## The automorphism group of 1'1* and the subgroups thereof

$\mathbf{1 ' 1}^{*}$ is isomorphic to $\mathbb{Z}_{2} \times \mathbb{Z}_{2}$. We can use this isomorphism to derive $\operatorname{Aut}\left(\mathbf{1}^{\prime} \mathbf{1}^{*}\right)$. Let $m$ be the isomorphism of $\mathbf{1}^{\prime} \mathbf{1}^{*}$ onto $\mathbb{Z}_{2} \times \mathbb{Z}_{2}$ which maps 1 to $(0,0), 1^{\prime}$ to $(1,0), 1^{*}$ to $(0,1)$, and $1^{\prime *}$ to $(1,1)$. Using this isomorphism, the automorphism group of $\mathbf{1 ' 1}^{*}$ can be solved for from the automorphism group of $\mathbb{Z}_{2} \times \mathbb{Z}_{2}$ :

$$
\operatorname{Aut}\left(\mathbf{1}^{\prime} \mathbf{1}^{*}\right)=\left\{m^{-1} b m: b \in \operatorname{Aut}\left(\mathbb{Z}_{2} \times \mathbb{Z}_{2}\right)\right\}
$$

The automorphism group of $\mathbb{Z}_{2} \times \mathbb{Z}_{2}$ is $\mathbf{G L}\left(2, \mathbb{Z}_{2}\right)$, the general linear group of degree 2 over the field of 2 elements (or more precisely is isomorphic to it). $\mathbf{G L}\left(2, \mathbb{Z}_{2}\right)$ is represented by these six matrices: $\left\{\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right),\left(\begin{array}{ll}1 & 1 \\ 1 & 0\end{array}\right),\left(\begin{array}{ll}0 & 1 \\ 1 & 1\end{array}\right),\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right),\left(\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right),\left(\begin{array}{ll}1 & 0 \\ 1 & 1\end{array}\right)\right\}$. Using $m$, we can solve for the $\mathbf{1}^{\prime} \mathbf{1}^{*}$ automorphism implied by each of these matrices. As an example, consider the $\mathbf{1}^{\prime} \mathbf{1}^{*}$ automorphism, $m^{-1}\left(\begin{array}{ll}1 & 1 \\ 1 & 0\end{array}\right) m$, implied by the $\mathbb{Z}_{2} \times \mathbb{Z}_{2}$ automorphism $\left(\begin{array}{ll}1 & 1 \\ 1 & 0\end{array}\right)$ :

$$
\begin{aligned}
& (0,0) \xrightarrow{\left(\begin{array}{ll}
1 & 1 \\
1 & 0
\end{array}\right)}(0,0) \leftrightarrow 1 \xrightarrow{m^{-1}\left(\begin{array}{ll}
1 & 1 \\
1 & 0
\end{array}\right) m} 1 \\
& (1,0) \xrightarrow{\left(\begin{array}{ll}
1 & 1 \\
1 & 0
\end{array}\right)}(1,1) \leftrightarrow 1^{\prime} \xrightarrow{m^{-1}\left(\begin{array}{ll}
1 & 1 \\
1 & 0
\end{array}\right) m} 1^{\prime *} \\
& (0,1) \xrightarrow{\left(\begin{array}{ll}
1 & 1 \\
1 & 0
\end{array}\right)}(1,0) \leftrightarrow 1^{*} \xrightarrow{m^{-1}\left(\begin{array}{ll}
1 & 1 \\
1 & 0
\end{array}\right) m} 1^{\prime} \\
& (1,1) \xrightarrow{\left(\begin{array}{ll}
1 & 1 \\
1 & 0
\end{array}\right)}(0,1) \leftrightarrow 1^{\prime *} \xrightarrow{m^{-1}\left(\begin{array}{ll}
1 & 1 \\
1 & 0
\end{array}\right) m} 1^{*}
\end{aligned}
$$

Thus, $m^{-1}\left(\begin{array}{ll}1 & 1 \\ 1 & 0\end{array}\right) m$ is the permutation $\left(\begin{array}{lll}1^{\prime} & 1^{*} & 1^{\prime *} \\ 1^{\prime *} & 1^{\prime} & 1^{*}\end{array}\right)$. Or using one-line notation, $\left(1^{\prime *}, 1^{\prime}, 1^{*}\right)$. Applying this to each element of $\mathbf{G L}\left(2, \mathbb{Z}_{2}\right)$, we find that $\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right),\left(\begin{array}{ll}1 & 1 \\ 1 & 0\end{array}\right),\left(\begin{array}{ll}0 & 1 \\ 1 & 1\end{array}\right),\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right),\left(\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right)$, and $\left(\begin{array}{ll}1 & 0 \\ 1 & 1\end{array}\right)$ imply $\left(1^{\prime}, 1^{*}, 1^{* *}\right),\left(1^{\prime *}, 1^{\prime}, 1^{*}\right),\left(1^{*}, 1^{*}, 1^{\prime}\right),\left(1^{*}, 1^{\prime}, 1^{\prime *}\right),\left(1^{\prime}, 1^{\prime *}, 1^{*}\right)$, and $\left(1^{*}, 1^{*}, 1^{\prime}\right)$ respectively.

