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Supporting information for article:

***Polynator* – A Tool to Identify and Quantitatively Evaluate Polyhedra and Other Shapes in Crystal Structures**

Lukas Link and Rainer Niewa

For more comprehensive information on the working process, mathematical aspects and the actual code, see the *Polynator* manual.

S1. Methodology

The data presented in Tab. 3 and Fig. 11 is based on a selection of .cif files downloaded from the International Crystal Structure Database (ICSD). It goes without saying that this comes with several pitfalls. To avoid those as well as possible, the files were picked according to a strict set of rules. Each unique composition was selected only once to avoid biases due to some compounds being uploaded multiple times. Only ternary perovskites and ternary or quaternary elpasolites were selected for our analysis. Only ambient pressure data marked as high quality by the ICSD was admitted. Room temperature data was preferred when available. Data with clearly anomalous behavior, such as space groups apparently too low in symmetry, was discarded. *Polynator* made this step much easier, as perfect or near perfect fits to models with unexpectedly high symmetry point groups were readily detected. Finally, structures with significant octahedral distortions ($\delta > 3.2$) were discarded. To allow for sufficient sample sizes, cations with the potential for electronic effects, such as Pb^{2+} and V^{4+} , were admitted as long as they did not violate the previous rule. In accordance with these rules, the sample sizes given in Tab. 3 were achieved. The majority of the data set is composed of oxides, but also includes some sulfides and halides.

User Manual for Polynator 1.4

Contents

1	Introduction	1
2	Getting Started	1
3	Using Polynator	2
3.1	The Graphical User Interface (GUI)	2
3.2	Coordination Environments	4
3.3	Molecules and Cages	7
3.4	Selecting Specific Models	8
3.5	Custom Model Axis	9
3.6	The Settings Menu	9
3.7	The Custom Model Construction Menu	10
4	Output Files	12
4.1	Data Tables (.csv)	13
4.2	General Output Files (.out)	13
4.3	Atom Specific Output Files (.aso)	14
4.4	Minimal Coordinate Files (.cif)	14
4.5	Report Files (.log)	15
5	Mathematical Aspects	15
5.1	Overview	15
5.2	Pairing Up Atom and Model Vertices	17
5.3	Centering the Model	20
5.4	Optimization of Shape Parameters	21
5.5	Generic Distortion Values	23
6	Navigating the Code	24
6.1	General Structure	24
6.2	Classes	24
6.3	Global Constants	25
6.4	Utility Functions	26
6.5	Model Parameters	26

1 Introduction

Polynator is a computer program written in Python. Its main purpose is to fit models to atom arrangements. An atom arrangement is a set of atoms found in a crystal structure, typically a coordination environment or a molecule. A model represents a specific geometric shape, e.g. an octahedron or a pentagonal prism. Models are defined by a set of vertices which may be manipulated according to a set of rules specific to each model. In the context of computations, atoms and model vertices are internally represented by vectors. A model may be rigid or dynamic. Rigid models have fixed proportions, meaning they can be rotated and resized, but not deformed in any other way. Dynamic models can be deformed in a variety of ways (e.g. stretching, twisting, puckering...). Fitting a model to an atom arrangement allows for the quantification of the distortion of that atom arrangement with respect to that model. This allows the user to more precisely describe a coordination environment or molecule, compare distortions between similar structures, or find trends within structural families. Polynator can also be utilized to construct Voronoi polyhedra and to calculate angles, volumes and surface areas. In addition, it provides a convenient way to extract structural units from larger crystal structures. This documentation is intended to assist the user in navigating these features. It also gives an overview of the code structure and the mathematical methods fueling Polynator.

2 Getting Started

The program is available as a free download from <https://www.iac.uni-stuttgart.de/forschung/akniewa/downloads/>. This website currently gives the options of downloading the Python script or a .zip folder containing a Windows executable file. The Python version provides full access to the source code. It requires Python 3.9 or a newer version of Python 3 (slightly older versions might be fine too). The graphical user interface is accessed by running the `polynator_gui.py` script. The main program is contained in `polynator_main.py` and can be run independently. All imported modules are part of the Python standard library. The executable version requires Windows 10 or a newer version of Microsoft Windows. To get started with the latter, unpack the .zip folder wherever you would store applications on your computer. Then start Polynator by executing `polynator.exe`. The folder extracted from the .zip file is self-contained, it doesn't create configura-

tion files anywhere else on your computer. Therefore, you can uninstall Polynator by simply deleting the folder. For convenience, you may want to create a shortcut to polynator.exe somewhere easy to reach on your computer. Polynator comes with a graphical user interface (GUI). The main window is created when starting the program. With the .exe version in particular, this may take a few seconds. If the window doesn't appear in the tab bar after 10 seconds, check if the system requirements are met and that no changes have been made to the folder.

3 Using Polynator

3.1 The Graphical User Interface (GUI)

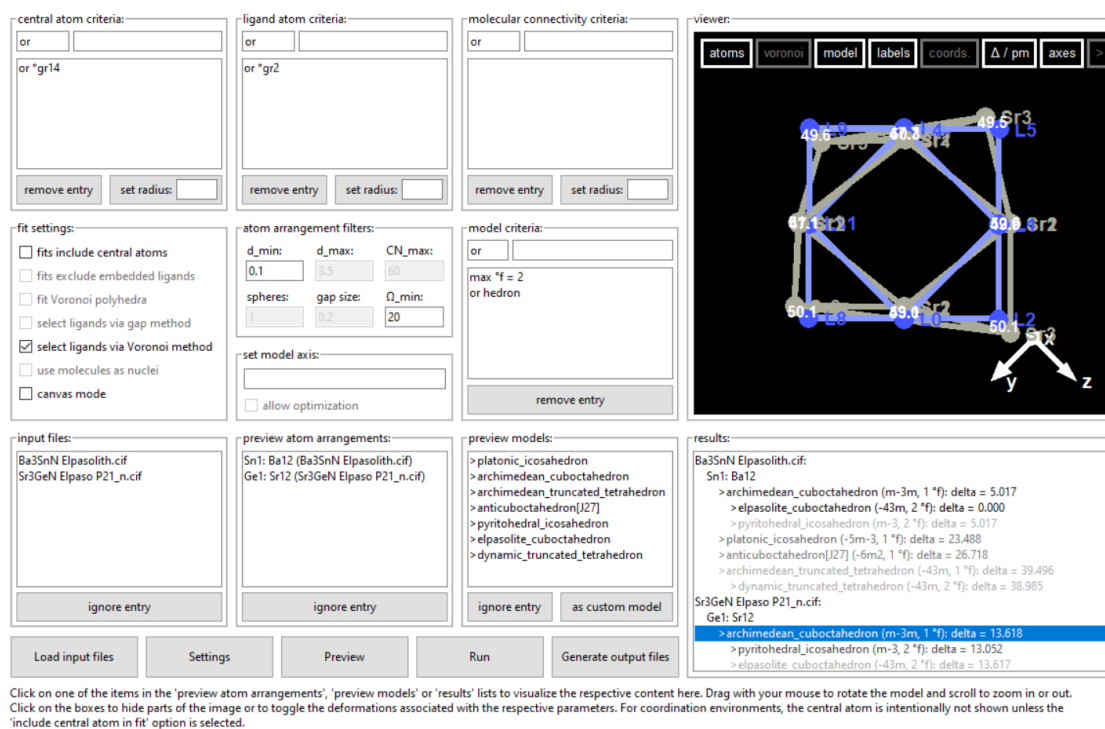


Figure 1: Screenshot of the main window of the GUI.

The GUI comprises a total of four windows:

- The main window is opened when starting the program. It allows the user to find atom arrangements, fit models and generate output files.
- The settings window is accessible from the main window via the *Settings* button. It allows for modifications to some aspects of the program.

- The custom model construction window allows the user to create new models. It is accessible from the main window via *as custom model*.
- The small custom carving instructions window allows the user to define new carving schemes (see section 3.3). It is accessible via *Settings* → *define new carving scheme*.

Each window contains several framed, labelled panels. Except for the custom carving instructions window, each window has a *viewer* panel at the top right position. On the main window, this allows for the visualization of atom arrangements, models and fit results by clicking on entries of the *preview atom arrangements*, *preview models*, or *results* panels, respectively. The visualizations for models include the deformations caused by free parameters. These can be toggled separately at the bottom left corner of the *viewer* panel. Visualizations of results show an overlay of atom arrangement and fitted model, each of which can be hidden separately by clicking on the appropriate boxes at the top of the panel. They also show the distances between paired atom and model vectors (Δ/pm).

