-2k+2t)\pi+(\frac{s-2b-2k+2t}{n})\pi]\sin(s-2b-2k+2t)\pi}{\sin(\frac{s-2b-2k+2t}{n})\pi}$$

Since s-2b-2k+2t is even

$$= \frac{\sin(\frac{s-2b-2k+2t}{n})\pi \sin(s-2b-2k+2t)\pi}{\sin(\frac{s-2b-2k+2t}{n})\pi} = 0 \ .$$

Therefore:

$$\sum_{j=1}^{n} \cos^{s-b}\frac{2\pi}{n}j \sin^{b}\frac{2\pi}{n}j = 0 \quad \text{when } b\ne 0 \text{ or } s,\ \text{ s-b odd and b odd.}$$

8) <u>b≠0 or s   s-b odd, b even</u>

$$\sum_{j=1}^{n} \cos^{s-b}\frac{2\pi}{n}j \sin^{b}\frac{2\pi}{n}j = \sum_{j=1}^{n} \sum_{k=0}^{(s-b-1)/2} D_k \cos(s-b-2k)\frac{2\pi}{n}j \left(A_{oo} + \sum_{t=0}^{b/2-1} A_t \cos(b-2t)\frac{2\pi}{n}j \right)$$

$$= A_{oo}\sum_{j=1}^{n} \sum_{k=0}^{(s-b-1)/2} D_k \cos(s-b-2k)\frac{2\pi}{n}j$$

$$+ \sum_{j=1}^{n} \sum_{k=0}^{(s-b-1)/2} D_k \cos(s-b-2k)\frac{2\pi}{n}j \sum_{t=0}^{b/2-1} A_t \cos(b-2t)\frac{2\pi}{n}j$$

$$= A_{oo}\sum_{k=0}^{(s-b-1)/2} D_k \left(\sum_{j=1}^{n} \cos(s-b-2k)\frac{2\pi}{n}j\right)$$

$$+ \sum_{k=0}^{(s-b-1)/2} D_k \sum_{t=0}^{b/2-1} A_t \left(\sum_{j=1}^{n} \cos(s-b-2k)\frac{2\pi}{n}j \cos(b-2t)\frac{2\pi}{n}j\right)$$

In the first term

s-b-2k≠0 because s-b is odd. Therefore:

$$\sum_{j=1}^{n} \cos(s-b-2k)\frac{2\pi}{n}j = \frac{\cos[(s-b-2k)\pi + (\frac{s-b-2k}{n})\pi] \sin(s-b-2k)\pi}{\sin(\frac{s-b-2k}{n})\pi}$$

Since s-b-2k is odd

$$= - \frac{\cos(\frac{s-b-2k}{n})\pi \sin(s-b-2k)\pi}{\sin(\frac{s-b-2k}{n})\pi} = 0.$$

In the second term

$$\sum_{j=1}^{n} \cos(s-b-2k)\frac{2\pi}{n}j\cos(b-2t)\frac{2\pi}{n} =$$

$$=½\sum_{j=1}^{n} \cos(s-2b-2k+2t)\frac{2\pi}{n}j \;+½\sum_{j=1}^{n} \cos(s-2k-2t)\frac{2\pi}{n}j$$

In the first part

s-2b-2k+2t ≠ 0 since s is odd. Therefore

$$\sum_{j=1}^{n} \cos(s-2b-2k+2t)\frac{2\pi}{n}j =$$

$$= \frac{\cos[(s-2b-2k+2t)\pi+(\frac{s-2b-2k+2t}{n})\pi]\sin(s-2b-2k+2t)\pi}{\sin(\frac{s-2b-2k+2t}{n})\pi}$$

and since s-2b-2k+2t is odd

$$= - \frac{\cos(\frac{s-2b-2k+2t}{n})\pi\sin(s-2b-2k+2t)\pi}{\sin(\frac{s-2b-2k+2t}{n})\pi} = 0 .$$

Therefore:

$$\sum_{j=1}^{n}\cos^{s-b}\frac{2\pi}{n}j\sin^{b}\frac{2\pi}{n}j = 0$$ when b≠0 or s, s-b odd and b even.