

Small-angle scattering behaviour of thread-like and film-like systems
by

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We report below the MATHEMATICA codes that we used to get the CF of the right circular cylinder and the CF of a cubic surface.

Derivation of equations (102)-(109)

RE-CALCULATION OF THE CORRELATION FUNCTION OF THE CYLINDER WITH RADIUS R, DIAMETER D AND HEIGHT H.

```
(* SetDirectory["C:\\\\Users\\\\salvino\\\\Documents\\\\SALVIN\\\\LAVORI\\\\CYLINDER"] ; HP *)
SetDirectory["/Users/salvino/Desktop/WORK_IN_PRGS/THREAD_FILM_LIKE_SYSTEMS"];
Directory[]

/Users/salvino/Desktop/WORK_IN_PRGS/THREAD_FILM_LIKE_SYSTEMS

a := H / D
The variable x = r / D ranges within [0, Rmax = Sqrt[1 + a^2]].
This interval splits into three intervals :
[0, min[a, 1]] [min[a, 1], max[a, 1]] and
[max[a, 1], Rmax] with
```

The evalution proceeds by evaluating the overlapping volume between the cylinder and that translated by r . We assume that the plane $z=0$ cuts the cylinder halfway and that the z -axis coincides with the cylinder-s axis.

One sets $\vec{r}=\{r \cdot \text{Cos}[\varphi] \cdot \text{Sin}[\theta], r \cdot \text{Sin}[\varphi] \cdot \text{Sin}[\theta], r \cdot \text{Cos}[\theta]\}$. with $0 < \alpha < \pi/2$

The overlapping area of two circle of radius R and centers separated by x is

$$\text{Sovrlp}[R_, x_]:= 2 R^2 [\text{ArcCos}[\frac{x}{2*R}] - \frac{x}{2*R} \sqrt{1 - (\frac{x}{2*R})^2}] .$$

The separation bewteen the centers is $x=r_\perp=r \cdot \text{Sin}[\theta]$.

The height of the overlapping volume is $Hovrl=H - r_\parallel = H - r \cdot \text{Cos}[\theta]$

Hence the overlapping volume is

$$\text{Vovrlpng}[R_, x_, t_]:= \text{Sovrlp} * \text{Hovrl}$$

DEFINTION OF THE INTEGRAND FOR THE CASE WE ARE INTERESTED IN

VOLUME OF THE OVERLAPPING REGION

$$\text{Sovrlp}[R_, r_, \theta_] := 2 R^2 \left[\text{ArcCos} \left[\frac{r * \text{Sin}[\theta]}{2 * R} \right] - \frac{r * \text{Sin}[\theta]}{2 * R} \sqrt{1 - \left(\frac{r * \text{Sin}[\theta]}{2 * R} \right)^2} \right];$$

$$\text{Vovrlpng}[R_, H_, r_, \theta_] := \text{Sovrlp}[R, r, \theta] * (H - r * \text{Cos}[\theta]);$$

The integral is

$$2 \int_0^{2\pi} \int_0^{\pi/2} \text{Vovrlpng}[R, H, r, \theta] * \text{Sin}[\theta] d\theta d\varphi = \\ 4 \pi \int_0^{\pi/2} \text{Vovrlpng}[R, H, r, \theta] * \text{Sin}[\theta] d\theta = 4 \pi \int_0^1 \text{Vovrlpng}[R, H, r, \theta] d\cos[\theta]$$

$$4 * \pi * \text{Sin}[\theta] * \text{Vovrlpng}[R, H, r, \theta] \\ 8 \pi R^2 (H - r \text{Cos}[\theta]) \text{Sin}[\theta] \left(\text{ArcCos} \left[\frac{r \text{Sin}[\theta]}{2 R} \right] - \frac{r \text{Sin}[\theta] \sqrt{1 - \frac{r^2 \text{Sin}[\theta]^2}{4 R^2}}}{2 R} \right)$$

the above function is correctly normalized. In fact its integral, divide by (4π) , must reproduce the cylinder volume in the limit $r \rightarrow 0$ as it happens. See below

$$\text{Integrate}[(4 * \pi * \text{Sin}[\theta] * \text{Vovrlpng}[R, H, r, \theta]) /. \{r \rightarrow 0\}, \{\theta, 0, \pi/2\}] / (4 \pi)$$

$$H \pi R^2$$

The integrand is obtained putting $\text{Cos}[\theta] \rightarrow t$ and reads

$$2 \pi R (H - r t) \left(-r \sqrt{1 - t^2} \sqrt{\frac{-r^2 + 4 R^2 + r^2 t^2}{R^2}} + 4 R \text{ArcCos} \left[\frac{r \sqrt{1 - t^2}}{2 R} \right] \right)$$

and it is split into the sum of three terms.

$$\text{Simplify}[(4 * \pi * \text{Sin}[\theta] * \text{Vovrlpng}[R, H, r, \theta]) /. \{\text{Cos}[\theta] \rightarrow t, \text{Sin}[\theta] \rightarrow \sqrt{1 - t^2}\}] / \sqrt{1 - t^2},$$

$$\text{Assumptions} \rightarrow \{0 < t < 1\}$$

$$2 \pi R (H - r t) \left(-r \sqrt{-\frac{(-1 + t^2) (4 R^2 + r^2 (-1 + t^2))}{R^2}} + 4 R \text{ArcCos} \left[\frac{r \sqrt{1 - t^2}}{2 R} \right] \right)$$

```

integrndTot[R_, H_, r_, t_] =
  2 π R (H - r t) 
$$\left( -r \sqrt{1-t^2} \sqrt{\frac{-r^2 + 4 R^2 + r^2 t^2}{R^2}} + 4 R \text{ArcCos}\left[\frac{r \sqrt{1-t^2}}{2 R}\right] \right);$$

integrndaa[R_, H_, r_, t_] :=
  2 π r^2 R t 
$$\sqrt{-\frac{(-1+t^2) (4 R^2+r^2 (-1+t^2))}{R^2}} - 8 \pi r R^2 t \text{ArcCos}\left[\frac{r \sqrt{1-t^2}}{2 R}\right];$$

integrndbb[R_, H_, r_, t_] := -2 π H r * 
$$\sqrt{-(-1+t^2) (4 R^2+r^2 (-1+t^2))};$$

integrndcc[R_, H_, r_, t_] := 8 π H R^2 
$$\text{ArcCos}\left[\frac{r \sqrt{1-t^2}}{2 R}\right];$$

FullSimplify[(integrndaa[R, H, r, t] + integrndbb[R, H, r, t] + integrndcc[R, H, r, t]) -
  integrndTot[R, H, r, t], Assumptions → {0 < t < 1 && R > 0 && H > 0}]

```

0

integrndaa[R, H, r, t] can explicitly be integrated. One find

```

Simplify[(FullSimplify[
  Integrate[integrndaa[R, H, r, t], t, Assumptions → {0 < t < 1 && H > 0 && R > 0 && r > 0}],
  Assumptions → {0 < t < 1 && H > 0 && R > 0 && r > 0}) /.
  {ArcCsc[ $\frac{2 R}{r \sqrt{1-t^2}}$ ] → ArcSin[ $\frac{r \sqrt{1-t^2}}{2 R}$ ], ArcSec[ $\frac{2 R}{r \sqrt{1-t^2}}$ ] →  $\pi / 2 - \text{ArcSin}\left[\frac{r \sqrt{1-t^2}}{2 R}\right]$ ,
   ArcTan[ $\frac{r}{\sqrt{-\frac{4 R^2+r^2 (-1+t^2)}{-1+t^2}}}$ ] → ArcSin[ $\frac{r \sqrt{1-t^2}}{2 R}$ ],
    $\sqrt{-\frac{(-1+t^2) (4 R^2+r^2 (-1+t^2))}{R^2}}$  →  $\frac{\sqrt{-(-1+t^2) (4 R^2+r^2 (-1+t^2))}}{R}$ },
  Assumptions → {0 < t < 1 && H > 0 && R > 0 && r > 0}]]

-  $\frac{1}{2 r} \pi \left( r \left( 4 \pi r R^2 (-1+t^2) + (2 R^2 - r^2 (-1+t^2)) \sqrt{-(-1+t^2) (4 R^2+r^2 (-1+t^2))} \right) - \right.$ 

$$8 R^2 (R^2+r^2 (-1+t^2)) \text{ArcSin}\left[\frac{r \sqrt{1-t^2}}{2 R}\right]$$

FullSimplify[Sin[ArcCsc[ $\frac{2 R}{r \sqrt{1-t^2}}$ ]]]
FullSimplify[Sin[ $\pi / 2 - \text{ArcSec}\left[\frac{2 R}{r \sqrt{1-t^2}}\right]$ ]]
FullSimplify[Sin[ $\pi / 2 - \text{ArcCos}\left[\frac{r \sqrt{1-t^2}}{2 R}\right]$ ]]

```

```

FullSimplify[TrigExpand[Sin[ArcTan[ $\frac{r}{\sqrt{-\frac{4 R^2+r^2 (-1+t^2)}{-1+t^2}}}$ ]]], Assumptions -> {0 < t < 1 && H > 0 && R > 0 && r > 0 && 4 R^2 + r^2 (-1 + t^2) > 0}]

```

One performs an integration of `integrndcc[R, H, r, t]` by parts

```

Expand[(integrndcc[R, H, r, t]) /. {ArcCos[r Sqrt[1 - t^2]/(2 R)] -> \[Pi]/2 - ArcSin[r Sqrt[1 - t^2]/(2 R)]}]
4 H \[Pi]^2 R^2 - 8 H \[Pi] R^2 ArcSin[r Sqrt[1 - t^2]/(2 R)]
FullSimplify[-8 H \[Pi] R^2 * t * D[ArcSin[r Sqrt[1 - t^2]/(2 R)], t], Assumptions -> {0 < t < 1 && R > 0 && r > 0}]
8 H \[Pi] r R^2 t^2
────────────────────────────────
Sqrt[-(-1 + t^2) (4 R^2 + r^2 (-1 + t^2))]

```

Primitive of the integrable term plus the first term of the integration by parts.

```

primitvAA[R_, H_, r_, t_] :=
Simplify[(FullSimplify[Integrate[integrndaa[R, H, r, t], t, Assumptions -> {0 < t < 1 && H > 0 && R > 0 && r > 0}], Assumptions -> {0 < t < 1 && H > 0 && R > 0 && r > 0}]) /.

{ArcCsc[2 R / r Sqrt[1 - t^2]] -> ArcSin[r Sqrt[1 - t^2] / 2 R], ArcSec[2 R / r Sqrt[1 - t^2]] ->

π/2 - ArcSin[r Sqrt[1 - t^2] / 2 R], ArcTan[r / Sqrt[-(4 R^2 + r^2 (-1 + t^2)) / (-1 + t^2)]] -> ArcSin[r Sqrt[1 - t^2] / 2 R],


Sqrt[-(((-1 + t^2) (4 R^2 + r^2 (-1 + t^2))) / R^2)] -> Sqrt[-((-1 + t^2) (4 R^2 + r^2 (-1 + t^2))) / R]}],


Assumptions -> {0 < t < 1 && H > 0 && R > 0 && r > 0}] +


4 H π^2 R^2 - 8 H π R^2 ArcSin[r Sqrt[1 - t^2] / 2 R] * t;

```

check

```

FullSimplify[D[primitvAA[R, H, r, t], t] - 
  FullSimplify[
    -8 H \pi R^2 * t * D[ArcSin[(r \sqrt[3]{1 - t^2}) / (2 R)], t], Assumptions -> {0 < t < 1 && R > 0 && r > 0}] - 
  (integrndaa[R, H, r, t] + integrndcc[R, H, r, t]), 
  Assumptions -> {0 < t < 1 && H > 0 && R > 0 && r > 0}]

```

```

Limit[primitvAA[R, H, r, t], t → √(r^2 - 4 R^2) / r, Direction → -1]
Series[FullSimplify[Limit[primitvAA[R, H, r, t], t → 1, Direction → 1] -
Limit[primitvAA[R, H, r, t], t → √(r^2 - 4 R^2) / r, Direction → -1],
Assumptions → {0 < 2 * R < r < H}], {R, 0, 2}]

```

Remaining contribution to be integrated

```

Factor[Together[integrndbb[R, H, r, t] - 
FullSimplify[
-8 H π R^2 * t * D[ArcSin[r √(1 - t^2) / (2 R)], t], Assumptions → {0 < t < 1 && R > 0 && r > 0}]]]

```

$$\text{rmngintegrnd}[R_-, H_-, r_-, t_-] := \frac{2 H \pi r (-r - 2 R + r t^2) (-r + 2 R + r t^2)}{\sqrt{-(-1 + t^2) (-r^2 + 4 R^2 + r^2 t^2)}};$$

Further Check that the outset integrand be reproduced (OK)

```

FullSimplify[D[primitvAA[R, H, r, t], t] + rmngintegrnd[R, H, r, t] - integrndTot[R, H, r, t],
Assumptions → {0 < t < 1 && H > 0 && R > 0 && r > 0}]
Expand[(-r - 2 R + r t^2) (-r + 2 R + r t^2)]

```

The remaining integrand "rmngintegrnd[R_, H_, r_, t_]" must be written as a sum of elliptical canonical integrals.

```
Factor[rmngintegrnd[R, H, r, t]]
```

$$\begin{aligned} \text{rmngintegrnd44}[R_-, H_-, r_-, t_-] &:= \frac{t^4}{\sqrt{-(-1 + t^2) (-r^2 + 4 R^2 + r^2 t^2)}}; \\ \text{rmngintegrnd22}[R_-, H_-, r_-, t_-] &:= \frac{t^2}{\sqrt{-(-1 + t^2) (-r^2 + 4 R^2 + r^2 t^2)}}; \\ \text{rmngintegrnd00}[R_-, H_-, r_-, t_-] &:= \frac{1}{\sqrt{-(-1 + t^2) (-r^2 + 4 R^2 + r^2 t^2)}}; \\ \text{Simplify}[rmngintegrnd[R, H, r, t] - (2 H \pi r) * ((r^2 - 4 R^2) * rmngintegrnd00[R, H, r, t] + (-2 r^2) * rmngintegrnd22[R, H, r, t] + (r^2) * rmngintegrnd44[R, H, r, t])] \end{aligned}$$

By the following identities it is possible to write the integrand as a sum of the canonical elliptic forms

```

D[t / rmngintegrnd00[R, H, r, t], t] → XXXX;
rmngintegrnd22XX →  $\frac{1}{r^2} \left( \frac{\sqrt{-r^2 + 4 R^2 + r^2 t^2}}{\sqrt{1-t^2}} - (-r^2 + 4 R^2) * rmngintegrnd00XX \right);$ 
rmngintegrnd44XX -

$$\frac{1}{r^2} \left( \frac{(r^2 - 4 R^2) * (3 r^2 - 8 R^2) rmngintegrnd00XX}{3 r^2} + \frac{4 (r^2 - 2 R^2) \sqrt{-r^2 + 4 R^2 + r^2 t^2}}{3 r^2 \sqrt{1-t^2}} - \frac{XXXX}{3} \right)$$


```

Proof of the identities

```

FullSimplify[rmngintegrnd22[R, H, r, t] -

$$\frac{1}{r^2} \left( \frac{\sqrt{-r^2 + 4 R^2 + r^2 t^2}}{\sqrt{1-t^2}} - (-r^2 + 4 R^2) * rmngintegrnd00[R, H, r, t] \right),$$

Assumptions → {0 < t^2 < 1 && -r^2 + 4 R^2 + r^2 t^2 > 0}]

FullSimplify[
rmngintegrnd44[R, H, r, t] -  $\frac{1}{r^2} \left( \frac{(r^2 - 4 R^2) * (3 r^2 - 8 R^2) rmngintegrnd00[R, H, r, t]}{3 r^2} + \right.$ 

$$\left. \frac{4 (r^2 - 2 R^2) \sqrt{-r^2 + 4 R^2 + r^2 t^2}}{3 r^2 \sqrt{1-t^2}} - \frac{D[t / rmngintegrnd00[R, H, r, t], t]}{3} \right),$$

Assumptions → {0 < t^2 < 1 && -r^2 + 4 R^2 + r^2 t^2 > 0}]

FullSimplify[rmngintegrnd[R, H, r, t] -  $\left( -\frac{16 H \pi R^2 (r^2 - 4 R^2)}{3 r} * rmngintegrnd00[R, H, r, t] - \right.$ 

$$\left. \frac{2 H \pi r * D[t / rmngintegrnd00[R, H, r, t], t]}{3} - \frac{4 H \pi (r^2 + 4 R^2) \sqrt{-r^2 + 4 R^2 + r^2 t^2}}{3 r \sqrt{1-t^2}} \right),$$

Assumptions → {0 < t^2 < 1 && -r^2 + 4 R^2 + r^2 t^2 > 0}]

```

The integrand in the canonical form becomes

```

rmngintegrnd[R, H, r, t] →  $\left( -\frac{16 H \pi R^2 (r^2 - 4 R^2)}{3 r} * rmngintegrnd00[R, H, r, t] - \right.$ 

$$\left. \frac{2 H \pi r * D[t / rmngintegrnd00[R, H, r, t], t]}{3} - \frac{4 H \pi (r^2 + 4 R^2) \sqrt{-r^2 + 4 R^2 + r^2 t^2}}{3 r \sqrt{1-t^2}} \right)$$


```

```

intgrndFFF[R_, r_, t_] := 
$$\frac{1}{\sqrt{(1-t^2)(-r^2+4R^2+r^2t^2)}};$$

intgrndEEE[R_, r_, t_] := 
$$\frac{\sqrt{-r^2+4R^2+r^2t^2}}{\sqrt{1-t^2}};$$


rmnFFF[R_, H_, r_, t_] := -
$$\frac{16H\pi R^2(r^2-4R^2)}{3r} * \text{intgrndFFF}[R, r, t];$$

rmnEEE[R_, H_, r_, t_] := -
$$\frac{4H\pi(r^2+4R^2)\text{intgrndEEE}[R, r, t]}{3r};$$

(* ATTENTION THIS IS A PRIMITIVE *)
rmngprmtv[R_, H_, r_, t_] := -
$$\frac{2}{3}H\pi r*t / \text{rmngintegrnd00}[R, H, r, t];$$


```

check

```

FullSimplify[rmngintegrnd[R, H, r, t] -
  (rmnFFF[R, H, r, t] + rmnEEE[R, H, r, t] + D[rmngprmtv[R, H, r, t], t]), 
  Assumptions → {0 < t < 1 && -r^2 + 4 R^2 + r^2 t^2 > 0 && R > 0 && r > 0 && H > 0}]

```

The integration of intgrndFFF and intgrndEEE involves the elliptic integrals.
The integration will be done by using some formulae reported by Gradshteyn and Ryzhik.
We need first to define the integration domains. These are obtained by solving the inequalities

Disk : $0 < H < 2 * R, 0 < t < 1, -r^2 + 4 R^2 + r^2 t^2 > 0, H - r t > 0 ;$

DscAA : $0 < r < H < 2 R$ and $0 < t < 1 ;$

DscBB : $0 < H < r < 2 R$ and $0 < t < \frac{H}{r} ;$

DscCC : $2 R < r < \sqrt{H^2 + 4 R^2}$ and $\sqrt{\frac{r^2 - 4 R^2}{r^2}} < t < \frac{H}{r} ;$

Needle : $0 < 2 * R < H, 0 < t < 1, -r^2 + 4 R^2 + r^2 t^2 > 0, H - r t > 0 ;$

NdlAA : $0 < r \leq 2 R$ and $0 < t < 1 ;$

NdlBB : $2 R < r \leq H$ && $\sqrt{\frac{r^2 - 4 R^2}{r^2}} < t < 1 ;$

NdlCC : $H < r < \sqrt{H^2 + 4 R^2}$ && $\sqrt{\frac{r^2 - 4 R^2}{r^2}} < t < \frac{H}{r} .$

To integrate intgrndFFF[R, r, t] and intgrndEEE[R, r, t] one needs to distinguish the cases aa : $r < 2 * R$ from bb : $2 * R < r$

$$\text{Reduce}\left[\left\{0 < 2 * R < H, \quad 0 < t < 1, \quad -r^2 + 4 R^2 + r^2 t^2 > 0,\right.\right.$$

$$\left.\left.H - r t > 0, \quad 0 < r < \sqrt{H^2 + 4 R^2}\right\}, \{H, R, r, t\}, \text{Reals}\right]$$

$$\begin{aligned} H > 0 \&& 0 < R < \frac{H}{2} \&& \left(0 < r \leq 2R \&& 0 < t < 1\right) \mid | \\ \left(2R < r \leq H \&& \sqrt{\frac{r^2 - 4R^2}{r^2}} < t < 1\right) \mid | \mid \left(H < r < \sqrt{H^2 + 4R^2} \&& \sqrt{\frac{r^2 - 4R^2}{r^2}} < t < \frac{H}{r}\right) \end{aligned}$$

Case (aa) : $r < 2*R$

$$\text{One finds that } \sqrt{-r^2 + 4R^2 + r^2 t^2} \rightarrow r \sqrt{t^2 + AA^2} \text{ with } AA \rightarrow \frac{\sqrt{4R^2 - r^2}}{r}$$

For the FF using the identity 3.152 .3 del G&R one gets

$$\int_0^u \text{intgrndFFF}[R, r, t] dt \rightarrow \int_0^u \frac{1}{\sqrt{(1-t^2)(-r^2+4R^2+r^2t^2)}} dt = \int_0^u \frac{1}{r \sqrt{(1-t^2)(t^2+a^2)}} dt =$$

$$\frac{1}{r} \frac{1}{\sqrt{1+a^2}} F\left[\text{ArcSin}\left[u \sqrt{\frac{1+a^2}{u^2+a^2}}\right], \frac{1}{\sqrt{1+a^2}}\right] \rightarrow \frac{1}{r} \frac{1}{\sqrt{1+a^2}} \text{EllipticF}\left[\text{ArcSin}\left[u \sqrt{\frac{1+a^2}{u^2+a^2}}\right], \frac{1}{1+a^2}\right];$$

For the EEE using the identity Using eq .3 .169 .3 one gets

$$\int_0^u \text{intgrndEEE}[R, r, t] dt \rightarrow \int_0^u \frac{\sqrt{-r^2 + 4R^2 + r^2 t^2}}{\sqrt{1-t^2}} dt \rightarrow$$

$$r \int_0^u \frac{\sqrt{aa^2 + t^2}}{\sqrt{1-t^2}} dt \rightarrow r \left(\sqrt{a^2 + 1} E\left[\text{ArcSin}\left[u \sqrt{\frac{a^2 + 1}{a^2 + u^2}}\right], \frac{1}{\sqrt{a^2 + 1}}\right] - u * \sqrt{\frac{1 - u^2}{a^2 + u^2}} \right) \rightarrow$$

$$r \left(\sqrt{a^2 + 1} \text{EllipticE}\left[\text{ArcSin}\left[u \sqrt{\frac{a^2 + 1}{a^2 + u^2}}\right], \frac{1}{a^2 + 1}\right] - u * \sqrt{\frac{1 - u^2}{a^2 + u^2}} \right)$$

```

prmtvFFFaa[R_, r_, u_] :=
FullSimplify[ $\left(\frac{1}{r} \frac{1}{\sqrt{1+a^2}} \text{EllipticF}\left[\text{ArcSin}\left[u \sqrt{\frac{1+a^2}{u^2+a^2}}\right], \frac{1}{1+a^2}\right]\right) /. \{a \rightarrow \frac{\sqrt{4R^2 - r^2}}{r}\},$ 
Assumptions \rightarrow {0 < r < 2 * R};

prmtvEEEaa[R_, r_, u_] := FullSimplify[
 $\left(\left(r \left(\sqrt{a^2 + 1} \text{EllipticE}\left[\text{ArcSin}\left[u \sqrt{\frac{a^2 + 1}{a^2 + u^2}}\right], \frac{1}{a^2 + 1}\right] - u * \sqrt{\frac{1 - u^2}{a^2 + u^2}}\right)\right) /. \{a \rightarrow \frac{\sqrt{4R^2 - r^2}}{r}\},$ 
Assumptions \rightarrow {0 < r < 2 * R}\];

```

$$\begin{aligned}
\text{prmtvFFFaa}[R_, r_, u_] &:= \frac{\text{EllipticF}\left[\text{ArcSin}\left[2 R t \sqrt{\frac{1}{4 R^2 + r^2 (-1+t^2)}}\right], \frac{r^2}{4 R^2}\right]}{2 R}; \\
\text{prmtvEEEaa}[R_, r_, u_] &:= \\
&\quad -r^2 t \sqrt{\frac{1-t^2}{4 R^2 + r^2 (-1+t^2)}} + 2 R \text{EllipticE}\left[\text{ArcSin}\left[2 R t \sqrt{\frac{1}{4 R^2 + r^2 (-1+t^2)}}\right], \frac{r^2}{4 R^2}\right];
\end{aligned}$$

CHECKS on the derivatives

```

FullSimplify[
FullSimplify[D[prmtvFFFaa[R, r, t], t], Assumptions -> {0 < r < 2 * R && 0 < t < 1}] -
intgrndFFF[R, r, t], Assumptions -> {0 < r < 2 * R && 0 < t < 1}]

FullSimplify[
FullSimplify[D[prmtvEEEaa[R, r, t], t], Assumptions -> {0 < r < 2 * R && 0 < t < 1}] -
intgrndEEE[R, r, t], Assumptions -> {0 < r < 2 * R && 0 < t < 1}]

intgrndFFF[R, r, t]
intgrndEEE[R, r, t]

```

Case (bb): $2*R < r$

$$\text{One finds that } \sqrt{-r^2 + 4 R^2 + r^2 t^2} \rightarrow r \sqrt{t^2 - b^2} \text{ with } b \rightarrow \frac{\sqrt{r^2 - 4 R^2}}{r}$$

For the FF using the identity 3.152.9 del G&R one gets

$$\begin{aligned}
(* b < u < 1 = a *) \int_b^u \frac{1}{\sqrt{(1-t^2)(-r^2+4R^2+r^2t^2)}} dt &\rightarrow \int_b^u \frac{1}{r \sqrt{(1-t^2)(t^2-b^2)}} dt \rightarrow \\
&\frac{1}{r} F\left[\text{ArcSin}\left[\frac{1}{u} \sqrt{\frac{u^2-b^2}{1-b^2}}\right], \sqrt{1-b^2}\right] \rightarrow \frac{1}{r} \text{EllipticF}\left[\text{ArcSin}\left[\frac{1}{u} \sqrt{\frac{u^2-b^2}{1-b^2}}\right], 1-b^2\right]
\end{aligned}$$

For the EE using the equation 3.169 .11 of G&R

$$\begin{aligned}
\int_b^u \frac{\sqrt{-r^2 + 4 R^2 + r^2 t^2}}{\sqrt{1-t^2}} dt &\rightarrow \int_b^u \frac{r \sqrt{-b^2 + t^2}}{\sqrt{1-t^2}} dt \rightarrow \\
r \int_b^u \frac{\sqrt{-b^2 + t^2}}{\sqrt{1-t^2}} dt &\rightarrow r \left(E\left[\text{ArcSin}\left[\frac{1}{u} \sqrt{\frac{u^2-b^2}{1-b^2}}\right], \sqrt{1-b^2}\right] - \right.
\end{aligned}$$

$$\begin{aligned}
& b^2 F \left[\text{ArcSin} \left[\frac{1}{u} \sqrt{\frac{u^2 - b^2}{1 - b^2}} \right], \sqrt{1 - b^2} \right] - \frac{1}{u} \sqrt{(1 - u^2) (u^2 - b^2)} \Bigg) \rightarrow \\
& r \left(\text{EllipticE} \left[\text{ArcSin} \left[\frac{1}{u} \sqrt{\frac{u^2 - b^2}{1 - b^2}} \right], 1 - b^2 \right] - \right. \\
& \left. b^2 \text{EllipticF} \left[\text{ArcSin} \left[\frac{1}{u} \sqrt{\frac{u^2 - b^2}{1 - b^2}} \right], 1 - b^2 \right] - \frac{1}{u} \sqrt{(1 - u^2) (u^2 - b^2)} \right) \\
\\
\text{prmtvFFFbb[R_, r_, u_]} & := \\
& \text{FullSimplify} \left[\left(\frac{1}{r} \text{EllipticF} \left[\text{ArcSin} \left[\frac{1}{u} \sqrt{\frac{u^2 - b^2}{1 - b^2}} \right], 1 - b^2 \right] \right) / . \left\{ b \rightarrow \frac{\sqrt{r^2 - 4 R^2}}{r} \right\}, \right. \\
& \left. \text{Assumptions} \rightarrow \left\{ 0 < 2 * R < r \& \& \frac{\sqrt{r^2 - 4 R^2}}{r} < u < 1 \right\} \right]; \\
\\
\text{prmtvEEEbb[R_, r_, u_]} & := \\
& \text{FullSimplify} \left[\left(\left(r \left(\text{EllipticE} \left[\text{ArcSin} \left[\frac{1}{u} \sqrt{\frac{u^2 - b^2}{1 - b^2}} \right], 1 - b^2 \right] - b^2 \text{EllipticF} \left[\right. \right. \right. \right. \right. \right. \\
& \left. \left. \left. \left. \left. \left. \text{ArcSin} \left[\frac{1}{u} \sqrt{\frac{u^2 - b^2}{1 - b^2}} \right], 1 - b^2 \right] - \frac{1}{u} \sqrt{(1 - u^2) (u^2 - b^2)} \right) \right) / . \right. \right. \\
& \left. \left. \left. \left. \left. \left. \left\{ b \rightarrow \frac{\sqrt{r^2 - 4 R^2}}{r} \right\} \right), \text{Assumptions} \rightarrow \left\{ 0 < 2 * R < r \& \& \frac{\sqrt{r^2 - 4 R^2}}{r} < u < 1 \right\} \right] \right] \right];
\end{aligned}$$

$$\begin{aligned}
& \text{EllipticF} \left[\text{ArcCsc} \left[\frac{2 R t}{\sqrt{4 R^2 + r^2 (-1 + t^2)}} \right], \frac{4 R^2}{r^2} \right] \\
\text{prmtvFFFbb[R_, r_, t_]} & := \frac{\text{prmtvEEEbb[R_, r_, t_]}}{r}; \\
\text{prmtvEEEbb[R_, r_, t_]} & := \frac{1}{r t} \\
& \left(-r \sqrt{-(-1 + t^2) (4 R^2 + r^2 (-1 + t^2))} + r^2 t \text{EllipticE} \left[\text{ArcCsc} \left[\frac{2 R t}{\sqrt{4 R^2 + r^2 (-1 + t^2)}} \right], \frac{4 R^2}{r^2} \right] - \right. \\
& \left. (r^2 - 4 R^2) t \text{EllipticF} \left[\text{ArcCsc} \left[\frac{2 R t}{\sqrt{4 R^2 + r^2 (-1 + t^2)}} \right], \frac{4 R^2}{r^2} \right] \right);
\end{aligned}$$

CHECKS on the derivatives

```

FullSimplify[
  FullSimplify[D[prmtvFFFbb[R, r, t], t], Assumptions -> {0 < 2*R < r &&  $\frac{\sqrt{r^2 - 4 R^2}}{r} < t < 1}]]

intgrndFFF[R, r, t], Assumptions -> {0 < 2*R < r &&  $\frac{\sqrt{r^2 - 4 R^2}}{r} < t < 1}\]

FullSimplify[
  FullSimplify[D[prmtvEEEbb[R, r, t], t], Assumptions -> {0 < 2*R < r &&  $\frac{\sqrt{r^2 - 4 R^2}}{r} < t < 1}]]

intgrndEEE[R, r, t], Assumptions -> {0 < 2*R < r &&  $\frac{\sqrt{r^2 - 4 R^2}}{r} < t < 1}\]$$$$ 
```

The Needle cases

CASE AA $0 < r < 2*R < H$

```

FullSimplify[FullSimplify[primitvAA[R, H, r, t] +
  rmngprmtv[R, H, r, t] +  $\left( -\frac{16 H \pi R^2 (r^2 - 4 R^2)}{3 r} \right) * prmtvFFFaa[R, r, t] +$ 
   $\left( -\frac{4 H \pi (r^2 + 4 R^2)}{3 r} \right) * prmtvEEEaa[R, r, t], Assumptions -> \{0 < r < 2*R < H \&& 0 < t < 1\}] -$ 
  PrmtvCFNdlAA[R, H, r, t],
  Assumptions -> {-1 < t < 1 && R > 0 && r > 0 && 4 R^2 + r^2 (-1 + t^2) > 0 && 1 - t^2 > 0}\]

```

0

```

PrmtvCFNdlAA[R_, H_, r_, t_] :=
  -  $\frac{1}{6 r^2} \pi \left( r \left( r \left( -12 \pi R^2 (r + 2 H t - r t^2) + \sqrt{\frac{1 - t^2}{4 R^2 + r^2 (-1 + t^2)}} (24 R^4 -$ 
     $16 H r R^2 t + 4 H r^3 t (-3 + t^2) - 6 r^2 R^2 (-1 + t^2) - 3 r^4 (-1 + t^2)^2) \right) -$ 
     $24 R^2 (R^2 - 2 H r t + r^2 (-1 + t^2)) \text{ArcSin}\left[\frac{r \sqrt{1 - t^2}}{2 R}\right]\right) +$ 
   $16 H r R (r^2 + 4 R^2) \text{EllipticE}\left[\text{ArcSin}\left[\frac{2 R t}{\sqrt{4 R^2 + r^2 (-1 + t^2)}}\right], \frac{r^2}{4 R^2}\right] +$ 
   $32 H R^2 (r^2 - 4 R^2) \text{EllipticF}\left[\text{ArcSin}\left[\frac{r t}{\sqrt{4 R^2 + r^2 (-1 + t^2)}}\right], \frac{4 R^2}{r^2}\right];$ 

```

Derivative check

$$\text{Simplify}\left[\left(\text{FullSimplify}\left[\text{D}[\text{PrmtvCFNdlAA}[R, H, r, t], t] - \text{integrndTot}[R, H, r, t]\right], \text{Assumptions} \rightarrow \{0 < r < 2 * R < H \& 0 < t < 1 \& 0 < t^2 < 1 \& 4 R^2 + r^2 (-1 + t^2) > 0\}\right]\right] / .$$

$$\left\{\sqrt{-(-1 + t^2)^3 (4 R^2 + r^2 (-1 + t^2))} \rightarrow (1 - t^2)^2 \sqrt{-(-1 + t^2) (4 R^2 + r^2 (-1 + t^2))}\right\}$$

evaluation of the CF

$$\begin{aligned} &\text{Simplify}\left[\left(\text{FullSimplify}\left[\text{Limit}\left[\left(\text{FullSimplify}\left[\text{ExpandAll}\left[\left(\text{PrmtvCFNdlAA}[R, H, r, t]\right)\right] / . \{t \rightarrow 1 - \epsilon\}\right]\right], \text{Assumptions} \rightarrow \{0 < r < 2 * R < H \& 0 < \epsilon < r < 2 * R < H\}\}, \epsilon \rightarrow 0, \text{Direction} \rightarrow -1\right], \text{Assumptions} \rightarrow \{R > 0\}\right] - \text{Limit}\left[\text{PrmtvCFNdlAA}[R, H, r, t], t \rightarrow 0, \text{Direction} \rightarrow -1\right], \text{Assumptions} \rightarrow \{0 < r < 2 * R < H\}\right] / . \left\{\text{ArcCsc}\left[\frac{2 R}{r}\right] \rightarrow \text{ArcSin}\left[\frac{r}{2 R}\right], \text{ArcSec}\left[\frac{2 R}{r}\right] \rightarrow \left(\frac{\pi}{2} - \text{ArcSin}\left[\frac{r}{2 R}\right]\right)\right\}\right] \\ &\text{Apart}\left[-\frac{1}{6 r} \pi \left(-3 \left(r \left(8 H \pi R^2 - 4 \pi r R^2 + (r^2 + 2 R^2) \sqrt{-r^2 + 4 R^2}\right) + 8 R^2 (r^2 - R^2) \text{ArcSin}\left[\frac{r}{2 R}\right]\right) + 16 H R (r^2 + 4 R^2) \text{EllipticE}\left[\frac{r^2}{4 R^2}\right] + 16 H R (r^2 - 4 R^2) \text{EllipticK}\left[\frac{r^2}{4 R^2}\right]\right)\right] \\ &\text{Simplify}\left[\text{FullSimplify}\left[\text{Together}\left[-2 \pi^2 (-2 H + r) R^2 + \frac{1}{2} \pi r^2 \sqrt{-r^2 + 4 R^2} + \pi R^2 \sqrt{-r^2 + 4 R^2}\right]\right] + \frac{1}{3 r} 4 \pi R \text{Simplify}\left[\left(3 r^2 R \text{ArcSin}\left[\frac{r}{2 R}\right] - 3 R^3 \text{ArcSin}\left[\frac{r}{2 R}\right] - 2 H r^2 \text{EllipticE}\left[\frac{r^2}{4 R^2}\right] - 8 H R^2 \text{EllipticE}\left[\frac{r^2}{4 R^2}\right] - 2 H r^2 \text{EllipticK}\left[\frac{r^2}{4 R^2}\right] + 8 H R^2 \text{EllipticK}\left[\frac{r^2}{4 R^2}\right]\right] - \text{CFNdlAA}[R, H, r]\right] \end{aligned}$$

$$\begin{aligned} \text{CFNdlAA}[R_, H_, r_] := & 2 \pi^2 R^2 (2 H - r) + \frac{\pi (r^2 + 2 R^2)}{2} \sqrt{-r^2 + 4 R^2} + \frac{4 \pi R}{3 r} \left(3 R (r^2 - R^2) \text{ArcSin}\left[\frac{r}{2 R}\right] - 2 H (r^2 + 4 R^2) \text{EllipticE}\left[\frac{r^2}{4 R^2}\right] - 2 H (r^2 - 4 R^2) \text{EllipticK}\left[\frac{r^2}{4 R^2}\right]\right); \end{aligned}$$

$$\begin{aligned} &\text{FullSimplify}\left[\text{Limit}\left[\text{CFNdlAA}[R, H, r], r \rightarrow 0, \text{Direction} \rightarrow -1\right] / (4 \pi), \text{Assumptions} \rightarrow \{0 < 2 * R < H\}\right] \\ &\text{FullSimplify}\left[\text{Limit}\left[\text{CFNdlAA}[R, H, r], r \rightarrow 2 * R, \text{Direction} \rightarrow 1\right], \text{Assumptions} \rightarrow \{0 < 2 * R < H\}\right] \\ &\text{FullSimplify}\left[\text{Series}\left[\text{CFNdlAA}[R, H, \delta], \{\delta, 0, 6\}\right], \text{Assumptions} \rightarrow \{R > 0 \& H > 0 \& 2 * R < H\}\right] \end{aligned}$$

$$\begin{aligned} \text{SerisCFNdlAALwrBnd}[R_, H_, \delta_] := & 4 H \pi^2 R^2 - 2 (\pi^2 R (H + R)) \delta + \frac{8}{3} \pi R \delta^2 + \frac{H \pi^2 \delta^3}{16 R} - \frac{\pi \delta^4}{15 R} + \frac{H \pi^2 \delta^5}{512 R^3}; \end{aligned}$$

$$\begin{aligned} &\text{FullSimplify}\left[\text{Series}\left[\text{FullSimplify}\left[\text{ExpandAll}\left[\left(\text{CFNdlAA}[R, H, r]\right)\right] / . \{r \rightarrow (2 * R - \delta)\}\right], \text{Assumptions} \rightarrow \{R > 0 \& H > 0 \& 2 * R < H \& \delta > 0\}, \{\delta, 0, 4\}\right], \text{Assumptions} \rightarrow \{R > 0 \& H > 0 \& 2 * R < H \& \delta > 0\}\right] \end{aligned}$$

$$\begin{aligned}
\text{SerisCFNdlAAUpBnd}[R_-, H_-, \delta_-] := & -\frac{1}{3} \pi R^2 (4 H (8 - 3 \pi) + 3 \pi R) + \\
& \frac{1}{6} \pi R (16 H - 3 \pi R) \delta + \left(-\frac{\pi^2 R}{4} + H \pi \left(-\frac{7}{6} + \text{Log}[16] \right) + H \pi \text{Log}\left[\frac{R}{\delta}\right] \right) \delta^2 + \\
& \frac{32}{15} \pi \sqrt{R} \delta^{5/2} + \frac{\pi \left(-3 \pi R + H (-13 + 24 \text{Log}[2]) + 6 H \text{Log}\left[\frac{R}{\delta}\right] \right) \delta^3}{24 R} + \\
& \frac{4 \pi \delta^{7/2}}{21 \sqrt{R}} + \frac{\pi \left(-96 \pi R + H (-371 + 720 \text{Log}[2]) + 180 H \text{Log}\left[\frac{R}{\delta}\right] \right) \delta^4}{1536 R^2};
\end{aligned}$$

CASE BB: $0 < 2*R < r < H$

$$\begin{aligned}
\text{Simplify} \left[\left(\text{FullSimplify} [\text{primitvAA}[R, H, r, t] + \right. \right. \\
& \text{rmngprmtv}[R, H, r, t] + \left(-\frac{16 H \pi R^2 (r^2 - 4 R^2)}{3 r} \right) * \text{prmtvFFFbb}[R, r, t] + \\
& \left. \left. \left(-\frac{4 H \pi (r^2 + 4 R^2)}{3 r} \right) * \text{prmtvEEEbb}[R, r, t], \right. \right. \\
& \left. \left. \text{Assumptions} \rightarrow \left\{ 0 < 2 * R < r < H \&& \frac{\sqrt{r^2 - 4 R^2}}{r} < t < 1 \right\} \right] \right] / . \\
& \left\{ \text{ArcCsc} \left[\frac{2 R}{r \sqrt{1 - t^2}} \right] \rightarrow \text{ArcSin} \left[\frac{r \sqrt{1 - t^2}}{2 R} \right], \text{ArcSec} \left[\frac{2 R}{r \sqrt{1 - t^2}} \right] \rightarrow \left(\frac{\pi}{2} - \text{ArcSin} \left[\frac{r \sqrt{1 - t^2}}{2 R} \right] \right) \right\}, \\
& \text{Assumptions} \rightarrow \left\{ 0 < 2 * R < r < H \&& \frac{\sqrt{r^2 - 4 R^2}}{r} < t < 1 \right\}
\end{aligned}$$

$$\begin{aligned}
\text{PrmtvCFNdlBB}[R_-, H_-, r_-, t_-] := & \frac{1}{6 r t} \pi \left(12 \pi r^2 R^2 t + \right. \\
& \sqrt{-(-1 + t^2) (4 R^2 + r^2 (-1 + t^2))} (-6 r R^2 t + 3 r^3 t (-1 + t^2) + H (32 R^2 - 4 r^2 (-2 + t^2))) + \\
& 24 R^2 t \left(-\frac{1}{2} \pi r t (-2 H + r t) + (R^2 - 2 H r t + r^2 (-1 + t^2)) \text{ArcSin} \left[\frac{r \sqrt{1 - t^2}}{2 R} \right] \right) + \\
& 8 H r t \left(-(r^2 + 4 R^2) \text{EllipticE} \left[\text{ArcSin} \left[\frac{\sqrt{4 R^2 + r^2 (-1 + t^2)}}{2 R t} \right], \frac{4 R^2}{r^2} \right] + \right. \\
& \left. \left. (r^2 - 4 R^2) \text{EllipticF} \left[\text{ArcSin} \left[\frac{\sqrt{4 R^2 + r^2 (-1 + t^2)}}{2 R t} \right], \frac{4 R^2}{r^2} \right] \right);
\end{aligned}$$

Derivative check

```
FullSimplify[D[
  PrmtvCFNdlBB[R, H, r, t], t] - integrndTot[R, H, r, t],
  Assumptions → {0 < 2 * R < r < H &&  $\frac{\sqrt{r^2 - 4 R^2}}{r} < t < 1}]]$ 
```

evaluation of the CF

```
Simplify[Simplify[Limit[PrmtvCFNdlBB[R, H, r, t], t → 1, Direction → 1] -
  Limit[PrmtvCFNdlBB[R, H, r, t], t →  $\frac{\sqrt{r^2 - 4 R^2}}{r}$ , Direction → -1],
  Assumptions → {0 < 2 * R < r < H}] - CFNdlBB[R, H, r]]]
```

$$\text{CFNdlBB}[R_-, H_-, r_-] := \frac{2 \pi^2 R^2 (2 H r - R^2)}{r} - \frac{4 \pi H (r^2 + 4 R^2)}{3} \text{EllipticE}\left[\frac{4 R^2}{r^2}\right] + \frac{4 \pi H (r^2 - 4 R^2)}{3} \text{EllipticK}\left[\frac{4 R^2}{r^2}\right];$$

$$\begin{aligned} \text{SerisCFNdlBBLwrBnd}[R_-, H_-, \delta_-] := \\ -\frac{1}{3} \pi R^2 (4 H (8 - 3 \pi) + 3 \pi R) + \frac{1}{6} \pi R (-16 H + 3 \pi R) \delta + \\ \left(-\frac{\pi^2 R}{4} + H \pi \left(-\frac{7}{6} + \text{Log}[16]\right) + H \pi \text{Log}\left[\frac{R}{\delta}\right]\right) \delta^2 + \frac{\pi (3 \pi R + H (13 - 24 \text{Log}[2]) + 6 H \text{Log}\left[\frac{\delta}{R}\right]) \delta^3}{24 R} + \\ \frac{\pi (-96 \pi R + H (-371 + 720 \text{Log}[2]) + 180 H \text{Log}\left[\frac{R}{\delta}\right]) \delta^4}{1536 R^2}; \end{aligned}$$

```
Simplify[FullSimplify[Series[FullSimplify[ExpandAll[(CFNdlBB[R, H, r] /. {r → 2 * R + δ})]], Assumptions → {0 < 2 * R < H && δ > 0}], {δ, 0, 4}], Assumptions → {0 < 2 * R < H && δ > 0}] - SerisCFNdlBBLwrBnd[R, H, δ], Assumptions → {0 < 2 * R < H && δ > 0}]
```

$$\begin{aligned} \text{SerisCFNdlBBUpBnd}[R_-, H_-, \delta_-] := \\ \frac{1}{3 H} 2 \pi \left(6 H^2 \pi R^2 - 3 \pi R^4 + 2 H^2 \left(-(H^2 + 4 R^2) \text{EllipticE}\left[\frac{4 R^2}{H^2}\right] + (H^2 - 4 R^2) \text{EllipticK}\left[\frac{4 R^2}{H^2}\right]\right)\right) + \\ \frac{1}{3 H^2} 2 \pi \left(-3 \pi R^4 + 4 H^2 \left((H^2 - 2 R^2) \text{EllipticE}\left[\frac{4 R^2}{H^2}\right] - (H^2 - 4 R^2) \text{EllipticK}\left[\frac{4 R^2}{H^2}\right]\right)\right) \delta + \\ \frac{1}{3 H^3} 2 \pi \left(-3 \pi R^4 + 2 H^2 \left(-(H^2 + 4 R^2) \text{EllipticE}\left[\frac{4 R^2}{H^2}\right] + (H^2 + 2 R^2) \text{EllipticK}\left[\frac{4 R^2}{H^2}\right]\right)\right) \delta^2 + \\ \frac{1}{3 H^4} \frac{2 \pi R^2}{(H^2 - 4 R^2)} \\ \left(-4 (H^4 - 8 H^2 R^2) \text{EllipticE}\left[\frac{4 R^2}{H^2}\right] + (H^2 - 4 R^2) \left(-3 \pi R^2 + 4 H^2 \text{EllipticK}\left[\frac{4 R^2}{H^2}\right]\right)\right) \delta^3; \end{aligned}$$

the limit of the cylinder of negligible diameter

```
FullSimplify[Series[CFNdlBB[R, H, r], {R, 0, 6}], Assumptions → {0 < r < H}]
```

$$\frac{2 \pi^2 (H - r) R^4}{r^2} + \frac{H \pi^2 R^6}{r^4} + O[R]^7$$

This is the sought for result. It shows that the CF behaves as $\frac{1}{r^2}$ in the limit $r \rightarrow 0$

The discontinuity of the CF as $r \rightarrow 2R$

the sign of δ must be changed in one of the two contributions because
 $r = 2R + \delta$ in the first and $r = 2R - \delta$ in the second.