Each window has some space on the bottom for the display of comments and error messages. When hovering over a widget, a comment about that widget is displayed there in most cases. This feature can be disabled in the settings menu. When clicking on an entry in the preview models panel, some information about that model is displayed in the comment bar.

To start working with the program, click on *Load input files* on the bottom left corner of the main window and locate any number of .cif or .xyz files on your computer. A click on *Run* starts Polynator's main process. However, depending on the crystal structures and the type of atom arrangement which you are interested in, some additional input may be required, as explained following within this chapter.

3.2 Coordination Environments

The screenshot shows the Polynator software interface. The 'central atom criteria' panel has 'or' selected and 'ca' entered. The 'ligand atom criteria' panel has 'or' selected and 'o' entered. The 'atom arrangement filters' panel has 'd_max' set to 3.0. The 'input files' panel has 'CaFeTi2O6 P42_nmc_3.cif' entered. The 'results' panel shows a list of coordination environments with their delta values, with the top result being '>sphenocorona[J86] [mm2, 1 *f]: delta = 6.121'.

Figure 2: Evaluating a coordination environment in Polynator. In this screenshot, user inputs are marked by black arrows. Only calcium and oxygen atoms are valid as central atoms and ligands, respectively. The distance from the central atom is capped at 3 Å for each ligand. The central atom contributes to the fit, its corresponding model vector is set at the model centroid.

The criterion panels at the top of the window allow for the selection of specific central and ligand atoms. To do this, first select a logic operator (and / or / not), then type in or select from the dropdown menu (accessible via the ↓ key) any valid entry for that box. Such an entry may be an element symbol, a specific atom name (e.g. Ca1) or a wildcard (e.g. '*M' for any metal, see tab. 1). Press Enter to insert your entry into the box directly below. You may also leave these boxes empty, in which case all possible central and ligand atoms will be evaluated.

Table 1: Wildcards for groups of elements in the atom criteria panels (not case sensitive).

wildcard	corresponding elements
*	all elements
* <i>Grn</i>	all elements in periodic table group n
* <i>M</i>	all metals
* <i>TM</i>	all transition metals (except rare earth metals and actinoids)
* <i>RE</i>	all rare earth elements including Sc, Y, lanthanoids and actinoids.
* <i>E</i>	all main group elements
* <i>Ln</i>	lanthanoids including La and Lu
* <i>An</i>	actinoids including Ac and Lr
* <i>X</i>	typical anions (N, P, O, S, Se, F, Cl, Br and I)

By default, any atom with a distance from the central atom larger than $d_{\min} = 0.1$ Å and smaller than $d_{\max} = 3.5$ Å will qualify as a ligand. These values can be freely adjusted in the *atom arrangement filters* panel. The same is true for the maximal coordination number CN_{\max} . If necessary, the most distant ligands in excess of this number will be dropped until the number of ligands is equal to CN_{\max} .

As alternatives to these simple cap values, the *fit settings* panel gives access to two algorithms which select the ligands for a coordination environment according to specific rules. The *gap method* looks for gaps in the distance distribution of ligands from the central atom. If a gap between two consecutive ligands is larger than a threshold value, this is considered the beginning of a new coordination sphere. The threshold value is calculated as the product of the smallest central–ligand distance times a coefficient with a default value of 0.2 (*gap size*). The gap method often allows easy access to any of the first few coordination spheres. However, for more disordered coordination environments, it becomes much less useful, as coordination spheres aren’t as neatly separated anymore. The *Voronoi method* is based on the construction of a Voronoi polyhedron around the central atom. Atoms are considered ligands if their respective Voronoi polyhedron shares a face with the Voronoi polyhedron of the central atom and this shared face subtends a solid angle greater than a given threshold (20° by default) from the perspective of the central atom. A similar algorithm is implemented in *ChemEnv* by Waroquiers et al. [1], who took inspiration from O’Keeffe [2]. Note that all methods to select ligands come with their own biases and will work better

for some types of coordination environments than for others. That said, the Voronoi method generally yields reasonable results as long as the coordination environment is non-planar.

Atomic radii allow for more selectivity when it comes to manually defining a coordination environment. If atomic radii are defined for the central and/or the ligand atoms and the sum of both is smaller than d_{\max} , it replaces d_{\max} for this combination of atoms. To use atomic radii, first select one or more entries in either central atom or ligand atom criterion box, then enter a number into the respective *set radius* field and press enter.

Polynator will by default not include the central atom in the arrangement of atoms which are to be fitted. To include the central atom, check the box *fits include central atoms* in the *fit settings* panel. In this case, the central atom is fitted against the centroid of all atom vectors and will also factor into the centering of the model. This panel also gives you the option to fit the vertices of a Voronoi polyhedron constructed around the central atom instead of the actual ligands.

3.3 Molecules and Cages

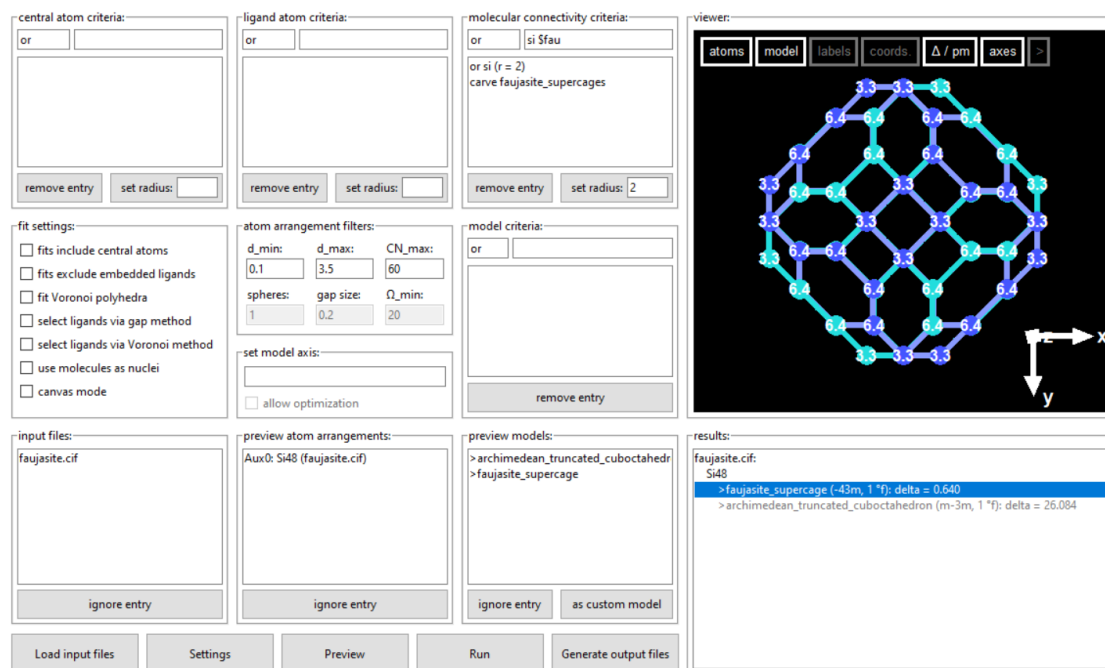


Figure 3: Analyzing a faujasite supercage in Polynator. The cage is formed by silicon atoms, each at the center of a SiO_4 tetrahedron. They are connected by specifying a radius of 2 Å for each silicon atom in the 'molecular connectivity criteria' panel. This will produce an infinite framework of silicon atoms. To separate one cage, the '\$faujasite_supercage' carving scheme is added.

To find molecules and empty cages in a crystal structure, we recommend to use the *molecular connectivity* panel. This will connect selected atoms based on covalent radii specified by the user or otherwise taken from a default table. You may also choose from among a number of topological schemes to conveniently isolate a particular subunit from a larger molecule or an infinite framework of connected atoms. These schemes are most easily accessible from the dropdown menu by typing '\$' into the input field at the top right of the panel. They allow you to isolate e.g. a sodalite cage from the infinite framework of silicon atoms in a zeolite structure or to obtain all hexagonal rings from an organic molecule. While most schemes will only look for one specific shape, some are intentionally designed to find a slightly wider variety. For example, the scheme for fullerenes will find any polyhedron with only pentagonal and hexagonal faces as long as its dual is convex. Keep in mind that appropriate atom radii are a prerequisite for finding the desired shape! Also be aware that a polyhedron with a central atom which is connected to the rest of the graph will not be found by this method (use the regular central

atom and ligand criteria for that). Additional schemes can be defined via *Settings* → *define new carving scheme*.