The subgroups of $\mathbf{G L}\left(2, \mathbb{Z}_{2}\right)$ can be used to find the subgroups of $\operatorname{Aut}\left(\mathbf{1}^{\prime} \mathbf{1}^{*}\right)$. The subgroups of $\mathbf{G L}\left(2, \mathbb{Z}_{2}\right)$ :

Index $=6($ order $=1):\left\{\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right)\right\}$
Index $=3$ (order=2): $\left\{\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right),\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right)\right\},\left\{\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right),\left(\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right)\right\}$, and $\left\{\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right),\left(\begin{array}{ll}1 & 0 \\ 1 & 1\end{array}\right)\right\}$
Index $=2(\operatorname{order}=3):\left\{\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right),\left(\begin{array}{ll}1 & 1 \\ 1 & 0\end{array}\right),\left(\begin{array}{ll}0 & 1 \\ 1 & 1\end{array}\right)\right\}$

Index=1 (order=6): $\left\{\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right),\left(\begin{array}{ll}1 & 1 \\ 1 & 0\end{array}\right),\left(\begin{array}{ll}0 & 1 \\ 1 & 1\end{array}\right),\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right),\left(\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right),\left(\begin{array}{ll}1 & 0 \\ 1 & 1\end{array}\right)\right\}$
imply the following $\operatorname{Aut}\left(\mathbf{1}^{\prime} \mathbf{1}^{*}\right)$ subgroups:
Index $=6($ order $=1):\left\{\left(1^{\prime}, 1^{*}, 1^{*}\right)\right\}$
Index=3 (order=2): $\left\{\left(1^{\prime}, 1^{*}, 1^{\prime *}\right),\left(1^{\prime}, 1^{\prime *}, 1^{*}\right)\right\},\left\{\left(1^{\prime}, 1^{*}, 1^{*}\right),\left(1^{*}, 1^{\prime}, 1^{\prime *}\right)\right\}$, and $\left\{\left(1^{\prime}, 1^{*}, 1^{*}\right),\left(1^{*}, 1^{*}, 1^{\prime}\right)\right\}$
Index=2 (order=3): $\left\{\left(1^{\prime}, 1^{*}, 1^{*}\right),\left(1^{*}, 1^{\prime *}, 1^{\prime}\right),\left(1^{*}, 1^{\prime}, 1^{*}\right)\right\}$
Index=1 (order=6): $\left\{\left(1^{\prime}, 1^{*}, 1^{* *}\right),\left(1^{\prime}, 1^{*}, 1^{*}\right),\left(1^{*}, 1^{\prime}, 1^{* *}\right),\left(1^{*}, 1^{*}, 1^{\prime}\right),\left(1^{*}, 1^{\prime}, 1^{*}\right),\left(1^{*}, 1^{*}, 1^{\prime}\right)\right\}$
For each line of generators, Zamorzaev \& Palistrant (Zamorzaev \& Palistrant, 1980) give the number of types represented by that line, which corresponds to the order of a subgroup of $\operatorname{Aut}\left(\mathbf{1}^{\prime} \mathbf{1}^{*}\right)$. Zamorzaev \& Palistrant do not give which automorphisms need to be applied to generate these types. However, any valid partition of $\operatorname{Aut}\left(\mathbf{1}^{\prime} \mathbf{1}^{*}\right)$ by generated type must have an associated subgroup by which the elements of the members of the partition must be related, i.e. said partition is a coset decomposition of $\operatorname{Aut}\left(\mathbf{1}^{\prime} \mathbf{1}^{*}\right)$.

Applying this understanding, we find that for lines of generators which result in 1,2 , or 6 types, there is only one valid partition of $\operatorname{Aut}\left(\mathbf{1}^{\prime} \mathbf{1}^{*}\right)$ by generated type because there is only one coset decomposition for index 1, 2, and 6 respectively. For lines of generators which result in 3 types, there are three possible coset decompositions (technically three left and three right, but we can only choose to use left or right multiplication on the generating set, not both).

The results from this analysis for a line of generators can be summarized as follows:

| $\left(1^{\prime}, 1^{*}, 1^{\prime *}\right)$ | $\left(1^{*}, 1^{\prime *}, 1^{\prime}\right)$ | $\left(1^{\prime *}, 1^{\prime}, 1^{*}\right)$ |
| :--- | :--- | :--- |
| $\left(1^{\prime}, 1^{*}, 1^{*}\right)$ | $\left(1^{*}, 1^{\prime}, 1^{\prime *}\right)$ | $\left(1^{\prime *}, 1^{*}, 1^{\prime}\right)$ |

1 type $=$ all permutations yield groups of the same type
2 types $=$ permutations in the same row yield groups of the same type
3 types $=$ permutations in the same row yield distinct types of groups
6 types $=$ all permutations are distinct types of groups
This method can be generalized to multiple antisymmetry and other color symmetries. For double antisymmetry, it would have been simpler to check which permutations of anti-identities preserved the group structure of $\mathbf{1}^{\prime} \mathbf{1}^{*}$. As it turns out, they all do, i.e. $\operatorname{Aut}\left(\mathbf{1}^{\prime} \mathbf{1}^{*}\right) \cong \mathbf{S}_{\mathbf{3}}$. This will not work $l$-tuple antisymmetry for $l>2$. For $l$-tuple antisymmetry for $l>2$ : the automorphism group of $\mathbb{Z}_{2}^{n}$ is isomorphic
to general linear group of degree $n$ over the field of 2 elements, $\mathbf{G L}\left(n, \mathbb{Z}_{2}\right)$, and a similar method can thus be employed.