In fact if one expands $f[x]=x$, using $x \rightarrow x+\delta$ one finds $x+\delta$ and using
 $x \rightarrow x-\delta$ one finds $x-\delta$. So, before subtracting the two expansion one must
change the sign of δ in one of the two expansions.

One sees that the leading term is

$$-\delta^2 H \pi \text{Log}[\delta] \text{ as } \delta \rightarrow 0^+$$

and

$$\delta^2 H \pi \text{Log}[|\delta|] \text{ as } \delta \rightarrow 0^-.$$

```
SerisCFNdlAAUpBnd[R, H, δ]
```

```
SerisCFNdlBBLwrBnd[R, H, δ]
```

```
Simplify[(SerisCFNdlBBLwrBnd[R, H, δ] - SerisCFNdlAAUpBnd[R, H, -δ])]
```

CASE CC: $0 < 2R < H < r < \sqrt{4R^2 + H^2}$

$$\begin{aligned} H < r < \sqrt{H^2 + 4R^2} \quad \& \quad \sqrt{\frac{r^2 - 4R^2}{r^2}} < t < \frac{H}{r} \\ \text{Simplify}\left[\text{Simplify}\left[\left(\text{FullSimplify}\left[\text{primitvAA}[R, H, r, t] + \right.\right.\right.\right. \right. \\ \left. \left. \left. \left. \left. \text{rmngprmtv}[R, H, r, t] + \left(-\frac{16H\pi R^2(r^2 - 4R^2)}{3r}\right) * \text{prmtvFFFbb}[R, r, t] + \right.\right.\right. \\ \left. \left. \left. \left. \left. \left(-\frac{4H\pi(r^2 + 4R^2)}{3r}\right) * \text{prmtvEEEbb}[R, r, t], \text{Assumptions} \rightarrow \right.\right.\right. \\ \left. \left. \left. \left. \left. \left\{0 < 2R < H < r < \sqrt{R^2 + H^2} \quad \& \quad \frac{\sqrt{r^2 - 4R^2}}{r} < t < \frac{H}{r}\right\}\right]\right)\right] / . \\ \left\{\text{ArcCsc}\left[\frac{2Rt}{\sqrt{4R^2 + r^2(-1 + t^2)}}\right] \rightarrow \text{ArcSin}\left[\frac{\sqrt{4R^2 + r^2(-1 + t^2)}}{2Rt}\right], \right. \\ \left. \text{ArcCsc}\left[\frac{2R}{r\sqrt{1 - t^2}}\right] \rightarrow \text{ArcSin}\left[\frac{r\sqrt{1 - t^2}}{2R}\right], \right. \\ \left. \text{ArcSec}\left[\frac{2R}{r\sqrt{1 - t^2}}\right] \rightarrow \left(\frac{\pi}{2} - \text{ArcSin}\left[\frac{r\sqrt{1 - t^2}}{2R}\right]\right)\right\} - \text{PrmtvCFNdlCC}[R, H, r, t] \end{aligned}$$

$$\begin{aligned}
\text{PrmtvCFNdlCC}[\text{R}_-, \text{H}_-, \text{r}_-, \text{t}_-] := & \frac{1}{6 \text{r} \text{t}} \pi \left(12 \pi \text{r}^2 \text{R}^2 \text{t} + \right. \\
& \sqrt{-(-1+\text{t}^2) (4 \text{R}^2 + \text{r}^2 (-1+\text{t}^2))} (-6 \text{r} \text{R}^2 \text{t} + 3 \text{r}^3 \text{t} (-1+\text{t}^2) + \text{H} (32 \text{R}^2 - 4 \text{r}^2 (-2+\text{t}^2))) + \\
& 24 \text{R}^2 \text{t} \left(-\frac{1}{2} \pi \text{r} \text{t} (-2 \text{H} + \text{r} \text{t}) + (\text{R}^2 - 2 \text{H} \text{r} \text{t} + \text{r}^2 (-1+\text{t}^2)) \text{ArcSin}\left[\frac{\text{r} \sqrt{1-\text{t}^2}}{2 \text{R}}\right] \right) + \\
& 8 \text{H} \text{r} \text{t} \left(-(\text{r}^2 + 4 \text{R}^2) \text{EllipticE}\left[\text{ArcSin}\left[\frac{\sqrt{4 \text{R}^2 + \text{r}^2 (-1+\text{t}^2)}}{2 \text{R} \text{t}}\right], \frac{4 \text{R}^2}{\text{r}^2}\right] + \right. \\
& \left. \left. (\text{r}^2 - 4 \text{R}^2) \text{EllipticF}\left[\text{ArcSin}\left[\frac{\sqrt{4 \text{R}^2 + \text{r}^2 (-1+\text{t}^2)}}{2 \text{R} \text{t}}\right], \frac{4 \text{R}^2}{\text{r}^2}\right]\right);
\end{aligned}$$

DERIVATIVE CHECK

$$\begin{aligned}
& \text{FullSimplify}[\text{FullSimplify}[\text{FullSimplify}[\text{D}[\text{PrmtvCFNdlCC}[\text{R}, \text{H}, \text{r}, \text{t}], \text{t}], \text{Assumptions} \rightarrow \\
& \left\{ 0 < 2 * \text{R} < \text{H} < \text{r} < \sqrt{4 \text{R}^2 + \text{H}^2} \&& \frac{\sqrt{\text{r}^2 - 4 \text{R}^2}}{\text{r}} < \text{t} < \frac{\text{H}}{\text{r}} \right\}] - \text{integrndTot}[\text{R}, \text{H}, \text{r}, \text{t}], \\
& \text{Assumptions} \rightarrow \left\{ 0 < 2 * \text{R} < \text{H} < \text{r} < \sqrt{4 \text{R}^2 + \text{H}^2} \&& \frac{\sqrt{\text{r}^2 - 4 \text{R}^2}}{\text{r}} < \text{t} < \frac{\text{H}}{\text{r}} \right\}, \text{Assumptions} \rightarrow \\
& \left\{ 0 < 2 * \text{R} < \text{H} < \text{r} < \sqrt{\text{R}^2 + \text{H}^2} \&& \frac{\sqrt{\text{r}^2 - 4 \text{R}^2}}{\text{r}} < \text{t} < \frac{\text{H}}{\text{r}} \&& 1 - \text{t}^2 > 0 \&& 4 \text{R}^2 + \text{r}^2 (-1 + \text{t}^2) > 0 \right\}] \\
& \text{Simplify}[\text{Simplify}[\text{Limit}[\text{PrmtvCFNdlCC}[\text{R}, \text{H}, \text{r}, \text{t}], \text{t} \rightarrow \frac{\text{H}}{\text{r}}, \text{Direction} \rightarrow 1] - \\
& \text{Limit}[\text{PrmtvCFNdlCC}[\text{R}, \text{H}, \text{r}, \text{t}], \text{t} \rightarrow \sqrt{\frac{\text{r}^2 - 4 \text{R}^2}{\text{r}^2}}, \text{Direction} \rightarrow -1], \\
& \text{Assumptions} \rightarrow \left\{ 0 < \text{R} \&& 0 < \text{H} \&& 2 * \text{R} < \text{H} \&& \text{H} < \text{r} < \sqrt{4 \text{R}^2 + \text{H}^2} \right\} - \text{CFNdlCC}[\text{R}, \text{H}, \text{r}]
\end{aligned}$$

$$\begin{aligned}
\text{CFNdlCC}[\text{R}_-, \text{H}_-, \text{r}_-] := & \frac{1}{6 \text{r}} \pi \left(12 \text{H}^2 \pi \text{R}^2 + 12 \pi \text{r}^2 \text{R}^2 - 12 \pi \text{R}^4 - \text{H}^2 \sqrt{(-\text{H}^2 + \text{r}^2) (\text{H}^2 - \text{r}^2 + 4 \text{R}^2)} + \right. \\
& 5 \text{r}^2 \sqrt{(-\text{H}^2 + \text{r}^2) (\text{H}^2 - \text{r}^2 + 4 \text{R}^2)} + 26 \text{R}^2 \sqrt{(-\text{H}^2 + \text{r}^2) (\text{H}^2 - \text{r}^2 + 4 \text{R}^2)} - \\
& 24 \text{H}^2 \text{R}^2 \text{ArcSin}\left[\frac{\sqrt{-\text{H}^2 + \text{r}^2}}{2 \text{R}}\right] - 24 \text{r}^2 \text{R}^2 \text{ArcSin}\left[\frac{\sqrt{-\text{H}^2 + \text{r}^2}}{2 \text{R}}\right] + 24 \text{R}^4 \text{ArcSin}\left[\frac{\sqrt{-\text{H}^2 + \text{r}^2}}{2 \text{R}}\right] - \\
& 8 \text{H} \text{r} (\text{r}^2 + 4 \text{R}^2) \text{EllipticE}\left[\text{ArcSin}\left[\frac{\text{r} \sqrt{\text{H}^2 - \text{r}^2 + 4 \text{R}^2}}{2 \text{H} \text{R}}\right], \frac{4 \text{R}^2}{\text{r}^2}\right] + \\
& \left. 8 \text{H} \text{r} (\text{r}^2 - 4 \text{R}^2) \text{EllipticF}\left[\text{ArcSin}\left[\frac{\text{r} \sqrt{\text{H}^2 - \text{r}^2 + 4 \text{R}^2}}{2 \text{H} \text{R}}\right], \frac{4 \text{R}^2}{\text{r}^2}\right]\right);
\end{aligned}$$

```

FullSimplify[FullSimplify[
  Limit[CFNdlCC[R, H, r], r → H, Direction → -1], Assumptions → {0 < R && 0 < H && 2 * R < H}] -
  
$$\left( \frac{1}{3H} 2\pi \left( 6H^2\pi R^2 - 3\pi R^4 + 2H^2 \left( - (H^2 + 4R^2) \text{EllipticE}\left[\frac{4R^2}{H^2}\right] + (H^2 - 4R^2) \text{EllipticK}\left[\frac{4R^2}{H^2}\right] \right) \right) \right),$$

  Assumptions → {0 < R && 0 < H && 2 * R < H}]

FullSimplify[Limit[CFNdlCC[R, H, r], r → √(H^2 + 4 * R^2), Direction → 1],
  Assumptions → {0 < R && 0 < H && 2 * R < H}]

SerisCFNdlCCUpBnd[R_, H_, δ_] := 
$$\frac{16\sqrt{2}\pi(H^2 + 4R^2)^{5/4}\delta^{7/2}}{105H^2R} + \frac{1}{8505\sqrt{2}H^8R^7}\pi(H^2 + 4R^2)^{3/4}$$


$$(H^{12} - 102H^{10}R^2 + 78H^8R^4 + 4700H^6R^6 + 18564H^4R^8 + 96H^2R^{10} + 64R^{12})\delta^{9/2} +$$


$$\frac{1}{1871100\sqrt{2}H^{10}R^9}\pi(H^2 + 4R^2)^{1/4}(161H^{16} - 16073H^{14}R^2 + 22208H^{12}R^4 + 1001914H^{10}R^6 +$$


$$2932760H^8R^8 + 2006164H^6R^{10} + 5589728H^4R^{12} + 177472H^2R^{14} + 167936R^{16})\delta^{11/2};$$


Simplify[FullSimplify[Series[FullSimplify[((CFNdlCC[R, H, r]) /. r → √(H^2 + 4 * R^2) - δ),
  Assumptions → {0 < R && 0 < H && 2 * R < H && δ > 0}], {δ, 0, 6}],
  Assumptions → {0 < R && 0 < H && 2 * R < H && δ > 0}] - SerisCFNdlCCUpBnd[R, H, δ]]

Simplify[FullSimplify[Series[FullSimplify[((CFNdlCC[R, H, r]) /. r → H + δ),
  Assumptions → {0 < R && 0 < H && 2 * R < H && δ > 0}], {δ, 0, 4}],
  Assumptions → {0 < R && 0 < H && 2 * R < H && δ > 0}] - SerisCFNdlCCLwrBnd[R, H, δ]]

SerisCFNdlCCLwrBnd[R_, H_, δ_] :=

$$\frac{1}{3H} 2\pi \left( 6H^2\pi R^2 - 3\pi R^4 + 2H^2 \left( - (H^2 + 4R^2) \text{EllipticE}\left[\frac{4R^2}{H^2}\right] + (H^2 - 4R^2) \text{EllipticK}\left[\frac{4R^2}{H^2}\right] \right) \right) +$$


$$\frac{1}{3H^2} 2\pi \left( 3\pi R^4 + 4H^2 \left( - (H^2 - 2R^2) \text{EllipticE}\left[\frac{4R^2}{H^2}\right] + (H^2 - 4R^2) \text{EllipticK}\left[\frac{4R^2}{H^2}\right] \right) \right) \delta + \frac{1}{3H^3} 2\pi$$


$$\left( 3\pi(H - R)R^2(H + R) + 2H^2 \left( - (H^2 + 4R^2) \text{EllipticE}\left[\frac{4R^2}{H^2}\right] + (H^2 + 2R^2) \text{EllipticK}\left[\frac{4R^2}{H^2}\right] \right) \right) \delta^2 -$$


$$\frac{32(\sqrt{2}\pi R)\delta^{5/2}}{15\sqrt{H}} + \frac{1}{3H^4(H^2 - 4R^2)} 2\pi R^2 \left( 4(H^4 - 8H^2R^2) \text{EllipticE}\left[\frac{4R^2}{H^2}\right] -$$


$$(H - 2R)(H + 2R) \left( 3\pi(H - R)(H + R) + 4H^2 \text{EllipticK}\left[\frac{4R^2}{H^2}\right] \right) \delta^3 -$$


$$\frac{2(\sqrt{2}\pi(H^8 + 20H^6R^2 - 87H^4R^4 - 790H^2R^6 - 8R^8))\delta^{7/2}}{945(H^{7/2}R^5)} + \frac{1}{3H^5(H^2 - 4R^2)^2} 2\pi R^2$$


$$\left( 3\pi(H - R)(H + R)(H^2 - 4R^2)^2 + H^2 \left( (-3H^4 + 52H^2R^2 - 128R^4) \text{EllipticE}\left[\frac{4R^2}{H^2}\right] + \right.$$


$$\left. (3H^4 - 28H^2R^2 + 64R^4) \text{EllipticK}\left[\frac{4R^2}{H^2}\right] \right) \delta^4;$$


```

SOME PLOTS

```

NeedlIntPlt =
With[{H = 10, R = 1}, ParametricPlot[{{r, r^2 * CFNdlAA[R, H, r]}}, {r, 0, 2 R}, PlotStyle ->
{Thickness[0.004], Blue}, AspectRatio -> 1, AxesLabel -> {"r", "r^2 \gamma(r)"}]];

NeedlMedPlt = With[{H = 10, R = 1}, ParametricPlot[{{r, r^2 * CFNdlBB[R, H, r]}}, {r, 2 R, H}, PlotStyle -> {Thickness[0.002], Magenta}, AspectRatio -> 1]];

NeedlOutrPlt = With[{H = 10, R = 1}, ParametricPlot[{{r, r^2 * CFNdlCC[R, H, r]}}, {r, H, Sqrt[4 R^2 + H^2]}, PlotStyle -> {Thickness[0.006], Green}, AspectRatio -> 1]];

Show[NeedlIntPlt, NeedlMedPlt, NeedlOutrPlt, PlotRange -> {{0, 11}, {0, 250}}]

Show[NeedlIntPlt, NeedlMedPlt, PlotRange -> {{1.8, 2.2}, {0, 230}}];

Show[NeedlMedPlt, NeedlOutrPlt, PlotRange -> {{9.8, 10.5}, {0, 3}}];

```

1st ORDER DERIVATIVE IN THE NEEDLE CASE

```

DNeedlIntPlt = With[{H = 10, R = 1}, ParametricPlot[{{r, DCFNdlAA[R, H, r]}}, {r, 0, 2 R}, PlotStyle -> {Thickness[0.004], Blue}, AspectRatio -> 1, AxesLabel -> {"r", "\gamma'(r)"}];

DNeedlMedPlt = With[{H = 10, R = 1}, ParametricPlot[{{r, DCFNdlBB[R, H, r]}}, {r, 2 R, H}, PlotStyle -> {Thickness[0.002], Magenta}, AspectRatio -> 1];

DNeedlOutrPlt =
With[{H = 10, R = 1}, ParametricPlot[{{r, DCFNdlCC[R, H, r]}}, {r, H, Sqrt[4 R^2 + H^2]}, PlotStyle -> {Thickness[0.006], Green}, AspectRatio -> 1, PlotPoints -> 100]];

Show[DNeedlIntPlt, DNeedlMedPlt, DNeedlOutrPlt, PlotRange -> {{0, 11}, {-270, 5}}]

```

2nd ORDER DERIVATIVE IN THE NEEDLE CASE

```

DDNeedlIntPlt = With[{H = 10, R = 1}, ParametricPlot[
{{r, DDCFNdlAA[R, H, r]}}, {r, 10^(-2), 2 R}, PlotStyle -> {Thickness[0.004], Blue}, AspectRatio -> 1, AxesLabel -> {"r", "\gamma''(r)"}, PlotPoints -> 200]];
DDNeedlMedPlt = With[{H = 10, R = 1}, ParametricPlot[{{r, DDCFNdlBB[R, H, r]}}, {r, 2 R, H}, PlotStyle -> {Thickness[0.002], Magenta}, PlotPoints -> 2000, AspectRatio -> 1]];
DDNeedlOutrPlt = With[{H = 10, R = 1}, ParametricPlot[{{r, DDCFNdlCC[R, H, r]}}, {r, H, Sqrt[4 R^2 + H^2]}, PlotStyle -> {Thickness[0.006], Green}, PlotPoints -> 200, AspectRatio -> 1]];

Show[DDNeedlIntPlt, DDNeedlMedPlt, PlotRange -> {{1.9, 2.1}, {0, 1000}}]

Show[DDNeedlMedPlt, DDNeedlOutrPlt, PlotRange -> {{9.7, 10.5}, {-0.1, 5}}]

Show[DDNeedlIntPlt, DDNeedlMedPlt, DDNeedlOutrPlt, PlotRange -> {{0, 11}, {0, 250}}]

```

first derivatives

```

FullSimplify[D[CFNdlAA[R, H, r], r], Assumptions -> {0 < r < 2 R < H}]
FullSimplify[D[CFNdlBB[R, H, r], r], Assumptions -> {0 < 2 R < r < H}]
FullSimplify[D[CFNdlCC[R, H, r], r], Assumptions -> {0 < 2 R < H < r < Sqrt[4 R^2 + H^2]}]

```

second derivatives

```
FullSimplify[D[DCFNdlAA[R, H, r], r], Assumptions -> {0 < r < 2 R < H}]
```

```
FullSimplify[D[DCFNd1BB[R, H, r], r], Assumptions -> {0 < 2 R < r < H}]
```

```
FullSimplify[D[DCFNd1CC[R, H, r], r], Assumptions -> {0 < 2 R < H < r < Sqrt[4 R^2 + H^2]}]
```

FIRST AND SECOND DERIVATIVE EXPRESSIONS NEEDLE Case

$$\begin{aligned}
& \text{DCFNd1AA}[R_-, H_-, r_-] := \\
& \frac{1}{6 r^2} \pi \left(3 \left(8 R^4 \text{ArcCsc}\left[\frac{2 R}{r}\right] + r \left((3 r^2 - 2 R^2) \sqrt{-r^2 + 4 R^2} - 8 r R^2 \text{ArcSec}\left[\frac{2 R}{r}\right] \right) \right) + \\
& 16 H R \left(-2 (r^2 - 2 R^2) \text{EllipticE}\left[\frac{r^2}{4 R^2}\right] + (r^2 - 4 R^2) \text{EllipticK}\left[\frac{r^2}{4 R^2}\right] \right); \text{DCFNd1BB}[R_-, H_-, \\
& r_-] := \frac{1}{3 r^2} 2 \pi \left(3 \pi R^4 - 4 H r (r^2 - 2 R^2) \text{EllipticE}\left[\frac{4 R^2}{r^2}\right] + 4 H r (r^2 - 4 R^2) \text{EllipticK}\left[\frac{4 R^2}{r^2}\right] \right); \\
& \text{DCFNd1CC}[R_-, H_-, r_-] := \frac{1}{6 r^2} \pi \left((H^2 + 7 r^2 - 26 R^2) \sqrt{-(H - r) (H + r) (H^2 - r^2 + 4 R^2)} + \right. \\
& 24 R^2 (-H^2 + r^2 + R^2) \text{ArcSec}\left[\frac{2 R}{\sqrt{-H^2 + r^2}}\right] + \\
& 16 H r \left(- (r^2 - 2 R^2) \text{EllipticE}\left[\text{ArcCsc}\left[\frac{2 H R}{r \sqrt{H^2 - r^2 + 4 R^2}}\right], \frac{4 R^2}{r^2}\right] + \right. \\
& \left. \left. (r^2 - 4 R^2) \text{EllipticF}\left[\text{ArcCsc}\left[\frac{2 H R}{r \sqrt{H^2 - r^2 + 4 R^2}}\right], \frac{4 R^2}{r^2}\right] \right); \\
& \text{DDCFNd1AA}[R_-, H_-, r_-] := \frac{1}{3 r^3} \pi \left(3 r (3 r^2 + 2 R^2) \sqrt{-r^2 + 4 R^2} - 24 R^4 \text{ArcCsc}\left[\frac{2 R}{r}\right] + \right. \\
& 8 H R \left(-2 (r^2 + 4 R^2) \text{EllipticE}\left[\frac{r^2}{4 R^2}\right] + (r^2 + 8 R^2) \text{EllipticK}\left[\frac{r^2}{4 R^2}\right] \right); \text{DDCFNd1BB}[R_-, H_-, \\
& r_-] := -\frac{1}{3 r^3} 4 \pi \left(3 \pi R^4 + 2 H r (r^2 + 4 R^2) \text{EllipticE}\left[\frac{4 R^2}{r^2}\right] - 2 H r (r^2 + 2 R^2) \text{EllipticK}\left[\frac{4 R^2}{r^2}\right] \right); \\
& \text{DDCFNd1CC}[R_-, H_-, r_-] := \left(\pi \left((H - r) (H + r) (H^2 + r^2 - 26 R^2) (H^2 - r^2 + 4 R^2) + \right. \right. \\
& 24 (H - R) R^2 (H + R) \sqrt{-(H - r) (H + r) (H^2 - r^2 + 4 R^2)} \text{ArcSec}\left[\frac{2 R}{\sqrt{-H^2 + r^2}}\right] + \\
& 8 H r \sqrt{-(H - r) (H + r) (H^2 - r^2 + 4 R^2)} \\
& \left. \left. \left(- (r^2 + 4 R^2) \text{EllipticE}\left[\text{ArcCsc}\left[\frac{2 H R}{r \sqrt{H^2 - r^2 + 4 R^2}}\right], \frac{4 R^2}{r^2}\right] + (r^2 + 2 R^2) \text{EllipticF}\left[\text{ArcCsc}\left[\frac{2 H R}{r \sqrt{H^2 - r^2 + 4 R^2}}\right], \frac{4 R^2}{r^2}\right] \right) \right) / \left(3 r^3 \sqrt{(-H^2 + r^2) (H^2 - r^2 + 4 R^2)} \right);
\end{aligned}$$

value of the discontinuity as $r \rightarrow 2 R$

```

FullSimplify[
Series[Simplify[(DDCFNd1BB[R, H, r]) /. {r → 2 R + δ}, Assumptions → {0 < δ < 2 * R < H}], {δ, 0, 0}], Assumptions → {0 < δ < 2 * R < H}]

FullSimplify[
Series[Simplify[(DDCFNd1AA[R, H, r]) /. {r → 2 R - δ}, Assumptions → {0 < δ < 2 * R < H}], {δ, 0, 0}], Assumptions → {0 < δ < 2 * R < H}]

```

LEADING ASYMPTOTIC TERM AS $r \rightarrow 2R$ in the NEEDLE case.

$$\begin{aligned}
\text{RightLimitCFNd12R}[\delta] &:= -\frac{1}{6} \pi (3 \pi R - 16 H (-2 + \log[8]) + 12 H \log[\delta] - 12 H \log[R]); \\
\text{LeftLimitCFNd12R}[\delta] &:= \\
&\quad -\frac{1}{6} \pi (3 \pi R - 16 H (-2 + \log[8]) + 12 H \log[|\delta|] - 12 H \log[R]) + 8 \pi \sqrt{R} \sqrt{|\delta|};
\end{aligned}$$

REMARK: the right and left limit do not exist!.

```

Limit[DDCFNd1BB[R, H, r], r → 2 R, Direction → -1]

Limit[-\frac{4 \pi \left(3 \pi R^4 + 2 H r \left(r^2 + 4 R^2\right) \text{EllipticE}\left[\frac{4 R^2}{r^2}\right] - 2 H r \left(r^2 + 2 R^2\right) \text{EllipticK}\left[\frac{4 R^2}{r^2}\right]\right)}{3 r^3},
r → 2 R, Direction → -1]

Limit[DDCFNd1AA[R, H, r], r → 2 R, Direction → 1]

DirectedInfinity[H]

```

value of the discontinuity as $r \rightarrow H$

```

Limit[DDCFNd1CC[R, H, r], r → H, Direction → -1]

FullSimplify[(Limit[DDCFNd1CC[R, H, r], r → H, Direction → -1] -
Limit[DDCFNd1BB[R, H, r], r → H, Direction → 1]), Assumptions → {0 < 2 R < H}]

Limit[DDCFNd1CC[R, H, r], r → H, Direction → -1]

FullSimplify[
Series[Simplify[(DDCFNd1CC[R, H, r]) /. {r → H + δ}, Assumptions → {0 < δ < 2 * R < H}], {δ, 0, 0}] - Series[Simplify[(DDCFNd1BB[R, H, r]) /. {r → H - δ}, Assumptions → {0 < δ < 2 * R < H}], {δ, 0, 0}], Assumptions → {0 < 2 R < H}]

```

The Disc cases

CASE AA $0 < r < H < 2R$

```

Simplify[FullSimplify[primitvAA[R, H, r, t] +
rmngprmtv[R, H, r, t] + \left(-\frac{16 H \pi R^2 (r^2 - 4 R^2)}{3 r}\right) * prmtvFFFaa[R, r, t] +
\left(-\frac{4 H \pi (r^2 + 4 R^2)}{3 r}\right) * prmtvEEEaa[R, r, t], Assumptions → {0 < r < H < 2 * R && 0 < t < 1}] -
PrmtvCFDskAA[R, H, r, t]]

```

$$\begin{aligned}
\text{PrmtvCFDskAA}[\text{R}_-, \text{H}_-, \text{r}_-, \text{t}_-] := & \\
& \frac{1}{6 \text{r}} \pi \left(12 \pi \text{R}^4 + \text{r} \left(3 (-2 \text{R}^2 + \text{r}^2 (-1 + \text{t}^2)) \sqrt{-(-1 + \text{t}^2) (4 \text{R}^2 + \text{r}^2 (-1 + \text{t}^2))} + \right. \right. \\
& \left. \left. 4 \text{H} \text{r} \text{t} \left(2 (\text{r}^2 + 4 \text{R}^2) \sqrt{\frac{1 - \text{t}^2}{4 \text{R}^2 + \text{r}^2 (-1 + \text{t}^2)}} - \sqrt{-(-1 + \text{t}^2) (4 \text{R}^2 + \text{r}^2 (-1 + \text{t}^2))} \right) \right) - \\
& 24 \text{R}^2 (\text{R}^2 - 2 \text{H} \text{r} \text{t} + \text{r}^2 (-1 + \text{t}^2)) \text{ArcSec}\left[\frac{2 \text{R}}{\text{r} \sqrt{1 - \text{t}^2}}\right] + \\
& 16 \text{H} \text{R} \left(-(\text{r}^2 + 4 \text{R}^2) \text{EllipticE}\left[\text{ArcSin}\left[\frac{2 \text{R} \text{t}}{\sqrt{4 \text{R}^2 + \text{r}^2 (-1 + \text{t}^2)}}\right], \frac{\text{r}^2}{4 \text{R}^2}\right] - \right. \\
& \left. (\text{r}^2 - 4 \text{R}^2) \text{EllipticF}\left[\text{ArcSin}\left[\frac{2 \text{R} \text{t}}{\sqrt{4 \text{R}^2 + \text{r}^2 (-1 + \text{t}^2)}}\right], \frac{\text{r}^2}{4 \text{R}^2}\right] \right);
\end{aligned}$$

Derivative check

$$\begin{aligned}
& \text{Simplify}\left[(\text{FullSimplify}[D[\text{PrmtvCFDskAA}[\text{R}, \text{H}, \text{r}, \text{t}], \text{t}] - \text{integrndTot}[\text{R}, \text{H}, \text{r}, \text{t}], \right. \\
& \quad \text{Assumptions} \rightarrow \{0 < \text{r} < \text{H} < 2 * \text{R} \& 0 < \text{t} < 1 \& 0 < \text{t}^2 < 1 \& 4 \text{R}^2 + \text{r}^2 (-1 + \text{t}^2) > 0\}) / . \\
& \quad \left. \left\{ \sqrt{-(-1 + \text{t}^2)^3 (4 \text{R}^2 + \text{r}^2 (-1 + \text{t}^2))} \rightarrow (1 - \text{t}^2) \sqrt{-(-1 + \text{t}^2) (4 \text{R}^2 + \text{r}^2 (-1 + \text{t}^2))} \right\} \right] \\
& \text{Simplify}\left[\text{Simplify}\left[\left(\text{FullSimplify}[\text{Limit}[\text{PrmtvCFDskAA}[\text{R}, \text{H}, \text{r}, \text{t}], \text{t} \rightarrow 1, \text{Direction} \rightarrow 1] - \text{Limit}[\right. \right. \right. \\
& \quad \text{PrmtvCFDskAA}[\text{R}, \text{H}, \text{r}, \text{t}], \text{t} \rightarrow 0, \text{Direction} \rightarrow -1], \text{Assumptions} \rightarrow \\
& \quad \{0 < \text{r} < \text{H} < 2 * \text{R}\}) / . \left\{ \text{ArcCsc}\left[\frac{2 \text{R}}{\text{r}}\right] \rightarrow \text{ArcSin}\left[\frac{\text{r}}{2 \text{R}}\right], \text{ArcSec}\left[\frac{2 \text{R}}{\text{r}}\right] \rightarrow \frac{\pi}{2} - \text{ArcSin}\left[\frac{\text{r}}{2 \text{R}}\right] \right\}, \\
& \quad \left. \text{Assumptions} \rightarrow \{0 < \text{r} < \text{H} < 2 * \text{R}\} \right] - \text{CFDskAA}[\text{R}, \text{H}, \text{r}], \text{Assumptions} \rightarrow \{0 < \text{r} < \text{H} < 2 * \text{R}\} \left. \right]
\end{aligned}$$

$$\begin{aligned}
\text{CFDskAANotNorm}[\text{R}_-, \text{H}_-, \text{r}_-] := & \\
& -\frac{1}{6 \text{r}} \pi \left(-3 \left(\text{r} \left(8 \text{H} \pi \text{R}^2 - 4 \pi \text{r} \text{R}^2 + (\text{r}^2 + 2 \text{R}^2) \sqrt{-\text{r}^2 + 4 \text{R}^2} \right) + 8 \text{R}^2 (\text{r}^2 - \text{R}^2) \text{ArcSin}\left[\frac{\text{r}}{2 \text{R}}\right] \right) + \right. \\
& \left. 16 \text{H} \text{R} (\text{r}^2 + 4 \text{R}^2) \text{EllipticE}\left[\frac{\text{r}^2}{4 \text{R}^2}\right] + 16 \text{H} \text{R} (\text{r}^2 - 4 \text{R}^2) \text{EllipticK}\left[\frac{\text{r}^2}{4 \text{R}^2}\right] \right);
\end{aligned}$$

$$\begin{aligned}
& \text{Simplify}[\text{Limit}[\text{CFDskAANotNorm}[\text{R}, \text{H}, \text{r}], \text{r} \rightarrow 0, \text{Direction} \rightarrow -1], \\
& \quad \text{Assumptions} \rightarrow \{0 < \text{r} < \text{H} < 2 \text{R}\}]
\end{aligned}$$

The above function coincides with that relevant to the Needle case in the range $0 < \text{r} < \text{Min}[\text{H}, 2 * \text{R}]$.

$$\begin{aligned}
\text{SerisCFDskAALwrBnd}[\text{R}_-, \text{H}_-, \delta_-] := & \\
& 4 \text{H} \pi^2 \text{R}^2 - 2 (\pi^2 \text{R} (\text{H} + \text{R})) \delta + \frac{8}{3} \pi \text{R} \delta^2 + \frac{\text{H} \pi^2 \delta^3}{16 \text{R}} - \frac{\pi \delta^4}{15 \text{R}} + \frac{\text{H} \pi^2 \delta^5}{512 \text{R}^3};
\end{aligned}$$

$$\begin{aligned}
& \text{Simplify}[\text{FullSimplify}[\text{Series}[\text{CFDskAANotNorm}[\text{R}, \text{H}, \delta], \{\delta, 0, 5\}], \\
& \quad \text{Assumptions} \rightarrow \{\text{R} > 0 \& \text{H} > 0 \& 2 * \text{R} > \text{H}\}] - \text{SerisCFDskAALwrBnd}[\text{R}, \text{H}, \delta]
\end{aligned}$$

```

SerisCFDskAAUpBnd[R_, H_, δ_] :=

$$\frac{1}{6 H} \pi \left( 3 \left( H \left( 4 H \pi R^2 + H^2 \sqrt{-H^2 + 4 R^2} + 2 R^2 \sqrt{-H^2 + 4 R^2} \right) + 8 (H - R) R^2 (H + R) \text{ArcCsc}\left[\frac{2 R}{H}\right] \right) - \right.$$


$$16 H R (H^2 + 4 R^2) \text{EllipticE}\left[\frac{H^2}{4 R^2}\right] - 16 H R (H^2 - 4 R^2) \text{EllipticK}\left[\frac{H^2}{4 R^2}\right] \Big) +$$


$$\frac{1}{6 H^2} \pi \left( -24 R^4 \text{ArcCsc}\left[\frac{2 R}{H}\right] + 3 H \left( (-3 H^2 + 2 R^2) \sqrt{-H^2 + 4 R^2} + 8 H R^2 \text{ArcSec}\left[\frac{2 R}{H}\right] \right) + \right.$$


$$32 H R (H^2 - 2 R^2) \text{EllipticE}\left[\frac{H^2}{4 R^2}\right] - 16 H R (H^2 - 4 R^2) \text{EllipticK}\left[\frac{H^2}{4 R^2}\right] \Big) \delta +$$


$$\frac{1}{6 H^3} \pi \left( 3 H (3 H^2 + 2 R^2) \sqrt{-H^2 + 4 R^2} - 24 R^4 \text{ArcCsc}\left[\frac{2 R}{H}\right] - \right.$$


$$16 H R (H^2 + 4 R^2) \text{EllipticE}\left[\frac{H^2}{4 R^2}\right] + 8 H R (H^2 + 8 R^2) \text{EllipticK}\left[\frac{H^2}{4 R^2}\right] \Big) \delta^2 +$$


$$\frac{1}{6 H^4 (H^2 - 4 R^2)} \pi \left( H \sqrt{-H^2 + 4 R^2} (-3 H^4 + 2 H^2 R^2 - 24 R^4) - 24 R^4 (H^2 - 4 R^2) \text{ArcCsc}\left[\frac{2 R}{H}\right] + \right.$$


$$32 R^3 \left( - (H^3 - 8 H R^2) \text{EllipticE}\left[\frac{H^2}{4 R^2}\right] + 2 H (H^2 - 4 R^2) \text{EllipticK}\left[\frac{H^2}{4 R^2}\right] \right) \Big) \delta^3 +$$


$$\frac{1}{3 H^5 (H^2 - 4 R^2)^2} \pi R^2 \left( -H \sqrt{-H^2 + 4 R^2} (H^4 + 16 H^2 R^2 - 48 R^4) - 12 R^2 (H^2 - 4 R^2)^2 \text{ArcCsc}\left[\frac{2 R}{H}\right] - \right.$$


$$4 (3 H^5 R - 52 H^3 R^3 + 128 H R^5) \text{EllipticE}\left[\frac{H^2}{4 R^2}\right] +$$


$$4 (9 H^5 R - 68 H^3 R^3 + 128 H R^5) \text{EllipticK}\left[\frac{H^2}{4 R^2}\right] \Big) \delta^4; SerisCFDskAAUpBnd[R, H, δ];$$


```

```

Simplify[
FullSimplify[Series[FullSimplify[ExpandAll[(CFDskAANotNorm[R, H, r]) /. {r → (H - δ)}]], Assumptions → {R > 0 && H > 0 && 2 * R > H && δ > 0}], {δ, 0, 4}],
Assumptions → {R > 0 && H > 0 && 2 * R > H && δ > 0}] - SerisCFDskAAUpBnd[R, H, δ]

```

CASE BB: $0 < H < r < 2 * R$ $\&\&$ $0 < H < r < 2 R$ and $0 < t < \frac{H}{r}$

```

FullSimplify[FullSimplify[primitvAA[R, H, r, t] +
rmngprmtv[R, H, r, t] +  $\left( -\frac{16 H \pi R^2 (r^2 - 4 R^2)}{3 r} \right) * \text{prmtvFFFaa}[R, r, t] +$ 
 $\left( -\frac{4 H \pi (r^2 + 4 R^2)}{3 r} \right) * \text{prmtvEEEaa}[R, r, t], \text{Assumptions} \rightarrow$ 
 $\left\{ 0 < H < r < 2 * R \&\& 0 < t < \frac{H}{r} \&\& 4 R^2 + r^2 (-1 + t^2) > 0 \&\& 1 - t^2 > 0 \right\} -$ 
PrmtvCFDskBB[R, H, r, t], \text{Assumptions} \rightarrow
 $\left\{ 0 < H < r < 2 * R \&\& 0 < t < \frac{H}{r} \&\& 4 R^2 + r^2 (-1 + t^2) > 0 \&\& 1 - t^2 > 0 \right\} ]$ 

```

$$\begin{aligned}
\text{PrmtvCFDskBB}[\text{R}_-, \text{H}_-, \text{r}_-, \text{t}_-] := & \\
& \frac{1}{6 \text{r}} \pi \left(12 \pi \text{R}^4 + \text{r} \sqrt{\frac{1 - \text{t}^2}{4 \text{R}^2 + \text{r}^2 (-1 + \text{t}^2)}} \right) \left(-24 \text{R}^4 + 16 \text{H} \text{r} \text{R}^2 \text{t} - 4 \text{H} \text{r}^3 \text{t} (-3 + \text{t}^2) + 6 \text{r}^2 \text{R}^2 (-1 + \text{t}^2) + \right. \\
& \left. 3 \text{r}^4 (-1 + \text{t}^2)^2 \right) - 24 \text{R}^2 \left(\text{R}^2 - 2 \text{H} \text{r} \text{t} + \text{r}^2 (-1 + \text{t}^2) \right) \text{ArcSec} \left[\frac{2 \text{R}}{\text{r} \sqrt{1 - \text{t}^2}} \right] + \\
& 16 \text{H} \text{R} \left(\left(-\text{r}^2 - 4 \text{R}^2 \right) \text{EllipticE} \left[\text{ArcSin} \left[\frac{2 \text{R} \text{t}}{\sqrt{4 \text{R}^2 + \text{r}^2 (-1 + \text{t}^2)}} \right], \frac{\text{r}^2}{4 \text{R}^2} \right] - \right. \\
& \left. \left. \left(\text{r}^2 - 4 \text{R}^2 \right) \text{EllipticF} \left[\text{ArcSin} \left[\frac{2 \text{R} \text{t}}{\sqrt{4 \text{R}^2 + \text{r}^2 (-1 + \text{t}^2)}} \right], \frac{\text{r}^2}{4 \text{R}^2} \right] \right);
\end{aligned}$$

Derivative check

```

FullSimplify[D[
  PrmtvCFDskBB[R, H, r, t], t] - integrndTot[R, H, r, t],
Assumptions -> {0 < H < r < 2 * R && 0 < t < H/r}]
FullSimplify[Limit[PrmtvCFDskBB[R, H, r, t], t -> H/r, Direction -> 1] -
Limit[PrmtvCFDskBB[R, H, r, t], t -> 0, Direction -> -1], Assumptions -> {0 < H < r < 2 * R}]
FullSimplify[FullSimplify[Limit[PrmtvCFDskBB[R, H, r, t], t -> H/r, Direction -> 1] -
Limit[PrmtvCFDskBB[R, H, r, t], t -> 0, Direction -> -1], Assumptions -> {0 < H < r < 2 * R}] -
CFDskBB[R, H, r], Assumptions -> {0 < H < r < 2 * R}]
Clear[CFDskBOLD]; Clear[CFDskBB];

```

$$\begin{aligned}
\text{CFDskBOLD}[\text{R}_-, \text{H}_-, \text{r}_-] := & \\
& \frac{1}{6 \text{r}} \pi \left(12 \pi \text{R}^4 + \sqrt{\frac{-\text{H}^2 + \text{r}^2}{\text{H}^2 - \text{r}^2 + 4 \text{R}^2}} \right) \left(-\text{H}^4 + \text{H}^2 (6 \text{r}^2 + 22 \text{R}^2) + 3 (\text{r}^4 - 2 \text{r}^2 \text{R}^2 - 8 \text{R}^4) \right) - \\
& 3 \left(4 \pi \text{R}^4 - \text{r} (\text{r}^2 + 2 \text{R}^2) \sqrt{-\text{r}^2 + 4 \text{R}^2} + 8 (\text{r} - \text{R}) \text{R}^2 (\text{r} + \text{R}) \text{ArcCos} \left[\frac{\text{r}}{2 \text{R}} \right] \right) + 24 \text{R}^2 (\text{H}^2 + \text{r}^2 - \text{R}^2) \\
& \text{ArcCos} \left[\frac{\sqrt{-\text{H}^2 + \text{r}^2}}{2 \text{R}} \right] + 16 \text{H} \text{R} \left(-(\text{r}^2 + 4 \text{R}^2) \text{EllipticE} \left[\text{ArcSin} \left[\frac{2 \text{H} \text{R}}{\text{r} \sqrt{\text{H}^2 - \text{r}^2 + 4 \text{R}^2}} \right], \frac{\text{r}^2}{4 \text{R}^2} \right] - \right. \\
& \left. \left. \frac{2 \text{R} (\text{r}^2 - 4 \text{R}^2) \text{EllipticF} \left[\text{ArcSin} \left[\frac{\text{H}}{\sqrt{\text{H}^2 - \text{r}^2 + 4 \text{R}^2}} \right], \frac{4 \text{R}^2}{\text{r}^2} \right]}{\text{r}} \right) \right);
\end{aligned}$$

The elliptic integrals F has modulus equal $2R/r$, which is greater than one. One uses eq.s 8.127 of G&R to transfrom the into the elliptic F-integral with modulus smaller than 1.