Table 2: List of predefined carving scheme names (various sizes for schemes with *).

rings*	cubes	sodalite cages
pyramids*	icosahedra	ZTA-type supercages
bipyramids*	pentagonal dodecahedra	faujasite-type supercages
prisms*	rhombic dodecahedra	fulvalene skeletons
antiprisms*	ethene units	biphenyl skeletons
deltahedra	cuboctahedra	norbornene skeletons
fullerenes	anticuboctahedra	barrelene skeletons
tetrahedra	snub cubes	naphthalene skeletons
octahedra	adamantane cages	porphyrin skeletons

3.4 Selecting Specific Models

By default, all available models with the respectively appropriate number of vertices will be fitted to each atom arrangement. The name of each model contains some clarifying information in the square brackets. Platonic, Archimedean, Catalan and Johnson solids, as well as uniform prisms and antiprisms are marked as such. Dynamic models have their point group in square brackets if they represent the set of all shapes with that point group and that topology. If 'rigid' or 'dynamic' is in square brackets, the model has a specific shape which is in most cases explained in the comment line upon selecting such a model. The *model criteria* panel allows you to select or exclude specific models. This panel will also accept fragments of valid entries. This is in contrast to atom criterion panels discussed above (otherwise 'N' would also find Nb, Ni, Zn etc.). A model may have more than one valid name. These synonyms are displayed in the comment line upon selecting a model in the *model preview* panel. Models can also be filtered by tags (see tab. 3) and by point group. You may select the 'of_max' mode and enter an integer to exclude all models with more degrees of freedom than that number (not counting the three translational and two rotational degrees of freedom available to every model). For example, entering '1' will exclude all dynamic models, leaving only rigid models such as the Platonic and Archimedean solids. The *model exclusion criteria* in the *Settings* menu serves a similar function, but will exclude unwanted models permanently.

Table 3: Model tags. In contrast to earlier versions, tags do not bestow any properties onto a model. However, they may hint at specific behaviors. For example, models with the *#prolate* or *#oblate* tags use strategies 2 and 3 for their assignment, respectively.

<i>#rigid</i>	<i>#dynamic</i>	<i>#symmetry_aligned</i>
<i>#prolate</i>	<i>#oblate</i>	<i>#constrained_parameters</i>
<i>#molecule</i>	<i>#cage</i>	<i>#occupied</i>
<i>#pseudopolyhedron</i>	<i>#chiral</i>	<i>#essential</i>
<i>#equilateral</i>	<i>#equidistant</i>	<i>#planar</i>
<i>#regular_polygon</i>	<i>#platonic</i>	<i>#archimedean</i>
<i>#johnson</i>	<i>#catalan</i>	<i>#deltahedron</i>
<i>#fullerene</i>	<i>#frank_kasper</i>	<i>#capped_cube</i>
<i>#pyramid</i>	<i>#bipyramid</i>	<i>#heterobipyramid</i>
<i>#scalenoedron</i>	<i>#prism</i>	<i>#antiprism</i>
<i>#twisted_prism</i>	<i>#frustum</i>	<i>#antifrustum</i>
<i>#equator-capped_prism</i>	<i>#axis-capped_prism</i>	<i>#fully_capped_prism</i>
<i>#capped_frustum</i>	<i>#capped_antifrustum</i>	

3.5 Custom Model Axis

The user has the option of manually entering a model axis (see chapter 5, fig. 6). This bypasses the automatic belt assignment step and forces Polynator to assign belts according to this axis. This might be useful if an automatic belt assignment appears suboptimal. An axis is entered in the form of fractional coordinates, separated by commas or spaces. Unless the *allow optimization* box is ticked, this axis will remain unchanged throughout the entire fitting process.

3.6 The Settings Menu

The settings menu comprises several options to customize Polynator's behaviour. Changes made in this menu are remembered between sessions (they are stored in .cfg files located in the same folder as the main .exe or .py file.). The available options are as follows:

- Output behavior: Checkboxes allow the user to select which types of output files to produce. They also give the options to overwrite existing output files, to automatically produce outputs after every fit and to have only the best fitting model for each atom arrangement appear in the output.

- Skipping redundant models: If a model turns out to fit perfectly, versions of that model with additional degrees of freedom are not evaluated if the *skip redundant models* box is checked.
- Fit metric: Distortion values can be displayed either in the native δ metric or as CS(h)M-type values S (See section 5.1). The fitting procedure is not altered by either choice, since both values are minimized in the same fashion.
- Point group notation: The user is given the choice to have point group symbols displayed in Hermann-Mauguin (default) or Schönflies notation.
- Maximal δ value: Provides an upper threshold for the distortion. Models with a higher value do simply not appear in the output or the results box.
- Variable tax: The internal parameter δ_{taxed} is obtained by adding a small number (the variable tax) to the neutral δ value of a fitted model for each free variable that model has. The value of δ_{taxed} is not displayed anywhere, does not appear in the output files and does not influence the fitting procedure. However, it is consulted when deciding on the best fitting model, it influences the grayscale in the *results* box and determines the order of the list of models in the .out files.
- Model filter: The *active models* panel makes it possible to customize the set of models Polynator is actively using without having to specify preferences in the *model criteria* panel of the main window every time. For example, if 'exotic' model polyhedra are generally not useful to you, it may be beneficial to enter and confirm the exclusion criterion *not #essential* (see tab. 3 for a list of such model tags).

3.7 The Custom Model Construction Menu

The custom model construction window is accessible from the main window via *as custom model* (select a model in the box above to use it as a starting point) or via *Settings* \rightarrow *new custom model*. It allows for the creation of new models. The central component of this is worked out in the *belts* panel, where the user can add or remove belts (see fig. 6) with a number of vertices n . There is also a number of optional parameters which can be applied to a belt (see section 5.4). These parameters are perhaps best understood by selecting various model polyhedra in the *preview models* box at the bottom of the main window and observing them in

the *viewer* panel. Click at the boxes at the bottom of that panel to toggle the effects of each parameter separately. Alternatively, it is highly recommended to just play around and create your own models. The comment panel at the bottom of the window will guide you to some extent.

In order to guarantee a valid optimization procedure, some limits are placed on the customizability of a model. Firstly, the centroid of each belt must rest on the model axis. This means it is not possible to define e.g. a capped pentagonal pyramid (with the cap on a triangular face). Allowing imbalanced belts would introduce a nontrivial optimization problem for the centering of the model. For similar reasons, the centroid of all model vertices must rest on the origin at all times. Secondly, the frequency of each modulated parameter in a belt must be a product of one or more prime factors of the number of vertices n in that belt. Otherwise, imbalances and coupling between the optimization of orientation and shape would arise. If a model contains one or more s parameters, the average distance from the origin to the model vertices scaled by that parameter must be exactly 1.0. This ensures that the optimized parameter value matches the actual size of the model. Finally, if a belt contains a φ parameter, there must either be another belt with the same number of vertices and the same φ parameter in counterrotation (e.g. '-phi1'), or another belt with a φ parameter entitled 'phi*'. This eliminates or at least minimizes coupling between the optimization of orientation and those torsion angles, which would lead to slow convergence.

The new model also needs a name. Additionally, you have the option of assigning a point group, a symmetry operation pertaining to the model axis, a list of parent models, i.e. preexisting models with higher symmetry or fewer degrees of freedom, as well as a list of search terms or categories for the new model. However, these have no functionality other than being displayed at various points and allowing the user to search for them.

Lastly, constraints ('bundles') may be added. These are useful for dynamic models which do not have all of the degrees of freedom their point group would allow. Defining these is not always a simple task and may require some calculations. Some guidelines will be given here, but it is recommended to look at existing models with constrained parameters (They all share the *#constrained_parameters* tag). There are three types of models with constrained parameters:

- The first type is based on a rigid base polyhedron, which is modified by a number of constrained parameters which depend on a single variable $v1$. There is also a scaling parameter that is independent from $v1$. Modifying the value of $v1$ may change the shape of the model in a variety of ways. However, the average distance of the vertices from the center must be invariant under such modifications (otherwise calculations will be inaccurate, potentially in a subtle way). The parameters for this type of model typically work together to manipulate the angular components of the spherical vertex coordinates. This type is best suited for models with a cubic point group. Models of this type include the pyritohedral icosahedron and the elpasolite cuboctahedron.
- Some models have rigid components (e.g. vertically oriented regular polygons), but are not altogether rigid. This includes the fulvalene and biphenyl skeletons, among others. In these cases, the \dot{h} parameter type is useful for constraining the vertical expansion of the rigid parts. However, to fully preserve the shape of such a rigid part while allowing it to change size, the value of the \dot{h} parameter must be coupled with that of a \dot{w} parameter. The parameter values are linear functions of the variable $v1$ in these cases.
- The third type of model is entirely managed by a number of variables $v1$, $v2$, etc. Each parameter tends to depend on more than one variable with this type of model, which makes them the most computationally expensive models defined in Polynator. There is currently only one built-in example; the truncated hexagonal trapezohedron, which requires constrained parameters to keep its pentagonal faces planar.

4 Output Files

Output files for all evaluated objects can be generated via *Generate output files*. They will be put into a subdirectory of the folder holding the input files that were evaluated. There are seven types of output files, the generation of which can be toggled in the *Settings* menu: two types of data tables, general output files, atom specific output files, two types of minimal .cif files and .log files.

4.1 Data Tables (.csv)

Two data tables may be generated as single .csv files which contain the most relevant information about all atom arrangements and fitted models, respectively. The `real_atom_arrangements.csv` file contains the composition, volume and number of atoms for each atom arrangement, as well as δ values referring to a the closest line, plane, or sphere containing all model vectors (see section 5.5). This file can be generated without fitting any models. The `model_fits.csv` file includes the distortion value, the volumes of the convex hulls of atom arrangements and model polyhedra, the averaged linear distance between paired atom and model vectors, the radial and angular portions of said distance and the free and constrained parameter values for each fitted model.

4.2 General Output Files (.out)

General output files are purely text-based. One such file is generated for each atom arrangement. It contains general information about the atom arrangement at the top and a section with information about each individual fit after that. These sections are separated from each other by wide horizontal lines. Each section is further divided into paragraphs.

The paragraph at the very top should mostly be self-explanatory. It contains general information for the evaluated atom arrangement, such as the number of atoms, chemical composition and volume. The excentricity vector is the distance between the central atom and the centroid of all atoms. The three special δ values are explained in section 5.5. The second paragraph contains a Python dictionary with all input instructions to document the setup for this batch of results.

The first model-specific paragraph gives general information such as the name, point group, distortion value and volume of the model polyhedron. The third block holds some statistical information. This includes averages for the linear difference measures from the previous block. The standard deviation given for the length of the difference vector should not be confused with a quantification of measurement errors; Polynator's statistical errors are negligible (many tests suggest the same is true for systematic errors). In addition, this block gives the δ value for a central projection of all atom and model vectors onto a unit sphere and

for a cylindrical projection onto the model axis. The model constructor gives a Python dictionary of the model belts and their parameters. For more information on this entry, see sections 5.2 and 5.4. It might help to try out the custom model construction window. Lastly, the .out file lists the parameter values broken down into free and constrained parameters. These can be quite useful to measure e.g. the average height of a coordination environment or the torsion angle between the hexagonal rings in a biphenyl unit. However, it is important to be aware of prefactors! For instance, φ parameters often apply to two counterrotating belts, as marked by a minus sign before one of the 'phi' parameter names in the 'model constructor' block above. In that case, the value given for the φ parameter must be doubled to obtain the correct torsion angle between the two belts.

4.3 Atom Specific Output Files (.aso)

These are structured similarly to the .out files, but they contain information on the level of individual atoms and model vertices. Each file starts by listing the coordinates of all fitted atoms and their distance from the central atom (or from the centroid if a molecule or cage was evaluated). If Voronoi vertices were fitted, a similar list is generated for them.

The file continues with model-specific vectors and related information. All coordinates given are fractional (the same format you would find in a .cif file). Each atom vector comes with a vertical component (parallel to the model axis) and a horizontal component (orthogonal to the model axis). The 'angular difference' refers to the angle a given ligand vector is displaced from its model counterpart, from the perspective of the model center. 'Radial difference' means the difference between the distances of that atom vector and its model counterpart from the centroid. The 'angular difference' is just the angle between atom and model vertex from the perspective of the centroid. It is split into the spherical coordinate components 'phi difference' and 'theta difference'.

4.4 Minimal Coordinate Files (.cif)

The minimal .cif files Polynator creates are intended mostly for visualization and perhaps verification in an external program. They have space group $P1$ regardless of the original space group and contain only the atoms and model vertices involved in the respective fit. There are two versions of these files: fractional and cartesian.

Both are true to the size and proportions of the original coordination environment, but only the fractional version is also true to the original unit cell. However, that version may have inconvenient placement and overlaps between translated coordination environments, which can be avoided by using the cartesian version.

4.5 Report Files (.log)

The .log file mostly tries to record if something went wrong during the calculations. If you observe unexpected behaviour, it might be worthwhile to look at this file.

5 Mathematical Aspects

5.1 Overview

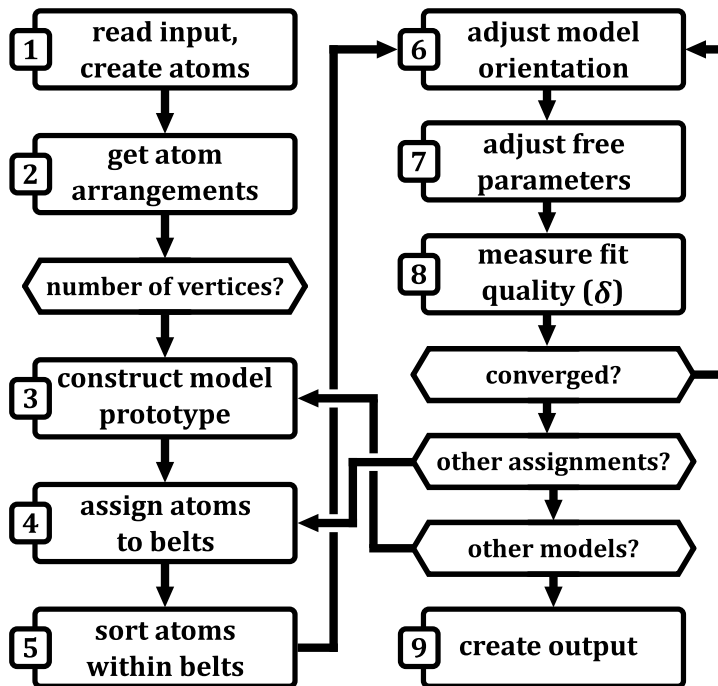


Figure 4: Flow chart of Polynators working process.

Polynator seeks to minimize the deviation δ , which is a least squares based metric for the purely geometrical dissimilarity between a set of pairs of atom and model vectors \vec{a}_i and \vec{v}_i defined as

$$\delta = 100 \cdot \sqrt{\frac{\sum_i |\vec{a}_i - \vec{v}_i|^2}{\sum_i |\vec{a}_i - \vec{c}|^2}}. \quad (1)$$

where \vec{c} is the centroid of all atom vectors. A perfect fit between model and atom vectors yields $\delta = 0$. Larger values indicate stronger distortions, up to a theoretical upper bound of $\delta = 100$. In practice, values above $\delta \approx 30$ should typically not be interpreted as validation for describing an atom arrangement in terms of the fitted model. Values in the range $18 < \delta < 30$ indicate strong distortions, while lower values correspond to moderate or small distortions.

The δ metric is closely related to the continuous symmetry measure (CSM) originally defined by Zabrodsky, Peleg and Avnir [3]. Values can be converted to and from the CSM-type S parameter via

$$S = 0.01 \cdot \delta^2 \quad \text{or} \quad \delta = 10 \cdot \sqrt{S} \quad (2)$$

However, there are several reasons for preferring the δ parameter. First of all, even though there are at least four types of values in the CSM family (CSM, CShM, SOM and CCM), dynamic models are not adequately covered by any of them. Secondly, CSM-type values increase approximately quadratically with increasing distortion, while δ increases approximately linearly, making it much more intuitive. Lastly, due to their quadratic nature, it is not uncommon for small distortions to register only in the third or fourth decimal place of CSM-type values, making them somewhat unhandy. With δ values, two decimal places are virtually always sufficient to distinguish between a perfect fit by symmetry and a tiny distortion.

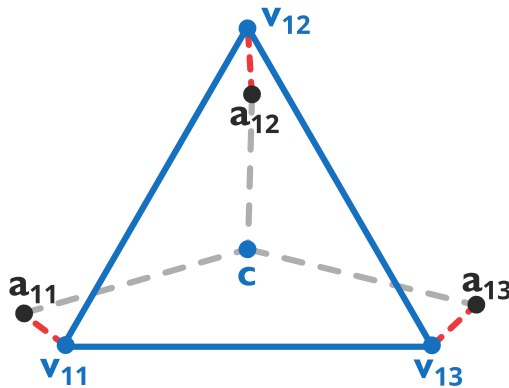


Figure 5: Measuring the value of δ for a regular triangular model (blue) fitted to three atom vectors (black). Red dotted lines contribute to the numerator and grey dotted lines to the denominator of equation 1.

To minimize δ , Polynator internally has to solve four main problems:

1. Pairing up atom and model vertices in an optimal fashion.

2. Finding the optimal coordinates to center the model at.
3. Finding the optimal orientation of the model. This has a reliable solution in the Kabsch algorithm [5], which will not be discussed here.
4. Optimizing one or more parameters to obtain the ideal shape of the model.