The transformation is

$$\{\text{EllipticF}\left[\text{ArcSin}\left[\frac{H}{\sqrt{H^2 - r^2 + 4 R^2}}\right], \frac{4 R^2}{r^2}\right] \rightarrow \left(\frac{r}{2 R} \text{EllipticF}\left[\text{ArcSin}\left[\frac{2 R H}{r \sqrt{H^2 - r^2 + 4 R^2}}\right], \frac{r^2}{4 R^2}\right]\right)\}$$

checks

$$\begin{aligned} & \text{FullSimplify}\left[\text{EllipticF}\left[\text{ArcSin}\left[\frac{H}{\sqrt{H^2 - r^2 + 4 R^2}}\right], \frac{4 R^2}{r^2}\right] - \right. \\ & \left. \left(\frac{r}{2 R} \text{EllipticF}\left[\text{ArcSin}\left[\frac{2 R H}{r \sqrt{H^2 - r^2 + 4 R^2}}\right], \frac{r^2}{4 R^2}\right]\right), \text{Assumptions} \rightarrow \{0 < H < r < 2 R\} \right] \\ & \text{With}\left[\{H = 1, R = 3\}, \text{ParametricPlot}\left[\left\{r, \text{EllipticF}\left[\text{ArcSin}\left[\frac{H}{\sqrt{H^2 - r^2 + 4 R^2}}\right], \frac{4 R^2}{r^2}\right]\right\} - \right. \\ & \left. \left(\frac{r}{2 R} \text{EllipticF}\left[\text{ArcSin}\left[\frac{2 R H}{r \sqrt{H^2 - r^2 + 4 R^2}}\right], \frac{r^2}{4 R^2}\right]\right), \{r, H, 2 R\}, \right. \\ & \left. \text{PlotRange} \rightarrow \{\{H, 2 R\}, \{-10^{(-15)}, 10^{(-15)}\}\}, \text{AspectRatio} \rightarrow 1, \text{PlotPoints} \rightarrow 500\right] \\ & (\text{CFDskBBOLD}[R, H, r]) /. \\ & \left\{\text{EllipticF}\left[\text{ArcSin}\left[\frac{H}{\sqrt{H^2 - r^2 + 4 R^2}}\right], \frac{4 R^2}{r^2}\right] \rightarrow \left(\frac{r}{2 R} \text{EllipticF}\left[\text{ArcSin}\left[\frac{2 R H}{r \sqrt{H^2 - r^2 + 4 R^2}}\right], \frac{r^2}{4 R^2}\right]\right)\right\} \end{aligned}$$

$$\begin{aligned} & \text{CFDskBBNotNorm}[R, H, r] := \\ & \frac{1}{6 r} \pi \left(12 \pi R^4 + \sqrt{\frac{-H^2 + r^2}{H^2 - r^2 + 4 R^2}} (-H^4 + H^2 (6 r^2 + 22 R^2) + 3 (r^4 - 2 r^2 R^2 - 8 R^4)) - \right. \\ & 3 \left(4 \pi R^4 - r (r^2 + 2 R^2) \sqrt{-r^2 + 4 R^2} + 8 (r - R) R^2 (r + R) \text{ArcCos}\left[\frac{r}{2 R}\right] \right) + 24 R^2 (H^2 + r^2 - R^2) \\ & \text{ArcCos}\left[\frac{\sqrt{-H^2 + r^2}}{2 R}\right] + 16 H R \left((-r^2 - 4 R^2) \text{EllipticE}\left[\text{ArcSin}\left[\frac{2 H R}{r \sqrt{H^2 - r^2 + 4 R^2}}\right], \frac{r^2}{4 R^2}\right] - \right. \\ & \left. \left. (r^2 - 4 R^2) \text{EllipticF}\left[\text{ArcSin}\left[\frac{2 H R}{r \sqrt{H^2 - r^2 + 4 R^2}}\right], \frac{r^2}{4 R^2}\right]\right); \end{aligned}$$

$$\text{FullSimplify}[\text{CFDskBBOLD}[R, H, r] - \text{CFDskBBNotNorm}[R, H, r], \text{Assumptions} \rightarrow \{0 < H < r < 2 R\}]$$

$$\begin{aligned} & \text{FullSimplify}[\\ & \text{Series}[\text{Simplify}[(\text{CFDskBBNotNorm}[R, H, r]) /. \{r \rightarrow H + \delta\}, \text{Assumptions} \rightarrow \{\delta > 0 \&& 0 < H < 2 R\}], \\ & \{\delta, 0, 2\}], \text{Assumptions} \rightarrow \{\delta > 0 \&& 0 < H < 2 R\}] \end{aligned}$$

$$\begin{aligned}
& \text{SeriesCFDskBBLwrBnd}[R_, H_, \delta_] := \\
& \frac{1}{6 H} \pi \left(3 \left(H \left(4 H \pi R^2 + H^2 \sqrt{-H^2 + 4 R^2} + 2 R^2 \sqrt{-H^2 + 4 R^2} \right) + 8 (H - R) R^2 (H + R) \text{ArcCsc}\left[\frac{2 R}{H}\right] \right) - \right. \\
& 16 H R (H^2 + 4 R^2) \text{EllipticE}\left[\frac{H^2}{4 R^2}\right] - 16 H R (H^2 - 4 R^2) \text{EllipticK}\left[\frac{H^2}{4 R^2}\right] \Big) + \\
& \frac{1}{6 H^2} \pi \left(3 \left(4 \pi R^4 + H (3 H^2 - 2 R^2) \sqrt{-H^2 + 4 R^2} - 8 R^2 (H^2 + R^2) \text{ArcSec}\left[\frac{2 R}{H}\right] \right) - \right. \\
& 32 H R (H^2 - 2 R^2) \text{EllipticE}\left[\frac{H^2}{4 R^2}\right] + 16 H R (H^2 - 4 R^2) \text{EllipticK}\left[\frac{H^2}{4 R^2}\right] \Big) \delta + \\
& \frac{1}{6 H^3} \pi \left(3 \left(4 H^2 \pi R^2 + 3 H^3 \sqrt{-H^2 + 4 R^2} + 2 H R^2 \sqrt{-H^2 + 4 R^2} - 8 R^4 \text{ArcCsc}\left[\frac{2 R}{H}\right] \right) - \right. \\
& 16 H R (H^2 + 4 R^2) \text{EllipticE}\left[\frac{H^2}{4 R^2}\right] + 8 H R (H^2 + 8 R^2) \text{EllipticK}\left[\frac{H^2}{4 R^2}\right] \Big) \delta^2;
\end{aligned}$$

Check of the continuity at $r = H$

```
Simplify[FullSimplify[Limit[CFDskBBNotNorm[R, H, r], r → H, Direction → -1], Assumptions → {0 < H < 2 R}] - FullSimplify[Limit[CFDskAANotNorm[R, H, r], r → H, Direction → 1], Assumptions → {0 < H < 2 R}]]
```

UPPER BOUND BEHAVIOUR

After applying the transformation $\{r \rightarrow 2*R - \delta\}$, MATHEMATICA is not able to work out the series expansion around $\{\delta = 0\}$.

The problem lies in the elliptic integrals E and F because the moduli approach to 1.

The limit value of the second derivatives are discussed later

CASE CC: $[0 < H < 2*R < r < \sqrt{4 R^2 + H^2}]$

$$\begin{aligned}
& \text{Simplify}\left[\text{Simplify}\left[\text{FullSimplify}\left[\text{primitvAA}[R, H, r, t] + \right. \right. \right. \\
& \text{rmngprmtv}[R, H, r, t] + \left(-\frac{16 H \pi R^2 (r^2 - 4 R^2)}{3 r} \right) * \text{prmtvFFFbb}[R, r, t] + \\
& \left. \left. \left. \left(-\frac{4 H \pi (r^2 + 4 R^2)}{3 r} \right) * \text{prmtvEEEbb}[R, r, t], \text{Assumptions} \rightarrow \right. \right. \right. \\
& \left. \left. \left. \left\{ 0 < H < 2 * R < r < \sqrt{4 R^2 + H^2} \& \frac{\sqrt{r^2 - 4 R^2}}{r} < t < \frac{H}{r} \right\} \right] \right] / . \\
& \left\{ \text{ArcCsc}\left[\frac{2 R}{r \sqrt{1 - t^2}}\right] \rightarrow \text{ArcSin}\left[\frac{r \sqrt{1 - t^2}}{2 R}\right], \text{ArcSec}\left[\frac{2 R}{r \sqrt{1 - t^2}}\right] \rightarrow \frac{\pi}{2} - \text{ArcSin}\left[\frac{r \sqrt{1 - t^2}}{2 R}\right], \right. \\
& \left. \left. \left. \text{ArcCsc}\left[\frac{2 R t}{\sqrt{4 R^2 + r^2 (-1 + t^2)}}\right] \rightarrow \text{ArcSin}\left[\frac{\sqrt{4 R^2 + r^2 (-1 + t^2)}}{2 R t}\right]\right] \right] - \text{PrmtvCFDskCC}[R, H, r, t], \\
& \text{Assumptions} \rightarrow \left\{ 0 < H < 2 * R < r < \sqrt{4 R^2 + H^2} \& \frac{\sqrt{r^2 - 4 R^2}}{r} < t < \frac{H}{r} \right\}
\end{aligned}$$

$$\begin{aligned}
\text{PrmtvCFDskCC}[\text{R}_-, \text{H}_-, \text{r}_-, \text{t}_-] := & \frac{1}{6 \text{r} \text{t}} \pi \left(12 \pi \text{r}^2 \text{R}^2 \text{t} + \right. \\
& \sqrt{-(-1+\text{t}^2) (4 \text{R}^2 + \text{r}^2 (-1+\text{t}^2))} (-6 \text{r} \text{R}^2 \text{t} + 3 \text{r}^3 \text{t} (-1+\text{t}^2) + \text{H} (32 \text{R}^2 - 4 \text{r}^2 (-2+\text{t}^2))) + \\
& 24 \text{R}^2 \text{t} \left(-\frac{1}{2} \pi \text{r} \text{t} (-2 \text{H} + \text{r} \text{t}) + (\text{R}^2 - 2 \text{H} \text{r} \text{t} + \text{r}^2 (-1+\text{t}^2)) \text{ArcSin}\left[\frac{\text{r} \sqrt{1-\text{t}^2}}{2 \text{R}}\right] \right) + \\
& 8 \text{H} \text{r} \text{t} \left(-(\text{r}^2 + 4 \text{R}^2) \text{EllipticE}\left[\text{ArcSin}\left[\frac{\sqrt{4 \text{R}^2 + \text{r}^2 (-1+\text{t}^2)}}{2 \text{R} \text{t}}\right], \frac{4 \text{R}^2}{\text{r}^2}\right] + \right. \\
& \left. \left. (\text{r}^2 - 4 \text{R}^2) \text{EllipticF}\left[\text{ArcSin}\left[\frac{\sqrt{4 \text{R}^2 + \text{r}^2 (-1+\text{t}^2)}}{2 \text{R} \text{t}}\right], \frac{4 \text{R}^2}{\text{r}^2}\right]\right);
\end{aligned}$$

DERIVATIVE CHECK

$$\begin{aligned}
& \text{FullSimplify}[\text{FullSimplify}[\text{FullSimplify}[\text{D}[\text{PrmtvCFDskCC}[\text{R}, \text{H}, \text{r}, \text{t}], \text{t}], \text{Assumptions} \rightarrow \\
& \left\{ 0 < \text{H} < 2 * \text{R} < \text{r} < \sqrt{4 \text{R}^2 + \text{H}^2} \& \& \frac{\sqrt{\text{r}^2 - 4 \text{R}^2}}{\text{r}} < \text{t} < \frac{\text{H}}{\text{r}} \right\}] - \text{integrndTot}[\text{R}, \text{H}, \text{r}, \text{t}], \\
& \text{Assumptions} \rightarrow \left\{ 0 < \text{H} < 2 * \text{R} < \text{r} < \sqrt{4 \text{R}^2 + \text{H}^2} \& \& \frac{\sqrt{\text{r}^2 - 4 \text{R}^2}}{\text{r}} < \text{t} < \frac{\text{H}}{\text{r}} \right\}, \text{Assumptions} \rightarrow \\
& \left\{ 0 < \text{H} < 2 * \text{R} < \text{r} < \sqrt{4 \text{R}^2 + \text{H}^2} \& \& \frac{\sqrt{\text{r}^2 - 4 \text{R}^2}}{\text{r}} < \text{t} < \frac{\text{H}}{\text{r}} \& \& 1 - \text{t}^2 > 0 \& \& 4 \text{R}^2 + \text{r}^2 (-1 + \text{t}^2) > 0 \right\} \\
& \text{Simplify}[\text{Simplify}[\text{Limit}[\text{PrmtvCFDskCC}[\text{R}, \text{H}, \text{r}, \text{t}], \text{t} \rightarrow \frac{\text{H}}{\text{r}}, \text{Direction} \rightarrow 1]] - \\
& \text{Limit}[\text{PrmtvCFDskCC}[\text{R}, \text{H}, \text{r}, \text{t}], \text{t} \rightarrow \sqrt{\frac{\text{r}^2 - 4 \text{R}^2}{\text{r}^2}}, \text{Direction} \rightarrow -1], \\
& \text{Assumptions} \rightarrow \left\{ 0 < \text{H} < 2 * \text{R} < \text{r} < \sqrt{4 \text{R}^2 + \text{H}^2} \& \& 2 * \text{R} < \text{r} < \sqrt{4 \text{R}^2 + \text{H}^2} \right\} - \\
& \text{CFDskCC}[\text{R}, \text{H}, \text{r}], \text{Assumptions} \rightarrow \left\{ 0 < \text{H} < 2 * \text{R} < \text{r} < \sqrt{4 \text{R}^2 + \text{H}^2} \& \& 2 * \text{R} < \text{r} < \sqrt{4 \text{R}^2 + \text{H}^2} \right\}]
\end{aligned}$$

$$\begin{aligned}
\text{CFDskCC}[\text{R}_-, \text{H}_-, \text{r}_-] := & \frac{1}{6 \text{r}} \pi \left(12 \text{H}^2 \pi \text{R}^2 + 12 \pi \text{r}^2 \text{R}^2 - 12 \pi \text{R}^4 - \text{H}^2 \sqrt{(-\text{H}^2 + \text{r}^2) (\text{H}^2 - \text{r}^2 + 4 \text{R}^2)} + \right. \\
& 5 \text{r}^2 \sqrt{(-\text{H}^2 + \text{r}^2) (\text{H}^2 - \text{r}^2 + 4 \text{R}^2)} + 26 \text{R}^2 \sqrt{(-\text{H}^2 + \text{r}^2) (\text{H}^2 - \text{r}^2 + 4 \text{R}^2)} - \\
& 24 \text{H}^2 \text{R}^2 \text{ArcSin}\left[\frac{\sqrt{-\text{H}^2 + \text{r}^2}}{2 \text{R}}\right] - 24 \text{r}^2 \text{R}^2 \text{ArcSin}\left[\frac{\sqrt{-\text{H}^2 + \text{r}^2}}{2 \text{R}}\right] + 24 \text{R}^4 \text{ArcSin}\left[\frac{\sqrt{-\text{H}^2 + \text{r}^2}}{2 \text{R}}\right] - \\
& 8 \text{H} \text{r} (\text{r}^2 + 4 \text{R}^2) \text{EllipticE}\left[\text{ArcSin}\left[\frac{\text{r} \sqrt{\text{H}^2 - \text{r}^2 + 4 \text{R}^2}}{2 \text{H} \text{R}}\right], \frac{4 \text{R}^2}{\text{r}^2}\right] + \\
& \left. 8 \text{H} \text{r} (\text{r}^2 - 4 \text{R}^2) \text{EllipticF}\left[\text{ArcSin}\left[\frac{\text{r} \sqrt{\text{H}^2 - \text{r}^2 + 4 \text{R}^2}}{2 \text{H} \text{R}}\right], \frac{4 \text{R}^2}{\text{r}^2}\right]\right);
\end{aligned}$$

BEHAVIOUR AROUND THE LOWER r VALUE in the DISK CC case

this case is discussed later

BEHAVIOUR AROUND THE UPPER r VALUE in the DISK CC case

$$\text{FullSimplify}\left[\text{FullSimplify}\left[\text{Series}\left[\text{FullSimplify}\left[\left((\text{CFDskCC}[R, H, r]) / . r \rightarrow \sqrt{H^2 + 4 * R^2} - \delta\right), \{\delta, 0, 4\}\right], \{\delta, 0, 4\}\right], \{\delta, 0, 4\}\right] - \text{SerisCFDskCCUpBnd}[R, H, \delta]\right]$$

$$-\frac{2 \left(\sqrt{2} \pi \left(H^2+4 R^2\right)^{9/4} \left(H^6-18 H^4 R^2+3 H^2 R^4+4 R^6\right)\right) \delta^{7/2}}{945 \left(H^6 R^5\right)} + O[\delta]^{9/2}$$

$$\text{SerisCFDskCCUpBnd}[R, H, \delta] := \frac{16 \sqrt{2} \pi \left(H^2+4 R^2\right)^{5/4} \delta^{7/2}}{105 H^2 R} + \frac{1}{8505 \sqrt{2} H^8 R^7} \pi \left(H^2+4 R^2\right)^{3/4}$$

$$\left(H^{12}-102 H^{10} R^2+78 H^8 R^4+4700 H^6 R^6+18564 H^4 R^8+96 H^2 R^{10}+64 R^{12}\right) \delta^{9/2} +$$

$$\frac{1}{1871100 \sqrt{2} H^{10} R^9} \pi \left(H^2+4 R^2\right)^{1/4} \left(161 H^{16}-16073 H^{14} R^2+22208 H^{12} R^4+1001914 H^{10} R^6+\right.$$

$$\left.2932760 H^8 R^8+2006164 H^6 R^{10}+5589728 H^4 R^{12}+177472 H^2 R^{14}+167936 R^{16}\right) \delta^{11/2};$$

$$\text{FullSimplify}\left[\text{FullSimplify}\left[\text{Series}\left[\text{FullSimplify}\left[\left((\text{CFDskCC}[R, H, r]) / . r \rightarrow \sqrt{H^2 + 4 * R^2} - \delta\right), \{\delta, 0, 6\}\right], \{\delta, 0, 6\}\right], \{\delta, 0, 6\}\right] - \text{SerisCFDskCCUpBnd}[R, H, \delta]\right]$$

EVALUATION OF THE DERIVATIVES in the DISK CASE

$$\text{FullSimplify}[D[\text{CFDskAA}[R, H, r], r] - \text{DCFDskAA}[R, H, r], \text{Assumptions} \rightarrow \{0 < r < H < 2 R\}]$$

$$\text{FullSimplify}[D[\text{DCFDskAA}[R, H, r], r] - \text{DDCFDskAA}[R, H, r], \text{Assumptions} \rightarrow \{0 < r < H < 2 R\}]$$

$$\text{FullSimplify}[D[\text{CFDskBB}[R, H, r], r] - \text{DCFDskBB}[R, H, r], \text{Assumptions} \rightarrow \{0 < H < r < 2 R \&& H^2 - r^2 + 4 R^2 > 0 \&& -H^2 + r^2 > 0\}]$$

$$\text{FullSimplify}[D[\text{DCFDskBB}[R, H, r], r] - \text{DDCFDskBB}[R, H, r], \text{Assumptions} \rightarrow \{0 < H < r < 2 R \&& H^2 - r^2 + 4 R^2 > 0 \&& -H^2 + r^2 > 0\}]$$

$$\text{FullSimplify}[D[\text{CFDskCC}[R, H, r], r] - \text{DCFDskCC}[R, H, r], \text{Assumptions} \rightarrow \{0 < H < 2 R < r < \sqrt{H^2 + 4 R^2}\}]$$

$$\text{FullSimplify}[D[\text{DCFDskCC}[R, H, r], r] - \text{DDCFDskCC}[R, H, r], \text{Assumptions} \rightarrow \{0 < H < 2 R < r < \sqrt{H^2 + 4 R^2}\}]$$

```

DCFDSkAA[R_, H_, r_] :=


$$\frac{1}{6 r^2} \pi \left( 3 \left( 8 R^4 \text{ArcCsc}\left[\frac{2 R}{r}\right] + r \left( (3 r^2 - 2 R^2) \sqrt{-r^2 + 4 R^2} - 8 r R^2 \text{ArcSec}\left[\frac{2 R}{r}\right] \right) \right) +$$


$$16 H R \left( -2 (r^2 - 2 R^2) \text{EllipticE}\left[\frac{r^2}{4 R^2}\right] + (r^2 - 4 R^2) \text{EllipticK}\left[\frac{r^2}{4 R^2}\right] \right);$$


DCFDSkBB[R_, H_, r_] := -  $\frac{1}{6 r^3 (-H^2 + r^2 - 4 R^2)} \pi$ 


$$\left( r \left( 9 H^2 r^3 \sqrt{-r^2 + 4 R^2} - 9 r^5 \sqrt{-r^2 + 4 R^2} - 6 H^2 r R^2 \sqrt{-r^2 + 4 R^2} +\right.$$


$$42 r^3 R^2 \sqrt{-r^2 + 4 R^2} - 24 r R^4 \sqrt{-r^2 + 4 R^2} + H^4 \sqrt{(-H^2 + r^2) (H^2 - r^2 + 4 R^2)} +$$


$$6 H^2 r^2 \sqrt{(-H^2 + r^2) (H^2 - r^2 + 4 R^2)} + 9 r^4 \sqrt{(-H^2 + r^2) (H^2 - r^2 + 4 R^2)} -$$


$$22 H^2 R^2 \sqrt{(-H^2 + r^2) (H^2 - r^2 + 4 R^2)} - 42 r^2 R^2 \sqrt{(-H^2 + r^2) (H^2 - r^2 + 4 R^2)} +$$


$$24 R^4 \sqrt{(-H^2 + r^2) (H^2 - r^2 + 4 R^2)} - 24 R^2 (r^2 + R^2) (H^2 - r^2 + 4 R^2) \text{ArcSec}\left[\frac{2 R}{r}\right] -$$


$$24 R^2 (H^4 + r^4 - 3 r^2 R^2 - 4 R^4 + H^2 (-2 r^2 + 3 R^2)) \text{ArcSec}\left[\frac{2 R}{\sqrt{-H^2 + r^2}}\right] \right) -$$


$$32 H r R (r^2 - 2 R^2) (H^2 - r^2 + 4 R^2) \text{EllipticE}\left[\text{ArcSin}\left[\frac{2 H R}{r \sqrt{H^2 - r^2 + 4 R^2}}\right], \frac{r^2}{4 R^2}\right] -$$


$$32 H R^2 (-r^2 + 4 R^2) (H^2 - r^2 + 4 R^2) \text{EllipticF}\left[\text{ArcSin}\left[\frac{H}{\sqrt{H^2 - r^2 + 4 R^2}}\right], \frac{4 R^2}{r^2}\right];$$


DCFDSkCC[R_, H_, r_] :=


$$\frac{1}{6 r^2} \pi \left( (H^2 + 7 r^2 - 26 R^2) \sqrt{-(H - r) (H + r) (H^2 - r^2 + 4 R^2)} + 24 R^2 (-H^2 + r^2 + R^2)$$


$$\text{ArcSec}\left[\frac{2 R}{\sqrt{-H^2 + r^2}}\right] + 16 H r \left( - (r^2 - 2 R^2) \text{EllipticE}\left[\text{ArcCsc}\left[\frac{2 H R}{r \sqrt{H^2 - r^2 + 4 R^2}}\right], \frac{4 R^2}{r^2}\right] +\right.$$


$$\left. (r^2 - 4 R^2) \text{EllipticF}\left[\text{ArcCsc}\left[\frac{2 H R}{r \sqrt{H^2 - r^2 + 4 R^2}}\right], \frac{4 R^2}{r^2}\right] \right);$$


```

$$\begin{aligned}
\text{DDCFDskAA}[\text{R}_-, \text{H}_-, \text{r}_-] &:= \frac{1}{3 \text{r}^3} \pi \left(3 \text{r} (3 \text{r}^2 + 2 \text{R}^2) \sqrt{-\text{r}^2 + 4 \text{R}^2} - 24 \text{R}^4 \text{ArcCsc}\left[\frac{2 \text{R}}{\text{r}}\right] + \right. \\
&\quad \left. 8 \text{H} \text{R} \left(-2 (\text{r}^2 + 4 \text{R}^2) \text{EllipticE}\left[\frac{\text{r}^2}{4 \text{R}^2}\right] + (\text{r}^2 + 8 \text{R}^2) \text{EllipticK}\left[\frac{\text{r}^2}{4 \text{R}^2}\right] \right) \right); \\
\text{DDCFDskBB}[\text{R}_-, \text{H}_-, \text{r}_-] &:= \frac{1}{3 \text{r}^4 (-\text{H}^2 + \text{r}^2 - 4 \text{R}^2)} \pi \left(\text{r} \left(\text{H}^4 \sqrt{(-\text{H}^2 + \text{r}^2) (\text{H}^2 - \text{r}^2 + 4 \text{R}^2)} - \right. \right. \\
&\quad \left. \left. 3 (3 \text{r}^4 - 10 \text{r}^2 \text{R}^2 - 8 \text{R}^4) \left(-\text{r} \sqrt{-\text{r}^2 + 4 \text{R}^2} + \sqrt{-(\text{H} - \text{r}) (\text{H} + \text{r}) (\text{H}^2 - \text{r}^2 + 4 \text{R}^2)} \right) - \right. \right. \\
&\quad \left. \left. \text{H}^2 \left(9 \text{r}^3 \sqrt{-\text{r}^2 + 4 \text{R}^2} + 2 \text{R}^2 \left(3 \text{r} \sqrt{-\text{r}^2 + 4 \text{R}^2} + 11 \sqrt{(-\text{H}^2 + \text{r}^2) (\text{H}^2 - \text{r}^2 + 4 \text{R}^2)} \right) \right) + \right. \right. \\
&\quad \left. \left. 24 \text{R}^2 (\text{H}^2 - \text{r}^2 + 4 \text{R}^2) \left(-\text{R}^2 \text{ArcSec}\left[\frac{2 \text{R}}{\text{r}}\right] + (-\text{H}^2 + \text{R}^2) \text{ArcSec}\left[\frac{2 \text{R}}{\sqrt{-\text{H}^2 + \text{r}^2}}\right] \right) \right) + \right. \\
&\quad \left. 16 \text{H} \text{R} (\text{H}^2 - \text{r}^2 + 4 \text{R}^2) \left(\text{r}^2 \text{R} \text{EllipticE}\left[\text{ArcSin}\left[\frac{\text{H}}{\sqrt{\text{H}^2 - \text{r}^2 + 4 \text{R}^2}}\right], \frac{4 \text{R}^2}{\text{r}^2}\right] + \right. \right. \\
&\quad \left. \left. (\text{r}^2 + 2 \text{R}^2) \left(\text{r} \text{EllipticE}\left[\text{ArcSin}\left[\frac{2 \text{H} \text{R}}{\text{r} \sqrt{\text{H}^2 - \text{r}^2 + 4 \text{R}^2}}\right], \frac{\text{r}^2}{4 \text{R}^2}\right] - \right. \right. \right. \\
&\quad \left. \left. \left. 2 \text{R} \text{EllipticF}\left[\text{ArcSin}\left[\frac{\text{H}}{\sqrt{\text{H}^2 - \text{r}^2 + 4 \text{R}^2}}\right], \frac{4 \text{R}^2}{\text{r}^2}\right] \right) \right) \right); \\
\text{DDCFDskCC}[\text{R}_-, \text{H}_-, \text{r}_-] &:= \left(\pi \left((\text{H} - \text{r}) (\text{H} + \text{r}) (\text{H}^2 + \text{r}^2 - 26 \text{R}^2) (\text{H}^2 - \text{r}^2 + 4 \text{R}^2) + \right. \right. \\
&\quad \left. \left. 24 (\text{H} - \text{R}) \text{R}^2 (\text{H} + \text{R}) \sqrt{-(\text{H} - \text{r}) (\text{H} + \text{r}) (\text{H}^2 - \text{r}^2 + 4 \text{R}^2)} \text{ArcSec}\left[\frac{2 \text{R}}{\sqrt{-\text{H}^2 + \text{r}^2}}\right] + \right. \right. \\
&\quad \left. \left. 8 \text{H} \text{r} \sqrt{-(\text{H} - \text{r}) (\text{H} + \text{r}) (\text{H}^2 - \text{r}^2 + 4 \text{R}^2)} \right. \right. \\
&\quad \left. \left. \left(-(\text{r}^2 + 4 \text{R}^2) \text{EllipticE}\left[\text{ArcCsc}\left[\frac{2 \text{H} \text{R}}{\text{r} \sqrt{\text{H}^2 - \text{r}^2 + 4 \text{R}^2}}\right], \frac{4 \text{R}^2}{\text{r}^2}\right] + (\text{r}^2 + 2 \text{R}^2) \text{EllipticF}\left[\text{ArcCsc}\left[\frac{2 \text{H} \text{R}}{\text{r} \sqrt{\text{H}^2 - \text{r}^2 + 4 \text{R}^2}}\right], \frac{4 \text{R}^2}{\text{r}^2}\right] \right) \right) \right) \Bigg) \Bigg) / \left(3 \text{r}^3 \sqrt{(-\text{H}^2 + \text{r}^2) (\text{H}^2 - \text{r}^2 + 4 \text{R}^2)} \right);
\end{aligned}$$

DISCONTINUITY VALUE AT $r = H$

```

FullSimplify[
Simplify[(DDCFDskBB[R, H, r]) /. {r → H}, Assumptions → {0 < H < 2 R && H > 0 && R > 0}] -
Simplify[((DDCFDskAA[R, H, r]) /. {r → H}), Assumptions → {0 < H < 2 R && H > 0 && R > 0}],
Assumptions → {0 < H < 2 R && H > 0 && R > 0}]

```

DISCONTINUITY VALUE AT $r = 2R$

```
(* the function EllipticF[φ,k2] is named F[φ,k] by G&R *)
```

G&R reports the following expansion as k→1.

$$(eq. 8.118.1) F[\varphi, k] \rightarrow -\frac{2}{\pi} * F[\pi/2, \sqrt{1 - k^2}] \operatorname{Log}[\operatorname{Cot}[\varphi/2 + \pi/4]] + ..$$

$$(eq. 8.118.2) E[\varphi, k] \rightarrow -\frac{2}{\pi} * (F[\pi/2, \sqrt{1 - k^2}] - E[\pi/2, \sqrt{1 - k^2}] \operatorname{Log}[\operatorname{Cot}[\varphi/2 + \pi/4]]) + ..$$

It is also recalled (see below) that MATHEMATICA's function $\operatorname{EllipticF}[\varphi, k^2]$ is named $F[\varphi, k]$ by G & R
Thus one gets

$$\operatorname{Integrate}\left[\frac{1}{\sqrt{1 - k^2 * \sin[\alpha]^2}}, \{\alpha, 0, \varphi\}, \operatorname{Assumptions} \rightarrow \{0 < k < 1 \&& 0 < \varphi < \pi/2\}\right]$$

(* $\operatorname{EllipticF}[\varphi, k^2]$ that is named $F[\varphi, k]$ by G&R *)

IDENTITIES

$$\begin{aligned} & \left\{ \operatorname{EllipticF}\left[\operatorname{ArcCsc}\left[\frac{2 H R}{r \sqrt{H^2 - r^2 + 4 R^2}}\right], \frac{4 R^2}{r^2}\right] \rightarrow \right. \\ & -\frac{2}{\pi} * \operatorname{EllipticF}\left[\pi/2, 1 - \frac{4 R^2}{r^2}\right] * \operatorname{Log}\left[\operatorname{Cot}\left[\operatorname{ArcCsc}\left[\frac{2 H R}{r \sqrt{H^2 - r^2 + 4 R^2}}\right]\right] / 2 + \pi/4\right], \\ & \operatorname{EllipticE}\left[\operatorname{ArcCsc}\left[\frac{2 H R}{r \sqrt{H^2 - r^2 + 4 R^2}}\right], \frac{4 R^2}{r^2}\right] \rightarrow \\ & -\frac{2}{\pi} * \left(\operatorname{EllipticF}\left[\pi/2, 1 - \frac{4 R^2}{r^2}\right] - \operatorname{EllipticE}\left[\pi/2, 1 - \frac{4 R^2}{r^2}\right] \right) * \\ & \left. \operatorname{Log}\left[\operatorname{Cot}\left[\operatorname{ArcCsc}\left[\frac{2 H R}{r \sqrt{H^2 - r^2 + 4 R^2}}\right]\right] / 2 + \pi/4\right] \right\} \\ & \left\{ \operatorname{Cot}\left[\frac{1}{4} \left(\pi + 2 \operatorname{ArcCsc}\left[\frac{2 H R}{(2 R + \delta) \sqrt{H^2 - \delta (4 R + \delta)}}\right] \right)\right] \rightarrow \right. \\ & \left. \frac{4 H R \sqrt{\delta (-H + 2 R + \delta) (H + 2 R + \delta) (4 R + \delta)}}{8 H^2 R^2 + 8 H R^2 \sqrt{H^2 - \delta (4 R + \delta)} + 4 H R \delta \sqrt{H^2 - \delta (4 R + \delta)}} \right\} \end{aligned}$$

LEADING TERM AS $r \rightarrow 2 R + \delta$

One isolates the leading term of $\operatorname{DDCFDskCC}[R, H, r]$ as $r \rightarrow 2R^+$.

To this aim one splits the function into the sum of three functions.

The one uses the G&R formulae to rewrite the elliptic integral E and F. One isolates the logarithmic divergent term and finally one gets the leading term.

$$\begin{aligned}
\text{DDCFDskCCaa}[\text{R}_-, \text{H}_-, \text{r}_-] &:= \pi \left((\text{H} - \text{r}) (\text{H} + \text{r}) (\text{H}^2 + \text{r}^2 - 26 \text{R}^2) (\text{H}^2 - \text{r}^2 + 4 \text{R}^2) + \right. \\
&\quad \left. 24 (\text{H} - \text{R}) \text{R}^2 (\text{H} + \text{R}) \sqrt{(-\text{H} + \text{r}) (\text{H} + \text{r}) (\text{H}^2 - \text{r}^2 + 4 \text{R}^2)} \text{ArcSec}\left[\frac{2 \text{R}}{\sqrt{-\text{H}^2 + \text{r}^2}}\right]\right)/ \\
&\quad \left(3 \text{r}^3 \sqrt{(-\text{H}^2 + \text{r}^2) (\text{H}^2 - \text{r}^2 + 4 \text{R}^2)}\right); \text{DDCFDskCCbb}[\text{R}_-, \text{H}_-, \text{r}_-] := \\
&\quad -\frac{1}{3 \text{r}^2} 8 \text{H} \pi (\text{r}^2 + 4 \text{R}^2) \text{EllipticE}\left[\text{ArcCsc}\left[\frac{2 \text{H} \text{R}}{\text{r} \sqrt{\text{H}^2 - \text{r}^2 + 4 \text{R}^2}}\right], \frac{4 \text{R}^2}{\text{r}^2}\right]; \\
&\quad \frac{8 \text{H} \pi (\text{r}^2 + 2 \text{R}^2) \text{EllipticF}\left[\text{ArcCsc}\left[\frac{2 \text{H} \text{R}}{\text{r} \sqrt{\text{H}^2 - \text{r}^2 + 4 \text{R}^2}}\right], \frac{4 \text{R}^2}{\text{r}^2}\right]}{3 \text{r}^2}; \\
\text{DDCFDskCCcc}[\text{R}_-, \text{H}_-, \text{r}_-] &:= \text{Simplify}[\text{DDCFDskCC}[\text{R}, \text{H}, \text{r}] - \text{DDCFDskCCaa}[\text{R}, \text{H}, \text{r}] - \text{DDCFDskCCbb}[\text{R}, \text{H}, \text{r}] - \\
&\quad \text{DDCFDskCCcc}[\text{R}, \text{H}, \text{r}]];
\end{aligned}$$

By the following three identities

$$\begin{aligned}
&\text{FullSimplify}\left[\text{TrigToExp}\left[\text{TrigExpand}\left[\cot\left[\frac{1}{4} \left(\pi + 2 \text{ArcCsc}\left[\frac{2 \text{H} \text{R}}{(2 \text{R} + \delta) \sqrt{\text{H}^2 - \delta (4 \text{R} + \delta)}}\right]\right)\right]\right]\right] \\
&- \frac{2 \text{H} \text{R} + i \sqrt{-(\text{H} - 2 \text{R} - \delta) \delta (\text{H} + 2 \text{R} + \delta) (4 \text{R} + \delta)} - (2 \text{R} + \delta) \sqrt{\text{H}^2 - \delta (4 \text{R} + \delta)}}{2 i \text{H} \text{R} + \sqrt{-(\text{H} - 2 \text{R} - \delta) \delta (\text{H} + 2 \text{R} + \delta) (4 \text{R} + \delta)} + i (2 \text{R} + \delta) \sqrt{\text{H}^2 - \delta (4 \text{R} + \delta)}}, \\
&\text{Assumptions} \rightarrow \{0 < \delta < \text{H} < 2 \text{R}\}], \\
&\text{FullSimplify}\left[\frac{2 \text{H} \text{R} + i \sqrt{-(\text{H} - 2 \text{R} - \delta) \delta (\text{H} + 2 \text{R} + \delta) (4 \text{R} + \delta)} - (2 \text{R} + \delta) \sqrt{\text{H}^2 - \delta (4 \text{R} + \delta)}}{2 i \text{H} \text{R} + \sqrt{-(\text{H} - 2 \text{R} - \delta) \delta (\text{H} + 2 \text{R} + \delta) (4 \text{R} + \delta)} + i (2 \text{R} + \delta) \sqrt{\text{H}^2 - \delta (4 \text{R} + \delta)}} - \right. \\
&\quad \left. \frac{4 \text{H} \text{R} \sqrt{\delta (-\text{H} + 2 \text{R} + \delta) (\text{H} + 2 \text{R} + \delta) (4 \text{R} + \delta)}}{8 \text{H}^2 \text{R}^2 + 8 \text{H} \text{R}^2 \sqrt{\text{H}^2 - \delta (4 \text{R} + \delta)} + 4 \text{H} \text{R} \delta \sqrt{\text{H}^2 - \delta (4 \text{R} + \delta)}}, \text{Assumptions} \rightarrow \{0 < \delta < \text{H} < 2 \text{R}\}\right], \\
&\text{FullSimplify}\left[\text{Log}\left[\frac{4 \text{H} \text{R} \sqrt{\delta (-\text{H} + 2 \text{R} + \delta) (\text{H} + 2 \text{R} + \delta) (4 \text{R} + \delta)}}{8 \text{H}^2 \text{R}^2 + 8 \text{H} \text{R}^2 \sqrt{\text{H}^2 - \delta (4 \text{R} + \delta)} + 4 \text{H} \text{R} \delta \sqrt{\text{H}^2 - \delta (4 \text{R} + \delta)}}\right] - \right. \\
&\quad \left. \left(\frac{1}{2} \text{Log}[\delta] + \text{Log}\left[\frac{4 \text{H} \text{R} \sqrt{(-\text{H} + 2 \text{R} + \delta) (\text{H} + 2 \text{R} + \delta) (4 \text{R} + \delta)}}{8 \text{H}^2 \text{R}^2 + 8 \text{H} \text{R}^2 \sqrt{\text{H}^2 - \delta (4 \text{R} + \delta)} + 4 \text{H} \text{R} \delta \sqrt{\text{H}^2 - \delta (4 \text{R} + \delta)}}\right]\right)\right], \\
&\text{Assumptions} \rightarrow \{0 < \delta < \text{H} < 2 \text{R}\}]
\end{aligned}$$

it follows that the G&R relations become

$$\begin{aligned} & \left\{ \text{EllipticF} \left[\text{ArcCsc} \left[\frac{2 H R}{r \sqrt{H^2 - r^2 + 4 R^2}} \right], \frac{4 R^2}{r^2} \right] \rightarrow -\frac{2}{\pi} * \text{EllipticF} \left[\pi / 2, 1 - \frac{4 R^2}{r^2} \right] * \right. \\ & \left. \left(\frac{1}{2} \text{Log} [\delta] + \text{Log} \left[\frac{4 H R \sqrt{(-H + 2 R + \delta) (H + 2 R + \delta) (4 R + \delta)}}{8 H^2 R^2 + 8 H R^2 \sqrt{H^2 - \delta (4 R + \delta)} + 4 H R \delta \sqrt{H^2 - \delta (4 R + \delta)}} \right] \right) \right\}, \\ & \left\{ \text{EllipticE} \left[\text{ArcCsc} \left[\frac{2 H R}{r \sqrt{H^2 - r^2 + 4 R^2}} \right], \frac{4 R^2}{r^2} \right] \rightarrow \right. \\ & -\frac{2}{\pi} * \left(\text{EllipticF} \left[\pi / 2, 1 - \frac{4 R^2}{r^2} \right] - \text{EllipticE} \left[\pi / 2, 1 - \frac{4 R^2}{r^2} \right] \right) * \\ & \left. \left(\frac{1}{2} \text{Log} [\delta] + \text{Log} \left[\frac{4 H R \sqrt{(-H + 2 R + \delta) (H + 2 R + \delta) (4 R + \delta)}}{8 H^2 R^2 + 8 H R^2 \sqrt{H^2 - \delta (4 R + \delta)} + 4 H R \delta \sqrt{H^2 - \delta (4 R + \delta)}} \right] \right) \right\} \end{aligned}$$

The expansions, as $r \rightarrow 2 R + \delta$, of the three terms are

```
Expand[Simplify[Series[((DDCFDskCCaa[R, H, r]) /. {r → 2 R + δ}), {δ, 0, 0}], Assumptions → {0 < δ < H < 2 R}]]
```

```
FullSimplify[
Series[(((DDCFDskCCbb[R, H, r]) /. {EllipticE[ArcCsc[(2 H R)/(r Sqrt[H^2 - r^2 + 4 R^2]), 4 R^2/r^2]] →
-2/π * (EllipticF[π/2, 1 - 4 R^2/r^2] - EllipticE[π/2, 1 - 4 R^2/r^2]) * (1/2 Log[δ] + Log[(4 H R Sqrt[(-H + 2 R + δ) (H + 2 R + δ) (4 R + δ)]/(8 H^2 R^2 + 8 H R^2 Sqrt[H^2 - δ (4 R + δ)] + 4 H R δ Sqrt[H^2 - δ (4 R + δ)]))])/.{r → 2 R + δ}, {δ, 0, 0}], Assumptions → {0 < δ < H < 2 R}], {r → 2 R + δ}], {δ, 0, 0}], Assumptions → {0 < δ < H < 2 R}]
Expand[FullSimplify[Series[((DDCFDskCCcc[R, H, r]) /.
{EllipticF[ArcCsc[(2 H R)/(r Sqrt[H^2 - r^2 + 4 R^2]), 4 R^2/r^2]] → -2/π * (EllipticF[π/2, 1 - 4 R^2/r^2]) * (1/2 Log[δ] + Log[(4 H R Sqrt[(-H + 2 R + δ) (H + 2 R + δ) (4 R + δ)]/(8 H^2 R^2 + 8 H R^2 Sqrt[H^2 - δ (4 R + δ)] + 4 H R δ Sqrt[H^2 - δ (4 R + δ)]))])/.{r → 2 R + δ}, {δ, 0, 0}], Assumptions → {0 < δ < H < 2 R}], {r → 2 R + δ}], {δ, 0, 0}], Assumptions → {0 < δ < H < 2 R}]
FullSimplify[Expand[-2 (H π (-2 Log[H] + Log[-(H^2 δ)/(4 R) + R δ]))] - 2 H π (Log[(4 H^2)/(4 R^2 - H^2)] + Log[R/δ]), Assumptions → {0 < δ < H < 2 R}]]
```

and collecting the three results one finds the leading term contribution

$$\text{DDCFDskCCLwrBnd}[R_, H_, \delta_] :=$$

$$\left(\frac{\pi \left(H^5 - 26 H^3 R^2 + 88 H R^4 + 24 R^2 (H^2 - R^2) \sqrt{-H^2 + 4 R^2} \text{ArcCos}\left[\frac{\sqrt{-H^2 + 4 R^2}}{2 R}\right] \right)}{24 R^3 \sqrt{-H^2 + 4 R^2}} \right) +$$

$$\left(2 H \pi \left(\text{Log}\left[\frac{4 H^2}{4 R^2 - H^2}\right] + \text{Log}\left[\frac{R}{\delta}\right] \right) \right);$$

LEADING TERM AS $r \rightarrow 2R - \delta$

Function DDCFDskBB[R, H, r] involves elliptic integrals with modulus $\frac{4 R^2}{r^2}$ that is greater than 1 in the range $r < 2R$.