Table 4: Mathematical symbols presented in this chapter do not always match their counterparts in the code or the GUI. This table translates the code names.

symbol	code name	location in the code
\vec{a}_{ij}	vec_real	ModelFit.belts_real (a list of lists of vectors)
\vec{v}_{ij}	vec_model	ModelFit.belts_model (a list of lists of vectors)
\mathbf{M}	var_matrix	get_covariance_matrix_eigenvectors function
\hat{m}	model_axis	ModelFit.model_axis
S	HALF_SPHERE	assign_to_belts_from_scratch function
q_1	estimated_cost	AtomArrangement.get_assignment_cost _estimate method
q_2	total_cost	ModelFit.match_real_vecs_to_model method
P_{ij}	(various names)	ModelFit.adjust...() methods
s	'sc'	ModelFit.belt_dicts, ModelFit.dict_parameters
\hat{h}, \hat{w}	'>h', '>w'	ModelFit.belt_dicts, ModelFit.dict_parameters

5.2 Pairing Up Atom and Model Vertices

This problem is in theory easily solved by just checking every permutation of vertex pairings. However, since the number of permutations grows factorially with the number of vertices and each 'checking' step comes with a significant computational cost, this approach was discarded. As far as we are aware, there is no substitute that is determined to yield the perfect solution for any distribution of atom vectors while allowing for a computationally cheap implementation. However, some observations about the typical shape of coordination environments and simple molecules can be exploited to develop strategies that come very close to this goal. Using these strategies, deviations from the optimal solution of the assignment problem will sometimes occur if the atom arrangement is unrecognizably dissimilar from the model, but are extremely unlikely if the

atom arrangement can reasonably be described as a distorted version of the model.

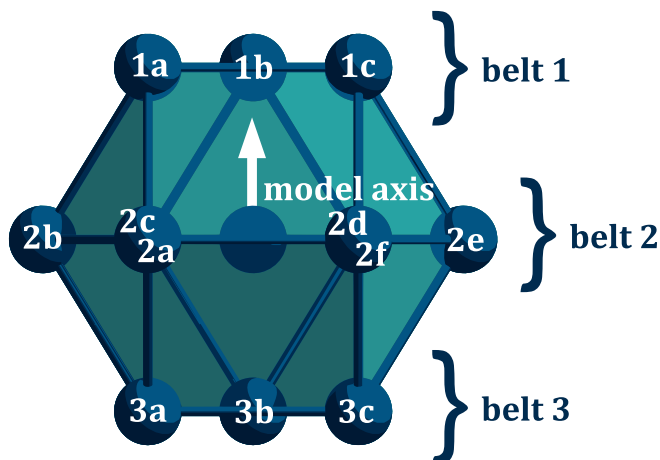


Figure 6: Separation of an anticuboctahedron into belts.

The first step towards this is to separate the model into subsections, which we will call belts. This separation into belts is specific to each model. Belts can be thought of as rings of vertices which lie perpendicular to the model axis (see fig. 6). The model axis is a high symmetry axis of the model (in some cases, the choice may be ambiguous, e.g. either 3- or 4-axes are valid for models with a cubic point group). Depending on the model, vertices are assigned to belts following one out of several strategies. In all cases, the origin is set to the centroid of all atoms which contribute to the fit.

For models representing coordination polyhedra or cages, vertices are assigned to belts based on strategy 1, which assumes that the atom arrangement is roughly spherical. The atom vectors are projected onto the surface of a unit sphere around the centroid. To assign each atom vector to a model belt, (not yet to a specific vertex), a tentative model axis vector is chosen from among a predefined set S of 92 vectors, which are distributed almost evenly on a sphere surface (they correspond to the vertices of a fully capped truncated icosahedron). The atom vectors are ranked according to their dot product with this axis vector and the belts are filled up in this order. This process is repeated for each remaining axis vector in S . Models with a rotoinversion center or horizontal mirror plane require only the 46 axis vectors corresponding to one hemisphere. Duplicate belt assignments are discarded. The remainder is ranked according to the estimated assignment cost q_1 . To obtain it, the model axis for each provisional

assignment is first refined as follows:

1. The centroid \vec{c}_i of the atom vertices in each belt i is calculated and subtracted from each atom vector \vec{a}_{ij} in the respective belts to obtain an auxiliary vector \vec{b} :

$$\vec{b}_{ij} = \vec{a}_{ij} - \vec{c}_i. \quad (3)$$

2. A covariance matrix \mathbf{M} is constructed from the cartesian coordinates x , y and z of the \vec{b}_{ij} vectors:

$$\mathbf{M} = \begin{pmatrix} \sum_{ij} x_{ij}^2 & \sum_{ij} x_{ij} \cdot y_{ij} & \sum_{ij} x_{ij} \cdot z_{ij} \\ \sum_{ij} y_{ij} \cdot x_{ij} & \sum_{ij} y_{ij}^2 & \sum_{ij} y_{ij} \cdot z_{ij} \\ \sum_{ij} z_{ij} \cdot x_{ij} & \sum_{ij} z_{ij} \cdot y_{ij} & \sum_{ij} z_{ij}^2 \end{pmatrix}. \quad (4)$$

3. The unit eigenvector of \mathbf{M} with the smallest eigenvalue is chosen as the model axis \hat{m} .
4. The estimated cost q_1 for this assignment can now be calculated as

$$q_1 = \sum_i |\vec{c}_i \times \hat{m}|^2 \cdot n_i + \sum_{ij} (\vec{a}_{ij} \cdot \hat{m})^2 \quad (5)$$

where n_i is the number of vertices in belt i . The first term represents horizontal displacements of the belts, the second vertical displacements of the individual vertices within each belt.

The four belt assignments with the smallest cost q_1 are evaluated further, the rest is discarded.

Some models, for example those for the biphenyl and porphyrin skeletons, do not work very well with strategy 1. Instead, atom vertices are assigned to their belts based on the much simpler strategies 2 or 3, respectively (this is encoded by a 'shape_type' entry in the dict_info of those model blueprints). Strategy 2 is designed to assign atom vectors to the belts of very prolate models. To do this, the most prolate axis of the atom arrangement has to be identified first. Thankfully, this is easily achieved: it is the eigenvector with the largest eigenvalue of the covariance matrix constructed from all atom vectors. The atom vertices are then simply ranked according to their dot product with the prolate axis and sequentially filled into belts in this order. Strategy 3, designed for complex oblate

models, such as the porphyrin skeleton, works similarly. The most oblate axis of the atom arrangement is obtained as the eigenvector with the smallest eigenvalue of the covariance matrix constructed from all atom vectors. Atom vertices are ranked according to their cross product with this axis and sequentially assigned to belts.

Some models can be thought of as derivative of one or more other models. For example, the tetragonal prism can be derived from the cube by adding a degree of freedom (independent height and width parameters instead of a single scaling parameter). For such a derivative model, the belt assignment step can effectively be skipped if the parent model fits sufficiently well. The belt assignment is then simply inherited from the parent model according to instructions encoded for each derivative model.

To pair up atom and model vectors, the combinations that maintain the order of dihedral angles (up to one per atom) are ranked by their estimated cost q_2 . To do this, the model is first rotated around \hat{m} so that the atom vector \vec{a}_{11} lies in the plane containing \hat{m} and an arbitrarily picked model vertex \vec{v}_{tare} . For each atom and model vector, the dihedral angle α in relation to \vec{v}_0 is measured (\hat{m} serves as the hinge). Atom and model vectors are then (separately) ordered within the individual belts according to their dihedral angles and tentatively paired up in this order. The estimated cost q_2 is measured as

$$q_2 = \sum_{ij} |\vec{v}_{ij} \times \hat{m}| \cdot |\Delta\alpha_{ij}| \quad (6)$$

where $\Delta\alpha_{ij}$ is the angular difference between the paired vectors with belt and vector numbers i and j . This is repeated until each atom vector \vec{a}_{ij} has been in the $\alpha = 0$ position once. The pairing scheme with the smallest q_2 is evaluated further, the rest is discarded.

5.3 Centering the Model

Each models in Polynator are defined such that the centroid of all model vertices always rests on the origin. Due to the nature of the least squares fit, the centroid of all fitted atom vectors must also be located at the origin to achieve an optimal fit.

The properties of the least squares fit entail that the optimal orientation and shape of the model vertices can be computed independently from the centering. Thus, the atom vectors could in principle be kept in place, correcting for the misalignment of the two centroids only retroactively. However, Polynator translates the atom vectors such that the two centroids coincide before starting the fitting procedure.

5.4 Optimization of Shape Parameters

In the simplest case, the model is a rigid body and the only parameter to be optimized is its size s . To do this, the length P of each atom vector \vec{a}_{ij} when projected onto its corresponding model vector \vec{v}_{ij} is acquired as

$$P_{ij} = \vec{a}_{ij} \cdot \hat{v}_{ij}. \quad (7)$$

If all model vectors have the same length, the problem can be solved analytically by taking the arithmetic mean of all lengths P :

$$s_{\text{opt}} = \frac{\sum_{ij} P_{ij}}{\sum_{ij} 1}. \quad (8)$$

Otherwise it is solved by iteratively minimizing the sum of the squared differences between $|\vec{v}_{ij}|$ and P_{ij} . This is still very straightforward, as the corresponding function is always a parabola, so there are no problems with local minima or discontinuities. Sometimes, for example with the dynamic model of the fully capped cube, there are two or more size parameters (one for the cube vertices, the other for the caps). These can just be solved separately with the same methods.

Many models, such as dynamic prisms, antiprisms, pyramids..., require the separate optimization of height and width parameters h and w (instead of a single s parameter, not in addition). Thanks to the Pythagorean theorem, this is easily done. To optimize h , instead of projecting the atom vectors onto the model vectors, as was done before, they are projected onto the model axis \hat{m} :

$$P_{ij} = \vec{a}_{ij} \cdot \hat{m}. \quad (9)$$

Similarly, to optimize w , they are projected onto the normal plane of \hat{m} :

$$P_{ij} = \vec{a}_{ij} \cdot \hat{n}_{ij} \quad \text{where} \quad \vec{n}_{ij} = \vec{a}_{ij} - (\vec{a}_{ij} \cdot \hat{m}) \cdot \hat{m}. \quad (10)$$

As with the s parameter, the optimal values of h and w can usually be obtained analytically by computing the arithmetic mean of the P values. However, there are the special variants \tilde{h} and \tilde{w} which may have different proportionality constants for different belts. These, again, need to be solved iteratively.

Parameters φ for rotations (used e.g. in twisted prisms) are also straightforward, but computationally more expensive, as they require vector operations during iterations. In each cycle of such an iteration, the model vectors are incrementally rotated around \hat{m} . Subsequently, the sum of the squared distances between atom and model vectors is measured. If the fit deteriorated compared to the last cycle, the increment is multiplied with $-\frac{1}{2}$. Once the magnitude of the increment falls below $10^{-(4+n)}$, where n is the loop count of the overall optimization (`ModelFit.loop_count`), the iteration is terminated. To make the computation slightly more efficient, the atom and model vectors are first transformed into two-dimensional cartesian and polar coordinates, respectively (as evident from the Pythagorean theorem, the component in the direction of \hat{m} doesn't affect the result). In the built-in models, φ is always balanced by a counterrotation φ^* in a different belt in order to minimize the dependence on the model orientation.

There are also modulating versions of h , w and φ which allow the modelling of normal mode 'vibrations' of orders higher than 0. As an example, a square is modelled by a single belt containing four vertices, with a single w parameter. By adding \tilde{h} , \tilde{w} or $\tilde{\varphi}$, the disphenoid, rhombus or rectangle, respectively, can be derived from it. Modulated parameters are not applied equally to all vertices in a belt, but rather with a prefactor generated by a sinusoidal function (cos for \tilde{h} and \tilde{w} , sin for $\tilde{\varphi}$). The frequency f and offset σ of these functions are determined when a model is defined and never changed or optimized. Hence, the value of modulating parameters corresponds solely to the amplitude of the respective sinusoidal function. The prefactor g for each vertex is obtained as a function of its position p in an n -membered belt (belts are sorted according to the dihedral angles of their member vertices around \hat{m} , enumeration starts at 0):

$$g(p) = \begin{cases} \tilde{h}, \tilde{w} & \rightarrow \cos\left(\frac{2\pi \cdot f \cdot (p+\sigma)}{n}\right) \\ \tilde{\varphi} & \rightarrow \sin\left(\frac{2\pi \cdot f \cdot (p+\sigma)}{n}\right) \end{cases} \quad (11)$$

The divergent choice of basic trigonometric functions may seem odd, but in practice makes it easier to think about these, due to their transversal (\tilde{h} , \tilde{w}) and longitudinal ($\tilde{\varphi}$) nature. Optimization of the modulating parameters follows the same principles as for their non-modulating counterparts.

Lastly, there is the option to bind multiple parameters to freely definable functions of one or more newly defined variables, effectively constraining them. This allows for models such as the elpasolite cuboctahedron, the pyritohedron and pyritohedral icosahedron, as well as symmetry-preserving non-equilateral versions of archimedean solids and more. Optimization of these variables always involves an iterative process where each parameter bound to the variable is newly calculated and applied to the model in each iteration.

After all shape parameters have been optimized, the orientation is adjusted again, taking into account the new shape of the model. This process is repeated for up to ten cycles or until the δ value ceases to improve by more than 0.000003 between cycles.

5.5 Generic Distortion Values

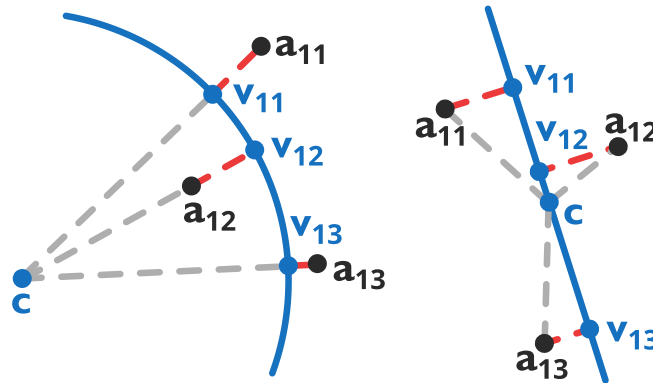


Figure 7: Measuring the value of $\delta_{\text{spherical}}$ (left) and δ_{linear} (or δ_{planar} , right). Red dotted lines contribute to the numerator and grey dotted lines to the denominator of equation 1.

For each atom arrangement, Polynator calculates three generic distortion values: δ_{linear} , δ_{planar} and $\delta_{\text{spherical}}$. A 'model' for this kind of fit is not constrained by specific proportions, but by an overall shape which must contain all its vertices. These shapes are respectively a line, a plane and the surface of a sphere. The most closely fitting line and plane normal are easily obtained as eigenvectors of

the covariance matrix of all atom vectors. For each atom vector, the corresponding 'model vector' is the nearest point on that line or plane. For $\delta_{\text{spherical}}$, the most closely fitting sphere surface centered at the centroid has a radius equal to the average distance of each atom vector from the centroid. Note that $\delta_{\text{spherical}}$ does not necessarily refer to the most closely fitting sphere surface with freely selected centering. This becomes obvious when evaluating the case of four randomly distributed atom vectors: absent degenerate cases, it is always possible to find a perfectly fitting sphere surface by centering it on the circumcenter of the four vectors, which generally does not coincide with the centroid.

6 Navigating the Code

6.1 General Structure

Polynator is written in a loosely object-oriented style. While most of the code consists of class definitions, there are also some global constants (located at the beginning of the script) and independent functions (towards the end). There are 17 classes relevant to the backend and 15 other classes which manage the graphical user interface (GUI). Other than the modules imported from the standard library, all of the code is contained in the `polynator_gui.py` and `polynator_main.py` files. This short guide will ignore the GUI and focus exclusively on the backend.

6.2 Classes

The `MainProcess` class orchestrates the backend at the highest level. It holds general information such as the user input and the blueprints used to construct models. Its `run_full` method starts the process by instantiating `Structure` objects. Each of these then parses one of the input files provided by the user with its `parse_cif` or `parse_xyz` methods. It performs symmetry operations on the raw atom vectors to generate one or more coordination environments or molecules. The latter may or may not be helped by the `TopologicalKnife` class, which connects atoms and employs a large number of methods to find specific molecular subunits or cages within the network. Each set of atoms is then further processed by an `AtomArrangement` object instantiated by the `MainProcess.create_atom_arrangements` method. An `AtomArrangement` object contains information about a single molecule, coordination environment or Voronoi polyhedron and is responsible

for administering models to it. After retrieving the appropriate ModelPrecursor containers from the `MainProcess.dict_model_precursors` dictionary, it starts to assign the atom vectors to the belts of the first model with its `assign_to_belts` method. Only after this assignment step is complete, a ModelFit object is instantiated by the `AtomArrangement.run_assignment` method. The actual fitting procedure is then carried out by this object.