Hence one uses identities 8.127 of G&R to convert to $k < 1$. The identities are

IDENTITIES

$$\left\{ \text{EllipticF} \left[\text{ArcSin} \left[\frac{H}{\sqrt{H^2 - r^2 + 4 R^2}} \right], \frac{4 R^2}{r^2} \right] \rightarrow \right.$$

$$\left. \left(\frac{r}{2 R} \right) * \left\{ \text{EllipticF} \left[\text{ArcSin} \left[\frac{2 R H}{r \sqrt{H^2 - r^2 + 4 R^2}} \right], \frac{r^2}{4 R^2} \right] \right\} \right\}$$

and

$$\left\{ \text{EllipticE} \left[\text{ArcSin} \left[\frac{H}{\sqrt{H^2 - r^2 + 4 R^2}} \right], \frac{4 R^2}{r^2} \right] \rightarrow \right.$$

$$\left. \left(\frac{2 R}{r} \right) * \left\{ \text{EllipticE} \left[\text{ArcSin} \left[\frac{2 R H}{r \sqrt{H^2 - r^2 + 4 R^2}} \right], \frac{r^2}{4 R^2} \right] - \right. \right.$$

$$\left. \left. \left(1 - \frac{r^2}{4 R^2} \right) * \text{EllipticF} \left[\text{ArcSin} \left[\frac{2 R H}{r \sqrt{H^2 - r^2 + 4 R^2}} \right], \frac{r^2}{4 R^2} \right] \right\} \right\}$$

PROOF OF THE IDENTITES

```

FullSimplify[EllipticF[ArcSin[H/Sqrt[H^2 - r^2 + 4 R^2]], 4 R^2/r^2] -
  (r/(2 R)) * EllipticF[ArcSin[2 R H/(r Sqrt[H^2 - r^2 + 4 R^2])], r^2/(4 R^2)],
  Assumptions -> {0 < H < r < 2 * R}]

FullSimplify[EllipticE[ArcSin[H/Sqrt[H^2 - r^2 + 4 R^2]], 4 R^2/r^2] -
  (2 R/r) * EllipticE[ArcSin[2 R H/(r Sqrt[H^2 - r^2 + 4 R^2])], r^2/(4 R^2)] -
  (1 - r^2/(4 R^2)) * EllipticF[ArcSin[2 R H/(r Sqrt[H^2 - r^2 + 4 R^2])], r^2/(4 R^2)],
  Assumptions -> {0 < H < r < 2 * R}]

H = 1; R = 2; Npoint = 50; step = (2 * R - H) / Npoint;
Do[r = H + (i - 1) * step; val = N[EllipticF[ArcSin[H/Sqrt[H^2 - r^2 + 4 R^2]], 4 R^2/r^2] -
  (r/(2 R)) * EllipticF[ArcSin[2 R H/(r Sqrt[H^2 - r^2 + 4 R^2])], r^2/(4 R^2)], 30];
Print[i, ", ", val];, {i, 1, Npoint}];

Clear[H]; Clear[Npoint]; Clear[step];
Clear[R]; Clear[r]; Clear[val];

H = 1; R = 2; Npoint = 50; step = (2 * R - H) / Npoint;
Do[r = H + (i - 1) * step; val = N[EllipticE[ArcSin[H/Sqrt[H^2 - r^2 + 4 R^2]], 4 R^2/r^2] -
  (2 R/r) * EllipticE[ArcSin[2 R H/(r Sqrt[H^2 - r^2 + 4 R^2])], r^2/(4 R^2)] -
  (1 - r^2/(4 R^2)) * EllipticF[ArcSin[2 R H/(r Sqrt[H^2 - r^2 + 4 R^2])], r^2/(4 R^2)], 30];
Print[i, ", ", val];, {i, 1, Npoint}];

Clear[H]; Clear[Npoint]; Clear[step]; Clear[R]; Clear[r]; Clear[val];

```

splitting of the function into the sum of two functions one of which containing the elliptic integrals and subsequent conversion of the elliptic integrals with moduli greater than one into those with moduli smaller than one.

```
DDCFDskBB[R, H, r]
```

```

DDCFDskBBaa[R_, H_, r_] :=

$$\frac{1}{3 r^4 (-H^2 + r^2 - 4 R^2)} \pi \left( r \left( H^4 \sqrt{(-H^2 + r^2) (H^2 - r^2 + 4 R^2)} - 3 (3 r^4 - 10 r^2 R^2 - 8 R^4) \right. \right.$$


$$\left. \left( -r \sqrt{-r^2 + 4 R^2} + \sqrt{(-H+r) (H+r) (H^2 - r^2 + 4 R^2)} \right) - \right.$$


$$H^2 \left( 9 r^3 \sqrt{-r^2 + 4 R^2} + 2 R^2 \left( 3 r \sqrt{-r^2 + 4 R^2} + 11 \sqrt{(-H^2 + r^2) (H^2 - r^2 + 4 R^2)} \right) \right) +$$


$$24 R^2 (H^2 - r^2 + 4 R^2) \left( -R^2 \text{ArcSec}\left[\frac{2 R}{r}\right] + (-H^2 + R^2) \text{ArcSec}\left[\frac{2 R}{\sqrt{-H^2 + r^2}}\right] \right) \right);$$

Simplify[(Simplify[DDCFDskBB[R, H, r] - DDCFDskBBaa[R, H, r]]) /.

$$\left\{ \text{EllipticF}\left[\text{ArcSin}\left[\frac{H}{\sqrt{H^2 - r^2 + 4 R^2}}\right], \frac{4 R^2}{r^2}\right] \rightarrow \right.$$


$$\left( \frac{r}{2 R} \right) * \left( \text{EllipticF}\left[\text{ArcSin}\left[\frac{2 R H}{r \sqrt{H^2 - r^2 + 4 R^2}}\right], \frac{r^2}{4 R^2}\right], \text{EllipticE}\left[ \right.$$


$$\text{ArcSin}\left[\frac{H}{\sqrt{H^2 - r^2 + 4 R^2}}\right], \frac{4 R^2}{r^2} \right) \rightarrow \left( \frac{2 R}{r} \right) * \left( \text{EllipticE}\left[\text{ArcSin}\left[\frac{2 R H}{r \sqrt{H^2 - r^2 + 4 R^2}}\right], \frac{r^2}{4 R^2}\right] - \right.$$


$$\left. \left( 1 - \frac{r^2}{4 R^2} \right) * \text{EllipticF}\left[\text{ArcSin}\left[\frac{2 R H}{r \sqrt{H^2 - r^2 + 4 R^2}}\right], \frac{r^2}{4 R^2}\right] \right) \}]$$

DDCFDskBBbb[R_, H_, r_] :=

$$-\frac{1}{3 r^3} 8 H \pi R \left( 2 (r^2 + 4 R^2) \text{EllipticE}\left[\text{ArcSin}\left[\frac{2 H R}{r \sqrt{H^2 - r^2 + 4 R^2}}\right], \frac{r^2}{4 R^2}\right] - \right.$$


$$\left. (r^2 + 8 R^2) \text{EllipticF}\left[\text{ArcSin}\left[\frac{2 H R}{r \sqrt{H^2 - r^2 + 4 R^2}}\right], \frac{r^2}{4 R^2}\right] \right);$$

EllipticF[ArcCsc[ $\frac{2 H R}{r \sqrt{H^2 - r^2 + 4 R^2}}$ ],  $\frac{4 R^2}{r^2}$ ]

```

check

```

Simplify[(Simplify[DDCFDskBB[R, H, r] - (DDCFDskBBaa[R, H, r] + DDCFDskBBbb[R, H, r])]) /.

{EllipticF[ArcSin[H/Sqrt[H^2 - r^2 + 4 R^2]], 4 R^2/r^2] \rightarrow

(r/(2 R)) * EllipticF[ArcSin[2 R H/(r Sqrt[H^2 - r^2 + 4 R^2])], r^2/(4 R^2)],

EllipticE[ArcSin[H/Sqrt[H^2 - r^2 + 4 R^2]], 4 R^2/r^2] \rightarrow

(2 R/r) * EllipticE[ArcSin[2 R H/(r Sqrt[H^2 - r^2 + 4 R^2])], r^2/(4 R^2)] -

(1 - r^2/(4 R^2)) * EllipticF[ArcSin[2 R H/(r Sqrt[H^2 - r^2 + 4 R^2])], r^2/(4 R^2)]}

Simplify[ExpandAll[DDCFDskBBbb[R, H, 2 R - δ]], Assumptions \rightarrow {0 < δ < H < 2 R}]

- 1/(3 (2 R - δ)^3) 8 H π R \left( 2 (8 R^2 - 4 R δ + δ^2) EllipticE[ArcSin[2 H R/((2 R - δ) Sqrt[H^2 + 4 R δ - δ^2])], (-2 R + δ)^2/(4 R^2)] - \right.

\left. (12 R^2 - 4 R δ + δ^2) EllipticF[ArcSin[2 H R/((2 R - δ) Sqrt[H^2 + 4 R δ - δ^2])], (-2 R + δ)^2/(4 R^2)] \right)

G&R reports the following expansion as k\rightarrow 1.

(eq. 8.118.1) F[φ, k] \rightarrow -2/(π) * F[π/2, √(1 - k^2)] Log[Cot[φ/2 + π/4]] + ..

(eq. 8.118.2) E[φ, k] \rightarrow -2/(π) * (F[π/2, √(1 - k^2)] - E[π/2, √(1 - k^2)] Log[Cot[φ/2 + π/4]]) + ..

It is also recalled (see below) that MATHEMATIC ' s function EllipticF[φ, k^2] is named F[φ, k] by G & R
Thus one gets

{EllipticE[ArcSin[2 H R/((2 R - δ) Sqrt[H^2 + 4 R δ - δ^2])], (-2 R + δ)^2/(4 R^2)] \rightarrow

- 2/(π) * \left( EllipticF[π/2, 1 - (-2 R + δ)^2/(4 R^2)] - EllipticE[π/2, 1 - (-2 R + δ)^2/(4 R^2)] \right) * Log[ArgLog[R, H, δ]],

EllipticF[ArcSin[2 H R/((2 R - δ) Sqrt[H^2 + 4 R δ - δ^2])], (-2 R + δ)^2/(4 R^2)] \rightarrow

- 2/(π) * \left( EllipticF[π/2, 1 - (-2 R + δ)^2/(4 R^2)] \right) * Log[ArgLog[R, H, δ]]}

FullSimplify[Together[

ExpandAll[FullSimplify[TrigExpand[Cot[ArcSin[2 H R/((2 R - δ) Sqrt[H^2 + 4 R δ - δ^2])]/(2 + π/4)]]], Assumptions \rightarrow {0 < δ < H < 2 R}], Assumptions \rightarrow {2 R - δ > 0 \&& H^2 + 4 R δ - δ^2 > 0}]]]

```

$$\begin{aligned}
\text{ArgLog}[\mathbf{R}_-, \mathbf{H}_-, \delta_-] := & \frac{-2 \mathbf{H} \mathbf{R} + \sqrt{-(\mathbf{H} + 2 \mathbf{R} - \delta) (4 \mathbf{R} - \delta) \delta (\mathbf{H} - 2 \mathbf{R} + \delta)} + (2 \mathbf{R} - \delta) \sqrt{\mathbf{H}^2 + (4 \mathbf{R} - \delta) \delta}}{2 \mathbf{H} \mathbf{R} + \sqrt{-(\mathbf{H} + 2 \mathbf{R} - \delta) (4 \mathbf{R} - \delta) \delta (\mathbf{H} - 2 \mathbf{R} + \delta)} + (2 \mathbf{R} - \delta) \sqrt{\mathbf{H}^2 + (4 \mathbf{R} - \delta) \delta}}, \\
\text{FullSimplify}[\text{FullSimplify}[& \text{ExpandAll}[\text{Series}[\text{ArgLog}[\mathbf{R}, \mathbf{H}, \delta], \{\delta, 0, 2\}]], \text{Assumptions} \rightarrow \{0 < \delta < \mathbf{H} < 2 \mathbf{R}\}] - \\
& \left(\frac{\sqrt{4 \mathbf{R}^2 - \mathbf{H}^2} \sqrt{\frac{\delta}{\mathbf{R}}}}{2 \mathbf{H}} \left(1 + \frac{(\mathbf{H}^4 + 4 \mathbf{H}^2 \mathbf{R}^2 + 32 \mathbf{R}^4) \delta}{8 \mathbf{H}^2 (\mathbf{H} - 2 \mathbf{R}) \mathbf{R} (\mathbf{H} + 2 \mathbf{R})} \right) \right), \text{Assumptions} \rightarrow \{0 < \delta < \mathbf{H} < 2 \mathbf{R}\}] \\
& \left(\text{FullSimplify}[& \left(\text{Simplify}[\text{ExpandAll}[\text{DDCFDskBBbb}[\mathbf{R}, \mathbf{H}, 2 \mathbf{R} - \delta]], \text{Assumptions} \rightarrow \{0 < \delta < \mathbf{H} < 2 \mathbf{R}\}] \right) / . \right. \\
& \left\{ \text{EllipticE}[\text{ArcSin}\left[\frac{2 \mathbf{H} \mathbf{R}}{(2 \mathbf{R} - \delta) \sqrt{\mathbf{H}^2 + 4 \mathbf{R} \delta - \delta^2}}\right], \frac{(-2 \mathbf{R} + \delta)^2}{4 \mathbf{R}^2}] \rightarrow \right. \\
& \left. - \frac{2}{\pi} * \left(\text{EllipticF}\left[\pi/2, 1 - \frac{(-2 \mathbf{R} + \delta)^2}{4 \mathbf{R}^2}\right] - \text{EllipticE}\left[\pi/2, 1 - \frac{(-2 \mathbf{R} + \delta)^2}{4 \mathbf{R}^2}\right] \right) * \right. \\
& \left. \text{Log}[\text{ArgLog}[\mathbf{R}, \mathbf{H}, \delta]], \right. \\
& \left. \text{EllipticF}[\text{ArcSin}\left[\frac{2 \mathbf{H} \mathbf{R}}{(2 \mathbf{R} - \delta) \sqrt{\mathbf{H}^2 + 4 \mathbf{R} \delta - \delta^2}}\right], \frac{(-2 \mathbf{R} + \delta)^2}{4 \mathbf{R}^2}] \rightarrow \right. \\
& \left. - \frac{2}{\pi} * \left(\text{EllipticF}\left[\pi/2, 1 - \frac{(-2 \mathbf{R} + \delta)^2}{4 \mathbf{R}^2}\right] \right) * \text{Log}[\text{ArgLog}[\mathbf{R}, \mathbf{H}, \delta]] \right\}, \\
& \left. \text{Assumptions} \rightarrow \{0 < \delta < \mathbf{H} < 2 \mathbf{R}\} \right] / . \\
& \left\{ \frac{-2 \mathbf{H} \mathbf{R} + \sqrt{(4 \mathbf{R} - \delta) \delta (-\mathbf{H} - 2 \mathbf{R} + \delta) (\mathbf{H} - 2 \mathbf{R} + \delta)} + (2 \mathbf{R} - \delta) \sqrt{\mathbf{H}^2 + (4 \mathbf{R} - \delta) \delta}}{2 \mathbf{H} \mathbf{R} + \sqrt{(4 \mathbf{R} - \delta) \delta (-\mathbf{H} - 2 \mathbf{R} + \delta) (\mathbf{H} - 2 \mathbf{R} + \delta)} + (2 \mathbf{R} - \delta) \sqrt{\mathbf{H}^2 + (4 \mathbf{R} - \delta) \delta}} \rightarrow \right. \\
& \left. \left(\frac{\sqrt{4 \mathbf{R}^2 - \mathbf{H}^2} \sqrt{\frac{\delta}{\mathbf{R}}}}{2 \mathbf{H}} \left(1 + \frac{(\mathbf{H}^4 + 4 \mathbf{H}^2 \mathbf{R}^2 + 32 \mathbf{R}^4) \delta}{8 \mathbf{H}^2 (\mathbf{H} - 2 \mathbf{R}) \mathbf{R} (\mathbf{H} + 2 \mathbf{R})} \right) \right) \right\}
\end{aligned}$$

```

FullSimplify[Series[-(1/(3(2R - δ)^3)) 16 H R
  (2 (8 R^2 - 4 R δ + δ^2) EllipticE[(4 R - δ) δ/(4 R^2)] - (-2 R + δ)^2 EllipticK[(4 R - δ) δ/(4 R^2)])
  Log[(Sqrt[-H^2 + 4 R^2] Sqrt[δ/R] (1 + ((H^4 + 4 H^2 R^2 + 32 R^4) δ)/(8 H^2 (H - 2 R) R (H + 2 R)))/
  2 H], {δ, 0, 0}] -
  (-4 H π Log[Sqrt[-H^2 + 4 R^2]/2 H] - 2 H π Log[δ/R]), Assumptions → {0 < δ < H < 2 R}]

FullSimplify[
  Series[FullSimplify[ExpandAll[DDCFDskBBaa[R, H, 2R - δ]], Assumptions → {0 < δ < H < 2 R}],
  {δ, 0, 0}], Assumptions → {0 < δ < H < 2 R}]

```

Collecting the leading terms of the above two expansions one
find the leading asymptotic term of the CF 2 nd derivative as $r \rightarrow (2R)^-$

```

DDCFDskBBUpBnd[R_, H_, δ_] :=
  π (H (H^2 - 22 R^2) Sqrt[-H^2 + 4 R^2] + 24 R^2 (-H^2 + R^2) ArcSec[2 R/Sqrt[-H^2 + 4 R^2]])/
  24 R^3 -
  4 H π Log[Sqrt[-H^2 + 4 R^2]/2 H] - 2 H π Log[δ/R];

```

Discontinuity value at $r = 2R$

```

DDCFDskCCLwrBnd[R, H, δ] - DDCFDskBBUpBnd[R, H, δ]
  π (H^5 - 26 H^3 R^2 + 88 H R^4 + 24 R^2 (H^2 - R^2) Sqrt[-H^2 + 4 R^2] ArcCos[Sqrt[-H^2 + 4 R^2]/2 R])/
  24 R^3 Sqrt[-H^2 + 4 R^2] +
  π (H (H^2 - 22 R^2) Sqrt[-H^2 + 4 R^2] + 24 R^2 (-H^2 + R^2) ArcSec[2 R/Sqrt[-H^2 + 4 R^2]])/
  24 R^3 + 4 H π Log[Sqrt[-H^2 + 4 R^2]/2 H] +
  2 H π Log[4 H^2/(-H^2 + 4 R^2)], Assumptions → {0 < δ < H < 2 R}] + 2 H π Log[R/δR] + 2 H π Log[δL/R]

```

CHECK: around $r = \sqrt{H^2 + 4 R^2}$ the expansions of the Neddle and Disc cases must coincide as it really happens.

```
Simplify[SerisCFDskCCUpBnd[R, H, δ] - SerisCFNdlCCUpBnd[R, H, δ]]
```

SOME PLOTS for the disk case

```

DiskIntPlt =
With[{H = 1, R = 5}, ParametricPlot[{{r, r * CFDskAA[R, H, r]}}, {r, 0, H}, PlotStyle ->
{Thickness[0.004], Blue}, AspectRatio -> 1, AxesLabel -> {"r", "r \gamma(r)"}]] ;

DiskMedPlt = With[{H = 1, R = 5}, ParametricPlot[{{r, r * CFDskBB[R, H, r]}}, {r, H, 2 R}, PlotStyle -> {Thickness[0.002], Magenta}, AspectRatio -> 1]];

DiskOutrPlt = With[{H = 1, R = 5}, ParametricPlot[{{r, r * CFDskCC[R, H, r]}}, {r, 2 R, Sqrt[4 R^2 + H^2]}, PlotStyle -> {Thickness[0.006], Green}, AspectRatio -> 1]];

Show[DiskIntPlt, DiskMedPlt, DiskOutrPlt, PlotRange -> {{0, 11}, {0, 450}}];

Show[DiskIntPlt, DiskMedPlt, PlotRange -> {{0.9, 1.1}, {0, 450}}];

Show[DiskMedPlt, DiskOutrPlt, PlotRange -> {{9.8, 10.2}, {0, 0.04}}];

```

1st ORDER DERIVATIVE IN THE NEEDLE CASE

```

DDiskIntPlt = With[{H = 1, R = 5},
ParametricPlot[{{r, DCFDskAA[R, H, r]}}, {r, 0, H}, PlotStyle -> {Thickness[0.004], Blue},
AspectRatio -> 1, AxesLabel -> {"r", "\gamma'(r)"}, PlotLabel -> "DISK: r=5, H=1"]]

DDiskMedPlt = With[{H = 1, R = 5}, ParametricPlot[{{r, DCFDskBB[R, H, r]}}, {r, H, 2 R}, PlotStyle -> {Thickness[0.002], Magenta}, AspectRatio -> 1]];

DDiskOutrPlt =
With[{H = 1, R = 5}, ParametricPlot[{{r, DCFDskCC[R, H, r]}}, {r, 2 R, Sqrt[4 R^2 + H^2]}, PlotStyle -> {Thickness[0.006], Green}, AspectRatio -> 1, PlotPoints -> 100]];

Show[DDiskIntPlt, DDiskMedPlt, DDiskOutrPlt, PlotRange -> {{0, 11}, {-600, 1}}]

```

2nd ORDER DERIVATIVE IN THE NEEDLE CASE

```

DDDDiskIntPlt = With[{H = 1, R = 5}, ParametricPlot[{{r, DDCFDskAA[R, H, r]}}, {r, 10^-3, H}, PlotStyle -> {Thickness[0.004], Blue}, AspectRatio -> 1, AxesLabel -> {"r", "\gamma''(r)"}, PlotPoints -> 200, PlotLabel -> "DISK: R=5, H=1"]]

DDDDiskMedPlt = With[{H = 1, R = 5}, ParametricPlot[{{r, DDCFDskBB[R, H, r]}}, {r, H + 1/1000000, 2 R - 1/1000000}, PlotStyle -> {Thickness[0.002], Magenta}, PlotPoints -> 2000, AspectRatio -> 1, PlotRange -> {{H, 2 R}, {0, 1100}}];

DDDDiskOutrPlt = With[{H = 1, R = 5}, ParametricPlot[{{r, DDCFDskCC[R, H, r]}}, {r, 2 R + 1/1000000, Sqrt[4 R^2 + H^2]}, PlotStyle -> {Thickness[0.006], Green}, PlotPoints -> 200, AspectRatio -> 1, PlotRange -> {{2 R, Sqrt[4 R^2 + H^2]}, {0, 100}}]];

Show[DDDDiskIntPlt, DDDDiskMedPlt, PlotRange -> {{0.98, 1.05}, {0, 1100}}];

Show[DDDDiskMedPlt, DDDDiskOutrPlt, PlotRange -> {{9.9, 10.1}, {-0.1, 200}}];

Show[DDDDiskIntPlt, DDDDiskMedPlt, DDDDiskOutrPlt, PlotRange -> {{0, 10.5}, {-1, 1100}}]

```

THE FINAL FORMULAE OF THE CYLINDER's CF THE NEEDLE CASE $0 < 2R < H$

```
FullSimplify[Limit[CFNdlAA[R, H, r], r -> 0, Direction -> -1], Assumptions -> {R > 0 && H > 0}]
```

The expressions of the Cf and its derivatives are normalized to 1 at r=0 dividing them by $(4 H \pi^2 R^2)$

```

CFNd1AA[R_, H_, r_] := 
  
$$\left( 2 \pi^2 R^2 (2 H - r) + \frac{\pi (r^2 + 2 R^2)}{2} \sqrt{-r^2 + 4 R^2} + \frac{4 \pi R}{3 r} \left( 3 R (r^2 - R^2) \operatorname{ArcSin}\left[\frac{r}{2 R}\right] - 2 H (r^2 + 4 R^2) \operatorname{EllipticE}\left[\frac{r^2}{4 R^2}\right] - 2 H (r^2 - 4 R^2) \operatorname{EllipticK}\left[\frac{r^2}{4 R^2}\right] \right) \right) / (4 H \pi^2 R^2);$$


CFNd1BB[R_, H_, r_] := 
  
$$\left( \frac{2 \pi^2 R^2 (2 H r - R^2)}{r} - \frac{4 \pi H (r^2 + 4 R^2)}{3} \operatorname{EllipticE}\left[\frac{4 R^2}{r^2}\right] + \frac{4 \pi H (r^2 - 4 R^2)}{3} \operatorname{EllipticK}\left[\frac{4 R^2}{r^2}\right] \right) / (4 H \pi^2 R^2);$$


CFNd1CC[R_, H_, r_] := 
  
$$\begin{aligned} & \left( \frac{1}{6 r} \pi \left( 12 H^2 \pi R^2 + 12 \pi r^2 R^2 - 12 \pi R^4 - H^2 \sqrt{(-H^2 + r^2) (H^2 - r^2 + 4 R^2)} + 5 r^2 \sqrt{(-H^2 + r^2) (H^2 - r^2 + 4 R^2)} + 26 R^2 \sqrt{(-H^2 + r^2) (H^2 - r^2 + 4 R^2)} - 24 H^2 R^2 \operatorname{ArcSin}\left[\frac{\sqrt{-H^2 + r^2}}{2 R}\right] - 24 r^2 R^2 \operatorname{ArcSin}\left[\frac{\sqrt{-H^2 + r^2}}{2 R}\right] + 24 R^4 \operatorname{ArcSin}\left[\frac{\sqrt{-H^2 + r^2}}{2 R}\right] - 8 H r (r^2 + 4 R^2) \operatorname{EllipticE}\left[\operatorname{ArcSin}\left[\frac{r \sqrt{H^2 - r^2 + 4 R^2}}{2 H R}\right], \frac{4 R^2}{r^2}\right] + 8 H r (r^2 - 4 R^2) \operatorname{EllipticF}\left[\operatorname{ArcSin}\left[\frac{r \sqrt{H^2 - r^2 + 4 R^2}}{2 H R}\right], \frac{4 R^2}{r^2}\right] \right) \right) / (4 H \pi^2 R^2); \\ & \text{CFTotNdl[R_, H_, r_]} := \Theta[2 * R - r] * CFNd1AA[R, H, r] + \Theta[H - r] * \Theta[r - 2 * R] * CFNd1BB[R, H, r] + \Theta[\sqrt{H^2 + 4 * R^2} - r] * \Theta[r - H] * CFNd1CC[R, H, r]; \end{aligned}$$


```

Simplify[ExpandAll[(CFNd1AA[R, H, r])]]

$$\begin{aligned} & \frac{1}{24 H \pi r R^2} \left(3 \left(r \left(\operatorname{Factor}[8 H \pi R^2 - 4 \pi r R^2] + (r^2 + 2 R^2) \sqrt{-r^2 + 4 R^2} \right) + 8 R^2 (r^2 - R^2) \operatorname{ArcSin}\left[\frac{r}{2 R}\right] \right) - 16 H R (r^2 + 4 R^2) \operatorname{EllipticE}\left[\frac{r^2}{4 R^2}\right] - 16 H R (r^2 - 4 R^2) \operatorname{EllipticK}\left[\frac{r^2}{4 R^2}\right] \right) \end{aligned}$$

Simplify[ExpandAll[(CFNd1BB[R, H, r])]]

$$\begin{aligned} & \frac{1}{6 H \pi r R^2} \left(\operatorname{Factor}[6 H \pi r R^2 - 3 \pi R^4] - 2 H r (r^2 + 4 R^2) \operatorname{EllipticE}\left[\frac{4 R^2}{r^2}\right] + 2 H r (r^2 - 4 R^2) \operatorname{EllipticK}\left[\frac{4 R^2}{r^2}\right] \right) \\ & \frac{1}{6 H \pi r R^2} \left(3 \pi R^2 (2 H r - R^2) - 2 H r (r^2 + 4 R^2) \operatorname{EllipticE}\left[\frac{4 R^2}{r^2}\right] + 2 H r (r^2 - 4 R^2) \operatorname{EllipticK}\left[\frac{4 R^2}{r^2}\right] \right) \end{aligned}$$

Simplify[ExpandAll[(CFNd1CC[R, H, r])]]

$$\frac{1}{24 H \pi r R^2} \left(\text{Factor} [12 H^2 \pi R^2 + 12 \pi r^2 R^2 - 12 \pi R^4] + \text{Factor} [-H^2 \sqrt{- (H^2 - r^2) (H^2 - r^2 + 4 R^2)} + \right.$$

$$5 r^2 \sqrt{- (H^2 - r^2) (H^2 - r^2 + 4 R^2)} + 26 R^2 \sqrt{- (H^2 - r^2) (H^2 - r^2 + 4 R^2)}] + \text{Factor} [$$

$$-24 H^2 R^2 \text{ArcSin} \left[\frac{\sqrt{-H^2 + r^2}}{2 R} \right] - 24 r^2 R^2 \text{ArcSin} \left[\frac{\sqrt{-H^2 + r^2}}{2 R} \right] + 24 R^4 \text{ArcSin} \left[\frac{\sqrt{-H^2 + r^2}}{2 R} \right]] -$$

$$8 H r (r^2 + 4 R^2) \text{EllipticE} \left[\text{ArcSin} \left[\frac{r \sqrt{H^2 - r^2 + 4 R^2}}{2 H R} \right], \frac{4 R^2}{r^2} \right] +$$

$$\left. 8 H r (r^2 - 4 R^2) \text{EllipticF} \left[\text{ArcSin} \left[\frac{r \sqrt{H^2 - r^2 + 4 R^2}}{2 H R} \right], \frac{4 R^2}{r^2} \right] \right)$$

DCFNd1AA[R_, H_, r_] :=

$$\left(\frac{1}{6 r^2} \pi \left(3 \left(8 R^4 \text{ArcCsc} \left[\frac{2 R}{r} \right] + r \left((3 r^2 - 2 R^2) \sqrt{-r^2 + 4 R^2} - 8 r R^2 \text{ArcSec} \left[\frac{2 R}{r} \right] \right) \right) + \right.$$

$$16 H R \left(-2 (r^2 - 2 R^2) \text{EllipticE} \left[\frac{r^2}{4 R^2} \right] + (r^2 - 4 R^2) \text{EllipticK} \left[\frac{r^2}{4 R^2} \right] \right) \left. \right) \Big/ (4 H \pi^2 R^2);$$

DCFNd1BB[R_, H_, r_] :=

$$\left(\frac{1}{3 r^2} 2 \pi \left(3 \pi R^4 - 4 H r (r^2 - 2 R^2) \text{EllipticE} \left[\frac{4 R^2}{r^2} \right] + \right. \right.$$

$$4 H r (r^2 - 4 R^2) \text{EllipticK} \left[\frac{4 R^2}{r^2} \right] \left. \right) \Big/ (4 H \pi^2 R^2);$$

DCFNd1CC[R_, H_, r_] :=

$$\left(\frac{1}{6 r^2} \pi \left((H^2 + 7 r^2 - 26 R^2) \sqrt{- (H - r) (H + r) (H^2 - r^2 + 4 R^2)} + \right. \right.$$

$$24 R^2 (-H^2 + r^2 + R^2) \text{ArcSec} \left[\frac{2 R}{\sqrt{-H^2 + r^2}} \right] +$$

$$16 H r \left(- (r^2 - 2 R^2) \text{EllipticE} \left[\text{ArcCsc} \left[\frac{2 H R}{r \sqrt{H^2 - r^2 + 4 R^2}} \right], \frac{4 R^2}{r^2} \right] + \right. \right.$$

$$\left. \left. (r^2 - 4 R^2) \text{EllipticF} \left[\text{ArcCsc} \left[\frac{2 H R}{r \sqrt{H^2 - r^2 + 4 R^2}} \right], \frac{4 R^2}{r^2} \right] \right) \Big) \Big) \Big/ (4 H \pi^2 R^2);$$

```

DDCFNd1AA[R_, H_, r_] := 
  
$$\left( \frac{1}{3 r^3} \pi \left( 3 r (3 r^2 + 2 R^2) \sqrt{-r^2 + 4 R^2} - 24 R^4 \text{ArcCsc}\left[\frac{2 R}{r}\right] + 8 H R \left( -2 (r^2 + 4 R^2) \text{EllipticE}\left[\frac{r^2}{4 R^2}\right] + (r^2 + 8 R^2) \text{EllipticK}\left[\frac{r^2}{4 R^2}\right] \right) \right) \right) / (4 H \pi^2 R^2);$$

  DDCFn1BB[R_, H_, r_] := 
  
$$\left( -\frac{1}{3 r^3} 4 \pi \left( 3 \pi R^4 + 2 H r (r^2 + 4 R^2) \text{EllipticE}\left[\frac{4 R^2}{r^2}\right] - 2 H r (r^2 + 2 R^2) \text{EllipticK}\left[\frac{4 R^2}{r^2}\right] \right) \right) / (4 H \pi^2 R^2);$$

  DDCFn1CC[R_, H_, r_] := 
  
$$\left( \left( \pi \left( (H - r) (H + r) (H^2 + r^2 - 26 R^2) (H^2 - r^2 + 4 R^2) + 24 (H - R) R^2 (H + R) \sqrt{-(H - r) (H + r) (H^2 - r^2 + 4 R^2)} \text{ArcSec}\left[\frac{2 R}{\sqrt{-H^2 + r^2}}\right] + 8 H r \sqrt{-(H - r) (H + r) (H^2 - r^2 + 4 R^2)} \left( - (r^2 + 4 R^2) \text{EllipticE}\left[\text{ArcCsc}\left[\frac{2 H R}{r \sqrt{H^2 - r^2 + 4 R^2}}\right]\right], \frac{4 R^2}{r^2}\right) + (r^2 + 2 R^2) \text{EllipticF}\left[\text{ArcCsc}\left[\frac{2 H R}{r \sqrt{H^2 - r^2 + 4 R^2}}\right], \frac{4 R^2}{r^2}\right] \right) \right) \right) / (3 r^3 \sqrt{(-H^2 + r^2) (H^2 - r^2 + 4 R^2)};$$


```

the limit CF of the cylinder of negligible diameter

```
FullSimplify[Series[CFN1BB[R, H, r], {R, 0, 6}], Assumptions -> {0 < r < H}]
```

$$\frac{(H - r) R^2}{2 H r^2} + \frac{R^4}{4 r^4} + \frac{5 R^6}{16 r^6} + O[R]^7$$

```
supefCFN1[R_, H_, r_] :=  $\frac{(H - r) R^2}{2 H r^2};$ 
```

THE DISK CASE $0 < H < 2R$

ATTENTION: THE PREVIOUS EXPRESSION OF THE CFs HAVE BEEN DIVIDED BY 4π TO GET THE CORRECT NORMALIZATION OF 1 AT R=0.

```
FullSimplify[Limit[CFDskAA[R, H, r], r -> 0, Direction -> -1], Assumptions -> {R > 0 && H > 0}]
```

```

Theta[x_] := If[x > 0, 1, 0];
NormFactCyl[R_, H_] := 4 * π^2 * R^2 * H; CFDskAA[R_, H_, r_] :=
  
$$\left( -\frac{1}{6r} \pi \left( -3 \left( r \left( 8H\pi R^2 - 4\pi r R^2 + (r^2 + 2R^2) \sqrt{-r^2 + 4R^2} \right) + 8R^2 (r^2 - R^2) \text{ArcSin}\left[\frac{r}{2R}\right] \right) + 16HR \right.$$


$$\left. (r^2 + 4R^2) \text{EllipticE}\left[\frac{r^2}{4R^2}\right] + 16HR (r^2 - 4R^2) \text{EllipticK}\left[\frac{r^2}{4R^2}\right] \right) \right) / (4H\pi^2 R^2); CFDskBB[$$

R_, H_, r_] := 
$$\left( \frac{1}{6r} \pi \left( 12\pi R^4 + \sqrt{\frac{-H^2 + r^2}{H^2 - r^2 + 4R^2}} (-H^4 + H^2 (6r^2 + 22R^2) + 3(r^4 - 2r^2R^2 - 8R^4)) - \right.$$


$$3 \left( 4\pi R^4 - r(r^2 + 2R^2) \sqrt{-r^2 + 4R^2} + 8(r - R)R^2(r + R) \text{ArcCos}\left[\frac{r}{2R}\right] \right) + 24R^2(H^2 + r^2 - R^2)$$


$$\text{ArcCos}\left[\frac{\sqrt{-H^2 + r^2}}{2R}\right] + 16HR \left( (-r^2 - 4R^2) \text{EllipticE}\left[\text{ArcSin}\left[\frac{2HR}{r\sqrt{H^2 - r^2 + 4R^2}}\right], \frac{r^2}{4R^2}\right] - \right.$$


$$\left. \left. (r^2 - 4R^2) \text{EllipticF}\left[\text{ArcSin}\left[\frac{2HR}{r\sqrt{H^2 - r^2 + 4R^2}}\right], \frac{r^2}{4R^2}\right] \right) \right) \right) / (4H\pi^2 R^2);$$

CFDskCC[R_, H_, r_] := 
$$\left( \frac{1}{6r} \pi \left( 12H^2\pi R^2 + 12\pi r^2 R^2 - 12\pi R^4 - H^2 \sqrt{(-H^2 + r^2)(H^2 - r^2 + 4R^2)} + \right.$$


$$5r^2 \sqrt{(-H^2 + r^2)(H^2 - r^2 + 4R^2)} + 26R^2 \sqrt{(-H^2 + r^2)(H^2 - r^2 + 4R^2)} -$$


$$24H^2R^2 \text{ArcSin}\left[\frac{\sqrt{-H^2 + r^2}}{2R}\right] - 24r^2R^2 \text{ArcSin}\left[\frac{\sqrt{-H^2 + r^2}}{2R}\right] + 24R^4 \text{ArcSin}\left[\frac{\sqrt{-H^2 + r^2}}{2R}\right] -$$


$$8Hr(r^2 + 4R^2) \text{EllipticE}\left[\text{ArcSin}\left[\frac{r\sqrt{H^2 - r^2 + 4R^2}}{2HR}\right], \frac{4R^2}{r^2}\right] +$$


$$\left. \left. 8Hr(r^2 - 4R^2) \text{EllipticF}\left[\text{ArcSin}\left[\frac{r\sqrt{H^2 - r^2 + 4R^2}}{2HR}\right], \frac{4R^2}{r^2}\right] \right) \right) / (4H\pi^2 R^2);$$

CFTotDsk[R_, H_, r_] := Theta[H - r] * CFTDskAA[R, H, r] + Theta[2 * R - r] * Theta[r - H] *
  CFTDskBB[R, H, r] + Theta[ $\sqrt{4 * R^2 + H^2} - r$ ] * Theta[r - 2 * R] * CFTDskAA[R, H, r];

```

The limit CF when the cylinder becomes a disk

N.B. One must multiply the cylinder CF by V, and then divide by H^2 and πR^2 (and not by $2\pi R^2$) because one must consider the area of S_1 , the limit of the cylinder.