Voronoi polyhedra are constructed by the `VoronoiDiagram` class and its subordinate `DelaunayTetrahedron`, `VoronoiPolyhedron` and `VoronoiFace` classes using a flip-based incremental insertion algorithm for the construction of the dual Delaunay tetrahedralization. Convex hulls are constructed by the `ConvexHull` class and its subordinate `ConvexFace` class via a divide and conquer algorithm. The information about each atom generated by the `Structure` class is stored in `Atom` containers. The `NetworkNode` and `Ring` classes store connections between atoms, which makes them useful to the `TopologicalKnife` class. To make the process of connecting atoms more efficient, the `CellSubdivision` class implements a spatial hash, which is accessed by the `Structure` and `TopologicalKnife` classes.

6.3 Global Constants

A notable constant is the rather large `DICT_BLUEPRINTS`, which has atom counts as keys and lists of tuples as values. The tuples are referred to as blueprints. They store all the information needed to construct the models. Each has three entries, which are referred to as `name`, `dict_info` and `belt_dicts`, respectively.

`DICT_CARVING_INSTRUCTIONS` holds information utilized by the `TopologicalKnife` class to carve various molecular subunits and cages out of a larger network of atoms. Each set of instructions is encoded as a dictionary. Each such dictionary (also found as the `target_dict` attribute of a `TopologicalKnife` object) has a 'strategy' key. This strategy determines, which method or combination of methods will be utilized by the `TopologicalKnife`.

`DICT_SYMMETRY_RELATIONS` encodes relationships between models. This means mostly symmetry relations, but also other relations between topologically equivalent models with different degrees of freedom (e.g. between a rigid and a dynamic trigonal prism). It allows a derivative model to inherit its belt

assignment from a previously fitted parent model (not to be confused with class inheritance). This is encoded in the encapsulated lists of lists of integers. Each integer is the index of an atom vector in the flattened `belts_real` attribute of the parent `ModelFit` object. If an entry is a 'same' string, the `belts_real` attribute of the parent `ModelFit` may just be copied outright when constructing the derivative `ModelFit`.

6.4 Utility Functions

The `parse_math()` function has the main purpose of processing the mathematical expressions binding parameters to variables. This could also be done using `eval()` or probably some function from a module, but the custom function allows for dealing with floating point issues (e.g. `arccos(1.0000000000000002)` nested somewhere in a mathematical expression).

The `atom_search` and `blueprint_search` functions process the raw strings input by the user to translate them into atoms found in crystal structures and valid model names, respectively. The majority of the other utility functions implement vector algebra and special iterators.

6.5 Model Parameters

All parameter types other than 'sc' have a version with an '_init' suffix, which signals a fixed parameter, as opposed to a free or dependent one. These '_init' values are applied when the model prototypes are constructed by the `ModelPrecursor.construct_prototype` function. They are useful for the construction of many rigid models, including the rigid base models for dynamic models with constrained parameters. The modulating parameters described in section 5.4 go along with suffixes '_freq' and '_off' for the frequency and offset of the sinusoidal wave function, respectively. If a model requires more than one modulated parameter of the same type in the same belt, the type name of the second parameter, as well as its '_freq' and '_off' supplements are prefixed with one or more additional leading '~' characters. As discussed in section 5.4, some models require multiple parameters to be constrained as a function of a single variable. If a blueprint contains one or more such 'parameter_bundles', they are defined in the `dict_info` section, along with a 'bundle_domains' dictionary, which defines the domain of each bundle variable as a tuple with the minimal, maximal and starting value, respectively.

References

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Appendix

Table 5: Names and free parameters for each model predefined in Polynator.

3	triangle[regular]	sc1
	linear_molecule[∞ /mmm]	sc1
	psi-1_planar_triangle[rigid]	sc1
	psi-2_tetrahedron[rigid]	sc1
	right_angle[rigid]	sc1
	isosceles_triangle[mm2]	h1, h*, w1
	kinked_molecule[mm2]	h1, h*, w1
	linear_molecule[∞ mm]	h1, >h1
	4	square[regular]
tetrahedron[platonic]		sc1
trigonal_star[-6m2]		sc1
psi-1_tetrahedron[rigid]		sc1
fac-psi-3_octahedron[rigid]		sc1
T_shape[rigid]		sc1
disphenoid[-42m]		h1, w1
rectangle[mmm]		w1, phi1
rhombus[mmm]		w1, w1
trigonal_pyramid[3m]		h1, h*, w1
twisted_disphenoid[222]		h1, phi1, w1
heterodisphenoid[mm2]		h1, w1, w2
trapezoid[mm2]		h1, w1, w2
parallelogram[2/m]		w1, phi1, w1
tripod[3m]		h1, h*, w1

5	pentagon[regular]	sc1
	tetragonal_star[4/mmm]	sc1
	trigonal_bipyramid[johnson]	sc1
	tetragonal_pyramid[johnson]	sc1
	trigonal_bipyramid[equidistant]	sc1
	tetragonal_pyramid[equidistant]	sc1
	psi-1_trigonal_bipyramid[rigid]	sc1
	berry_pseudorotation[dynamic]	sc1, v1
	trigonal_bipyramid[-6m2]	h1, w1
	tetragonal_pyramid[4mm]	h1, h*, w1
	tetrapod[4mm]	h1, h*, w1
	trigonal_heterobipyramid[3m]	h1, h2, h*, w1
	rhombic_pyramid[dynamic]	h1, h*, w1, w1
	rectangular_pyramid[mm2]	h1, h*, w1, phi1
6	hexagon[regular]	sc1
	octahedron[platonic]	sc1
	trigonal_prism[uniform]	sc1
	pentagonal_pyramid[johnson]	sc1
	pentagonal_pyramid[equidistant]	sc1
	psi-1_octahedron[rigid]	sc1
	bailar_twist[dynamic]	sc1, v1
	tetragonal_bipyramid[4/mmm]	h1, w1
	tetragonal_heterobipyramid[4mm]	h1, h2, h*, w1
	didigonal_scalenohedron[-42m]	h1, w1, h1
	rhombic_bipyramid[mmm]	h1, w1, w1
	rectangular_bipyramid[mmm]	h1, w1, phi1
	parallelogrammic_bipyramid[2/m]	h1, w1, phi1, w1
	trigonal_antiprism[-3m]	h1, w1
	twisted_trigonal_prism[32]	h1, phi1, w1
	trigonal_antifrustum[3m]	h1, w1, w2
	trigonal_prism[-6m2]	h1, w1
	trigonal_frustum[3m]	h1, w1, w2
	isosceles_wedge[mm2]	h1, h*, w1, w2, phi1
	pentagonal_pyramid[5m]	h1, h*, w1
	ethene_unit[dynamic]	h1, v1

7	heptagon[regular]	sc1
	pentagonal_bipyramid[johnson]	sc1
	monocapped_trigonal_prism[johnson]	sc1
	pentagonal_bipyramid[equidistant]	sc1
	hexagonal_pyramid[equidistant]	sc1
	psi-1_pentagonal_bipyramid[rigid]	sc1
	pentagonal_bipyramid[-10m2]	h1, w1
	pentagonal_heterobipyramid[5m]	h1, h2, h*, w1
	monocapped_trigonal_prism[dynamic]	v1, v2, v3
	monocapped_isosceles_wedge[mm2]	h1, h2, h*, w1, w2, phi1
	hexagonal_pyramid[6mm]	h1, h*, w1
	monocapped_trigonal_frustum[3m]	h1, h2, h*, w1, w2
	monocapped_trigonal_antifrustum[3m]	h1, h2, h*, w1, w2

8	octagon[regular]	sc1
	cube[platonic]	sc1
	tetragonal_antiprism[uniform]	sc1
	triangular_dodecahedron[johnson]	sc1
	gyrobifastigium[johnson]	sc1
	bicapped_trigonal_prism[johnson]	sc1
	hexagonal_bipyramid[equidistant]	sc1
	edge-sharing_tetrahedron_doublet[rigid]	sc1
	triakis_tetrahedron[-43m]	sc1, sc2
	tetragonal_prism[4/mmm]	h1, w1
	twisted_tetragonal_prism[422]	h1, phi1, w1
	tetragonal_frustum[4mm]	h1, w1, w2
	triangular_dodecahedron[-42m]	h1, w1, w2, h1
	anticuboid[-42m]	h1, w1, phi1
	cuboid[mmm]	h1, w1, phi1
	rhombic_prism[dynamic]	h1, w1, w1
	parallelogrammic_prism[dynamic]	h1, w1, phi1, w1
	rhombohedron[dynamic]	w1, >h1
	bicapped_trigonal_antiprism[-3m]	h1, h2, w1
	tetragonal_antiprism[-82m]	h1, w1
	tetragonal_antifrustum[4mm]	h1, w1, w2
	gyrobifastigium[-42m]	h1, w1, w2
	bicapped_trigonal_prism[dynamic]	v1, v2, v3
	biaugmented_isosceles_wedge[mm2]	h1, w1, w2, h1, phi1, w1
	hexagonal_bipyramid[6/mmm]	h1, w1
	hexagonal_heterobipyramid[6mm]	h1, h2, h*, w1
	elongated_trigonal_bipyramid[-6m2]	h1, h2, w1
	digonal_orthobicupola[mmm]	h1, w1, w2, phi1

9	tricapped_trigonal_prism[johnson]	sc1
	monocapped_cube[johnson]	sc1
	tridiminished_icosahedron[johnson]	sc1
	trigonal_cupola[johnson]	sc1
	monocapped_tetragonal_antiprism[johnson]	sc1
	tricapped_trigonal_prism[-6m2]	h1, w1, w2
	monocapped_tetragonal_frustum[4mm]	h1, h2, h*, w1, w2
	monocapped_tetragonal_antifrustum[4mm]	h1, h2, h*, w1, w2
	trigonal_cupola[3m]	h1, h*, w1, w2, phi1
10	pentagonal_prism[uniform]	sc1
	pentagonal_antiprism[uniform]	sc1
	bicapped_tetragonal_antiprism[johnson]	sc1
	trans-bicapped_cube[johnson]	sc1
	meta-bidiminished_icosahedron[johnson]	sc1
	sphenocorona[johnson]	sc1
	adamantane_cage[rigid]	sc1
	cis-bicapped_cube[rigid]	sc1
	naphthalene_skeleton[rigid]	sc1
	tetracapped_octahedron[-43m]	sc1, sc2
	pentagonal_prism[-10m2]	h1, w1
	pentagonal_antiprism[-5m]	h1, w1
	twisted_pentagonal_prism[52]	h1, phi1, w1
	bicapped_tetragonal_antiprism[-82m]	h1, h2, w1
	bicapped_tetragonal_prism[4/mmm]	h1, h2, w1
	monocapped_trigonal_cupola[3m]	h1, h2, h*, w1, w2, phi1
	fulvalene_skeleton[dynamic]	h1, v1
11	monodiminished_icosahedron[johnson]	sc1
	monocapped_pentagonal_prism[johnson]	sc1
	edshammar_polyhedron[rigid]	sc1
	fac-tricapped_cube[rigid]	sc1
	mer-tricapped_cube[rigid]	sc1
	face-sharing_octahedron_doublet[rigid]	sc1
	edshammar_polyhedron[dynamic]	w1, >h1
	fully_capped_trigonal_prism[-6m2]	h1, h2, w1, w2
	monocapped_pentagonal_frustum[5m]	h1, h2, h*, w1, w2
	monocapped_pentagonal_antifrustum[5m]	h1, h2, h*, w1, w2

12	icosahedron[platonic]	sc1
	cuboctahedron[archimedean]	sc1
	truncated_tetrahedron[archimedean]	sc1
	hexagonal_prism[uniform]	sc1
	hexagonal_antiprism[uniform]	sc1
	bicapped_pentagonal_prism[johnson]	sc1
	anticuboctahedron[johnson]	sc1
	tetracapped_cube[rigid]	sc1
	icosahedron[m-3]	sc1, v1
	icosahedron[-5m]	h1, h2, w1
	icosahedron[-3m]	h1, w1, w2, h1
	cuboctahedron[-43m]	sc1, v1
	cuboctahedron[4/mmm]	h1, w1, w2
	cuboctahedron[-3m]	h1, w1, w2
	cuboctahedron[422]	h1, phi1, w1, w2
	cuboctahedron[4mm]	h1, h2, h*, w1, w2, w3
	cuboctahedron[-42m_a]	h1, w1, w2, h1, w1
	cuboctahedron[-42m_b]	h1, w1, w2, h1, phi1
	cuboctahedron[4/m]	h1, phi*, phi1, w1, w2
	cuboctahedron[mmm_a]	h1, w1, w2, phi1, w1
	cuboctahedron[mmm_b]	h1, w1, w2, h1, phi1, w1
	cuboctahedron[32]	h1, phi1, w1, w2, w1
	cuboctahedron[3m]	h1, h2, h*, w1, w2, w3, phi1
	cuboctahedron[-3]	h1, phi*, phi1, w1, w2, h1
	truncated_tetrahedron[-43m]	sc1, v1
	hexagonal_prism[6/mmm]	h1, w1
	hexagonal_antiprism[-122m]	h1, w1
	anticuboctahedron[-6m2]	h1, w1, w2, phi1
	bicapped_pentagonal_prism[-10m2]	h1, h2, w1
	elongated_gyrobifastigium[dynamic]	>w1, h1, h2
	twisted_biphenyl_skeleton[dynamic]	h1, phi1, v1

¹⁾ There are two distinct dynamic cuboctahedra for each of the point groups $\bar{4}2m$ and mmm , labelled a and b, respectively.

13	pentacapped_cube[rigid]	sc1
	monocapped_hexagonal_frustum[6mm]	h1, h2, h*, w1, w2
	monocapped_hexagonal_antifrustum[6mm]	h1, h2, h*, w1, w2
14	rhombic_dodecahedron[catalan]	sc1
	fully_capped_cube[m-3m]	sc1, sc2
	fully_capped_tetragonal_prism[4/mmm]	h1, h2, w1, w2
	heptagonal_prism[-14m2]	h1, w1
	heptagonal_antiprism[-7m]	h1, w1
	bicapped_hexagonal_prism[6/mmm]	h1, h2, w1
	bicapped_hexagonal_antiprism[-122m]	h1, h2, w1
15	pentacapped_pentagonal_prism[-10m2]	h1, w1, w2
	tricapped_ditrigonal_prism[-6m2]	h1, h2, w1, w2, w3
16	snub_tetragonal_antiprism[johnson]	sc1
	disphenocingulum[johnson]	sc1
	hexakis_truncated_tetrahedron[rigid]	sc1
	hexakis_truncated_tetrahedron[-43m]	sc1, sc2, v1
	octagonal_prism[8/mmm]	h1, w1
	octagonal_antiprism[-162m]	h1, w1
17	fully_capped_pentagonal_prism[-10m2]	h1, h2, w1, w2
18	elongated_trigonal_orthobicupola[johnson]	sc1
	elongated_trigonal_gyrobicupola[johnson]	sc1
	gyroelongated_trigonal_bicupola[johnson]	sc1
	elongated_rhombic_dodecahedron[rigid]	sc1
	hexacapped_cuboctahedron[m-3m]	sc1, sc2
	elongated_rhombic_dodecahedron[dynamic]	h1, >h1, >w1
	hexacapped_hexagonal_prism[6/mmm]	h1, w1, w2
	18-crown-6_skeleton[-3m]	w1, h1, phi1, w1, h1
20	dodecahedron[platonic]	sc1
	pyritohedron[dynamic]	sc1, v1
	faceted_pyritohedron[m-3]	sc1, sc2, v1
	fully_capped_hexagonal_prism[6/mmm]	h1, h2, w1, w2

24	truncated_octahedron[archimedean]	sc1
	rhombicuboctahedron[archimedean]	sc1
	truncated_cube[archimedean]	sc1
	snub_cube[archimedean]	sc1
	elongated_tetragonal_gyrobicupola[johnson]	sc1
	porphyrin_skeleton[rigid]	sc1
	truncated_octahedron[m-3m]	sc1, v1
	rhombicuboctahedron[m-3m]	sc1, v1
	truncated_cube[m-3m]	sc1, v1
	twisted_rhombicuboctahedron[432]	sc1, v1, v2
	porphyrin_skeleton[dynamic]	sc1, sc2, v1
	truncated_hexagonal_trapezohedron[dynamic]	v1, v2, v3
26	deltoidal_icositetrahedron[catalan]	sc1
	fully_capped_cuboctahedron[m-3m]	sc1, sc2, sc3
28	truncated_triakis_tetrahedron[rigid]	sc1
	truncated_triakis_tetrahedron[dynamic]	sc1, v1
30	icosidodecahedron[archimedean]	sc1
	pentagonal_orthobirotunda[johnson]	sc1
	hexacapped_truncated_cube[m-3m]	sc1, sc2, v1
32	rhombic_triacontahedron[catalan]	sc1
	chamfered_cube[rigid]	sc1
	fully_capped_platonic_dodecahedron[-5m-3]	sc1, sc2
38	pentagonal_icositetrahedron[catalan]	sc1
48	truncated_cuboctahedron[archimedean]	sc1
	faujasite_supercage[rigid]	sc1
60	truncated_icosahedron[archimedean]	sc1