The approximation is

$$\text{Cylinder Cf} \approx \text{SuperCf} x \left(\frac{H^2 \pi R^2}{\pi H R^2} \right) = \text{SuperCf} x (H)$$

```

FullSimplify[Series[(CFDskBB[R, H, r] * π * R^2 * H / (H^2 * π * R^2)), {H, 0, 1}],
Assumptions → {0 < R && 0 < r < 2R}]

```

```

supefCFDisk[R_, r_] := 
$$\frac{-\frac{\sqrt{-r^2 + 4R^2}}{R^2} + \frac{4 \text{ArcSec}\left[\frac{2R}{r}\right]}{r}}{4\pi};$$

NrmsupefCFDisk[R_, H_, r_] := supefCFDisk[R, r] * (H^2 * π * R^2) / (π * R^2 * H);

```

CHECK

```

Simplify[Limit[CFDskAA[R, H, r], r → 0, Direction → -1], Assumptions → {R > 0 && H > 0}]

FullSimplify[Limit[CFDskBB[R, H, r], r → H, Direction → -1] -
  Limit[CFDskAA[R, H, r], r → H, Direction → 1], Assumptions → {R > 0 && H > 0 && H < 2 R}]

?? ??????

FullSimplify[Simplify[(CFDskBB[R, H, r]) /. {r → H}, Assumptions → {0 < H < 2 R}] - Simplify[
  (CFDskAA[R, H, r]) /. {r → H}, Assumptions → {0 < H < 2 R}], Assumptions → {0 < H < 2 R}]

FullSimplify[Simplify[(CFDskCC[R, H, r]) /. {r → 2 R}, Assumptions → {0 < H < 2 R}] - Simplify[
  (CFDskBB[R, H, r]) /. {r → 2 R}, Assumptions → {0 < H < 2 R}], Assumptions → {0 < H < 2 R}]

Simplify[(CFDskBB[R, H, r]) /. {r → 2 R}, Assumptions → {0 < H < 2 R}]

Series[FullSimplify[(CFDskCC[R, H, r]) /. {r → 2 R + δ},
  Assumptions → {0 < H < 2 R && δ > 0}], {δ, 0, 2}]

FullSimplify[Limit[CFDskCC[R, H, r], r → 2 R, Direction → -1] -
  Limit[CFDskBB[R, H, r], r → 2 R, Direction → 1], Assumptions → {R > 0 && H > 0 && H < 2 R}]

FullSimplify[Limit[CFDskCC[R, H, r], r → √(4 R² + H²), Direction → 1],
  Assumptions → {R > 0 && H > 0 && H < 2 R}]

```

simplification of CFDiskBB

```

FullSimplify[Sin[π / 2 - ArcSin[√(-H² + r²) / (2 R)] - ArcCos[√(-H² + r²) / (2 R)]],
  Assumptions → {0 < H < r < 2 R}]

FullSimplify[Sin[ArcCos[r / (2 R)] - (π / 2 - ArcSin[r / (2 R)])], Assumptions → {0 < H < r < 2 R}]

Simplify[ExpandAll[(CFDskBB[R, H, r]) /. {ArcCos[√(-H² + r²) / (2 R)] → π / 2 - ArcSin[√(-H² + r²) / (2 R)],
  ArcCos[r / (2 R)] → (π / 2 - ArcSin[r / (2 R)])}], Assumptions → {0 < H < r < 2 R}]

```

$$\begin{aligned}
& \frac{1}{24 H \pi r R^2} \left(12 H^2 \pi R^2 + \text{Factor} \left[3 r^3 \sqrt{-r^2 + 4 R^2} + 6 r R^2 \sqrt{-r^2 + 4 R^2} \right] + \right. \\
& \quad \text{Factor} \left[-H^4 \sqrt{\frac{-H^2 + r^2}{H^2 - r^2 + 4 R^2}} + 6 H^2 r^2 \sqrt{\frac{-H^2 + r^2}{H^2 - r^2 + 4 R^2}} + 3 r^4 \sqrt{\frac{-H^2 + r^2}{H^2 - r^2 + 4 R^2}} + \right. \\
& \quad 22 H^2 R^2 \sqrt{\frac{-H^2 + r^2}{H^2 - r^2 + 4 R^2}} - 6 r^2 R^2 \sqrt{\frac{-H^2 + r^2}{H^2 - r^2 + 4 R^2}} - 24 R^4 \sqrt{\frac{-H^2 + r^2}{H^2 - r^2 + 4 R^2}} \left. \right] + \\
& \quad \text{Factor} \left[24 r^2 R^2 \text{ArcSin} \left[\frac{r}{2 R} \right] - 24 R^4 \text{ArcSin} \left[\frac{r}{2 R} \right] \right] + \text{Factor} \left[\right. \\
& \quad - 24 H^2 R^2 \text{ArcSin} \left[\frac{\sqrt{-H^2 + r^2}}{2 R} \right] - 24 r^2 R^2 \text{ArcSin} \left[\frac{\sqrt{-H^2 + r^2}}{2 R} \right] + 24 R^4 \text{ArcSin} \left[\frac{\sqrt{-H^2 + r^2}}{2 R} \right] \left. \right] - \\
& \quad 16 H R (r^2 + 4 R^2) \text{EllipticE} \left[\text{ArcSin} \left[\frac{2 H R}{r \sqrt{H^2 - r^2 + 4 R^2}} \right], \frac{r^2}{4 R^2} \right] - \\
& \quad \left. 16 H R (r^2 - 4 R^2) \text{EllipticF} \left[\text{ArcSin} \left[\frac{2 H R}{r \sqrt{H^2 - r^2 + 4 R^2}} \right], \frac{r^2}{4 R^2} \right] \right)
\end{aligned}$$

simplification of CFDskCC

$$\begin{aligned}
& (\text{CFDskCC}[R, H, r]) \\
& \frac{1}{24 H \pi r R^2} \left(\text{Factor} \left[12 H^2 \pi R^2 + 12 \pi r^2 R^2 - 12 \pi R^4 \right] + \text{Factor} \left[-H^2 \sqrt{(-H^2 + r^2)(H^2 - r^2 + 4 R^2)} + \right. \right. \\
& \quad 5 r^2 \sqrt{(-H^2 + r^2)(H^2 - r^2 + 4 R^2)} + 26 R^2 \sqrt{(-H^2 + r^2)(H^2 - r^2 + 4 R^2)} \left. \right] + \text{Factor} \left[\right. \\
& \quad - 24 H^2 R^2 \text{ArcSin} \left[\frac{\sqrt{-H^2 + r^2}}{2 R} \right] - 24 r^2 R^2 \text{ArcSin} \left[\frac{\sqrt{-H^2 + r^2}}{2 R} \right] + 24 R^4 \text{ArcSin} \left[\frac{\sqrt{-H^2 + r^2}}{2 R} \right] \left. \right] - \\
& \quad 8 H r (r^2 + 4 R^2) \text{EllipticE} \left[\text{ArcSin} \left[\frac{r \sqrt{H^2 - r^2 + 4 R^2}}{2 H R} \right], \frac{4 R^2}{r^2} \right] + \\
& \quad \left. 8 H r (r^2 - 4 R^2) \text{EllipticF} \left[\text{ArcSin} \left[\frac{r \sqrt{H^2 - r^2 + 4 R^2}}{2 H R} \right], \frac{4 R^2}{r^2} \right] \right)
\end{aligned}$$

$$\begin{aligned}
\text{DCFDskAA}[\mathbf{R}_-, \mathbf{H}_-, \mathbf{r}_-] &:= \\
&\left(\frac{1}{6 \mathbf{r}^2} \pi \left(3 \left(8 \mathbf{R}^4 \text{ArcCsc}\left[\frac{2 \mathbf{R}}{\mathbf{r}} \right] + \mathbf{r} \left((3 \mathbf{r}^2 - 2 \mathbf{R}^2) \sqrt{-\mathbf{r}^2 + 4 \mathbf{R}^2} - 8 \mathbf{r} \mathbf{R}^2 \text{ArcSec}\left[\frac{2 \mathbf{R}}{\mathbf{r}} \right] \right) \right) + \right. \\
&\left. 16 \mathbf{H} \mathbf{R} \left(-2 (\mathbf{r}^2 - 2 \mathbf{R}^2) \text{EllipticE}\left[\frac{\mathbf{r}^2}{4 \mathbf{R}^2} \right] + (\mathbf{r}^2 - 4 \mathbf{R}^2) \text{EllipticK}\left[\frac{\mathbf{r}^2}{4 \mathbf{R}^2} \right] \right) \right) \Big/ (4 \mathbf{H} \pi^2 \mathbf{R}^2); \\
\text{DCFDskBB}[\mathbf{R}_-, \mathbf{H}_-, \mathbf{r}_-] &:= \\
&\left(-\frac{1}{6 \mathbf{r}^3 (-\mathbf{H}^2 + \mathbf{r}^2 - 4 \mathbf{R}^2)} \pi \right. \\
&\left(\mathbf{r} \left(9 \mathbf{H}^2 \mathbf{r}^3 \sqrt{-\mathbf{r}^2 + 4 \mathbf{R}^2} - 9 \mathbf{r}^5 \sqrt{-\mathbf{r}^2 + 4 \mathbf{R}^2} - 6 \mathbf{H}^2 \mathbf{r} \mathbf{R}^2 \sqrt{-\mathbf{r}^2 + 4 \mathbf{R}^2} + \right. \right. \\
&\quad 42 \mathbf{r}^3 \mathbf{R}^2 \sqrt{-\mathbf{r}^2 + 4 \mathbf{R}^2} - 24 \mathbf{r} \mathbf{R}^4 \sqrt{-\mathbf{r}^2 + 4 \mathbf{R}^2} + \mathbf{H}^4 \sqrt{(-\mathbf{H}^2 + \mathbf{r}^2) (\mathbf{H}^2 - \mathbf{r}^2 + 4 \mathbf{R}^2)} + \\
&\quad 6 \mathbf{H}^2 \mathbf{r}^2 \sqrt{(-\mathbf{H}^2 + \mathbf{r}^2) (\mathbf{H}^2 - \mathbf{r}^2 + 4 \mathbf{R}^2)} + 9 \mathbf{r}^4 \sqrt{(-\mathbf{H}^2 + \mathbf{r}^2) (\mathbf{H}^2 - \mathbf{r}^2 + 4 \mathbf{R}^2)} - \\
&\quad 22 \mathbf{H}^2 \mathbf{R}^2 \sqrt{(-\mathbf{H}^2 + \mathbf{r}^2) (\mathbf{H}^2 - \mathbf{r}^2 + 4 \mathbf{R}^2)} - 42 \mathbf{r}^2 \mathbf{R}^2 \sqrt{(-\mathbf{H}^2 + \mathbf{r}^2) (\mathbf{H}^2 - \mathbf{r}^2 + 4 \mathbf{R}^2)} + \\
&\quad 24 \mathbf{R}^4 \sqrt{(-\mathbf{H}^2 + \mathbf{r}^2) (\mathbf{H}^2 - \mathbf{r}^2 + 4 \mathbf{R}^2)} - 24 \mathbf{R}^2 (\mathbf{r}^2 + \mathbf{R}^2) (\mathbf{H}^2 - \mathbf{r}^2 + 4 \mathbf{R}^2) \text{ArcSec}\left[\frac{2 \mathbf{R}}{\mathbf{r}} \right] - \\
&\quad 24 \mathbf{R}^2 (\mathbf{H}^4 + \mathbf{r}^4 - 3 \mathbf{r}^2 \mathbf{R}^2 - 4 \mathbf{R}^4 + \mathbf{H}^2 (-2 \mathbf{r}^2 + 3 \mathbf{R}^2)) \text{ArcSec}\left[\frac{2 \mathbf{R}}{\sqrt{-\mathbf{H}^2 + \mathbf{r}^2}} \right] \Big) - \\
&\quad 32 \mathbf{H} \mathbf{r} \mathbf{R} (\mathbf{r}^2 - 2 \mathbf{R}^2) (\mathbf{H}^2 - \mathbf{r}^2 + 4 \mathbf{R}^2) \text{EllipticE}\left[\text{ArcSin}\left[\frac{2 \mathbf{H} \mathbf{R}}{\mathbf{r} \sqrt{\mathbf{H}^2 - \mathbf{r}^2 + 4 \mathbf{R}^2}} \right], \frac{\mathbf{r}^2}{4 \mathbf{R}^2} \right] - \\
&\quad 32 \mathbf{H} \mathbf{R}^2 (-\mathbf{r}^2 + 4 \mathbf{R}^2) (\mathbf{H}^2 - \mathbf{r}^2 + 4 \mathbf{R}^2) \\
&\quad \left. \text{EllipticF}\left[\text{ArcSin}\left[\frac{\mathbf{H}}{\sqrt{\mathbf{H}^2 - \mathbf{r}^2 + 4 \mathbf{R}^2}} \right], \frac{4 \mathbf{R}^2}{\mathbf{r}^2} \right] \right) \Big/ (4 \mathbf{H} \pi^2 \mathbf{R}^2); \\
\text{DCFDskCC}[\mathbf{R}_-, \mathbf{H}_-, \mathbf{r}_-] &:= \\
&\left(\frac{1}{6 \mathbf{r}^2} \pi \left((\mathbf{H}^2 + 7 \mathbf{r}^2 - 26 \mathbf{R}^2) \sqrt{-(\mathbf{H} - \mathbf{r}) (\mathbf{H} + \mathbf{r}) (\mathbf{H}^2 - \mathbf{r}^2 + 4 \mathbf{R}^2)} + 24 \mathbf{R}^2 (-\mathbf{H}^2 + \mathbf{r}^2 + \mathbf{R}^2) \right. \right. \\
&\quad \text{ArcSec}\left[\frac{2 \mathbf{R}}{\sqrt{-\mathbf{H}^2 + \mathbf{r}^2}} \right] + 16 \mathbf{H} \mathbf{r} \left(-(\mathbf{r}^2 - 2 \mathbf{R}^2) \text{EllipticE}\left[\text{ArcCsc}\left[\frac{2 \mathbf{H} \mathbf{R}}{\mathbf{r} \sqrt{\mathbf{H}^2 - \mathbf{r}^2 + 4 \mathbf{R}^2}} \right], \frac{4 \mathbf{R}^2}{\mathbf{r}^2} \right] + \right. \\
&\quad \left. \left. (\mathbf{r}^2 - 4 \mathbf{R}^2) \text{EllipticF}\left[\text{ArcCsc}\left[\frac{2 \mathbf{H} \mathbf{R}}{\mathbf{r} \sqrt{\mathbf{H}^2 - \mathbf{r}^2 + 4 \mathbf{R}^2}} \right], \frac{4 \mathbf{R}^2}{\mathbf{r}^2} \right] \right) \right) \Big/ (4 \mathbf{H} \pi^2 \mathbf{R}^2);
\end{aligned}$$

$$\begin{aligned}
\text{DDCFDskAA}[\text{R}_-, \text{H}_-, \text{r}_-] &:= \left(\frac{1}{3 \text{r}^3} \pi \left(3 \text{r} (3 \text{r}^2 + 2 \text{R}^2) \sqrt{-\text{r}^2 + 4 \text{R}^2} - 24 \text{R}^4 \text{ArcCsc}\left[\frac{2 \text{R}}{\text{r}}\right] + \right. \right. \\
&\quad \left. \left. 8 \text{H} \text{R} \left(-2 (\text{r}^2 + 4 \text{R}^2) \text{EllipticE}\left[\frac{\text{r}^2}{4 \text{R}^2}\right] + (\text{r}^2 + 8 \text{R}^2) \text{EllipticK}\left[\frac{\text{r}^2}{4 \text{R}^2}\right] \right) \right) \right) / (4 \text{H} \pi^2 \text{R}^2); \\
\text{DDCFDskBB}[\text{R}_-, \text{H}_-, \text{r}_-] &:= \left(\frac{1}{3 \text{r}^4 (-\text{H}^2 + \text{r}^2 - 4 \text{R}^2)} \pi \left(\text{r} \left(\text{H}^4 \sqrt{(-\text{H}^2 + \text{r}^2) (\text{H}^2 - \text{r}^2 + 4 \text{R}^2)} - \right. \right. \right. \right. \\
&\quad \left. \left. \left. \left. 3 (3 \text{r}^4 - 10 \text{r}^2 \text{R}^2 - 8 \text{R}^4) \left(-\text{r} \sqrt{-\text{r}^2 + 4 \text{R}^2} + \sqrt{-(\text{H} - \text{r}) (\text{H} + \text{r}) (\text{H}^2 - \text{r}^2 + 4 \text{R}^2)} \right) - \right. \right. \right. \\
&\quad \left. \left. \left. \text{H}^2 \left(9 \text{r}^3 \sqrt{-\text{r}^2 + 4 \text{R}^2} + 2 \text{R}^2 \left(3 \text{r} \sqrt{-\text{r}^2 + 4 \text{R}^2} + 11 \sqrt{(-\text{H}^2 + \text{r}^2) (\text{H}^2 - \text{r}^2 + 4 \text{R}^2)} \right) \right) + \right. \right. \\
&\quad \left. \left. \left. 24 \text{R}^2 (\text{H}^2 - \text{r}^2 + 4 \text{R}^2) \left(-\text{R}^2 \text{ArcSec}\left[\frac{2 \text{R}}{\text{r}}\right] + (-\text{H}^2 + \text{R}^2) \text{ArcSec}\left[\frac{2 \text{R}}{\sqrt{-\text{H}^2 + \text{r}^2}}\right] \right) \right) \right) + \right. \\
&\quad \left. 16 \text{H} \text{R} (\text{H}^2 - \text{r}^2 + 4 \text{R}^2) \left(\text{r}^2 \text{R} \text{EllipticE}\left[\text{ArcSin}\left[\frac{\text{H}}{\sqrt{\text{H}^2 - \text{r}^2 + 4 \text{R}^2}}\right], \frac{4 \text{R}^2}{\text{r}^2}\right] + \right. \right. \\
&\quad \left. \left. (\text{r}^2 + 2 \text{R}^2) \left(\text{r} \text{EllipticE}\left[\text{ArcSin}\left[\frac{2 \text{H} \text{R}}{\text{r} \sqrt{\text{H}^2 - \text{r}^2 + 4 \text{R}^2}}\right], \frac{\text{r}^2}{4 \text{R}^2}\right] - \right. \right. \right. \\
&\quad \left. \left. \left. 2 \text{R} \text{EllipticF}\left[\text{ArcSin}\left[\frac{\text{H}}{\sqrt{\text{H}^2 - \text{r}^2 + 4 \text{R}^2}}\right], \frac{4 \text{R}^2}{\text{r}^2}\right] \right) \right) \right) \right) / (4 \text{H} \pi^2 \text{R}^2); \\
\text{DDCFDskCC}[\text{R}_-, \text{H}_-, \text{r}_-] &:= \left(\left(\pi \left((\text{H} - \text{r}) (\text{H} + \text{r}) (\text{H}^2 + \text{r}^2 - 26 \text{R}^2) (\text{H}^2 - \text{r}^2 + 4 \text{R}^2) + \right. \right. \right. \right. \\
&\quad \left. \left. \left. 24 (\text{H} - \text{R}) \text{R}^2 (\text{H} + \text{R}) \sqrt{-(\text{H} - \text{r}) (\text{H} + \text{r}) (\text{H}^2 - \text{r}^2 + 4 \text{R}^2)} \text{ArcSec}\left[\frac{2 \text{R}}{\sqrt{-\text{H}^2 + \text{r}^2}}\right] + \right. \right. \right. \\
&\quad \left. \left. \left. 8 \text{H} \text{r} \sqrt{-(\text{H} - \text{r}) (\text{H} + \text{r}) (\text{H}^2 - \text{r}^2 + 4 \text{R}^2)} \right. \right. \right. \\
&\quad \left. \left. \left. \left(-(\text{r}^2 + 4 \text{R}^2) \text{EllipticE}\left[\text{ArcCsc}\left[\frac{2 \text{H} \text{R}}{\text{r} \sqrt{\text{H}^2 - \text{r}^2 + 4 \text{R}^2}}\right], \frac{4 \text{R}^2}{\text{r}^2}\right] + \right. \right. \right. \right. \\
&\quad \left. \left. \left. \left. (\text{r}^2 + 2 \text{R}^2) \text{EllipticF}\left[\text{ArcCsc}\left[\frac{2 \text{H} \text{R}}{\text{r} \sqrt{\text{H}^2 - \text{r}^2 + 4 \text{R}^2}}\right], \frac{4 \text{R}^2}{\text{r}^2}\right] \right) \right) \right) \right) / \right. \\
&\quad \left. \left(3 \text{r}^3 \sqrt{(-\text{H}^2 + \text{r}^2) (\text{H}^2 - \text{r}^2 + 4 \text{R}^2)} \right) \right) / (4 \text{H} \pi^2 \text{R}^2);
\end{aligned}$$

```

Expand[supefCFDisk[R, r]]
Simplify[Series[supefCFDisk[R, r], {r, 0, 2}], Assumptions -> {0 < H && 0 < R}]

```

FIGURES FOR THE DISK AND THE NEEDLE CASES

```

VolDisk[R_, H_] := π * R^2 * H;

PolynomApprNdl[R_, H_, r_] := 1 - (2 π R^2 + 2 π R H) * r / (4 * VolDisk[R, H]) +
r^3 (2 π R^2 + 2 π R H)
(3 * (2 π R H) / ((2 * R)^2)) / (16 (2 π R^2 + 2 π R H)) / (6 * VolDisk[R, H]);
PolynomApprDisk[R_, H_, r_] := PolynomApprNdl[R, H, r];

meanradiusAA[R_, H_] := 1 / Sqrt[((2 π R H) / (R)^2) / (2 π R^2 + 2 π R H)];

```

THE NEEDLE CASE

```

With[{R = 1 / 10, H = 1}, N[R / meanradiusAA[R, H]]]

FullSimplify[Limit[CFNdlAA[R, H, r], r → 0, Direction → -1], Assumptions → {R > 0 && H > 0}]

```

```

PlottSurfCF = With[{R = 1 / 100, H = 2}, ParametricPlot[{{r, supefCFDndl[R, H, r]}},
{r, 2 R, H}, PlotRange → {{0, 2.1}, {-0.02, 1.1}},
PlotStyle → {Thickness[0.002], Red}, AspectRatio → 1]];
PlottaAA = With[{R = 1 / 100, H = 2}, ParametricPlot[{{r, CFNdlAA[R, H, r]}},
{r, 1 / 2000000, 2 R - 1 / 2000000},
PlotRange → {{0, 2.1}, {-0.02, 1.1}}, PlotStyle → {Thickness[0.004], Blue},
AspectRatio → 1, AxesLabel → {"r", "γ(r)"}]];
PlottaAaaa = With[{R = 1 / 100, H = 2}, ParametricPlot[{{r, PolynomApprNdl[R, H, r]}},
{r, 1 / 2000000, 2 R - 1 / 2000000},
PlotRange → {{0, 2.1}, {-0.02, 1.1}}, PlotStyle → {Thickness[0.002], Green},
AspectRatio → 1, AxesLabel → {"r", "γ(r)"}]];
PlottaAB = With[{R = 1 / 100, H = 2}, ParametricPlot[{{r, CFNdlBB[R, H, r]}},
{r, 2 R - 1 / 2000000, H - 1 / 2000000}, PlotRange → {{0, 2.1}, {-0.02, 1.1}},
PlotStyle → {Thickness[0.004], Blue}, AspectRatio → 1]];
PlottaAC = With[{R = 1 / 100, H = 2}, ParametricPlot[{{r, CFNdlCC[R, H, r]}},
{r, H + 1 / 2000000, Sqrt[4 R^2 + H^2]}, PlotRange → {{0, 2.1}, {-0.02, 1.1}},
PlotStyle → {Thickness[0.004], Blue}, AspectRatio → 1]]

```

```

PlottRsupCF =
With[{R = 1 / 100, H = 2}, ParametricPlot[{{r, 10000 * (r^2 * supefCFDndl[R, H, r])}},
{r, H, 2 R}, PlotRange → {{0, 2.1}, {-0.02, 1.1}},
PlotStyle → {Thickness[0.002], Dashing[{0.04, 0.01}], Red}, AspectRatio → 1]];
PlottRsupCFaa = With[{R = 1 / 100, H = 2},
ParametricPlot[{{r, 10000 * (r^2 * PolynomApprNdl[R, H, r])}},
{r, 0, 2 R}, PlotRange → {{0, 2.1}, {-0.02, 1.1}}, PlotStyle →
{Thickness[0.002], Dashing[{0.04, 0.01}], Green}, AspectRatio → 1]]; PlottRCFNd1AA =
With[{R = 1 / 100, H = 2}, ParametricPlot[{{r, 10000 * (r^2 * CFNdlAA[R, H, r])}},
{r, 0, 2 R - 1 / 1000000}, PlotRange → {{0, 2.1}, {-0.02, 1.1}},
PlotStyle → {Thickness[0.002], Dashing[{0.04, 0.01}], Blue}, AspectRatio → 1]];
PlottRscFNd1BB = With[{R = 1 / 100, H = 2},
ParametricPlot[{{r, 10000 * (r^2 * CFNdlBB[R, H, r])}},
{r, 2 R + 1 / 1000000, H - 1 / 1000000}, PlotRange → {{0, 2.1}, {-0.02, 1.1}},
PlotStyle → {Thickness[0.002], Dashing[{0.04, 0.01}], Blue}, AspectRatio → 1]];
PlottRscFNd1CC = With[{R = 1 / 100, H = 2},
ParametricPlot[{{r, 10000 * r^2 * CFNdlCC[R, H, r]}},
{r, H + 1 / 1000000, √[4 R^2 + H^2]}, PlotRange → {{0, 2.1}, {-0.02, 1.1}},
PlotStyle → {Thickness[0.002], Dashing[{0.04, 0.01}], Blue}, AspectRatio → 1]];
Show[PlottRsupCF, PlottRCFNd1AA, PlottRscFNd1BB, PlottRsupCF, PlottRsupCFaa]

```

```

PlotDiffaa = With[{R = 1 / 200, H = 1},
  ParametricPlot[{{r, CFNdlAA[R, H, r] - PolynomApprNdl[R, H, r]}},
    {r, 0, H - 1 / 1000000}, PlotRange -> {{0, 1.1}, {-0.02, 1.1}},
    PlotStyle -> {Thickness[0.004], Dashing[{0.01, 0.01}], Magenta}, AspectRatio -> 1]];
PlotDiffbb = With[{R = 1 / 200, H = 1}, ParametricPlot[
  {{r, CFNdlBB[R, H, r] - supefCFDndl[R, H, r]}},
  {r, H + 1 / 1000000, 2 R - 1 / 1000000}, PlotRange -> {{0, 1.1}, {-0.02, 1.1}},
  PlotStyle -> {Thickness[0.004], Dashing[{0.01, 0.01}], Magenta}, AspectRatio -> 1]];

needlplot = Show[PlottaAA, PlottaBB, PlottaCC,
  PlottSurfCF, PlotDiffaa, PlotDiffbb, PlottRsupCF, PlottRsupCFaa,
  PlottaAaaa, PlottRCFNd1AA, PlottRsCFNdlBB, PlottRsCFNdlCC]

```

```

PlotSurfCF = With[{R = 1 / 20, H = 1}, ParametricPlot[{{r, supefCFDndl[R, H, r]}},
  {r, 2 R, H}, PlotRange -> {{0, 1.1}, {-0.02, 1.1}},
  PlotStyle -> {Thickness[0.002], Red}, AspectRatio -> 1];
PlottaAA = With[{R = 1 / 20, H = 1}, ParametricPlot[{{r, CFNdlAA[R, H, r]}},
  {r, 1 / 2000000, 2 R - 1 / 2000000},
  PlotRange -> {{0, 1.1}, {-0.02, 1.1}}, PlotStyle -> {Thickness[0.004], Blue},
  AspectRatio -> 1, AxesLabel -> {"r", "γ(r)"}];
PlottaAaaa = With[{R = 1 / 20, H = 1}, ParametricPlot[{{r, PolynomApprNdl[R, H, r]}},
  {r, 1 / 2000000, 2 R - 1 / 2000000}, PlotRange -> {{0, 1.1}, {-0.02, 1.1}}, PlotStyle ->
  {Thickness[0.002], Green}, AspectRatio -> 1, AxesLabel -> {"r", "γ(r)"}];
PlottaBB = With[{R = 1 / 20, H = 1}, ParametricPlot[{{r, CFNdlBB[R, H, r]}},
  {r, 2 R - 1 / 2000000, H - 1 / 2000000}, PlotRange -> {{0, 1.1}, {-0.02, 1.1}},
  PlotStyle -> {Thickness[0.004], Blue}, AspectRatio -> 1];
PlottaCC = With[{R = 1 / 20, H = 1}, ParametricPlot[{{r, CFNdlCC[R, H, r]}},
  {r, H + 1 / 2000000, Sqrt[4 R^2 + H^2]}, PlotRange -> {{0, 1.1}, {-0.02, 1.1}},
  PlotStyle -> {Thickness[0.004], Blue}, AspectRatio -> 1];

```

```
Show[PlottaAA, PlottaBB, PlottaCC, PlotSurfCF, PlottaAaaa]
```

```

PlotDiffaa = With[{R = 1 / 20, H = 1},
  ParametricPlot[{{r, CFNdlAA[R, H, r] - PolynomApprNdl[R, H, r]}},
    {r, 0, H - 1 / 1000000}, PlotRange -> {{0, 1.1}, {-0.02, 1.1}},
    PlotStyle -> {Thickness[0.004], Dashing[{0.01, 0.01}], Magenta}, AspectRatio -> 1];
PlotDiffbb = With[{R = 1 / 20, H = 1}, ParametricPlot[
  {{r, CFNdlBB[R, H, r] - supefCFDndl[R, H, r]}},
  {r, H + 1 / 1000000, 2 R - 1 / 1000000}, PlotRange -> {{0, 1.1}, {-0.02, 1.1}},
  PlotStyle -> {Thickness[0.004], Dashing[{0.01, 0.01}], Magenta}, AspectRatio -> 1]]

needlplot = Show[PlottaAA, PlottaBB, PlottaCC,
  PlottSurfCF, PlotDiffaa, PlotDiffbb, PlottRsupCF, PlottRsupCFaa,
  PlottaAaaa, PlottRCFNd1AA, PlottRsCFNdlBB, PlottRsCFNdlCC]

```

```

needlplot = Show[PlottaAA, PlottaBB, PlottaCC,
  PlottSurfCF, PlotDiffaa, PlotDiffbb, PlottRsupCF, PlottRsupCFaa,
  PlottaAaaa, PlottRCFNd1AA, PlottRsCFNdlBB, PlottRsCFNdlCC]

```

```
Export["FigNdl.eps", needlplot]
```

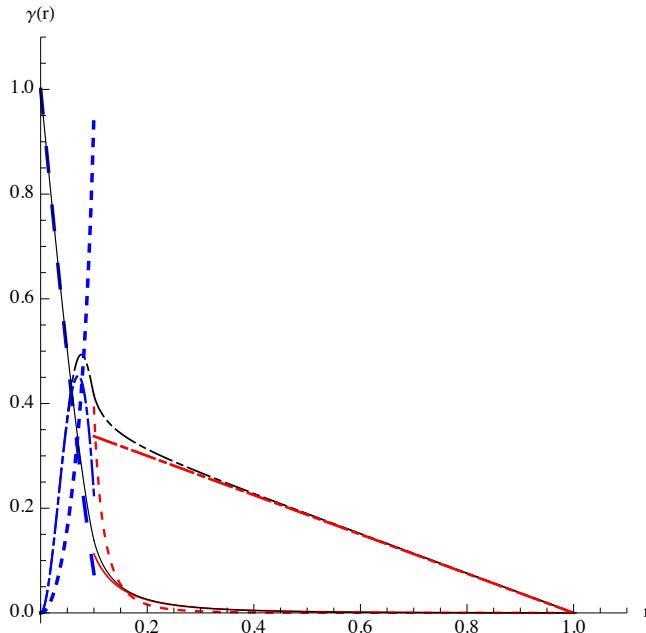
```
Show[PlottaAaaa, PlottaAA, PlotSurfCF, PlottaBB, PlottaCC]
```

```

PlottDiffaa = With[{R = 1 / 20, H = 1},
  ParametricPlot[{{r, 15 * (CFNdlAA[R, H, r] - PolynomApprNdl[R, H, r])}},
  {r, 0, H - 1 / 1000000}, PlotRange -> {{0, 1.1}, {-0.02, 1.1}},
  PlotStyle -> {Thickness[0.006], Dashing[{0.01, 0.02}], Blue}, AspectRatio -> 1]];
PlottDiffbb = With[{R = 1 / 20, H = 1}, ParametricPlot[
  {{r, 15 * (CFNdlBB[R, H, r] - supefCFDndl[R, H, r])}},
  {r, H + 1 / 1000000, 2 R - 1 / 1000000}, PlotRange -> {{0, 1.1}, {-0.02, 1.1}},
  PlotStyle -> {Thickness[0.004], Dashing[{0.01, 0.02}], Red}, AspectRatio -> 1]];
PlottSurfCF = With[{R = 1 / 20, H = 1}, ParametricPlot[{{r, supefCFDndl[R, H, r]}},
  {r, 2 R, H}, PlotRange -> {{0, 1.1}, {-0.02, 1.1}},
  PlotStyle -> {Thickness[0.003], Red}, AspectRatio -> 1];
PlottaAA = With[{R = 1 / 20, H = 1}, ParametricPlot[{{r, CFNdlAA[R, H, r]}},
  {r, 1 / 2000000, 2 R - 1 / 2000000},
  PlotRange -> {{0, 1.1}, {-0.02, 1.1}}, PlotStyle -> {Thickness[0.002], Black},
  AspectRatio -> 1, AxesLabel -> {"r", "Y(r)"}]; (* *)
PlottaAaaa = With[{R = 1 / 20, H = 1}, ParametricPlot[{{r, PolynomApprNdl[R, H, r]}},
  {r, 1 / 2000000, 2 R - 1 / 2000000}, PlotRange -> {{0, 1.1}, {-0.02, 1.1}},
  PlotStyle -> {Thickness[0.006], Blue,
  Dashing[{0.04, 0.06}]}
  , AspectRatio -> 1, AxesLabel -> {"r", "Y(r)"}]]; (* *)
PlottaaBB = With[{R = 1 / 20, H = 1}, ParametricPlot[{{r, CFNdlBB[R, H, r]}},
  {r, 2 R - 1 / 2000000, H - 1 / 2000000}, PlotRange -> {{0, 1.1}, {-0.02, 1.1}},
  PlotStyle -> {Thickness[0.002], Black}, AspectRatio -> 1]];

PlottaACC = With[{R = 1 / 20, H = 1}, ParametricPlot[{{r, CFNdlCC[R, H, r]}},
  {r, H + 1 / 2000000, Sqrt[4 R^2 + H^2]}, PlotRange -> {{0, 1.1}, {-0.02, 1.1}},
  PlotStyle -> {Thickness[0.002], Black}, AspectRatio -> 1]]; PlottRsupCF =
With[{R = 1 / 20, H = 1}, ParametricPlot[{{r, 300 * r^2 * supefCFDndl[R, H, r]}},
  {r, H, 2 R}, PlotRange -> {{0, 1.1}, {-0.02, 1.1}},
  PlotStyle -> {Thickness[0.004], Dashing[{0.04, 0.01, 0.01, 0.01}], Red},
  AspectRatio -> 1]]; PlottRsupCfaa =
With[{R = 1 / 20, H = 1}, ParametricPlot[{{r, 300 * r^2 * PolynomApprNdl[R, H, r]}},
  {r, 0, 2 R}, PlotRange -> {{0, 1.1}, {-0.02, 1.1}},
  PlotStyle -> {Thickness[0.004], Dashing[{0.04, 0.01, 0.01, 0.01}], Blue},
  AspectRatio -> 1]]; PlottRCFNd1AA =
With[{R = 1 / 20, H = 1}, ParametricPlot[{{r, 300 * r^2 * CFNdlAA[R, H, r]}},
  {r, 0, 2 R - 1 / 1000000}, PlotRange -> {{0, 1.1}, {-0.02, 1.1}}, PlotStyle ->
  {Thickness[0.003], Dashing[{0.04, 0.01, 0.01, 0.01}], Black}, AspectRatio -> 1]];
PlottRscFnldBB = With[{R = 1 / 20, H = 1}, ParametricPlot[
  {{r, 300 * r^2 * CFNdlBB[R, H, r]}},
  {r, 2 R + 1 / 1000000, H - 1 / 1000000}, PlotRange -> {{0, 1.1}, {-0.02, 1.1}},
  PlotStyle -> {Thickness[0.003], Dashing[{0.04, 0.01, 0.01, 0.01}], Black},
  AspectRatio -> 1]]; PlottRscFnldCC =
With[{R = 1 / 20, H = 1}, ParametricPlot[{{r, 300 * r^2 * CFNdlCC[R, H, r]}},
  {r, H + 1 / 1000000, Sqrt[4 R^2 + H^2]}, PlotRange -> {{0, 1.1}, {-0.02, 1.1}},
  PlotStyle -> {Thickness[0.002], Dashing[{0.04, 0.01}], Blue}, AspectRatio -> 1]];
needlplot = Show[PlottaAaaa, PlottaAA, PlottSurfCF, PlottaaBB, PlottaACC, PlottRsupCF,
  PlottRCFNd1AA, PlottRscFnldBB, PlottRsupCfaa, PlottDiffaa, PlottDiffbb]

```



```
Export["Fig3.eps", needlplot]
```

Fig3.eps

THE DISK CASE

```
With[{R = 1, H = 2 / 10}, N[H / meanradiusAA[R, H]]]

0.0816497

FullSimplify[Limit[CFDskAA[R, H, r], r → 0, Direction → -1], Assumptions → {R > 0 && H > 0}]

PlottSurfCF = With[{R = 1, H = 0.2}, ParametricPlot[{{r, supefCFDisk[R, H, r]}},
{r, H + 1 / 1000000, 2 R - 1 / 1000000}, PlotRange → {{0, 2.5}, {-0.02, 1.1}},
PlotStyle → {Thickness[0.004], Red}, AspectRatio → 1]];
PlottaAA = With[{R = 1, H = 0.2}, ParametricPlot[{{r, CFDskAA[R, H, r]}},
{r, 1 / 1000000, H - 1 / 1000000}, PlotRange → {{0, 2.5}, {-0.02, 1.1}},
PlotStyle → {Thickness[0.004], Blue}, AspectRatio → 1, AxesLabel → {"r", "γ(r)"}]];
PlottaAAaa = With[{R = 1, H = 0.2}, ParametricPlot[{{r, PolynomApprDisk[R, H, r]}},
{r, 1 / 1000000, H - 1 / 1000000}, PlotRange → {{0, 2.5}, {-0.02, 1.1}},
PlotStyle → {Thickness[0.004], Green}, AspectRatio → 1, AxesLabel → {"r", "γ(r)"}]];
PlottaAB = With[{R = 1, H = 0.2}, ParametricPlot[{{r, CFDskBB[R, H, r]}},
{r, H + 1 / 1000000, 2 R - 1 / 1000000}, PlotRange → {{0, 2.5}, {-0.02, 1.1}},
PlotStyle → {Thickness[0.004], Blue}, AspectRatio → 1]];
PlottaAC = With[{R = 1, H = 0.2}, ParametricPlot[{{r, CFDskCC[R, H, r]}},
{r, 2 R + 1 / 1000000, Sqrt[4 R^2 + H^2]}, PlotRange → {{0, 2.5}, {-0.02, 1.1}},
PlotStyle → {Thickness[0.004], Blue}, AspectRatio → 1]];

Show[PlottaAA, PlottaAB, PlottaAC, PlottSurfCF, PlottaAAaa];
```

```

PlottRsupCF = With[{R = 1, H = 0.2}, ParametricPlot[{{r, 10 * r * supefCFDisk[R, H, r]}},
{r, H, 2 R}, PlotRange -> {{0, 2.5}, {-0.02, 1.1}}, PlotStyle ->
{Thickness[0.002], Dashing[{0.04, 0.01}], Red}, AspectRatio -> 1]]; PlottRsupCFaa =
With[{R = 1, H = 0.2}, ParametricPlot[{{r, 10 * r * PolynomApprDisk[R, H, r]}},
{r, 0, H}, PlotRange -> {{0, 2.5}, {-0.02, 1.1}}, PlotStyle -> {Thickness[0.002], Dashing[{0.04, 0.01}], Green}, AspectRatio -> 1]];
PlottRCFDskAA = With[{R = 1, H = 0.2}, ParametricPlot[{{r, 10 * r * CFDskAA[R, H, r]}},
{r, 0, H}, PlotRange -> {{0, 2.5}, {-0.02, 1.1}}, PlotStyle -> {Thickness[0.002], Dashing[{0.04, 0.01}], Blue}, AspectRatio -> 1]];
PlottRsCFDskBB = With[{R = 1, H = 0.2}, ParametricPlot[{{r, 10 * r * CFDskBB[R, H, r]}},
{r, H, 2 R}, PlotRange -> {{0, 2.5}, {-0.02, 1.1}}, PlotStyle -> {Thickness[0.002], Dashing[{0.04, 0.01}], Blue}, AspectRatio -> 1]];
PlottRsCFDskCC = With[{R = 1, H = 0.2}, ParametricPlot[{{r, 10 * r * CFDskCC[R, H, r]}},
{r, 2 R, Sqrt[4 R^2 + H^2]}, PlotRange -> {{0, 2.5}, {-0.02, 1.1}}, PlotStyle -> {Thickness[0.002], Dashing[{0.04, 0.01}], Blue}, AspectRatio -> 1]];
Show[PlottRsupCF, PlottRCFDskAA, PlottRsCFDskBB, PlottRsCFDskCC, PlottRsupCFaa];

PlottDiffaa = With[{R = 1, H = 0.2},
ParametricPlot[{{r, (CFDskAA[R, H, r] - PolynomApprDisk[R, H, r])}},
{r, 0, H}, PlotRange -> {{0, 2.5}, {-0.02, 1.1}}, PlotStyle -> {Thickness[0.004], Dashing[{0.01, 0.01}], Magenta}, AspectRatio -> 1]];
PlottDiffbb = With[{R = 1, H = 0.2}, ParametricPlot[
{{r, (CFDskBB[R, H, r] - supefCFDisk[R, H, r])}},
{r, H, 2 R}, PlotRange -> {{0, 2.5}, {-0.02, 1.1}}, PlotStyle -> {Thickness[0.004], Dashing[{0.01, 0.01}], Magenta}, AspectRatio -> 1]];

diskplot = Show[PlottaAA, PlottaBB, PlottaCC, PlottSurfCF, PlottDiffaa, PlottDiffbb,
PlottRsupCF, PlottRsupCFaa, PlottaAAaa, PlottRCFDskAA, PlottRsCFDskBB, PlottRsCFDskCC]

Export["FigDisk.eps", diskplot]

FigDisk.eps

```

```

PlottSurfCF = With[{R = 1, H = 0.2}, ParametricPlot[{{r, NrmsupefCFDisk[R, H, r]}},
{r, H + 1/1000000, 2 R - 1/1000000}, PlotRange -> {{0, 2.5}, {-0.02, 1.1}},
PlotStyle -> {Thickness[0.004], Red}, AspectRatio -> 1];

PlottaAA = With[{R = 1, H = 0.2}, ParametricPlot[{{r, CFDskAA[R, H, r]}},
{r, 1/1000000, H - 1/1000000}, PlotRange -> {{0, 2.5}, {-0.02, 1.1}}, PlotStyle ->
{Thickness[0.002], Black}, AspectRatio -> 1, AxesLabel -> {"r", "y(r)"}];
PlottaAAaa = With[{R = 1, H = 0.2}, ParametricPlot[{{r, PolynomApprDisk[R, H, r]}},
{r, 1/1000000, H - 1/1000000},
PlotRange -> {{0, 2.5}, {-0.02, 1.1}}, PlotStyle -> {Thickness[0.006], Blue,
Dashing[{0.04, 0.06}]}, AspectRatio -> 1, AxesLabel -> {"r", "y(r)"}];
PlottaAB = With[{R = 1, H = 0.2}, ParametricPlot[{{r, CFDskBB[R, H, r]}},
{r, H + 1/1000000, 2 R - 1/1000000}, PlotRange -> {{0, 2.5}, {-0.02, 1.1}},
PlotStyle -> {Thickness[0.002], Black}, AspectRatio -> 1];
PlottaAC = With[{R = 1, H = 0.2}, ParametricPlot[{{r, CFDskCC[R, H, r]}},
{r, 2 R + 1/1000000, Sqrt[4 R^2 + H^2]}, PlotRange -> {{0, 2.5}, {-0.02, 1.1}},
PlotStyle -> {Thickness[0.002], Black}, AspectRatio -> 1];

Show[PlottSurfCF, PlottaAAaa, PlottaAA, PlottaAB, PlottaAC]

```

```

PlottRsupCF =
With[{R = 1, H = 0.2}, ParametricPlot[{{r, 10 * r * NrmsupefCFDisk[R, H, r]}},
{r, H, 2 R}, PlotRange -> {{0, 2.5}, {-0.02, 1.1}},
PlotStyle -> {Thickness[0.004], Dashing[{0.04, 0.01, 0.01, 0.01}], Red},
AspectRatio -> 1]];
PlottRsupCFaa =
With[{R = 1, H = 0.2}, ParametricPlot[{{r, 10 * r * PolynomApprDisk[R, H, r]}},
{r, 0, H}, PlotRange -> {{0, 2.5}, {-0.02, 1.1}}, PlotStyle ->
{Thickness[0.004], Dashing[{0.04, 0.01, 0.01, 0.01}], Blue}, AspectRatio -> 1]];
PlottRCFDskAA = With[{R = 1, H = 0.2}, ParametricPlot[{{r, 10 * r * CFDskAA[R, H, r]}},
{r, 0, H}, PlotRange -> {{0, 2.5}, {-0.02, 1.1}}, PlotStyle ->
{Thickness[0.003], Dashing[{0.04, 0.01, 0.01, 0.01}], Black}, AspectRatio -> 1]];
PlottRscFDskBB = With[{R = 1, H = 0.2}, ParametricPlot[{{r, 10 * r * CFDskBB[R, H, r]}},
{r, H, 2 R}, PlotRange -> {{0, 2.5}, {-0.02, 1.1}}, PlotStyle ->
{Thickness[0.003], Dashing[{0.04, 0.01, 0.01, 0.01}], Black}, AspectRatio -> 1]];
PlottRscFDskCC = With[{R = 1, H = 0.2}, ParametricPlot[{{r, 10 * r * CFDskCC[R, H, r]}},
{r, 2 R, Sqrt[4 R^2 + H^2]}, PlotRange -> {{0, 2.5}, {-0.02, 1.1}}, PlotStyle ->
{Thickness[0.003], Dashing[{0.04, 0.01, 0.01, 0.01}], Black}, AspectRatio -> 1]];
Show[PlottRsupCF, PlottRCFDskAA, PlottRscFDskBB, PlottRscFDskCC, PlottRsupCFaa];

```

```

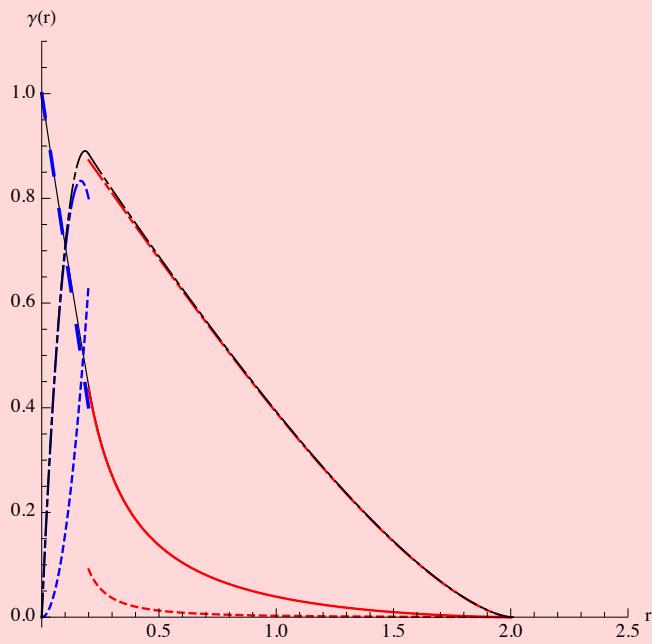
PlottDiffaa = With[{R = 1, H = 0.2},
ParametricPlot[{{r, 15 * (CFDskAA[R, H, r] - PolynomApprDisk[R, H, r])}},
{r, 0, H}, PlotRange -> {{0, 2.5}, {-0.02, 1.1}},
PlotStyle -> {Thickness[0.004], Dashing[{0.01, 0.01}], Blue}, AspectRatio -> 1];
PlottDiffbb = With[{R = 1, H = 0.2}, ParametricPlot[
{{r, 15 * (CFDskBB[R, H, r] - NrmsupefCFDisk[R, H, r])}},
{r, H, 2 R}, PlotRange -> {{0, 2.5}, {-0.02, 1.1}},
PlotStyle -> {Thickness[0.004], Dashing[{0.01, 0.01}], Red}, AspectRatio -> 1];

```

```

diskplot = Show[PlottaAA, PlottaBB, PlottaCC,
PlottSurfCF, PlottDiffaa, PlottDiffbb, PlottRsupCF, PlottRsupCFaa,
PlottaAAaa, PlottRCFDskAA, PlottRscFDskBB, PlottRscFDskCC]

```



```
Export["Fig2.eps", diskplot]
```

Fig2.eps

Merano October 16 2015

After receiving the email of Gille I check that eq.s (100) - (107), reported in the submitted ms, do really coincide with expressions reported above.

```

CFNdlAA[R_, H_, r_] := 
  
$$\left( 2 \pi^2 R^2 (2 H - r) + \frac{\pi (r^2 + 2 R^2)}{2} \sqrt{-r^2 + 4 R^2} + \frac{4 \pi R}{3 r} \left( 3 R (r^2 - R^2) \text{ArcSin}\left[\frac{r}{2 R}\right] - \right. \right. \\
  \left. \left. 2 H (r^2 + 4 R^2) \text{EllipticE}\left[\frac{r^2}{4 R^2}\right] - 2 H (r^2 - 4 R^2) \text{EllipticK}\left[\frac{r^2}{4 R^2}\right] \right) \right) / (4 H \pi^2 R^2);$$


CFNdlBB[R_, H_, r_] := 
  
$$\left( \frac{2 \pi^2 R^2 (2 H r - R^2)}{r} - \frac{4 \pi H (r^2 + 4 R^2)}{3} \text{EllipticE}\left[\frac{4 R^2}{r^2}\right] + \right. \\
  \left. \frac{4 \pi H (r^2 - 4 R^2)}{3} \text{EllipticK}\left[\frac{4 R^2}{r^2}\right] \right) / (4 H \pi^2 R^2);$$


CFNdlCC[R_, H_, r_] := 
  
$$\left( \frac{1}{6 r} \pi \left( 12 H^2 \pi R^2 + 12 \pi r^2 R^2 - 12 \pi R^4 - H^2 \sqrt{(-H^2 + r^2) (H^2 - r^2 + 4 R^2)} + \right. \right. \\
  5 r^2 \sqrt{(-H^2 + r^2) (H^2 - r^2 + 4 R^2)} + 26 R^2 \sqrt{(-H^2 + r^2) (H^2 - r^2 + 4 R^2)} - \\
  24 H^2 R^2 \text{ArcSin}\left[\frac{\sqrt{-H^2 + r^2}}{2 R}\right] - 24 r^2 R^2 \text{ArcSin}\left[\frac{\sqrt{-H^2 + r^2}}{2 R}\right] + 24 R^4 \text{ArcSin}\left[\frac{\sqrt{-H^2 + r^2}}{2 R}\right] - \\
  8 H r (r^2 + 4 R^2) \text{EllipticE}\left[\text{ArcSin}\left[\frac{r \sqrt{H^2 - r^2 + 4 R^2}}{2 H R}\right], \frac{4 R^2}{r^2}\right] + \\
  \left. \left. 8 H r (r^2 - 4 R^2) \text{EllipticF}\left[\text{ArcSin}\left[\frac{r \sqrt{H^2 - r^2 + 4 R^2}}{2 H R}\right], \frac{4 R^2}{r^2}\right] \right) \right) / (4 H \pi^2 R^2);$$


CFTotNdl[R_, H_, r_] := Theta[2 * R - r] * CFNdlAA[R, H, r] + Theta[H - r] * Theta[r - 2 * R] * \\
  CFNdlBB[R, H, r] + Theta[\sqrt{H^2 + 4 * R^2} - r] * Theta[r - H] * CFNdlCC[R, H, r];

```

```

Theta[x_] := If[x > 0, 1, 0];
NormFactCyl[R_, H_] := 4 * π^2 * R^2 * H; CFDskAA[R_, H_, r_] :=
  
$$\left( -\frac{1}{6r} \pi \left( -3 \left( r \left( 8H\pi R^2 - 4\pi r R^2 + (r^2 + 2R^2) \sqrt{-r^2 + 4R^2} \right) + 8R^2 (r^2 - R^2) \text{ArcSin}\left[\frac{r}{2R}\right] \right) + 16HR \right.$$


$$\left. (r^2 + 4R^2) \text{EllipticE}\left[\frac{r^2}{4R^2}\right] + 16HR (r^2 - 4R^2) \text{EllipticK}\left[\frac{r^2}{4R^2}\right] \right) \right) / (4H\pi^2 R^2); CFDskBB[$$

R_, H_, r_] := 
$$\left( \frac{1}{6r} \pi \left( 12\pi R^4 + \sqrt{\frac{-H^2 + r^2}{H^2 - r^2 + 4R^2}} (-H^4 + H^2 (6r^2 + 22R^2) + 3(r^4 - 2r^2R^2 - 8R^4)) - \right.$$


$$3 \left( 4\pi R^4 - r(r^2 + 2R^2) \sqrt{-r^2 + 4R^2} + 8(r - R)R^2(r + R) \text{ArcCos}\left[\frac{r}{2R}\right] \right) + 24R^2(H^2 + r^2 - R^2)$$


$$\text{ArcCos}\left[\frac{\sqrt{-H^2 + r^2}}{2R}\right] + 16HR \left( (-r^2 - 4R^2) \text{EllipticE}\left[\text{ArcSin}\left[\frac{2HR}{r\sqrt{H^2 - r^2 + 4R^2}}\right], \frac{r^2}{4R^2}\right] - \right.$$


$$\left. \left. (r^2 - 4R^2) \text{EllipticF}\left[\text{ArcSin}\left[\frac{2HR}{r\sqrt{H^2 - r^2 + 4R^2}}\right], \frac{r^2}{4R^2}\right] \right) \right) / (4H\pi^2 R^2);$$

CFDskCC[R_, H_, r_] := 
$$\left( \frac{1}{6r} \pi \left( 12H^2\pi R^2 + 12\pi r^2 R^2 - 12\pi R^4 - H^2 \sqrt{(-H^2 + r^2)(H^2 - r^2 + 4R^2)} + \right.$$


$$5r^2 \sqrt{(-H^2 + r^2)(H^2 - r^2 + 4R^2)} + 26R^2 \sqrt{(-H^2 + r^2)(H^2 - r^2 + 4R^2)} -$$


$$24H^2R^2 \text{ArcSin}\left[\frac{\sqrt{-H^2 + r^2}}{2R}\right] - 24r^2R^2 \text{ArcSin}\left[\frac{\sqrt{-H^2 + r^2}}{2R}\right] + 24R^4 \text{ArcSin}\left[\frac{\sqrt{-H^2 + r^2}}{2R}\right] -$$


$$8Hr(r^2 + 4R^2) \text{EllipticE}\left[\text{ArcSin}\left[\frac{r\sqrt{H^2 - r^2 + 4R^2}}{2HR}\right], \frac{4R^2}{r^2}\right] +$$


$$\left. \left. 8Hr(r^2 - 4R^2) \text{EllipticF}\left[\text{ArcSin}\left[\frac{r\sqrt{H^2 - r^2 + 4R^2}}{2HR}\right], \frac{4R^2}{r^2}\right] \right) \right) / (4H\pi^2 R^2);$$

CFTotDsk[R_, H_, r_] := Theta[H - r] * CFTDskAA[R, H, r] + Theta[2 * R - r] * Theta[r - H] *
CFTDskBB[R, H, r] + Theta[ $\sqrt{4 * R^2 + H^2} - r$ ] * Theta[r - 2 * R] * CFTDskAA[R, H, r];

```

```

Delta1[R_, r_] :=  $\sqrt{4 * R^2 - r^2}$ ; Delta2[r_, h_] :=  $\sqrt{r^2 - h^2}$ ;
Delta3[r_, R_, h_] :=  $\sqrt{4 * R^2 + h^2 - r^2}$ ;
csi[r_, R_, h_] := r / (2 * R);
zeta[r_, R_, h_] := 2 * h * R / (r * Delta3[r, R, h]) (* this was wrong *);
eta[r_, R_, h_] := Delta1[R, r] / (2 * R);
phi1[r_, R_, h_] := ArcSin[csi[r, R, h]];
phi2[r_, R_, h_] := ArcSin[eta[r, R, h]];
phi3[r_, R_, h_] := ArcSin[zeta[r, R, h]];
phi4[r_, R_, h_] := ArcSin[Delta2[r, h] / (2 * R)];
phi5[r_, R_, h_] := ArcSin[r * Delta3[r, R, h] / (2 * h * R)];
Vic[R_, h_] := π * R^2 * h;
GA[r_, R_, h_] := (1 / (24 * r * Vic[R, h])) *
(3 * (r * (4 * π * R^2 * (2 * h - r) +
(r^2 + 2 * R^2) * Delta1[R, r]) +
8 * R^2 * (r^2 - R^2) * phi1[r, R, h]) -
16 * h * R * ((r^2 + 4 * R^2) * EllipticE[csi[r, R, h]^2] +
(r^2 - 4 * R^2) * EllipticK[csi[r, R, h]^2]));

```

```

GC[r_, R_, h_] := (1 / (24 * r * Vic[R, h])) * (12 * π * R^2 * (h^2 + r^2 - R^2) -
(h^2 - 5 * r^2 - 26 * R^2) * Delta2[r, h] * Delta3[r, R, h] -
24 * R^2 * (h^2 + r^2 - R^2) * phi4[r, R, h] -
8 * h * r * ((r^2 + 4 * R^2) * EllipticE[phi5[r, R, h], 1 / csi[r, R, h]^2] -
(r^2 - 4 * R^2) * EllipticF[phi5[r, R, h], 1 / csi[r, R, h]^2]));

```

CHECK OF THE INNERMOST REGION

Eq. (104), i.e. $GA[r, R, h]$, coincides with the needle and the disk expressions of the cylinder CF

```

FullSimplify[CFNd1AA[R, h, r] - GA[r, R, h], Assumptions → {0 < r < 2R < h}]
FullSimplify[CFDskAA[R, h, r] - GA[r, R, h], Assumptions → {0 < r < 2R < h}]
0
0

```

CHECK OF THE OUTERMOST REGION

The corrected Eq. (104), i.e. the above $GC[r, R, h]$, coincides with the needle and the disk expressions of the cylinder CF

```

FullSimplify[CFNd1CC[R, h, r] - GC[r, R, h],
Assumptions → {0 < 2R < h < r < √(4 * R^2 + h^2)}]
FullSimplify[CFDskCC[R, h, r] - GC[r, R, h], Assumptions → {0 < h < 2R < r < √(4 * R^2 + h^2)}]
0
0

```

CHECK OF EQUATION (106)

(this was wrong. The correc equation is:)

```

Eq106NEW[R_, h_, r_] := (1 / (24 * r * Vic[R, h])) * (12 π * h^2 * R^2 +
3 * r * (r^2 + 2 * R^2) * Delta1[R, r] + (Delta2[r, h] / Delta3[r, R, h]) *
(3 * r^4 - h^4 + 6 * r^2 * (h^2 - R^2) + 22 * h^2 * R^2 - 24 * R^4) +
24 * R^2 * (r^2 - R^2) * phi1[r, R, h] -
24 * R^2 * (h^2 + r^2 - R^2) * phi4[r, R, h] -
16 * h * R * ((r^2 + 4 * R^2) * EllipticE[phi3[r, R, h], csi[r, R, h]^2] +
(r^2 - 4 * R^2) * EllipticF[phi3[r, R, h], csi[r, R, h]^2]));

```

```

FullSimplify[CFDskBB[R, h, r] - Eq106NEW[R, h, r], Assumptions → {0 < h < r < 2R}]
0

```

CHECK OF EQUATION (107)

(this is correct)

```

Eq107[R_, h_, r_] := (1 / (6 * r * Vic[R, h])) * (3 π * R^2 * (2 * h * r - R^2) -
2 * h * r * ((r^2 + 4 * R^2) * EllipticE[1 / csi[r, R, h]^2] -
(r^2 - 4 * R^2) * EllipticK[1 / csi[r, R, h]^2]));

```

```

FullSimplify[CFNd1BB[R, h, r] - Eq107[R, h, r], Assumptions → {0 < h < r < 2R}]
0

```

```

Delta1[R_, r_] := Sqrt[4 * R^2 - r^2]; Delta2[r_, h_] := Sqrt[r^2 - h^2];
Delta3[r_, R_, h_] := Sqrt[4 * R^2 + h^2 - r^2];
csi[r_, R_, h_] := r / (2 * R);
zeta[r_, R_, h_] := 2 * h * R / (r * Delta3[r, R, h]);
phi1[r_, R_, h_] := ArcSin[csi[r, R, h]];
phi2[r_, R_, h_] := ArcSin[zeta[r, R, h]];
phi3[r_, R_, h_] := ArcSin[Delta2[r, h] / (2 * R)];
phi4[r_, R_, h_] := ArcSin[r * Delta3[r, R, h] / (2 * h * R)];
Vic[R_, h_] := Pi * R^2 * h;
GA[r_, R_, h_] := (1 / (24 * r * Vic[R, h])) *
(3 * (r * (4 * Pi * R^2 (2 * h - r) +
(r^2 + 2 * R^2) * Delta1[R, r]) +
8 * R^2 * (r^2 - R^2) * phi1[r, R, h]) -
16 * h * R * ((r^2 + 4 * R^2) * EllipticE[csi[r, R, h]^2] +
(r^2 - 4 * R^2) * EllipticK[csi[r, R, h]^2]));

```

```

GC[r_, R_, h_] := (1 / (24 * r * Vic[R, h])) * (12 * Pi * R^2 * (h^2 + r^2 - R^2) -
(h^2 - 5 * r^2 - 26 * R^2) * Delta2[r, h] * Delta3[r, R, h] -
24 * R^2 * (h^2 + r^2 - R^2) * phi3[r, R, h] -
8 * h * r * ((r^2 + 4 * R^2) * EllipticE[phi4[r, R, h], 1/csi[r, R, h]^2] -
(r^2 - 4 * R^2) * EllipticF[phi4[r, R, h], 1/csi[r, R, h]^2]));

```

CHECK OF THE INNERMOST REGION

Eq. (104), i.e. $GA[r, R, h]$, coincides with the needle and the disk expressions of the cylinder CF

```

FullSimplify[CFNd1AA[R, h, r] - GA[r, R, h], Assumptions -> {0 < r < 2 R < h}]
FullSimplify[CFDskAA[R, h, r] - GA[r, R, h], Assumptions -> {0 < r < 2 R < h}]

```

0

0

CHECK OF THE OUTERMOST REGION

The corrected Eq. (104), i.e. the above $GC[r, R, h]$, coincides with the needle and the disk expressions of the cylinder CF

```

FullSimplify[CFNd1CC[R, h, r] - GC[r, R, h],
Assumptions -> {0 < 2 R < h < r < Sqrt[4 * R^2 + h^2]}]
FullSimplify[CFDskCC[R, h, r] - GC[r, R, h], Assumptions -> {0 < h < 2 R < r < Sqrt[4 * R^2 + h^2]}]

```

0

0

CHECK OF EQUATION (106)

(this was wrong. The correc equation is:)

```
Eq106NEW[R_, h_, r_] := (1 / (24 * r * Vic[R, h])) * (12 π * h^2 * R^2 +
 3 * r * (r^2 + 2 * R^2) * Delta1[R, r] + (Delta2[r, h] / Delta3[r, R, h]) *
 (3 * r^4 - h^4 + 6 * r^2 * (h^2 - R^2) + 22 * h^2 * R^2 - 24 * R^4) +
 24 * R^2 * (r^2 - R^2) * phi1[r, R, h] -
 24 * R^2 * (h^2 + r^2 - R^2) * phi3[r, R, h] -
 16 * h * R * ((r^2 + 4 * R^2) * EllipticE[phi2[r, R, h], csi[r, R, h]^2] +
 (r^2 - 4 * R^2) * EllipticF[phi2[r, R, h], csi[r, R, h]^2]));
```

```
FullSimplify[CFDskBB[R, h, r] - Eq106NEW[R, h, r], Assumptions → {0 < h < r < 2 R}]
```

0

CHECK OF EQUATION (107)

(this is correct)

```
Eq107[R_, h_, r_] := (1 / (6 * r * Vic[R, h])) * (3 π * R^2 * (2 * h * r - R^2) -
 2 * h * r * ((r^2 + 4 * R^2) * EllipticE[1 / csi[r, R, h]^2] -
 (r^2 - 4 * R^2) * EllipticK[1 / csi[r, R, h]^2]));
```

```
FullSimplify[CFNdlBB[R, h, r] - Eq107[R, h, r], Assumptions → {0 < h < r < 2 R}]
```

0

Derivation of equations (110)-(112)

The detailed evaluation of the CF of a cubic surface is reported below. What follows is the MATHEMATICA code that allowed us to carry out the analytic calculations of the CF.

The case of the overlapping along lines

Writing the cubic surface as the sum over the six faces one has $S = \sum_{i=1}^6 S_i$. The integral

takes the form $\gamma_S[r] = \frac{2}{4 \pi S} \sum_{i < j=1}^6 \int_{S_i} dS_1 \int_{S_j} dS_2 \int d\vec{\omega} \delta(\vec{r}_1 + r \vec{\omega} - \vec{r}_2)$.

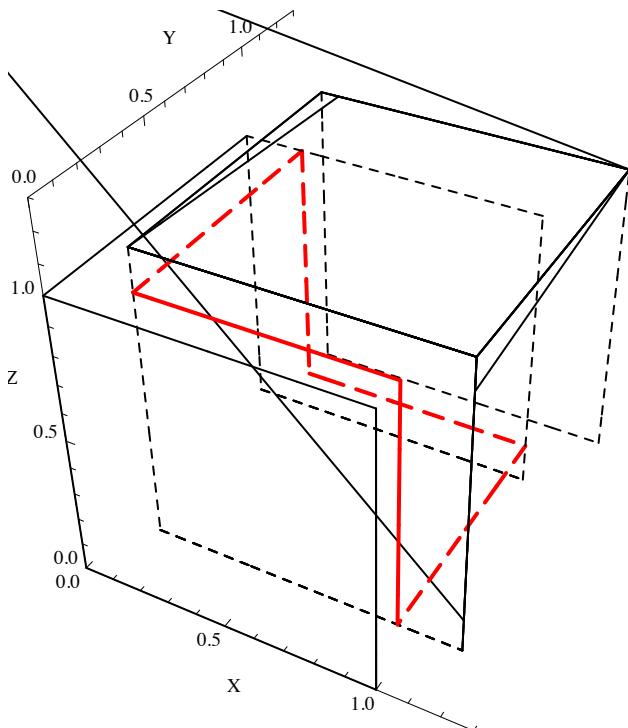
The cubic symmetry allows one to confine the angular integration to

$$\Omega = \left\{ \theta, \varphi \mid 0 \leq \theta \leq \frac{\pi}{2} \text{ and } 0 \leq \varphi \leq \frac{\pi}{2} \right\}.$$

Thus one finds that

$\gamma_S[r] = \frac{16}{4 \pi S} \sum_{i < j=1}^6 \int_{S_i} dS_1 \int_{S_j} dS_2 \int_{\Omega} d\vec{\omega} \delta(\vec{r}_1 + r \vec{\omega} - \vec{r}_2)$. One considers the case

where the intersections are as shown in the figure below (see the red polygon).



Since the surfaces intersecting themselves along the polygon are mutually orthogonal,
the integral along the surface reduces
to the length of the intersection and one finds that

$$\gamma_s[r] = \frac{2 * 8}{4 \pi S} \int_{\Omega} d\omega Fint[r, a, \theta, \varphi] \quad (1)$$

with

$$Fint[r, a, \theta, \varphi] = 2 ((a - r * \sin[\theta] \cos[\varphi]) + (a - r * \sin[\theta] \sin[\varphi]) + (a - r * \cos[\theta])). \quad (2)$$

Each of the three lengths : $(a - r * \sin[\theta] \cos[\varphi])$,
 $(a - r * \sin[\theta] \sin[\varphi])$ and $(a - r * \cos[\theta])$

must be positive and smaller than a . The set of these lengths does not change if one

makes the substitution $\varphi \rightarrow \frac{\pi}{2} - \varphi$. Thus one can restrict the φ -integration range to $[0, \frac{\pi}{4}]$

provided one multiplies the result of the integration by 2.

The aforesaid bounds on the three lengths imply that the integration range is
AA :

$$0 \leq \theta \leq \frac{\pi}{2} \quad \text{and} \quad 0 \leq \varphi \leq \frac{\pi}{4} \quad \text{if } 0 < r < a;$$

$$\text{BBA : } 0 < \varphi < \text{ArcSin}\left[\frac{\sqrt{-a^2 + r^2}}{r}\right], \quad \text{ArcSin}\left[\frac{\sqrt{-a^2 + r^2}}{r}\right] <$$

$$\theta < \text{ArcSin}\left[\frac{a}{r * \cos[\varphi]}\right] \text{ if } a < r < a\sqrt{2}$$

$$\text{BBb : } \text{ArcSin}\left[\frac{\sqrt{-a^2 + r^2}}{r}\right] < \varphi < \frac{\pi}{4},$$

$$\text{ArcSin}\left[\frac{\sqrt{-a^2 + r^2}}{r}\right] < \theta < \frac{\pi}{2} \text{ if } a < r < a\sqrt{2};$$

$$\text{CC : } \text{ArcSin}\left[\frac{\sqrt{r^2 - 2 * a^2}}{\sqrt{r^2 - a^2}}\right] < \varphi < \frac{\pi}{4},$$

$$\text{ArcSin}\left[\frac{\sqrt{r^2 - a^2}}{r}\right] < \theta < \text{ArcSin}\left[\frac{a}{r * \cos[\varphi]}\right] \text{ if } a\sqrt{2} < r < a\sqrt{3}$$

Integrand definition

In the case of the line intersection the normalization factor is
(8 is the octant number and 2 derives from the choice $0 < \varphi < \frac{\pi}{4}$)

```
Clear[a]; Clear[NormFactLin];
NormFactLin[a_] := 2 * 8 / (4 * π * 6 * a^2);
```

the integrand function is

```
Fint[r_, a_, θ_, φ_] :=
  2 ((a - r * Sin[θ] Cos[φ]) + (a - r * Sin[θ] Sin[φ]) + (a - r * Cos[θ])) Sin[θ];
```

```
TrigExpand[Integrate[Fint[r, a, θ, φ], θ]]
```

$$\begin{aligned}
& \frac{r}{2} - 6 a \cos[\theta] + \frac{1}{2} r \cos[\theta]^2 - r \theta \cos[\varphi] + \\
& r \cos[\theta] \cos[\varphi] \sin[\theta] - \frac{1}{2} r \sin[\theta]^2 - r \theta \sin[\varphi] + r \cos[\theta] \sin[\theta] \sin[\varphi]
\end{aligned}$$

```
Integrate[Fint[r, a, θ, φ], φ]
```

$$6 a \varphi \sin[\theta] - 2 r \varphi \cos[\theta] \sin[\theta] + 2 r \cos[\varphi] \sin[\theta]^2 - 2 r \sin[\theta]^2 \sin[\varphi]$$

Change of the integration variables

$$(\text{Fint}[r, a, \theta, \varphi]) /. \{\sin[\theta] \rightarrow t, \cos[\theta] \rightarrow \sqrt{1-t^2}, \sin[\varphi] \rightarrow u, \cos[\varphi] \rightarrow \sqrt{1-u^2}\}$$

$$2t \left(3a - r\sqrt{1-t^2} - rtu - r\sqrt{1-u^2} \right)$$

Determination of the integration variables ranges imposing the positiveness of the three factors present in the Fint[..] definition.

One confines himself to the range

$$0 < \varphi < \frac{\pi}{4} \text{ i.e. } 0 < u < \frac{1}{\sqrt{2}}$$

$$0 < \theta < \frac{\pi}{2} \text{ i.e. } 0 < t < 1$$

$$(a - r * \sin[\theta] * \cos[\varphi]) /. \{\sin[\theta] \rightarrow t, \cos[\theta] \rightarrow \sqrt{1-t^2}, \sin[\varphi] \rightarrow u, \cos[\varphi] \rightarrow \sqrt{1-u^2}\}$$

$$(a - r * \sin[\theta] \sin[\varphi]) /. \{\sin[\theta] \rightarrow t, \cos[\theta] \rightarrow \sqrt{1-t^2}, \sin[\varphi] \rightarrow u, \cos[\varphi] \rightarrow \sqrt{1-u^2}\}$$

$$(a - r * \cos[\theta]) /. \{\sin[\theta] \rightarrow t, \cos[\theta] \rightarrow \sqrt{1-t^2}, \sin[\varphi] \rightarrow u, \cos[\varphi] \rightarrow \sqrt{1-u^2}\}$$

Reduction of the inequalities in the region A : $0 < r < a$

$$\text{Reduce}[\{a - rt\sqrt{1-u^2} > 0, a - rtu > 0, a - r\sqrt{1-t^2} > 0, \\ a > 0, r > 0, r < a, 0 < t < 1, 0 < u < 1/\sqrt{2}\}, \{a, r, u, t\}, \text{Reals}]$$

AA REGION Def.

$$0 < \varphi < \frac{\pi}{4} \text{ and } 0 < \theta < \frac{\pi}{2} \quad \text{or}$$

$$0 < u < \frac{1}{\sqrt{2}} \text{ and } 0 < t < 1$$

Reduction of the inequalities in the region B : $a < r < a\sqrt{2}$

$$\sin[\theta] \rightarrow t, \sin[\varphi] \rightarrow u$$

$$\text{Reduce}[\{a - rt\sqrt{1-u^2} > 0, a - rtu > 0, a - r\sqrt{1-t^2} > 0, a > 0, \\ r > 0, a < r < a\sqrt{2}, 0 < t < 1, 0 < u < 1/\sqrt{2}\}, \{a, r, u, t\}, \text{Reals}]$$

BB REGION Definition $a < r < a\sqrt{2}$

$$\text{BBaa: } 0 < \varphi < \text{ArcSin}\left[\frac{\sqrt{-a^2+r^2}}{r}\right] \quad \&\& \quad \text{ArcSin}\left[\frac{\sqrt{-a^2+r^2}}{r}\right] < \theta < \text{ArcSin}\left[\frac{a}{r*\text{Cos}[\varphi]}\right] \quad \text{i.e.}$$

$$0 < u \leq \frac{\sqrt{-a^2+r^2}}{r} \quad \&\& \quad \frac{\sqrt{-a^2+r^2}}{r} < t < \frac{a}{r\sqrt{1-u^2}}$$

AND

$$\text{BBbb: } \text{ArcSin}\left[\frac{\sqrt{-a^2+r^2}}{r}\right] < \varphi < \frac{\pi}{4} \quad \&\& \quad \text{ArcSin}\left[\frac{\sqrt{-a^2+r^2}}{r}\right] < \theta < \frac{\pi}{2} \quad \text{i.e.} \quad \sqrt{\frac{-a^2+r^2}{r^2}} < u < \frac{1}{\sqrt{2}} \quad \&\& \quad \sqrt{\frac{-a^2+r^2}{r^2}} < t < 1$$

ALTERNATIVELY

$$\text{Reduce}\left[\left\{a - rt\sqrt{1-u^2} > 0, a - rtu > 0, a - r\sqrt{1-t^2} > 0, a > 0, r > 0, a < r < a\sqrt{2}, 0 < t < 1, 0 < u < 1/\sqrt{2}\right\}, \{a, r, t, u\}, \text{Reals}\right]$$

BB REGION DFN (alternative)

$$0 < \varphi < \frac{\pi}{4} \quad \&\& \quad \text{ArcSin}\left[\frac{\sqrt{-a^2+r^2}}{r}\right] < \theta < \text{ArcSin}\left[\frac{a}{r}\right] \quad \text{i.e.} \quad \sqrt{\frac{-a^2+r^2}{r^2}} < t \leq \frac{a}{r} \quad \&\& \quad 0 < u < \frac{1}{\sqrt{2}}$$

AND

$$\text{ArcSin}\left[\frac{a}{r}\right] < \theta < \frac{\pi}{2} \quad \&\& \quad \text{ArcSin}\left[\frac{\sqrt{-a^2+r^2}\sin[\theta]^2}{r\sin[\theta]}\right] < \varphi < \pi \quad \text{OR} \quad \frac{a}{r} < t < 1 \quad \&\& \quad \sqrt{\frac{-a^2+r^2 t^2}{r^2 t^2}} < u < \frac{1}{\sqrt{2}}$$

Reduction of the inequalities in the region C: $a\sqrt{2} < r < a\sqrt{3}$

$$\text{Reduce}\left[\left\{a - rt\sqrt{1-u^2} > 0, a - rtu > 0, a - r\sqrt{1-t^2} > 0, a > 0, r > 0, a\sqrt{2} < r < a\sqrt{3}, 0 < t < 1, 0 < u < 1/\sqrt{2}\right\}, \{a, r, u, t\}, \text{Reals}\right]$$

$$a > 0 \quad \&\& \quad \sqrt{2} < a < \sqrt{3} \quad a \quad \&\& \quad \sqrt{\frac{2a^2 - r^2}{a^2 - r^2}} < u < \frac{1}{\sqrt{2}} \quad \&\& \quad \sqrt{\frac{-a^2 + r^2}{r^2}} < t < \sqrt{-\frac{a^2}{r^2(-1+u^2)}}$$

$$\sin[\theta] \rightarrow t, \sin[\varphi] \rightarrow u$$

i.e. CC REGION DFN $a\sqrt{2} < r < a\sqrt{3}$

$$\text{ArcSin}\left[\frac{\sqrt{r^2-2*a^2}}{\sqrt{r^2-a^2}}\right] < \varphi < \frac{\pi}{4} \quad \&\& \quad \text{ArcSin}\left[\frac{\sqrt{r^2-a^2}}{r}\right] < \theta < \text{ArcSin}\left[\frac{a}{r*\text{Cos}[\varphi]}\right] \quad \text{or}$$

$$\sqrt{\frac{2a^2-r^2}{a^2-r^2}} < u < \frac{1}{\sqrt{2}} \quad \&\& \quad \frac{\sqrt{r^2-a^2}}{r} < t < \frac{a}{r*\sqrt{1-u^2}}$$

The angular bounds in the three cases are :

```

ThetaMaxBBaa[r_, a_, φ_] := ArcSin[a/(r * Cos[φ])];
ThetaMinBBaa[r_, a_] := ArcSin[Sqrt[r^2 - a^2]/r];
FiMaxBBaa[r_, a_] := ArcSin[Sqrt[-a^2 + r^2]/r]; FiMinBBaa[r_, a_] := 0;
ThetaMaxBBbb[r_, a_, φ_] := π/2; ThetaMinBBbb[r_, a_] := ArcSin[Sqrt[r^2 - a^2]/r];
FiMaxBBbb[r_, a_] := π/4; FiMinBBbb[r_, a_] := ArcSin[Sqrt[r^2 - a^2]/r];
ThetaMaxCC[r_, a_, φ_] := ArcSin[a/(r * Cos[φ])]; ThetaMinCC[r_, a_] := ArcSin[Sqrt[r^2 - a^2]/r];
FiMaxCC[r_, a_] := π/4; FiMinCC[r_, a_] := ArcSin[Sqrt[r^2 - 2*a^2]/Sqrt[r^2 - a^2]];

```

NUMERICAL INTEGRATION of the contributions relevant to the cases BBaa, BBbb and CC

```

With[{a = 1}, StepRAA = a / 10; StepRBB = a  $(\sqrt{2} - 1) / 10$ ; StepRCC = a  $(\sqrt{3} - \sqrt{2}) / 10$ ;
valAA = Table[0, {J, 1, 10}]; valBBaa = Table[0, {J, 1, 10}];
valBBbb = Table[0, {J, 1, 10}]; valCC = Table[0, {J, 1, 10}]; valDD = Table[0, {J, 1, 10}];
Do[ract = (J - 1 / 2) * StepRAA; valAA[[J]] = NormFactLin[a] *
NIntegrate[Fint[ract, a, θ, φ], {φ, 0, π/4}, {θ, 0, π/2}, PrecisionGoal → 8];
ract = 1 + (J - 1 / 2) * StepRBB; valBBaa[[J]] =
NormFactLin[a] * NIntegrate[NIntegrate[Fint[ract, a, θ, φ],
{θ, ArcSin[ $\frac{\sqrt{-a^2 + ract^2}}{ract}$ ], ArcSin[ $\frac{a}{ract * Cos[\phi]}$ ]}, WorkingPrecision → 30,
PrecisionGoal → 15], {φ, 0, ArcSin[ $\frac{\sqrt{-a^2 + ract^2}}{ract}$ ]}, PrecisionGoal → 8];
valBBbb[[J]] = NormFactLin[a] * NIntegrate[Fint[ract, a, θ, φ],
{φ, ArcSin[ $\frac{\sqrt{-a^2 + ract^2}}{ract}$ ], π/4}, {θ, ArcSin[ $\frac{\sqrt{-a^2 + ract^2}}{ract}$ ], π/2}, PrecisionGoal → 10];
ract =  $\sqrt{2} + (J - 1 / 2) * StepRCC$ ; valDD[[J]] =
NormFactLin[a] * NIntegrate[NIntegrate[Fint[ract, a, θ, φ],
{θ, ArcSin[ $\frac{\sqrt{-a^2 + ract^2}}{ract}$ ], ArcSin[ $\frac{a}{ract * Cos[\phi]}$ ]}, WorkingPrecision → 24,
PrecisionGoal → 12], {φ, ArcSin[ $\frac{\sqrt{ract^2 - 2 * a^2}}{\sqrt{ract^2 - a^2}}$ ], π/4}, PrecisionGoal → 8];
valCC[[J]] = NormFactLin[a] * NIntegrate[Fint[ract, a, θ, φ], {φ, ArcSin[ $\frac{\sqrt{ract^2 - 2 * a^2}}{\sqrt{ract^2 - a^2}}$ ],
π/4}, {θ, ArcSin[ $\frac{\sqrt{-a^2 + ract^2}}{ract}$ ], ArcSin[ $\frac{a}{ract * Cos[\phi]}$ ]}, PrecisionGoal → 8];
Print[J, ", ", valAA[[J]], ", ", valBBaa[[J]], ", ", valBBbb[[J]],
", ", valCC[[J]], ", ", valDD[[J]]], {J, 1, 10}]

```

Approximate value of the 0 - th moment of LINEAR CONTRIBUTION to the surface CF

```

With[{a = 1}, StepRAA = a / 10; StepRBB = a  $(\sqrt{2} - 1) / 10$ ;
StepRCC = a  $(\sqrt{3} - \sqrt{2}) / 10$ ; momntAA = 0; momntBB = 0; momntCC = 0;
Do[momntAA = momntAA + ((J - 1 / 2) * StepRAA)^2 * valAA[[J]];
momntBB = momntBB + (1 + (J - 1 / 2) * StepRBB)^2 * (valBBaa[[J]] + valBBbb[[J]]);
momntCC = momntCC +  $(\sqrt{2} + (J - 1 / 2) * StepRCC)^2 * (valCC[[J]])$ , {J, 1, 10}];
momntAA = momntAA * StepRAA;
momntBB = momntBB * StepRBB;
momntCC = momntCC * StepRCC;
momt = momntAA + momntBB + momntCC;
Print[momt, ", ", momntAA, ", ", momntBB, ", ", momntCC]

```

0.318109 - 1.50476×10^{-25} i, 0.208125, 0.106592 - 1.50476×10^{-25} i, 0.00339224

```
N[1 / 6]
0.166667
```

REGION A INTEGRAL

AA

$$0 < \varphi < \frac{\pi}{4} \text{ and } 0 < \theta < \frac{\pi}{2} \quad \text{or}$$

$$0 < u < \frac{1}{\sqrt{2}} \text{ and } 0 < t < 1$$

```
Expand[NormFactLin[a] * Integrate[
  Integrate[Fint[r, a, \theta, \varphi], {\varphi, 0, \frac{\pi}{4}}], {\theta, 0, \frac{\pi}{2}}, Assumptions \rightarrow {0 < a \&& 0 < r < a}]]
```

$$\text{CubSurfCFLnAA}[r_, a_] := \frac{1}{a} - \frac{r}{2 a^2};$$

The result coincides with that previously worked out. (See below)

numerical check (It's OK)

```
With[{a = 1}, Do[ract = (J - 1/2) * StepRAA;
  val = N[CubSurfCFLnAA[ract, a]]; Print[J, ", ", val, ", ", valAA[[J]]], {J, 1, 10}]

(* 0 < r < a *) CubSurfCFLnAAOLD[r_, a_] := \frac{1}{a} - \frac{r}{2 a^2};

MomlinAA = Simplify[Integrate[r^2 * CubSurfCFLnAA[r, a],
  {r, 0, a}, Assumptions \rightarrow {a > 0}], Assumptions \rightarrow {a > 0}]
```

REGION BB INTEGRAL

BB REGION Definition $a < r < a\sqrt{2}$

$$\text{BBaa: } 0 < \varphi < \text{ArcSin}\left[\frac{\sqrt{-a^2+r^2}}{r}\right] \&& \text{ArcSin}\left[\frac{\sqrt{-a^2+r^2}}{r}\right] < \theta < \text{ArcSin}\left[\frac{a}{r \cos[\varphi]}\right]$$

$$\text{i.e. } 0 < u \leq \frac{\sqrt{-a^2+r^2}}{r} \&& \frac{\sqrt{-a^2+r^2}}{r} < t < \frac{a}{r \sqrt{1-u^2}}$$

AND

$$\text{BBbb: } \text{ArcSin}\left[\frac{\sqrt{-a^2+r^2}}{r}\right] < \varphi < \frac{\pi}{4} \&& \text{ArcSin}\left[\frac{\sqrt{-a^2+r^2}}{r}\right] < \theta < \frac{\pi}{2}$$

$$\text{i.e. } \sqrt{\frac{-a^2+r^2}{r^2}} < u < \frac{1}{\sqrt{2}} \&& \sqrt{\frac{-a^2+r^2}{r^2}} < t < 1$$

INTEGRAL OVER THE REGION BBaa

$$0 < \varphi < \text{ArcSin}\left[\frac{\sqrt{-a^2+r^2}}{r}\right] \quad \&\& \quad \text{ArcSin}\left[\frac{\sqrt{-a^2+r^2}}{r}\right] < \theta < \text{ArcSin}\left[\frac{a}{r \cdot \cos[\varphi]}\right]$$

$$0 < u \leq \frac{\sqrt{-a^2+r^2}}{r} \quad \&\& \quad \frac{\sqrt{-a^2+r^2}}{r} < t < \frac{a}{r \sqrt{1-u^2}} \quad \&\& -a^2 + r^2 (1-u^2) > 0$$

the bounds are

$$\begin{aligned} \text{ThetaMaxBBaa}[r_, a_, \varphi_] &:= \text{ArcSin}\left[\frac{a}{r \cdot \cos[\varphi]}\right]; \quad \text{ThetaMinBBaa}[r_, a_] := \text{ArcSin}\left[\frac{\sqrt{r^2 - a^2}}{r}\right]; \\ \text{FiMaxBBaa}[r_, a_] &:= \text{ArcSin}\left[\frac{\sqrt{-a^2+r^2}}{r}\right]; \quad \text{FiMinBBaa}[r_, a_] := 0; \end{aligned}$$

First one evaluates with respect to θ

```
TrigExpand[Integrate[Fint[r, a, \theta, \varphi], \theta]]
```

$$\begin{aligned} \text{ThetaPrmtv}[r_, a_, \varphi_, \theta_] &:= \frac{r}{2} - 6 a \cos[\theta] + \frac{1}{2} r \cos[\theta]^2 - r \theta \cos[\varphi] + \\ &\quad r \cos[\theta] \cos[\varphi] \sin[\theta] - \frac{1}{2} r \sin[\theta]^2 - r \theta \sin[\varphi] + r \cos[\theta] \sin[\theta] \sin[\varphi]; \end{aligned}$$

```
Simplify[D[ThetaPrmtv[r, a, \varphi, \theta], \theta] - Fint[r, a, \theta, \varphi]]
```

0

```
Integrate[Simplify[ExpandAll[  
  \left( \left( Simplify[Limit[ThetaPrmtv[r, a, \varphi, \theta], \theta \rightarrow ThetaMaxBBaa[r, a, \varphi], Direction \rightarrow 1],  
    Assumptions \rightarrow \{a > 0 \&& a < r < a \sqrt{2} \&& FiMinBBaa[r, a] < \varphi < FiMaxBBaa[r, a]\}] \right) /.  
    \{Sin[\varphi] \rightarrow y, Cos[\varphi] \rightarrow \sqrt{1-y^2}, Sec[\varphi] \rightarrow \frac{1}{\sqrt{1-y^2}}, Tan[\varphi] \rightarrow \frac{y}{\sqrt{1-y^2}}\} \right) * \frac{1}{\sqrt{1-y^2}}],  
  Assumptions \rightarrow \{a > 0 \&& a < r < a \sqrt{2} \&& 0 < y < \frac{\sqrt{-a^2+r^2}}{r}\}],  
  \{y, 0, \frac{\sqrt{-a^2+r^2}}{r}\}, Assumptions \rightarrow \{a > 0 \&& a < r < a \sqrt{2}\}]
```

```

Integrate[Simplify[
  Simplify[Limit[ThetaPrmtv[r, a, φ, θ], θ → ThetaMinBBaa[r, a], Direction → -1],
    Assumptions → {a > 0 && a < r < a Sqrt[2] && FiMinBBaa[r, a] < φ < FiMaxBBaa[r, a]}]] /.
  {Sin[φ] → y, Cos[φ] → Sqrt[1 - y^2], Sec[φ] → 1/Sqrt[1 - y^2], Tan[φ] → y/Sqrt[1 - y^2]}]*1/Sqrt[1 - y^2],
  Assumptions → {a > 0 && a < r < a Sqrt[2] && 0 < y < Sqrt[-a^2 + r^2]/r}],
  {y, 0, Sqrt[-a^2 + r^2]/r}, Assumptions → {a > 0 && a < r < a Sqrt[2]}]

Simplify[Integrate[Simplify[ExpandAll[
  Simplify[Limit[ThetaPrmtv[r, a, φ, θ], θ → ThetaMaxBBaa[r, a, φ], Direction → 1],
    Assumptions → {a > 0 && a < r < a Sqrt[2] && FiMinBBaa[r, a] < φ < FiMaxBBaa[r, a]}]] /.
  {Sin[φ] → y, Cos[φ] → Sqrt[1 - y^2], Sec[φ] → 1/Sqrt[1 - y^2], Tan[φ] → y/Sqrt[1 - y^2]}]*1/Sqrt[1 - y^2],
  Assumptions → {a > 0 && a < r < a Sqrt[2] && 0 < y < Sqrt[-a^2 + r^2]/r}], {y, 0, Sqrt[-a^2 + r^2]/r},
  Assumptions → {a > 0 && a < r < a Sqrt[2]}] - Integrate[Simplify[
  Simplify[Limit[ThetaPrmtv[r, a, φ, θ], θ → ThetaMinBBaa[r, a], Direction → -1],
    Assumptions → {a > 0 && a < r < a Sqrt[2] && FiMinBBaa[r, a] < φ < FiMaxBBaa[r, a]}]] /.
  {Sin[φ] → y, Cos[φ] → Sqrt[1 - y^2], Sec[φ] → 1/Sqrt[1 - y^2], Tan[φ] → y/Sqrt[1 - y^2]}]*1/Sqrt[1 - y^2],
  Assumptions → {a > 0 && a < r < a Sqrt[2] && 0 < y < Sqrt[-a^2 + r^2]/r}], {y, 0, Sqrt[-a^2 + r^2]/r},
  Assumptions → {a > 0 && a < r < a Sqrt[2]}], Assumptions → {a > 0 && a < r < a Sqrt[2]}]

```

$$\begin{aligned} \text{Simplify}\left[-\frac{-5 a^2 \pi + 5 a \pi r + 4 a \sqrt{-a^2 + r^2} + \pi r \sqrt{-a^2 + r^2} - 2 r^2 \left(\text{ArcCos}\left[\frac{a}{r}\right] + \text{ArcCsc}\left[\frac{r}{\sqrt{-a^2+r^2}}\right] \right)}{2 r} + \right. \\ \left. \frac{a (a - r) \left(a + r + \sqrt{-a^2 + r^2}\right) + r \left(5 a^2 - a r + r \left(r + \sqrt{-a^2 + r^2}\right)\right) \text{ArcSin}\left[\frac{\sqrt{-a^2+r^2}}{r}\right]}{r^2} \right] / . \\ \left\{ \text{ArcCos}\left[\frac{a}{r}\right] \rightarrow \text{ArcSin}\left[\frac{\sqrt{-a^2+r^2}}{r}\right], \text{ArcCsc}\left[\frac{r}{\sqrt{-a^2+r^2}}\right] \rightarrow \text{ArcSin}\left[\frac{\sqrt{-a^2+r^2}}{r}\right] \right\}, \\ \text{Assumptions} \rightarrow \{a > 0 \&& a < r < a \sqrt{2}\} \end{aligned}$$

value of the contribution due to the BBaa region

$$\begin{aligned} \text{LinContrBBaa}[r_, a_] := \text{NormFactLin}[a] * \\ \left(\frac{1}{2 r^2} \left(2 a^3 - \pi r^2 \sqrt{-a^2 + r^2} + a^2 \left(5 \pi r + 2 \sqrt{-a^2 + r^2} \right) - a r \left(2 r + 5 \pi r + 6 \sqrt{-a^2 + r^2} \right) + \right. \right. \\ \left. \left. 2 r \left(5 a^2 - a r + r \left(3 r + \sqrt{-a^2 + r^2} \right) \right) \text{ArcSin}\left[\frac{\sqrt{-a^2+r^2}}{r}\right] \right) \right); \end{aligned}$$

$$\begin{aligned} \text{FullSimplify}\left[\text{TrigExpand}\left[\sin\left[\text{ArcCos}\left[\frac{a}{r}\right]\right] - \text{ArcSin}\left[\frac{\sqrt{-a^2+r^2}}{r}\right] \right] \right], \\ \text{Assumptions} \rightarrow \{a > 0 \&& a < r < a \sqrt{2}\} \end{aligned}$$

$$\begin{aligned} \text{FullSimplify}\left[\text{TrigExpand}\left[\cos\left[\text{ArcCos}\left[\frac{a}{r}\right]\right] - \text{ArcSin}\left[\frac{\sqrt{-a^2+r^2}}{r}\right] \right] \right], \\ \text{Assumptions} \rightarrow \{a > 0 \&& a < r < a \sqrt{2}\} \end{aligned}$$

$$\text{FullSimplify}\left[\text{ArcCsc}\left[\frac{r}{\sqrt{-a^2+r^2}}\right] - \text{ArcSin}\left[\frac{\sqrt{-a^2+r^2}}{r}\right], \text{Assumptions} \rightarrow \{a > 0 \&& a < r < a \sqrt{2}\} \right]$$

the values obtained by the above function are compared with the values obtained by the numerical integration of the outset integrand

$$\begin{aligned} \text{With}\left[\{a = 1\}, \text{StepRBB} = a (\sqrt{2} - 1) / 10; \right. \\ \text{Do}\left[\text{ract} = 1 + (J - 1 / 2) * \text{StepRBB}; \text{actval} = N[\text{LinContrBBaa}[\text{ract}, a]]; \right. \\ \left. \text{Print}[J, ", ", \text{valBBaa}[[J]], ", ", \text{actval}]; , \{J, 1, 10\} \right] \end{aligned}$$

the check is fully satisfactory

INTEGRAL OVER THE REGION BBbb

$$a < r < a\sqrt{2}$$

$$\text{BBbb: } \text{ArcSin}\left[\frac{\sqrt{-a^2+r^2}}{r}\right] < \varphi < \frac{\pi}{4} \quad \&\& \quad \text{ArcSin}\left[\frac{\sqrt{-a^2+r^2}}{r}\right] < \theta < \frac{\pi}{2}$$

$$\text{i.e. } \sqrt{\frac{-a^2+r^2}{r^2}} < u < \frac{1}{\sqrt{2}} \quad \&\& \quad \sqrt{\frac{-a^2+r^2}{r^2}} < t < 1$$

the bounds are

$$\begin{aligned} \text{ThetaMaxBBbb}[r, a, \varphi] &:= \frac{\pi}{2}; \quad \text{ThetaMinBBbb}[r, a] := \text{ArcSin}\left[\frac{\sqrt{r^2 - a^2}}{r}\right]; \\ \text{FiMaxBBbb}[r, a] &:= \frac{\pi}{4}; \quad \text{FiMinBBbb}[r, a] := \text{ArcSin}\left[\frac{\sqrt{r^2 - a^2}}{r}\right]; \end{aligned}$$

One evaluates the value of the θ - primitive at the upper and lower θ -bounds and then one integrates over φ with the integration variable change $\text{Sin}[\varphi] \rightarrow y$

$$\begin{aligned} \text{Integrate}\left[\text{FullSimplify}\left[\left(\text{Simplify}\left[\text{Limit}[\text{ThetaPrmtv}[r, a, \varphi, \theta], \theta \rightarrow \text{ThetaMaxBBbb}[r, a, \varphi], \text{Direction} \rightarrow 1], \text{Assumptions} \rightarrow \left\{a > 0 \&\& a < r < a\sqrt{2} \&\& \text{FiMinBBbb}[r, a] < \varphi < \text{FiMaxBBbb}[r, a]\right\}\right]\right) / . \right. \\ \left. \left\{\text{Sin}[\varphi] \rightarrow y, \text{Cos}[\varphi] \rightarrow \sqrt{1 - y^2}, \text{Sec}[\varphi] \rightarrow \frac{1}{\sqrt{1 - y^2}}, \text{Tan}[\varphi] \rightarrow \frac{y}{\sqrt{1 - y^2}}\right\} * \frac{1}{\sqrt{1 - y^2}}, \text{Assumptions} \rightarrow \left\{a > 0 \&\& a < r < a\sqrt{2} \&\& \frac{\sqrt{-a^2 + r^2}}{r} < y < \frac{1}{\sqrt{2}}\right\}\right], \\ \left\{y, \frac{\sqrt{-a^2 + r^2}}{r}, \frac{1}{\sqrt{2}}\right\}, \text{Assumptions} \rightarrow \left\{a > 0 \&\& a < r < a\sqrt{2}\right\} \right] \end{aligned}$$

```

Integrate[Simplify[
  Simplify[Limit[ThetaPrmtv[r, a, φ, θ], θ → ThetaMinBBbb[r, a], Direction → -1],
    Assumptions → {a > 0 && a < r < a √2 && FiMinBBbb[r, a] < φ < FiMaxBBbb[r, a]}]] /.
  {Sin[φ] → y, Cos[φ] → √(1 - y²), Sec[φ] → 1/√(1 - y²), Tan[φ] → y/√(1 - y²)}] * 1/√(1 - y²),
  Assumptions → {a > 0 && a < r < a √2 && √(-a² + r²)/r < y < 1/√2}]],
{y, √(-a² + r²)/r, 1/√2}, Assumptions → {a > 0 && a < r < a √2}]

FullSimplify[TrigExpand[Sin[ArcSec[r/√(-a² + r²)]] - (π/2 - ArcSin[√(-a² + r²)/r])]],
Assumptions → {a > 0 && a < r < a √2}]

FullSimplify[TrigExpand[Cos[ArcSec[r/√(-a² + r²)]] - (π/2 - ArcSin[√(-a² + r²)/r])]],
Assumptions → {a > 0 && a < r < a √2}]

FullSimplify[ArcCsc[r/√(-a² + r²)] - ArcSin[√(-a² + r²)/r], Assumptions → {a > 0 && a < r < a √2}]

```

```

Simplify[
  Simplify[Integrate[FullSimplify[
    Simplify[
      Limit[ThetaPrmtv[r, a, φ, θ], θ → ThetaMaxBBbb[r, a, φ], Direction → 1],
      Assumptions → {a > 0 && a < r < a √2 && FiMinBBbb[r, a] <
        φ < FiMaxBBbb[r, a]}]] /. {Sin[φ] → y,
    Cos[φ] → √(1 - y²), Sec[φ] → 1/√(1 - y²), Tan[φ] → y/√(1 - y²)}] * 1/√(1 - y²),
  Assumptions → {a > 0 && a < r < a √2 && √(-a² + r²)/r < y < 1/√2}],

{y, √(-a² + r²)/r, 1/√2}, Assumptions → {a > 0 && a < r < a √2}] - 

Integrate[Simplify[
  Simplify[Limit[ThetaPrmtv[r, a, φ, θ],
    θ → ThetaMinBBbb[r, a], Direction → -1],
    Assumptions → {a > 0 && a < r < a √2 && FiMinBBbb[r, a] <
      φ < FiMaxBBbb[r, a]}]] /. {Sin[φ] → y,
  Cos[φ] → √(1 - y²), Sec[φ] → 1/√(1 - y²), Tan[φ] → y/√(1 - y²)}] * 1/√(1 - y²),
  Assumptions → {a > 0 && a < r < a √2 && √(-a² + r²)/r < y < 1/√2}],

{y, √(-a² + r²)/r, 1/√2}, Assumptions → {a > 0 && a < r < a √2}],

Assumptions → {a > 0 && a < r < a √2}] /. 

{ArcSec[r/√(-a² + r²)] → π/2 - ArcSin[√(-a² + r²)/r],
  ArcCsc[r/√(-a² + r²)] → ArcSin[√(-a² + r²)/r]},
  Assumptions → {a > 0 && a < r < a √2}]

```

value of the contribution due to the BBbb region

```

LinContrBBbb[r_, a_] :=
  NormFactLin[a] * 
$$\left( \frac{1}{4 r^2} \left( -4 a^3 - 2 a (-2 + \pi) r^2 + 2 \pi r^2 \sqrt{-a^2 + r^2} + a^2 (5 \pi r - 4 \sqrt{-a^2 + r^2}) - 4 r (5 a^2 - a r + r \sqrt{-a^2 + r^2}) \text{ArcSin}\left[\frac{\sqrt{-a^2 + r^2}}{r}\right] \right) \right);$$


```

the values obtained by the above function are compared with the values obtained by the numerical integration of the outset integrand

```

With[{a = 1}, StepRBB = a  $(\sqrt{2} - 1)$  / 10;
Do[ract = 1 + (J - 1 / 2) * StepRBB; actval = N[LinContrBBbb[ract, a]];
Print[J, ", ", valBBbb[[J]], ", ", actval], {J, 1, 10}]

```

the check is fully satisfactory

the normalized total contribution of the linear intersections due to the BB - region is

```

Expand[
Simplify[LinContrBBbb[r, a] + LinContrBBaa[r, a], Assumptions -> {a > 0 && a < r < a  $\sqrt{2}$ }]]

```

$$\text{CubSurfCFLnBB}[r_, a_] := \frac{2}{a} + \frac{5}{2 r} - \frac{2 \sqrt{-a^2 + r^2}}{a \pi r} + \frac{2 r \text{ArcSin}\left[\frac{\sqrt{-a^2 + r^2}}{r}\right]}{a^2 \pi};$$

Check of the continuity at $r = a$

```

Simplify[Limit[CubSurfCFLnBB[r, a], r -> a, Direction -> -1] -
Limit[CubSurfCFLnAA[r, a], r -> a, Direction -> 1], Assumptions -> {a > 0}]
0

FullSimplify[Limit[CubSurfCFLnBB[r, a], r -> a  $\sqrt{2}$ , Direction -> 1], Assumptions -> {a > 0}]

$$-\frac{4 + (-7 + 4 \sqrt{2}) \pi}{2 \sqrt{2} a \pi}

\text{MomlinBB} = Simplify[Integrate[r^2 * CubSurfCFLnBB[r, a],
{r, a, a  $\sqrt{2}$ }, Assumptions -> {a > 0}], Assumptions -> {a > 0}]

$$-\frac{a^2 (16 + (-29 + 16 \sqrt{2}) \pi)}{12 \pi}

\text{MomlinAABB} = N[Simplify[MomlinAA + MomlinBB]]
0.314969 a^2$$$$

```

INTEGRAL OVER THE REGION CC

$$\begin{aligned}
 & a\sqrt{2} < r < a\sqrt{3} \\
 & \text{ArcSin}\left[\frac{\sqrt{r^2 - 2*a^2}}{\sqrt{r^2 - a^2}}\right] < \varphi < \frac{\pi}{4} \quad \&& \quad \text{ArcSin}\left[\frac{\sqrt{r^2 - a^2}}{r}\right] < \theta < \text{ArcSin}\left[\frac{a}{r*\cos[\varphi]}\right] \quad \text{or} \\
 & \sqrt{\frac{2 a^2 - r^2}{a^2 - r^2}} < u < \frac{1}{\sqrt{2}} \quad \&& \quad \frac{\sqrt{r^2 - a^2}}{r} < t < \frac{a}{r * \sqrt{1 - u^2}}
 \end{aligned}$$

the bounds are

$$\text{ThetaMaxCC}[r_, a_, \varphi_] := \text{ArcSin}\left[\frac{a}{r * \cos[\varphi]}\right]; \quad \text{ThetaMinCC}[r_, a_] := \text{ArcSin}\left[\frac{\sqrt{r^2 - a^2}}{r}\right];$$

$$\text{FiMaxCC}[r_, a_] := \frac{\pi}{4}; \quad \text{FiMinCC}[r_, a_] := \text{ArcSin}\left[\frac{\sqrt{r^2 - 2 * a^2}}{\sqrt{r^2 - a^2}}\right];$$

$$\begin{aligned}
 & \text{Simplify}\left[\text{Limit}[\text{ThetaPrmtv}[r, a, \varphi, \theta], \theta \rightarrow \text{ThetaMaxCC}[r, a, \varphi], \text{Direction} \rightarrow 1] - \right. \\
 & \quad \left. \text{ThetaPrmtv}[r, a, \varphi, \text{ThetaMaxCC}[r, a, \varphi]], \right. \\
 & \quad \left. \text{Assumptions} \rightarrow \{a > 0 \& \& a * \sqrt{2} < r < a * \sqrt{3} \& \& \text{FiMinCC}[r, a] < \varphi < \text{FiMaxCC}[r, a]\}\right]
 \end{aligned}$$

$$\begin{aligned}
 & \text{Simplify}\left[\text{Limit}[\text{ThetaPrmtv}[r, a, \varphi, \theta], \theta \rightarrow \text{ThetaMinCC}[r, a], \text{Direction} \rightarrow -1] - \right. \\
 & \quad \left. \text{ThetaPrmtv}[r, a, \varphi, \text{ThetaMinCC}[r, a]], \right. \\
 & \quad \left. \text{Assumptions} \rightarrow \{a > 0 \& \& a * \sqrt{2} < r < a * \sqrt{3} \& \& \text{FiMinCC}[r, a] < \varphi < \text{FiMaxCC}[r, a]\}\right]
 \end{aligned}$$

$$\begin{aligned}
 & \left(\text{FullSimplify}\left[\text{ThetaPrmtv}[r, a, \varphi, \text{ThetaMaxCC}[r, a, \varphi]] - \text{uprBndIntgrnd}[r, a, \varphi], \right. \right. \\
 & \quad \left. \left. \text{Assumptions} \rightarrow \{a * \sqrt{2} < r < a * \sqrt{3} \& \& \text{FiMinCC}[r, a] < \varphi < \text{FiMaxCC}[r, a] \& \& \cos[\varphi] > 0\}\right]\right) / . \\
 & \left[\sqrt{r^2 - a^2 \sec[\varphi]^2} \rightarrow \sqrt{-a^2 + r^2 \cos[\varphi]^2} \sec[\varphi]\right]
 \end{aligned}$$

$$\begin{aligned}
 & \text{FullSimplify}\left[\left(-\left(\sqrt{-a^2 + r^2 \cos[\varphi]^2} \sec[\varphi]\right)^2 + \left(\sqrt{r^2 - a^2 \sec[\varphi]^2}\right)^2\right), \right. \\
 & \quad \left. \text{Assumptions} \rightarrow \{a * \sqrt{2} < r < a * \sqrt{3} \& \& \text{FiMinCC}[r, a] < \varphi < \text{FiMaxCC}[r, a] \& \& \cos[\varphi] > 0\}\right]
 \end{aligned}$$

contribution due to the upper point of the θ - range

$$\begin{aligned}
 \text{uprBndIntgrnd}[r_, a_, \varphi_] := & \left(r - \frac{a^2 \sec[\varphi]^2}{r} - r * \text{ArcSin}\left[\frac{a}{r * \cos[\varphi]}\right] * \cos[\varphi] - \right. \\
 & \left. 5 a \frac{\sqrt{r^2 * \cos[\varphi]^2 - a^2}}{r * \cos[\varphi]} - r \text{ArcSin}\left[\frac{a}{r * \cos[\varphi]}\right] \sin[\varphi] + a \frac{\sqrt{r^2 * \cos[\varphi]^2 - a^2}}{r * \cos[\varphi]^2} * \sin[\varphi]\right);
 \end{aligned}$$

$$\begin{aligned}
 & \text{Simplify}\left[\text{ThetaPrmtv}[r, a, \varphi, \text{ThetaMinCC}[r, a]] - \text{lwrBndIntgrnd}[r, a, \varphi], \right. \\
 & \quad \left. \text{Assumptions} \rightarrow \{a * \sqrt{2} < r < a * \sqrt{3} \& \& \text{FiMinCC}[r, a] < \varphi < \text{FiMaxCC}[r, a] \& \& \cos[\varphi] > 0\}\right]
 \end{aligned}$$

contribution due to the lower point of the θ - range

$$\text{lwrBndIntgrnd}[r_, a_, \varphi_] := \frac{-r^2 \text{ArcSin}\left[\frac{\sqrt{-a^2+r^2}}{r}\right] (\text{Cos}[\varphi]+\text{Sin}[\varphi]) + a \left(-5 a+\sqrt{-a^2+r^2} \text{Cos}[\varphi]+\sqrt{-a^2+r^2} \text{Sin}[\varphi]\right)}{r};$$

resulting φ - integrand

$$\begin{aligned} \text{intgrndLinCC}[r_, a_, \varphi_] &:= \text{uprBndIntgrnd}[r, a, \varphi] - \text{lwrBndIntgrnd}[r, a, \varphi]; \\ \text{intgrndLinCC}[r, a, \varphi] \end{aligned}$$

MATHEmatica is unable to evaluate the φ - integrand (in less than one hour)

$$\begin{aligned} \text{Integrate}[\text{intgrndLinCC}[r, a, \varphi], \\ \{\varphi, \text{FiMinCC}[r, a], \text{FiMaxCC}[r, a]\}, \text{Assumptions} \rightarrow \{\sqrt{2} * a < r < \sqrt{3} * a\}] \\ \$\text{Aborted} \end{aligned}$$

Change of the integration variable

$$\begin{aligned} \text{Expand}\left[\left(\text{intgrndLinCC}[r, a, \varphi]\right) / . \right. \\ \left.\left\{\text{Sin}[\varphi] \rightarrow \sqrt{1-t^2}, \text{Cos}[\varphi] \rightarrow t, \text{Sec}[\varphi] \rightarrow \frac{1}{t}, \text{Tan}[\varphi] \rightarrow \frac{\sqrt{1-t^2}}{t}\right\} * \frac{1}{\sqrt{1-t^2}}\right] \end{aligned}$$

the new integrand

$$\begin{aligned} \text{intgrndLinCCtt}[r_, a_, t_] &:= -\frac{a \sqrt{-a^2+r^2}}{r} + \frac{5 a^2}{r \sqrt{1-t^2}} + \\ \frac{r}{\sqrt{1-t^2}} - \frac{a^2}{r t^2 \sqrt{1-t^2}} - \frac{a \sqrt{-a^2+r^2} t}{r \sqrt{1-t^2}} + \frac{a \sqrt{-a^2+r^2} t^2}{r t^2} - \frac{5 a \sqrt{-a^2+r^2} t^2}{r t \sqrt{1-t^2}} + \\ r \text{ArcSin}\left[\frac{\sqrt{-a^2+r^2}}{r}\right] + \frac{r t \text{ArcSin}\left[\frac{\sqrt{-a^2+r^2}}{r}\right]}{\sqrt{1-t^2}} - r \text{ArcSin}\left[\frac{a}{r t}\right] - \frac{r t \text{ArcSin}\left[\frac{a}{r t}\right]}{\sqrt{1-t^2}}; \end{aligned}$$

the primitive has an algebraic form

$$\text{Integrate}[\text{intgrndLinCCtt}[r, a, t], t]$$

The bounds with respect to the variable t are

$$\text{FullSimplify}[\text{Cos}[\text{FiMinCC}[r, a]], \text{Assumptions} \rightarrow \{a \sqrt{2} < r < a \sqrt{3}\}]$$

$$\text{FullSimplify}[\text{Cos}[\text{FiMaxCC}[r, a]], \text{Assumptions} \rightarrow \{a \sqrt{2} < r < a \sqrt{3}\}]$$

$$\text{TTMax}[r_, a_] := \frac{a}{\sqrt{-a^2+r^2}}; \text{TTMin}[r_, a_] := \frac{1}{\sqrt{2}};$$

Integration by MATHEMATICA is successful

```
Integrate[intgrndLinCCtt[r, a, t],
{t, TTMin[r, a], TTMax[r, a]}, Assumptions -> {a Sqrt[2] < r < a Sqrt[3]}]
```

value of the integral and normalization

```
LinContrCCNotSimpl[r_, a_] :=
NormFactLin[a] * 
$$\left( \frac{1}{4 r (a^2 - r^2)^2} \left( -12 a^6 + 15 a^6 \pi + 24 a^4 r^2 - 27 a^4 \pi r^2 - 12 a^2 r^4 + 9 a^2 \pi r^4 + 3 \pi r^6 + 12 a^5 \sqrt{-2 a^2 + r^2} - 24 a^3 r^2 \sqrt{-2 a^2 + r^2} + 12 a r^4 \sqrt{-2 a^2 + r^2} + 4 (a^2 - r^2)^2 (5 a^2 + r^2) \text{ArcSin}\left[\frac{a}{\sqrt{-a^2 + r^2}}\right] - 4 a^5 r \text{Log}[2] + 8 a^3 r^3 \text{Log}[2] - 4 a r^5 \text{Log}[2] + i a^5 r \text{Log}[4096] + i a r^5 \text{Log}[4096] - i a^3 r^3 \text{Log}[16777216] - 8 a r (a^2 - r^2)^2 \text{Log}[a] - 12 i a^5 r \text{Log}[-a^2 + r^2] + 24 i a^3 r^3 \text{Log}[-a^2 + r^2] - 12 i a r^5 \text{Log}[-a^2 + r^2] + 4 a^5 r \text{Log}\left[r - \sqrt{-2 a^2 + r^2}\right] - 8 a^3 r^3 \text{Log}\left[r - \sqrt{-2 a^2 + r^2}\right] + 4 a r^5 \text{Log}\left[r - \sqrt{-2 a^2 + r^2}\right] - 24 i a^5 r \text{Log}\left[r + i \sqrt{-2 a^2 + r^2}\right] + 48 i a^3 r^3 \text{Log}\left[r + i \sqrt{-2 a^2 + r^2}\right] - 24 i a r^5 \text{Log}\left[r + i \sqrt{-2 a^2 + r^2}\right] + 4 a^5 r \text{Log}\left[r + \sqrt{-2 a^2 + r^2}\right] - 8 a^3 r^3 \text{Log}\left[r + \sqrt{-2 a^2 + r^2}\right] + 4 a r^5 \text{Log}\left[r + \sqrt{-2 a^2 + r^2}\right] + 10 i a^6 \text{Log}\left[3 i a^2 - i r^2 - 2 a \sqrt{-2 a^2 + r^2}\right] - 18 i a^4 r^2 \text{Log}\left[3 i a^2 - i r^2 - 2 a \sqrt{-2 a^2 + r^2}\right] + 6 i a^2 r^4 \text{Log}\left[3 i a^2 - i r^2 - 2 a \sqrt{-2 a^2 + r^2}\right] + 2 i r^6 \text{Log}\left[3 i a^2 - i r^2 - 2 a \sqrt{-2 a^2 + r^2}\right] - 10 i a^6 \text{Log}\left[-3 i a^2 + i r^2 - 2 a \sqrt{-2 a^2 + r^2}\right] + 18 i a^4 r^2 \text{Log}\left[-3 i a^2 + i r^2 - 2 a \sqrt{-2 a^2 + r^2}\right] - 6 i a^2 r^4 \text{Log}\left[-3 i a^2 + i r^2 - 2 a \sqrt{-2 a^2 + r^2}\right] - 2 i r^6 \text{Log}\left[-3 i a^2 + i r^2 - 2 a \sqrt{-2 a^2 + r^2}\right] + 24 i a^5 r \text{Log}\left[i a^2 + r \sqrt{-2 a^2 + r^2}\right] - 48 i a^3 r^3 \text{Log}\left[i a^2 + r \sqrt{-2 a^2 + r^2}\right] + 24 i a r^5 \text{Log}\left[i a^2 + r \sqrt{-2 a^2 + r^2}\right] \right)$$

```

simplification of the above function

```
FullSimplify[LinContrCCNotSimpl[r, a], Assumptions -> {a Sqrt[2] < r < a Sqrt[3]}]
```

```
LinContrCCPartlySimpl[r_, a_] :=
1 / (6 a^2 \pi r) \left( a^2 (-12 + 5 \pi) + \pi r^2 + 12 a \sqrt{-2 a^2 + r^2} + 2 \left( 2 (5 a^2 + r^2) \text{ArcSin}\left[\frac{a}{\sqrt{-a^2 + r^2}}\right] + i \left( 6 a r \text{Log}\left[-\frac{2}{a^2 - r^2}\right] - 12 a r \text{Log}\left[r + i \sqrt{-2 a^2 + r^2}\right] + (5 a^2 + r^2) \text{Log}\left[3 i a^2 - i r^2 - 2 a \sqrt{-2 a^2 + r^2}\right] - \text{Log}\left[3 i a^2 - i r^2 + 2 a \sqrt{-2 a^2 + r^2}\right] \right) + 12 a r \text{Log}\left[i a^2 + r \sqrt{-2 a^2 + r^2}\right] \right) \right);
```

Analytic Checks. Continuity at the border points (satisfied)

```
FullSimplify[Limit[LinContrCCPartlySimpl[r, a], r → a √3, Direction → 1],
Assumptions → {a > 0}]
Simplify[(FullSimplify[Limit[LinContrCCPartlySimpl[r, a], r → a √2, Direction → -1],
Assumptions → {a > 0}] -
FullSimplify[Limit[CubSurfCFLnBB[r, a], r → a √2, Direction → 1],
Assumptions → {a > 0}])]
```

numerical check (it's OK)

```
With[{a = 1}, StepRCC = a (sqrt[3] - sqrt[2]) / 10;
Do[ract = sqrt[2] + (J - 1/2) * StepRCC; valreal =
N[(Re[Log[3 i a^2 - i r^2 - 2 a Sqrt[-2 a^2 + r^2]] - (Log[r^2 - a^2] + i * (-ArcSin[3 * a^2 - r^2 / (r^2 - a^2)] + π))]) /.
{r → ract}, 30]; valimag =
N[(Im[Log[3 i a^2 - i r^2 - 2 a Sqrt[-2 a^2 + r^2]] - (Log[r^2 - a^2] + i * (-ArcSin[3 * a^2 - r^2 / (r^2 - a^2)] + π))]) /.
{r → ract}, 30];
Print[J, ", ", valreal, ", ", valimag]], {J, 1, 10}]]
```

Further simplification of LinContrCCPartlySimpl[r, a] by the following identities

IDENTITIES (and their proofs)

$$\begin{aligned}\text{Log}[r + i \sqrt{-2 a^2 + r^2}] &\rightarrow \left(\text{Log}[\sqrt{2(r^2 - a^2)}] + i * \text{ArcSin}\left[\frac{\sqrt{-2 a^2 + r^2}}{\sqrt{2(r^2 - a^2)}}\right] \right), \\ \text{Log}[3 i a^2 - i r^2 - 2 a \sqrt{-2 a^2 + r^2}] &\rightarrow \left(\text{Log}[r^2 - a^2] + i * \left(-\text{ArcSin}\left[\frac{3 * a^2 - r^2}{r^2 - a^2}\right] + \pi\right) \right), \\ \text{Log}[3 i a^2 - i r^2 + 2 a \sqrt{-2 a^2 + r^2}] &\rightarrow \left(\text{Log}[r^2 - a^2] + i * \text{ArcSin}\left[\frac{3 * a^2 - r^2}{r^2 - a^2}\right] \right), \\ \text{Log}[i a^2 + r \sqrt{-2 a^2 + r^2}] &\rightarrow \left(\text{Log}[r^2 - a^2] + i * \text{ArcSin}\left[\frac{a^2}{r^2 - a^2}\right] \right)\end{aligned}$$

```
With[{a = 1}, ParametricPlot[
{{r, Re[Log[3 i a^2 - i r^2 - 2 a Sqrt[-2 a^2 + r^2]] - (Log[r^2 - a^2] + i * (-ArcSin[3 * a^2 - r^2 / (r^2 - a^2)] + π))]}},
{r, Im[Log[3 i a^2 - i r^2 - 2 a Sqrt[-2 a^2 + r^2]] - (Log[r^2 - a^2] + i * (-ArcSin[3 * a^2 - r^2 / (r^2 - a^2)] + π))]}},
{r, a √2, a √3},
PlotRange → {{a √2, a √3}, {-10^(-15), 10^(-15)}}, AspectRatio → 1]]
```

With[$\{a = 1\}$,

$$\text{ParametricPlot}\left[\left\{\left\{r, \operatorname{Re}\left[\operatorname{Log}\left[i a^2 + r \sqrt{-2 a^2 + r^2}\right]\right] - \left(\operatorname{Log}\left[r^2 - a^2\right] + i * \operatorname{ArcSin}\left[\frac{a^2}{r^2 - a^2}\right]\right)\right\}, \left\{r, \operatorname{Im}\left[\operatorname{Log}\left[i a^2 + r \sqrt{-2 a^2 + r^2}\right]\right] - \left(\operatorname{Log}\left[r^2 - a^2\right] + i * \operatorname{ArcSin}\left[\frac{a^2}{r^2 - a^2}\right]\right)\right\}\right], \{r, a \sqrt{2}, a \sqrt{3}\},$$

$$\text{PlotRange} \rightarrow \{\{a \sqrt{2}, a \sqrt{3}\}, \{-10^{(-15)}, 10^{(-15)}\}\}, \text{AspectRatio} \rightarrow 1\]$$

With[$\{a = 1\}$, ParametricPlot[

$$\left\{\left\{r, \operatorname{Re}\left[\operatorname{Log}\left[r + i \sqrt{-2 a^2 + r^2}\right]\right] - \left(\operatorname{Log}\left[\sqrt{2 (r^2 - a^2)}\right] + i * \operatorname{ArcSin}\left[\frac{\sqrt{-2 a^2 + r^2}}{\sqrt{2 (r^2 - a^2)}}\right]\right)\right\},$$

$$\left\{r, \operatorname{Im}\left[\operatorname{Log}\left[r + i \sqrt{-2 a^2 + r^2}\right]\right] - \left(\operatorname{Log}\left[\sqrt{2 (r^2 - a^2)}\right] + i * \operatorname{ArcSin}\left[\frac{\sqrt{-2 a^2 + r^2}}{\sqrt{2 (r^2 - a^2)}}\right]\right)\right\},$$

$$\{r, a \sqrt{2}, a \sqrt{3}\},$$

$$\text{PlotRange} \rightarrow \{\{a \sqrt{2}, a \sqrt{3}\}, \{-10^{(-15)}, 10^{(-15)}\}\}, \text{AspectRatio} \rightarrow 1\]$$

Further simplification of LinContrCCPartlySimpl[r, a] by the following identities

Expand[Simplify[(LinContrCCPartlySimpl[r, a]) /.

$$\left\{\operatorname{Log}\left[r + i \sqrt{-2 a^2 + r^2}\right] \rightarrow \left(\operatorname{Log}\left[\sqrt{2 (r^2 - a^2)}\right] + i * \operatorname{ArcSin}\left[\frac{\sqrt{-2 a^2 + r^2}}{\sqrt{2 (r^2 - a^2)}}\right]\right),$$

$$\operatorname{Log}\left[3 i a^2 - i r^2 - 2 a \sqrt{-2 a^2 + r^2}\right] \rightarrow \left(\operatorname{Log}\left[r^2 - a^2\right] + i * \left(-\operatorname{ArcSin}\left[\frac{3 * a^2 - r^2}{r^2 - a^2}\right] + \pi\right)\right),$$

$$\operatorname{Log}\left[3 i a^2 - i r^2 + 2 a \sqrt{-2 a^2 + r^2}\right] \rightarrow \left(\operatorname{Log}\left[r^2 - a^2\right] + i * \operatorname{ArcSin}\left[\frac{3 * a^2 - r^2}{r^2 - a^2}\right]\right),$$

$$\operatorname{Log}\left[i a^2 + r \sqrt{-2 a^2 + r^2}\right] \rightarrow \left(\operatorname{Log}\left[r^2 - a^2\right] + i * \operatorname{ArcSin}\left[\frac{a^2}{r^2 - a^2}\right]\right),$$

$$\text{Assumptions} \rightarrow \{a > 0 \&& a \sqrt{2} < r < a \sqrt{3}\}]$$

Simplify[$\frac{1}{6 a^2 \pi r}$

$$\left\{-12 a^2 - 5 a^2 \pi - \pi r^2 + 12 a \sqrt{-2 a^2 + r^2} + 24 a r \operatorname{ArcSin}\left[\frac{a^2}{a^2 - r^2}\right] + 4 (5 a^2 + r^2) \operatorname{ArcSin}\left[\frac{-3 a^2 + r^2}{a^2 - r^2}\right] +$$

$$24 a r \operatorname{ArcSin}\left[\frac{\sqrt{\frac{-2 a^2 + r^2}{-a^2 + r^2}}}{\sqrt{2}}\right] + 20 a^2 \operatorname{ArcSin}\left[\frac{a}{\sqrt{-a^2 + r^2}}\right] + 4 r^2 \operatorname{ArcSin}\left[\frac{a}{\sqrt{-a^2 + r^2}}\right]\right]$$

$$\begin{aligned}
& \frac{1}{6 a^2 \pi r} \left(-12 a^2 - 5 a^2 \pi - \pi r^2 + 12 a \sqrt{-2 a^2 + r^2} + 4 (5 a^2 + r^2) \operatorname{ArcSin}\left[\frac{-3 a^2 + r^2}{a^2 - r^2}\right] + \right. \\
& \operatorname{Factor}\left[24 a r \operatorname{ArcSin}\left[\frac{\sqrt{\frac{-2 a^2 + r^2}{-a^2 + r^2}}}{\sqrt{2}}\right] + 24 a r \operatorname{ArcSin}\left[\frac{a^2}{a^2 - r^2}\right]\right] + \\
& \left. \operatorname{Factor}\left[20 a^2 \operatorname{ArcSin}\left[\frac{a}{\sqrt{-a^2 + r^2}}\right] + 4 r^2 \operatorname{ArcSin}\left[\frac{a}{\sqrt{-a^2 + r^2}}\right]\right]\right)
\end{aligned}$$

$$\begin{aligned}
\text{LinContrCCFurtherSimpl}[r_, a_] := \frac{1}{6 a^2 \pi r} \\
& \left(-12 a^2 - 5 a^2 \pi - \pi r^2 + 12 a \sqrt{-2 a^2 + r^2} + 24 a r \left(\operatorname{ArcSin}\left[\frac{a^2}{a^2 - r^2}\right] + \operatorname{ArcSin}\left[\frac{\sqrt{\frac{-2 a^2 + r^2}{-a^2 + r^2}}}{\sqrt{2}}\right]\right) + \right. \\
& \left. \operatorname{Factor}\left[4 (5 a^2 + r^2) \operatorname{ArcSin}\left[\frac{a}{\sqrt{-a^2 + r^2}}\right] + 4 (5 a^2 + r^2) \operatorname{ArcSin}\left[\frac{-3 a^2 + r^2}{a^2 - r^2}\right]\right]\right);
\end{aligned}$$

$$\begin{aligned}
& \text{FullSimplify}\left[\text{Simplify}\left[(\text{LinContrCCPartlySimpl}[r, a]) /.\right.\right. \\
& \left.\left\{\operatorname{Log}\left[r + i \sqrt{-2 a^2 + r^2}\right] \rightarrow \left(\operatorname{Log}\left[\sqrt{2 (r^2 - a^2)}\right] + i * \operatorname{ArcSin}\left[\frac{\sqrt{-2 a^2 + r^2}}{\sqrt{2 (r^2 - a^2)}}\right]\right),\right.\right. \\
& \left.\left.\operatorname{Log}\left[3 i a^2 - i r^2 - 2 a \sqrt{-2 a^2 + r^2}\right] \rightarrow \left(\operatorname{Log}\left[r^2 - a^2\right] + i * \left(-\operatorname{ArcSin}\left[\frac{3 * a^2 - r^2}{r^2 - a^2}\right] + \pi\right)\right),\right.\right. \\
& \left.\left.\operatorname{Log}\left[3 i a^2 - i r^2 + 2 a \sqrt{-2 a^2 + r^2}\right] \rightarrow \left(\operatorname{Log}\left[r^2 - a^2\right] + i * \operatorname{ArcSin}\left[\frac{3 * a^2 - r^2}{r^2 - a^2}\right]\right),\right.\right. \\
& \left.\left.\operatorname{Log}\left[i a^2 + r \sqrt{-2 a^2 + r^2}\right] \rightarrow \left(\operatorname{Log}\left[r^2 - a^2\right] + i * \operatorname{ArcSin}\left[\frac{a^2}{r^2 - a^2}\right]\right)\right], \\
& \text{Assumptions} \rightarrow \{a > 0 \&& a \sqrt{2} < r < a \sqrt{3}\} - \\
& \text{LinContrCCFurtherSimpl}[r, a], \text{Assumptions} \rightarrow \{a > 0 \&& a \sqrt{2} < r < a \sqrt{3}\}]
\end{aligned}$$

Further Identities

$$\begin{aligned} \left(\text{ArcSin}\left[\frac{a^2}{a^2 - r^2}\right] + \text{ArcSin}\left[\frac{\sqrt{\frac{-2 a^2 + r^2}{-a^2 + r^2}}}{\sqrt{2}}\right] \right) &\rightarrow \left(\text{ArcSin}\left[\frac{\sqrt{\frac{-2 a^2 + r^2}{-a^2 + r^2}} (a^2 + r^2)}{\sqrt{2} (-a^2 + r^2)}\right] - \frac{\pi}{2} \right), \\ \left(\text{ArcSin}\left[\frac{-3 a^2 + r^2}{a^2 - r^2}\right] + \text{ArcSin}\left[\frac{a}{\sqrt{-a^2 + r^2}}\right] \right) &\rightarrow \left(\text{ArcSin}\left[\frac{7 a^3 - 3 a r^2}{(-a^2 + r^2)^{3/2}}\right] + \frac{\pi}{2} \right) \end{aligned}$$

With[{a = 1}, Plot[{ArcSin[a^2/(a^2 - r^2)] + ArcSin[-2 a^2 + r^2]/Sqrt[2]}, {r, a Sqrt[2], a Sqrt[3]}]]

$$\left(\text{ArcSin}\left[\frac{-3 a^2 + r^2}{a^2 - r^2}\right] + \text{ArcSin}\left[\frac{a}{\sqrt{-a^2 + r^2}}\right] \right) / \pi - 1/2, \{r, a \sqrt{2}, a \sqrt{3}\}]$$

FullSimplify[ExpandAll[TrigExpand[Sin[ArcSin[-3 a^2 + r^2]/(a^2 - r^2)] + ArcSin[a]/Sqrt[-a^2 + r^2]] - π/2]], Assumptions → {a > 0 && a Sqrt[2] < r < a Sqrt[3] && r^2 - a^2 > 0 && -2 a^2 + r^2 > 0}]

With[{a = 1}, ParametricPlot[{{r, ArcSin[-3 a^2 + r^2]/(a^2 - r^2) + ArcSin[a]/Sqrt[-a^2 + r^2]} - ArcSin[7 a^3 - 3 a r^2]/(a^2 - r^2)^{3/2} + π/2}, {r, a Sqrt[2], a Sqrt[3]}], PlotRange → {{a Sqrt[2], a Sqrt[3]}, {-10^(-15), 10^(-15)}}, AspectRatio → 1]]

With[{a = 1}, ParametricPlot[{{r, ArcSin[a^2/(a^2 - r^2)] + ArcSin[-2 a^2 + r^2]/Sqrt[2]} - ArcSin[-2 a^2 + r^2]/Sqrt[2] (-a^2 + r^2)} - π/2, {r, a Sqrt[2], a Sqrt[3]}], PlotRange → {{a Sqrt[2], a Sqrt[3]}, {-10^(-15), 10^(-15)}}, AspectRatio → 1]]

Simplify[-r^4/(-a^2 + r^2) + a^2 (r^2/(-a^2 + r^2) - 1)/Sqrt[2] (a^2 - r^2)] Sqrt[-2 a^2 + r^2]/(-a^2 + r^2)

FullSimplify[TrigExpand[Cos[ArcSin[-3 a^2 + r^2]/(a^2 - r^2)] + ArcSin[a]/Sqrt[-a^2 + r^2]]], Assumptions → {a > 0 && a Sqrt[2] < r < a Sqrt[3]}]

final simplification

```
Simplify[
```

$$\text{Expand}[\text{Simplify}[(\text{LinContrCCFurtherSimpl}[r, a]) / . \left\{ \text{ArcSin}\left[\frac{a^2}{a^2 - r^2}\right] + \text{ArcSin}\left[\frac{\sqrt{\frac{-2 a^2 + r^2}{-a^2 + r^2}}}{\sqrt{2}}\right] \rightarrow \right.$$

$$\left. \text{ArcSin}\left[\frac{\sqrt{\frac{-2 a^2 + r^2}{-a^2 + r^2}} (a^2 + r^2)}{\sqrt{2} (-a^2 + r^2)}\right] - \frac{\pi}{2} \right], \left\{ \text{ArcSin}\left[\frac{-3 a^2 + r^2}{a^2 - r^2}\right] + \text{ArcSin}\left[\frac{a}{\sqrt{-a^2 + r^2}}\right] \rightarrow \right.$$

$$\left. \left(\text{ArcSin}\left[\frac{7 a^3 - 3 a r^2}{(-a^2 + r^2)^{3/2}}\right] + \frac{\pi}{2} \right) \right] - \text{CubSurfCFLnCC}[r, a]$$

$$0$$

Final Expression

$$\text{CubSurfCFLnCC}[r, a] :=$$

$$\frac{2}{a} + \frac{-12 + 5 \pi}{6 \pi r} + \frac{r}{6 a^2} + \frac{2 \sqrt{-2 a^2 + r^2}}{a \pi r} + \frac{4}{a \pi} \text{ArcSin}\left[\sqrt{\frac{-2 a^2 + r^2}{-2 a^2 + 2 r^2}} * \frac{(a^2 + r^2)}{-a^2 + r^2}\right] +$$

$$\frac{2 (5 a^2 + r^2)}{3 a^2 \pi r} * \text{ArcSin}\left[\frac{7 a^3 - 3 a r^2}{(-a^2 + r^2)^{3/2}}\right];$$

Continuity Checks

$$\text{Simplify}[\text{Limit}[\text{CubSurfCFLnCC}[r, a], r \rightarrow a \sqrt{2}, \text{Direction} \rightarrow -1] -$$

$$\text{Limit}[\text{CubSurfCFLnBB}[r, a], r \rightarrow a \sqrt{2}, \text{Direction} \rightarrow 1], \text{Assumptions} \rightarrow \{a > 0\}]$$

$$\text{Simplify}[\text{Limit}[\text{CubSurfCFLnCC}[r, a], r \rightarrow a \sqrt{3}, \text{Direction} \rightarrow 1], \text{Assumptions} \rightarrow \{a > 0\}]$$

the values obtained by the above function are compared with the values obtained by the numerical integration of the outset integrand (it's OK)

$$\text{With}[\{a = 1\}, \text{StepRCC} = a (\sqrt{3} - \sqrt{2}) / 10;$$

$$\text{Do}[\text{ract} = \sqrt{2} + (J - 1/2) * \text{StepRCC}; \text{actval} = N[\text{CubSurfCFLnCC}[\text{ract}, a]]; \text{Print}[J, ", ", \text{valCC}[[J]], ", ", \text{actval}], \{J, 1, 10\}]$$

$$\text{MomlinCC} = \text{Simplify}[\text{Integrate}[r^2 * \text{CubSurfCFLnCC}[r, a],$$

$$\{r, a \sqrt{2}, a \sqrt{3}\}, \text{Assumptions} \rightarrow \{a > 0\}, \text{Assumptions} \rightarrow \{a > 0\}]$$

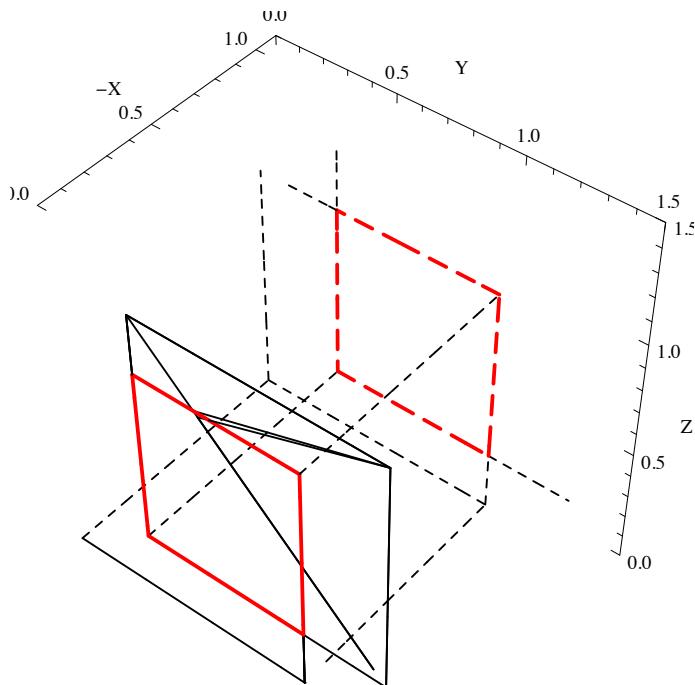
$$\text{MomlinCC} = \frac{a^2 (56 + (-63 + 32 \sqrt{2}) \pi)}{24 \pi};$$

```

Simplify[MomlinAA + MomlinBB +  $\frac{a^2 (56 + (-63 + 32 \sqrt{2}) \pi)}{24 \pi}]$ 
 $\frac{a^2}{\pi}$ 
MomlinAABBCC = N[Simplify[MomlinAA + MomlinBB + MomlinCC]]
0.31831 a2

```

EVALAUTION OF THE CF-CONTRIBUTION DUE TO THE SLIDING SURFACES



normalization factor and integrand with $0 < \varphi < \frac{\pi}{2}$

```

surfNormFact[r_, a_] := 2 * 3 *  $\frac{4}{r} * \frac{1}{4 * \pi * 6 * a^2}$ ;
surfIntgrnd[r_, a_,  $\varphi$ _] := (a - r * Cos[ $\varphi$ ]) * (a - r * Sin[ $\varphi$ ] );
(surfIntgrnd[r, a,  $\varphi$ ]) /. {Sin[ $\varphi$ ] → t, Cos[ $\varphi$ ] →  $\sqrt{1 - t^2}$ }
(a - r t) (a - r  $\sqrt{1 - t^2}$ )

```

Inequalities' reduction

```
Reduce[{{a > a - r t > 0, a > a - r Sqrt[1 - t^2] > 0, 0 < t < 1, a > 0, r > 0}, {a, r, t}, Reals]
```

$$a > 0 \&& \left((0 < r \leq a \&& 0 < t < 1) \mid\mid \left(a < r < \sqrt{2} \sqrt{a^2} \&& \sqrt{\frac{-a^2 + r^2}{r^2}} < t < \frac{a}{r} \right) \right)$$

Region AAsurf :

$$a > 0 \&& 0 < r < a \&& 0 < t < 1 \quad \text{or} \quad 0 < \varphi < \frac{\pi}{2}$$

Region BBsurf :

$$a > 0 \&& a < r < a * \sqrt{2} \&& \frac{\sqrt{-a^2 + r^2}}{r} < t < \frac{a}{r} \quad \text{or} \quad \text{ArcSin}\left[\frac{\sqrt{-a^2 + r^2}}{r}\right] < \varphi < \text{ArcSin}\left[\frac{a}{r}\right]$$

Evaluation of the Surface CF contribution over the AAsurf region

```
Integrate[surfIntgrnd[r, a, \varphi], {\varphi, 0, \frac{\pi}{2}}, Assumptions \rightarrow {a > 0 \&& 0 < r < a}]
```

$$\begin{aligned} &\text{Expand}\left[\text{surfNormFact}[r, a] * \frac{1}{2} (a^2 \pi - 4 a r + r^2)\right] \\ &- \frac{2}{a \pi} + \frac{1}{2 r} + \frac{r}{2 a^2 \pi} \end{aligned}$$

$$\text{surcContrCFAAs}[r, a] := -\frac{2}{a \pi} + \frac{1}{2 r} + \frac{r}{2 a^2 \pi}; \quad (* \quad 0 < r < a *)$$

Evaluation of the Surface CF contribution over the BBsurf region

```
FullSimplify[Integrate[surfIntgrnd[r, a, \varphi], {\varphi, \text{ArcSin}\left[\frac{\sqrt{-a^2 + r^2}}{r}\right], \text{ArcSin}\left[\frac{a}{r}\right]}], Assumptions \rightarrow {a > 0 \&& a < r < a * \sqrt{2}}]
```

$$-\frac{r^2}{2} + 2 a \sqrt{-a^2 + r^2} + a^2 \left(-1 - \text{ArcCsc}\left[\frac{r}{\sqrt{-a^2 + r^2}}\right] + \text{ArcSin}\left[\frac{a}{r}\right] \right)$$

```
FullSimplify[TrigExpand[\text{Sin}\left[\text{ArcCsc}\left[\frac{r}{\sqrt{-a^2 + r^2}}\right]\right] - \left(\frac{\pi}{2} - \text{ArcSin}\left[\frac{a}{r}\right]\right)]]]
```

$$\text{Assumptions} \rightarrow \{a > 0 \&& a < r < a * \sqrt{2}\}$$

```

FullSimplify[TrigExpand[Cos[ArcCsc[r/Sqrt[-a^2 + r^2]] - (π/2 - ArcSin[a/r])]], 
Assumptions → {a > 0 && a < r < a * Sqrt[2]}]

1

Expand[Simplify[surfNormFact[r, a] *
  FullSimplify[Integrate[surfIntgrnd[r, a, φ], {φ, ArcSin[Sqrt[-a^2 + r^2]/r], ArcSin[a/r]}], 
  Assumptions → {a > 0 && a < r < a * Sqrt[2]}, Assumptions → {a > 0 && a < r < a * Sqrt[2]}]] /.

{ArcCsc[r/Sqrt[-a^2 + r^2]] → (π/2 - ArcSin[a/r])}]

Simplify[Together[-1/(2 r) - 1/(π r) - r/(2 a^2 π) + 2 Sqrt[-a^2 + r^2]/(a π r) + 2 ArcSin[a/r]/(π r) - (surcContrCFBBs[r, a])]

0

surcContrCFBBs[r_, a_] := -2 + π/(2 π r) - r/(2 a^2 π) + 2 Sqrt[-a^2 + r^2]/(a π r) + 2 ArcSin[a/r]/(π r); (* a < r < a * Sqrt[2] *)

```

Evaluation of the moments of the two contributions

```
Simplify[Integrate[r^2 * surcContrCFAAs[r, a], {r, 0, a}, Assumptions → {a > 0}]]
```

$$\text{Momt00SupAA}[a_] := \frac{a^2 (-13 + 6 \pi)}{24 \pi};$$

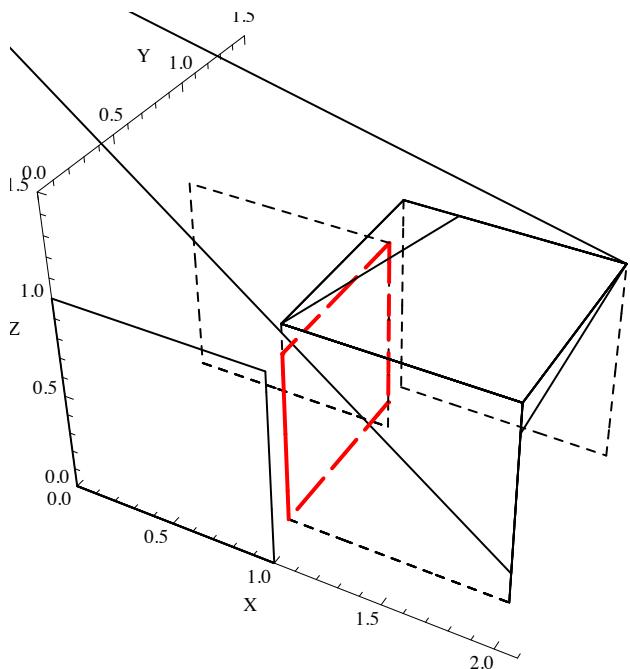
```
Simplify[Integrate[r^2 * surcContrCFBBs[r, a], {r, a, a * Sqrt[2]}, Assumptions → {a > 0}]]
```

$$\text{Momt00SupBB}[a_] := \frac{a^2 (19 - 6 \pi)}{24 \pi};$$

```
Simplify[Momt00SupAA[a] + Momt00SupBB[a]]
```

$$\text{Momt00Sup}[a_] := \frac{a^2}{4 \pi};$$

EVALUATION OF THE CONTRIBUTION TO THE CF DUE TO THE PAIRS OF OPPOSING FACES



The integral takes the form

$$2 \sum_{i < j=1}^6 \int_{S_i} d\mathbf{S}_1 \int_{S_i} d\mathbf{S}_2 \int d\vec{\omega} \delta(\vec{r}_1 + \vec{r}\vec{\omega} - \vec{r}_2)$$

One chooses the polar axis along axis x and the longitudinal plane along the (x, z) plane.

One consider the two faces

$S_1 = (0, y_1, z_1)$ and $S_2 = (a, y_1, z_1)$. Then

$\vec{r} = r (\cos[\theta], \sin[\theta] \sin[\varphi], \sin[\theta] \cos[\varphi])$ and the integral reads

$$\int dS_1 \int dS_2 \int d\omega \delta(r \cos[\theta] - a) \delta(y_1 + r \sin[\theta] \sin[\varphi] - y_2) \delta(z_1 + r \sin[\theta] \cos[\varphi] - z_2)$$

One confines himself to the angular ranges $(0 < \theta \leq \frac{\pi}{2})$ and $(0 < \varphi < \frac{\pi}{2})$.

The value of the resulting integral will then be multiplied by
2 (due to θ) \times 2 (due to φ) \times 3 (due to the three axes)

$$\times 2 \text{ (the factor in front of the integral)} \times \frac{1}{4\pi S}.$$

$$\text{Hence the normalization factor is } \frac{24}{4\pi S}$$

From the Dirac functions one gets

$$\cos[\theta] = \frac{a}{r}, \quad y_2 = y_1 + r \sin[\theta] \sin[\varphi], \quad z_2 = z_1 + r \sin[\theta] \cos[\varphi],$$

and the integral reads

$$\begin{aligned} \frac{1}{r} \int dy_1 \int dy_2 \int d\varphi \Theta(y_1 + r \sin[\theta] \sin[\varphi]) \Theta(z_1 + r \sin[\theta] \cos[\varphi]) = \\ \frac{1}{r} \int_0^{\pi/2} d\varphi (\min[a, a - r \sin[\theta] \sin[\varphi]] - \max[0, -r \sin[\theta] \sin[\varphi]]) \\ (\min[a, a - r \sin[\theta] \cos[\varphi]] - \max[0, -r \sin[\theta] \cos[\varphi]]) \end{aligned}$$

Putting $r = X * a$, one gets

$$\begin{aligned} \frac{1}{r} \int dy_1 \int dy_2 \int d\varphi \Theta(y_1 + r \sin[\theta] \sin[\varphi]) \Theta(z_1 + r \sin[\theta] \cos[\varphi]) = \\ \frac{a^2}{r} \int_0^{\pi/2} d\varphi (\min[1, 1 - X \sin[\theta] \sin[\varphi]] - \max[0, -X \sin[\theta] \sin[\varphi]]) \\ (\min[1, 1 - X \sin[\theta] \cos[\varphi]] - \max[0, -X \sin[\theta] \cos[\varphi]]) = \\ \frac{a^2}{r} \int_0^{\pi/2} d\varphi (1 - X \sin[\theta] \sin[\varphi]) (1 - X \sin[\theta] \cos[\varphi]) \end{aligned}$$

$$\sin\left[\text{ArcCos}\left[\frac{1}{X}\right]\right]$$

$$\sqrt{1 - \frac{1}{X^2}}$$

$$\sin[\varphi] \rightarrow t, \cos[\varphi] \rightarrow \sqrt{1 - t^2}, \sin[\theta] \rightarrow \frac{\sqrt{x^2 - 1}}{x}$$

$$((1 - X \sin[\theta] \sin[\varphi]) (1 - X \sin[\theta] \cos[\varphi])) / .$$

$$\left\{ \sin[\varphi] \rightarrow t, \cos[\varphi] \rightarrow \sqrt{1 - t^2}, \sin[\theta] \rightarrow \frac{\sqrt{x^2 - 1}}{x} \right\}$$

$$\left(1 - t \sqrt{-1 + X^2}\right) \left(1 - \sqrt{1 - t^2} \sqrt{-1 + X^2}\right)$$

Reduction of the inequalities and determination of the integration range

```
Reduce[{{1 > 1 - t Sqrt[-1 + x^2] > 0, 1 > 1 - Sqrt[1 - t^2] Sqrt[-1 + x^2] > 0, x > 1}, {x, t}, Reals]
```

$$\left(1 < x \leq \sqrt{2} \quad \& \quad 0 < t < 1\right) \quad \& \quad \left(\sqrt{2} < x < \sqrt{3} \quad \& \quad \sqrt{\frac{-2 + x^2}{-1 + x^2}} < t < \sqrt{\frac{1}{-1 + x^2}}\right)$$

integrand definition in terms of the new variables t and X

```
newCaseIntrg[x_, t_] := (1 - t Sqrt[-1 + x^2]) (1 - Sqrt[1 - t^2] Sqrt[-1 + x^2]) * 1/Sqrt[1 - t^2];
```

Integration over the first region

$$1 < x \leq \sqrt{2} \quad \& \quad 0 < t < 1$$

```
Integrate[newCaseIntrg[x, t], {t, 0, 1}, Assumptions -> {1 < x <= Sqrt[2]}]
```

$$\frac{1}{2} \left(-1 + \pi + x^2 - 4 \sqrt{-1 + x^2}\right)$$

```
FullSimplify[
```

$$\left(\text{Expand}[\text{FullSimplify}\left[\left(\frac{24}{4 \pi 6 * a^2} * \frac{a^2}{r} * \left(\frac{1}{2} \left(-1 + \pi + x^2 - 4 \sqrt{-1 + x^2}\right)\right)\right)] /.\{x \rightarrow \frac{r}{a}\}, \text{Assumptions} \rightarrow \{a < r < \sqrt{2} * a\}\right]\right) /.\left\{\frac{\sqrt{-1 + \frac{r^2}{a^2}}}{\pi r} \rightarrow \frac{\sqrt{r^2 - a^2}}{\pi a r}\right\} -$$

```
RmngContBB[r, a], Assumptions -> {a > 0 && a < r < a Sqrt[2]}]
```

```
RmngContBB[r_, a_] := (-1 + \pi)/(2 \pi r) + r/(2 a^2 \pi) - 2 \sqrt{-a^2 + r^2}/(a \pi r);
```

Integration over the second region

$$\sqrt{2} < x < \sqrt{3} \quad \& \quad \sqrt{\frac{-2 + x^2}{-1 + x^2}} < t < \sqrt{\frac{1}{-1 + x^2}}$$

```
Integrate[newCaseIntrg[x, t], t]
```

```
primitive3rdCas[x_, t_] := -t Sqrt[-1 + x^2] + Sqrt[1 - t^2] Sqrt[-1 + x^2] + 1/2 t^2 (-1 + x^2) + ArcSin[t];
```

```

Simplify[Limit[primitive3rdCas[X, t], t → √(1/(-1 + x^2)), Direction → 1] -
Limit[primitive3rdCas[X, t], t → √((-2 + x^2)/(-1 + x^2)), Direction → -1], Assumptions → {√2 < x < √3}]

Expand[Simplify[(24/(4 π 6 * a^2) * a^2/r * 
((1/2 - x^2/2 + 2 √(-2 + x^2) - ArcSin[√((-2 + x^2)/(-1 + x^2))] + ArcSin[1/√(-1 + x^2)]) /.
{x → r/a})],
Assumptions → {a √2 < r < a √3 && a > 0}]] /. {ArcSin[1/√(-1 + r^2/a^2)] → ArcSin[a/√(-a^2 + r^2)]}]

Simplify[
(((24/(4 π 6 * a^2) * a^2/r * Simplify[Limit[primitive3rdCas[X, t], t → √(1/(-1 + x^2)), Direction → 1] -
Limit[primitive3rdCas[X, t], t → √((-2 + x^2)/(-1 + x^2)), Direction → -1],
Assumptions → {√2 < x < √3}]) /.
{x → r/a}) - 
RmngContCC[r, a], Assumptions → {a √2 < r < √3 * a}]

0

RmngContCC[r_, a_] := -1/(2 π r) - r/(2 a^2 π) + (2 √(-2 a^2 + r^2)/(a π r) - ArcSin[√(2 a^2 - r^2)/(a^2 - r^2)]/π r + ArcSin[a/√(-a^2 + r^2)]/π r);

(* a < r < a*√2 *) RmngContBB[r_, a_] := (-1 + π)/(2 π r) + r/(2 a^2 π) - (2 √(-a^2 + r^2)/(a π r));
(* a*√2 < r < a*√3 *)
RmngContCC[r_, a_] := -1/(2 π r) - r/(2 a^2 π) + (2 √(-2 a^2 + r^2)/(a π r) - ArcSin[√(2 a^2 - r^2)/(a^2 - r^2)]/π r + ArcSin[a/√(-a^2 + r^2)]/π r);

Mom00RmngBB[a_] := Integrate[r^2 * RmngContBB[r, a], {r, a, a * √2}, Assumptions → {a > 0}];
Mom00RmngCC[a_] :=
Integrate[r^2 * RmngContCC[r, a], {r, a * √2, a * √3}, Assumptions → {a > 0}];

```

```

Mom00RmngBB[a]
Mom00RmngCC[a]
Simplify[Mom00RmngBB[a] + Mom00RmngCC[a], Assumptions -> {a > 0}]

```

Sum of the moments of the three contributions

$$\begin{aligned} & \text{Simplify}\left[\text{Simplify}\left[\text{MomlinAA} + \text{MomlinBB} + \frac{a^2 (56 + (-63 + 32 \sqrt{2}) \pi)}{24 \pi}\right] + \right. \\ & \quad \text{Simplify}[\text{Momt00SupAA}[a] + \text{Momt00SupBB}[a]] + \\ & \quad \left.\text{Simplify}[\text{Mom00RmngBB}[a] + \text{Mom00RmngCC}[a], \text{Assumptions} \rightarrow \{a > 0\}]\right] \end{aligned}$$

that, as required, is equal to $\frac{6 a^2}{4 \pi}$.

FINAL EXPRESSION

Contribution of the polygon intersection

$$\begin{aligned} \text{CubSurfCFLnAA}[r_, a_] &:= \frac{1}{a} - \frac{r}{2 a^2}; \quad (* \text{ If } 0 < r < a *) \\ \text{CubSurfCFLnBB}[r_, a_] &:= -\frac{2}{a} + \frac{5}{2 r} - \frac{2 \sqrt{-a^2 + r^2}}{a \pi r} + \frac{2 r \text{ArcSin}\left[\frac{\sqrt{-a^2+r^2}}{r}\right]}{a^2 \pi}; \\ (* \text{ If } a < r < a\sqrt{2} *) \text{CubSurfCFLnCC}[r_, a_] &:= -\frac{2}{a} + \frac{-12 + 5 \pi}{6 \pi r} + \frac{r}{6 a^2} + \frac{2 \sqrt{-2 a^2 + r^2}}{a \pi r} + \\ & \quad \frac{4}{a \pi} \text{ArcSin}\left[\sqrt{\frac{-2 a^2 + r^2}{-2 a^2 + 2 r^2}} * \frac{(a^2 + r^2)}{-a^2 + r^2}\right] + \frac{2 (5 a^2 + r^2)}{3 a^2 \pi r} * \text{ArcSin}\left[\frac{7 a^3 - 3 a r^2}{(-a^2 + r^2)^{3/2}}\right]; \\ (* \text{ If } a\sqrt{2} < r < a\sqrt{3} *) \end{aligned}$$

Contribution of the sliding surfaces

$$\begin{aligned} \text{surcContrCFAAs}[r_, a_] &:= -\frac{2}{a \pi} + \frac{1}{2 r} + \frac{r}{2 a^2 \pi}; \quad (* \text{ If } 0 < r < a *) \\ \text{surcContrCFBBS}[r_, a_] &:= -\frac{2 + \pi}{2 \pi r} - \frac{r}{2 a^2 \pi} + \frac{2 \sqrt{-a^2 + r^2}}{a \pi r} + \frac{2 \text{ArcSin}\left[\frac{a}{r}\right]}{\pi r}; \\ (* \text{ If } a < r < a*\sqrt{2} *) \end{aligned}$$

Contribution of the opposing surfaces

```

(* a < r < a*sqrt[2] *) RmngContBB[r_, a_] := 
$$\frac{-1 + \pi}{2\pi r} + \frac{r}{2a^2\pi} - \frac{2\sqrt{-a^2 + r^2}}{a\pi r};$$


(* a*sqrt[2] < r < a*sqrt[3] *)

RmngContCC[r_, a_] := 
$$-\frac{1}{2\pi r} - \frac{r}{2a^2\pi} + \frac{2\sqrt{-2a^2 + r^2}}{a\pi r} - \frac{\text{ArcSin}\left[\sqrt{\frac{2a^2 - r^2}{a^2 - r^2}}\right]}{\pi r} + \frac{\text{ArcSin}\left[\frac{a}{\sqrt{-a^2 + r^2}}\right]}{\pi r};$$


pltaa = With[{a = 1}, ParametricPlot[{r, CubSurfCFLnAA[r, a]}, {r, 0, a}, PlotRange -> {{0, a Sqrt[3]}, {0, 1.1}}, AspectRatio -> 1]];
pltbb = With[{a = 1}, ParametricPlot[{r, CubSurfCFLnBB[r, a]}, {r, a, a Sqrt[2]}]];
pltcc = With[{a = 1}, ParametricPlot[{r, CubSurfCFLnCC[r, a]}, {r, a Sqrt[2], a Sqrt[3]}]];
Show[pltaa, pltbb, pltcc]

pltaaslf = With[{a = 1}, ParametricPlot[{r, surcContrCFAAs[r, a]}, {r, 0, a}, PlotRange -> {{0, a Sqrt[3]}, {0, 1.1}}, AspectRatio -> 1]];
pltbbslf = With[{a = 1}, ParametricPlot[{r, surcContrCFBBS[r, a]}, {r, a, a Sqrt[2]}]];
Show[pltaaslf, pltbbslf]

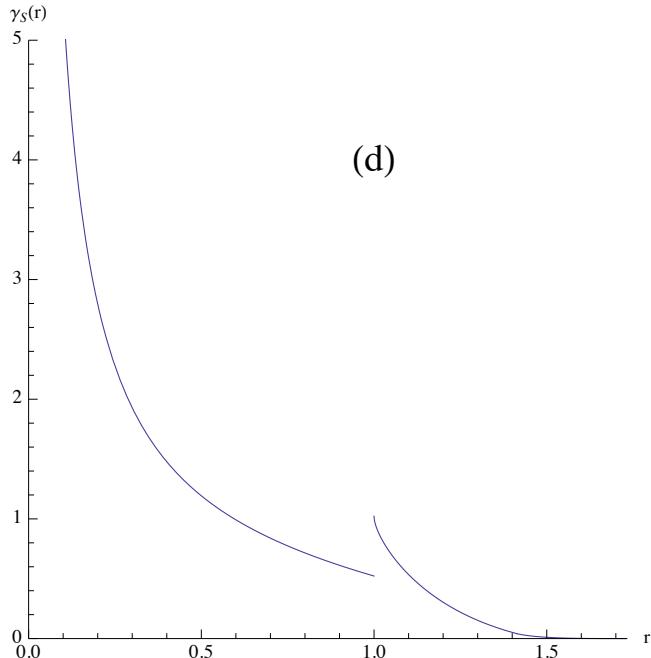
pltbkopf = With[{a = 1}, ParametricPlot[{r, RmngContBB[r, a]}, {r, a, a Sqrt[2]}, PlotRange -> {{0, a Sqrt[3]}, {0, 1.1}}, AspectRatio -> 1]];
pltccopf = With[{a = 1}, ParametricPlot[{r, RmngContCC[r, a]}, {r, a Sqrt[2], a Sqrt[3]}]];
Show[pltbkopf, pltccopf]

```

```

pltCSaa = With[{a = 1}, ParametricPlot[{r, CubSurfCFLnAA[r, a] + surcContrCFAAs[r, a]}, {r, 0, a}, PlotRange -> {{0, a Sqrt[3]}, {0, 5}}, AspectRatio -> 1, AxesLabel -> {"r", "γs(r)"}]];
pltCSbb = With[{a = 1}, ParametricPlot[
{r, CubSurfCFLnBB[r, a] + surcContrCFBBS[r, a] + RmngContBB[r, a]}, {r, a, a Sqrt[2]}]];
labels8D = Graphics[{Text[Style["(d)", FontSize -> 18], {1, 4}]}];
pltCScc = With[{a = 1},
ParametricPlot[{r, CubSurfCFLnCC[r, a] + RmngContCC[r, a]}, {r, a Sqrt[2], a Sqrt[3]}]];
Fig8D = Show[pltCSaa, pltCSbb, pltCScc, labels8D]

```



the discontinuity is OK since it arise from the contribution of the opposing faces

```

Export["Fig8D.eps", Fig8D]
Fig8D.eps

```

RELATIONS ON THE MOMENTS OF THE SURFACE CF

the zeroth moment of the surface CF obeys to

$$M_0 = \int r^2 * \gamma_s(r) dr = s / 4\pi \quad (1)$$

and the first moment is equal to

$$M_2 = \int r^4 * \gamma_s(r) dr = \frac{1}{2\pi} \int_S r^2 ds - \frac{1}{2\pi s} \int_S ds_1 \int_S ds_2 (r_1 \cdot r_2) \quad (2)$$

First moment of the surface CF of a spherical surface

```
sphereCF[r_, r_] :=  $\frac{1}{2 * r}$ ;
Simplify[Integrate[r^2 * sphereCF[r, r], {r, 0, 2 * R}], Assumptions -> {R > 0}] -
4 * π * R^2 / (4 * π)
Eq. (1) is obeyed
```

Second moment of the surface CF of a spherical surface

```
Simplify[Integrate[r^4 * sphereCF[r, r], {r, 0, 2 * R}], Assumptions -> {R > 0}] -
Integrate[Sin[θ] * Integrate[r^4, {φ, 0, 2 * π}], {θ, 0, π}] / (2 * π)
Eq. (2) is obeyed
```

First moment of the surface CF of a circle

```
CircleCF[R_, r_] := -  $\frac{\sqrt{4 * R^2 - r^2}}{4 * \pi * R^2} + \frac{1}{\pi * r} * \text{ArcCos}\left[\frac{r}{2 * R}\right]$ ;
```

```
Simplify[Integrate[r^2 * CircleCF[R, r], {r, 0, 2 * R}], Assumptions -> {R > 0}] -
(π * R^2) / (4 π)
```

Eq. (1) is obeyed

Second moment of the surface CF of a circle

```
Simplify[Integrate[r^4 * CircleCF[R, r], {r, 0, 2 * R}], Assumptions -> {R > 0}] -
Integrate[Integrate[x^3, {x, 0, R}], {φ, 0, 2 * π}] / (2 * π)
```

Eq. (2) is obeyed since $\frac{1}{2 \pi S} \int_S dS_1 \int_S dS_2 (r_1 . r_2)$ is equal to zero

```
Integrate[Integrate[Integrate[
  Integrate[x1 * x2 * (x1 * x2 * Cos[φ1] * Cos[φ2] + x1 * x2 * Sin[φ1] * Sin[φ2]), {φ1, 0, 2 * π}],
  {φ2, 0, 2 * π}], {x2, 0, R}], {x1, 0, R}] / (2 * π * π * R^2)
```

The case of a cubic surface

```
(* 0 < r < a *) CubSurfCFLnAAOld[r_, a_] :=  $\frac{1}{a} - \frac{r}{2 a^2}$ ;
(* a < r < a $\sqrt{2}$  *) CubSurfCFLnBBOld[r_, a_] :=  $\frac{2}{a} + \frac{5}{2 r} - \frac{2 \sqrt{-a^2 + r^2}}{a \pi r} + \frac{2 r \text{ArcCos}\left[\frac{a}{r}\right]}{a^2 \pi}$ ;
(* a $\sqrt{2}$  < r < a $\sqrt{3}$  *) CubSurfCFLnCCOld[r_, a_] :=

$$\begin{aligned}
& \frac{2}{a} + \frac{5}{6 r} - \frac{2}{\pi r} + \frac{r}{6 a^2} + \frac{2 \sqrt{-2 a^2 + r^2}}{a \pi r} - \frac{10 \text{ArcTan}\left[\frac{a (-7 a^2+3 r^2)}{(5 a^2-r^2) \sqrt{-2 a^2+r^2}}\right]}{3 \pi r} - \\
& \frac{2 r \text{ArcTan}\left[\frac{a (-7 a^2+3 r^2)}{(5 a^2-r^2) \sqrt{-2 a^2+r^2}}\right]}{3 a^2 \pi} - \frac{4 \text{ArcTan}\left[\frac{\sqrt{-2 a^2+r^2} (a^2+r^2)}{-3 a^2 r+r^3}\right]}{a \pi}
\end{aligned}$$

```

$$(* 0 < r < a *) \quad \text{CubSurfCFSupAAold}[r_, a_] := \frac{1}{2 r} - \frac{2}{a \pi} + \frac{r}{2 a^2 \pi};$$

(* a < r < a\sqrt{2} *)

$$\text{CubSurfCFSupBBold}[r_, a_] := -\frac{1}{2 r} - \frac{1}{\pi r} - \frac{r}{2 a^2 \pi} + \frac{2 \sqrt{-a^2 + r^2}}{a \pi r} + \frac{2 \text{ArcSin}\left[\frac{a}{r}\right]}{\pi r},$$

$$\begin{aligned} & \text{Simplify}\left[\text{Integrate}\left[r^2 * \text{CubSurfCFLnAA}[r, a], \{r, 0, a\}, \text{Assumptions} \rightarrow \{a > 0\}\right] + \right. \\ & \quad \text{Integrate}\left[r^2 * \text{CubSurfCFLnBB}[r, a], \{r, a, a\sqrt{2}\}, \text{Assumptions} \rightarrow \{a > 0\}\right] + \\ & \quad \text{Integrate}\left[r^2 * \text{CubSurfCFLnCC}[r, a], \{r, a\sqrt{2}, a\sqrt{3}\}, \text{Assumptions} \rightarrow \{a > 0\}\right] + \\ & \quad \text{Integrate}\left[r^2 * \text{CubSurfCFSupAA}[r, a], \{r, 0, a\}, \text{Assumptions} \rightarrow \{a > 0\}\right] + \\ & \quad \left.\text{Integrate}\left[r^2 * \text{CubSurfCFSupBB}[r, a], \{r, a, a\sqrt{2}\}, \text{Assumptions} \rightarrow \{a > 0\}\right]\right] \end{aligned}$$

$$\frac{5 a^2}{4 \pi}$$

the CF of the cubic surface is WRONG ! The above result

$$\text{ought to be } \frac{6 a^2}{4 \pi}$$

$$N\left[\frac{5 a^2}{4 \pi}\right]$$

$$0.397887 a^2$$

The chord length distribution of a spherical shell, according to Gille, is

$$\text{cld}[R_, r_] := (2 * r / (\pi * R^2)) * \text{ArcTan}\left[\sqrt{\frac{4 * R^2 - r^2}{r^2}}\right];$$

cld[R, r]

Simplify[Series[cld[R, r], {r, 0, 6}], Assumptions → {R > 0 && r > 0 && r < 2 * R}]

$$\frac{r}{R^2} - \frac{r^2}{\pi R^3} - \frac{r^4}{24 (\pi R^5)} - \frac{3 r^6}{640 (\pi R^7)} + O[r]^7$$

However, the surface CF does not coincide with the previous function because this contains the two angular factors $\sigma_1 \omega$ and $\sigma_2 \omega$.

These are equal to $(r/2R)$. Hence the surface CF is equal to

$$(cld[R, r] / (r / (2 R))^2) / 8$$

$$\frac{\text{ArcTan}\left[\sqrt{\frac{-r^2+4 R^2}{r^2}}\right]}{\pi r}$$

The expansion of this function is

```
Simplify[Series[(cld[R, r] / (r / (2 R))^2), {r, 0, 6}],
Assumptions -> {R > 0 && r > 0 && r < 2 * R}]
```

$$\frac{4}{r} - \frac{4}{\pi R} - \frac{r^2}{6(\pi R^3)} - \frac{3r^4}{160(\pi R^5)} + O[r]^6$$

FIGURES 8A, 8B AND 8C

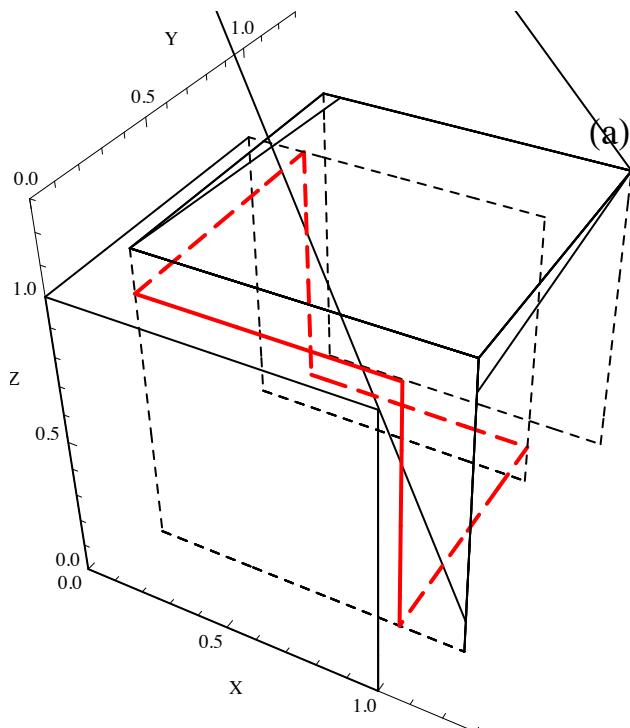
```
labels8A = Graphics3D[{Text[Style["(a)", FontSize -> 18], {1.1, 1.2, 1.2}]}];

AA = {0, 0, 0}; BB = {1, 0, 0}; CC = {1, 0, 1}; DD = {0, 0, 1};
Sh = {0, 1, 0};
AAP = AA + Sh; BBP = BB + Sh; CCP = CC + Sh; DDP = DD + Sh;
(* R={1,1/8,1/7};*) R = {1/5, 1/8, 1/7};
AAN = AA + R; BBN = BB + R; CCN = CC + R; DDN = DD + R;
AAPN = AAP + R; BBPN = BBP + R; CCPN = CCP + R; DDPN = DDP + R;
Rbis = {1/3, 0, 1/4};
AANbis = AA + Rbis; BBNbis = BB + Rbis; CCNbis = CC + Rbis; DDNbis = DD + Rbis;
AAPNbis = AAP + Rbis; BBPNbis = BBP + Rbis; CCPNbis = CCP + Rbis; DDPNbis = DDP + Rbis;
Rter = {1/2, 0, 0};
AANter = AA + Rter; BBNter = BB + Rter; CCNter = CC + Rter; DDNter = DD + Rter;
AAPNter = AAP + Rter; BBPNter = BBP + Rter; CCPNter = CCP + Rter;
DDPNter = DDP + Rter;
```

```

santaA = Line[{{AA, BB, CC, DD, AA}, {DD, DDP}}];
santbA = Line[{{AAP, BBP, CCP, DDP, AAP}, {AA, AAP}, {BB, BBP}, {CC, CCP}}];
gantaA = Graphics3D[{Thickness[0.003], Line[{{AA, BB, CC, DD, AA}, {DD, DDP}, {BB, CC}}]}, 
  Boxed -> False]; gantbA = Graphics3D[{Thickness[0.003], Dashed, 
  Line[{{AAP, BBP, CCP, DDP, AAP}, {AA, AAP}, {BB, BBP}, {CC, CCP}}]}, Boxed -> False];
cubeA = Show[gantaA, gantbA]; santNbA =
  Line[{{AAN, BBN, CCN, DDN, AAN}, {AAN, AAPN}, {BBN, BBPN}, {CCN, CCPN}, {DDN, DDPN}}];
santNbB = Line[{{AAPN, BBPN, CCPN, DDPN, AAPN}}];
gantNbA = Graphics3D[{Thickness[0.003], Dashed, santNbA}, Boxed -> False];
gantNbB = Graphics3D[{Thickness[0.003], Dashed, santNbB}, Boxed -> False];
cubeB = Show[gantNbA, gantNbB];
sup = Line[{{AAN[[1]], AAN[[2]], DDP[[3]]}, {BB[[1]], AAN[[2]], DDP[[3]]}}];
rgt = Line[{{BB[[1]], AAN[[2]], DDP[[3]]}, {BB[[1]], AAN[[2]], BBPN[[3]]}}];
suplft = Line[{{AAN[[1]], AAN[[2]], DD[[3]]}, {AAN[[1]], DDP[[2]], DD[[3]]}}];
bcklft = Line[{{AAN[[1]], DDP[[2]], DD[[3]]}, {AAN[[1]], DDP[[2]], AAPN[[3]]}}];
bckdwn = Line[{{AAN[[1]], DDP[[2]], AAPN[[3]]}, {BBP[[1]], BBP[[2]], AAPN[[3]]}}];
brgtdwn = Line[{{BBP[[1]], BBP[[2]], AAPN[[3]]}, {BBP[[1]], BBN[[2]], AAPN[[3]]}}];
intersct = Graphics3D[{{Thickness[0.006], Red, sup, rgt},
  {Dashing[{0.03, 0.02}], Thickness[0.006], Red, {suplft, bcklft, bckdwn, brgtdwn}}}],
Axes -> True, AxesLabel -> {"X", "Y", "Z"}, 
PlotRange -> {{0, 1.3}, {0, 1.3}, {0, 1.3}}, Boxed -> False]; CCNsh = {1, 1/8, 1/7};
DDNsh = {1/5, 1/8, 1}; CCSHo = {1, 1/8, 1}; DDPSh = {1/15, 1, 1}; BBSH = {1, 1/4.5, 0};
agg = Graphics3D[{Thickness[0.003],
  Line[{{DDN, CCN, CCPN, DDPN, DDN}, {CCN, BBN, BBPN, CCPN, CCN, BBN, CCNsh}, {DDN, DDNsh,
    DDN}, {CC, CCSHo, CC}, {DDP, DDPSh, DDP}, {BB, BBSH, BB}}}], Boxed -> False];
Fig8A = Show[intersct, cubeA, cubeB, agg, labels8A]

```

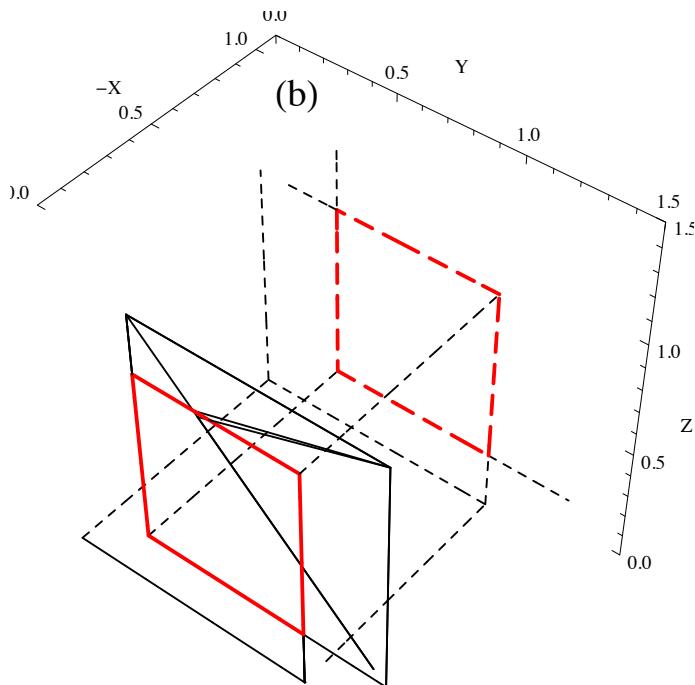


```
Export["Fig8A.eps", Fig8A];
```

```

labels8B = Graphics3D[{Text[Style["(b)", FontSize -> 18], {0, 1.2, 1.2}]}];
RbisP = {1/3, 1, 1/4}; BBupP = {1, 1, 1/4};
DDrgtP = {1/3, 1, 1}; BBNbisLft = {1, 1/9, 0};
Rbis = {1/3, 0, 1/4}; BBup = {1, 0, 1/4}; DDrgt = {1/3, 0, 1};
DDbck = {-1, 0, 1}; DDbckpp = {1/8, 1, 1};
contA =
  Show[Graphics3D[{Thickness[0.003],
    Line[{{BBup, BB}, {BB, AA, DD, DDrgt}, {DD, DDP, DDbckpp}, {BB, BBNbisLft}}]}],
    PlotRange -> {{0, 1.5}, {0, 1.1}, {0, 1.5}}, Axes -> True,
    AxesLabel -> {"Y", "-X", "Z"}, Boxed -> False]];
contB =
  Show[
    Graphics3D[{Thickness[0.003], Line[{{DDrgt, DDNbis, CCNbis, BBNbis, BBup}, {BBNbis,
      BBPNbis, CCPNbis, CCNbis}, {CCPNbis, DDPNbis, DDNbis}}]}], Boxed -> False]];
thickA = Show[Graphics3D[{Thickness[0.006], Red, Line[{{DDrgt, Rbis, BBup, CC, DDrgt}}]}],
  Boxed -> False];
dashB = Show[Graphics3D[{Thickness[0.006], Red, Dashing[{0.03, 0.02}],
  Line[{{DDrgtP, RbisP, BBupP, CCP, DDrgtP}}]}], Boxed -> False];
dottB = Show[Graphics3D[{Thickness[0.003], Dashed,
  Line[{{AANbis, AAPNbis}, {DDPNbis, DDrgtP}, {BBupP, BBPNbis}}]}], Boxed -> False];
dottA = Show[Graphics3D[{Thickness[0.003], Dashed,
  Line[{{AA, AAP}, {CC, CCP}, {BBNbisLft, BBP}, {AAP, DDP}, {AAP, BBP},
    {BBP, BBupP} (*,{DDP, DDrgtP}*)}, {DDbckpp, DDrgtP} }]}], Boxed -> False];
Fig8B = Show[contA, thickA, dashB, contB, dottA, dottB, labels8B,
  ViewPoint -> {2, -2.5, 3.2} ]

```



```
Export["Fig8B.eps", Fig8B];
```

```

AA = {0, 0, 0}; BB = {1, 0, 0}; CC = {1, 0, 1}; DD = {0, 0, 1};
Sh = {0, 1, 0};
AAP = AA + Sh; BBP = BB + Sh; CCP = CC + Sh; DDP = DD + Sh;
R = {1, 1 / 8, 1 / 7}; (* R={1/5,1/8,1/7}; *)
AAN = AA + R; BBN = BB + R; CCN = CC + R; DDN = DD + R;
AAPN = AAP + R; BBPN = BBP + R; CCPN = CCP + R; DDPN = DDP + R;
Rbis = {1 / 3, 0, 1 / 4};
AANbis = AA + Rbis; BBNbis = BB + Rbis; CCNbis = CC + Rbis; DDNbis = DD + Rbis;
AAPNbis = AAP + Rbis; BBPNbis = BBP + Rbis; CCPNbis = CCP + Rbis; DDPNbis = DDP + Rbis;
Rter = {1 / 2, 0, 0};
AANTer = AA + Rter; BBNter = BB + Rter; CCNter = CC + Rter; DDNter = DD + Rter;
AAPNter = AAP + Rter; BBPNter = BBP + Rter; CCPNter = CCP + Rter;
DDPNter = DDP + Rter;

CCPNew = {9 / 10, 1, 1};

DDN
DDNsh

{1, 1/8, 8/7}

{1, 1/8, 1}

```

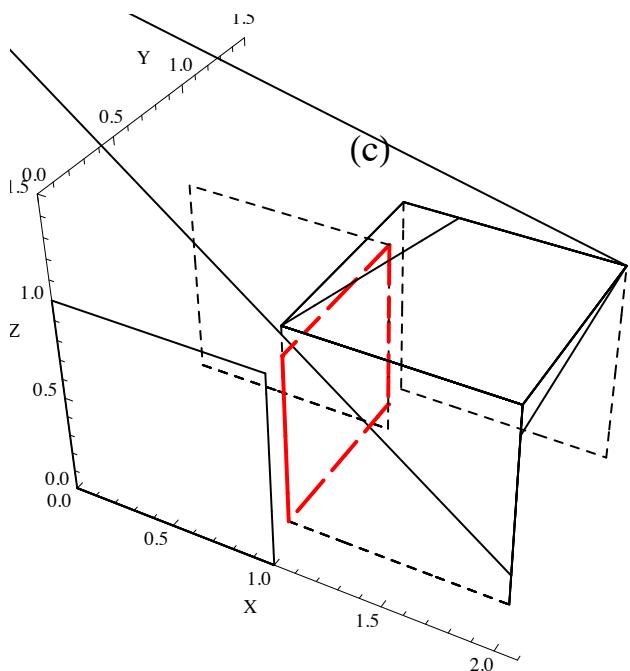
```

labels8C = Graphics3D[{Text[Style["(c)", FontSize -> 18], {0.8, 1.2, 1.3}]}];
AA = {0, 0, 0}; BB = {1, 0, 0}; CC = {1, 0, 1}; DD = {0, 0, 1};
Sh = {0, 1, 0};
AAP = AA + Sh; BBP = BB + Sh; CCP = CC + Sh; DDP = DD + Sh;
R = {1, 1/8, 1/7}; (* R={1/5,1/8,1/7}; *)
AAN = AA + R; BBN = BB + R; CCN = CC + R; DDN = DD + R;
AAPN = AAP + R; BBPN = BBP + R; CCPN = CCP + R; DDPN = DDP + R;
Rbis = {1/3, 0, 1/4};
AANbis = AA + Rbis; BBNbis = BB + Rbis; CCNbis = CC + Rbis; DDNbis = DD + Rbis;
AAPNbis = AAP + Rbis; BBPNbis = BBP + Rbis; CCPNbis = CCP + Rbis; DDPNbis = DDP + Rbis;
Rter = {1/2, 0, 0};
AANter = AA + Rter; BBNter = BB + Rter; CCNter = CC + Rter; DDNter = DD + Rter;
AAPNter = AAP + Rter; BBPNter = BBP + Rter; CCPNter = CCP + Rter;
DDPNter = DDP + Rter; CCPNew = {9/10, 1, 1};

santaA = Line[Thickness[0.002], {{AA, BB, CC, DD, AA}, {DD, DDP}}];
santbA = Line[{{AAP, BBP, CCP, DDP, AAP}, {AA, AAP}, {BB, BBP}, {CC, CCP}}];
gantaA = Graphics3D[
  {Thickness[0.003], Line[{{AA, BB, CC, DD, AA}, {DD, DDP}, {BB, CC}, {DDP, CCPNew}}]}, 
  Boxed -> False]; gantbA = Graphics3D[{Thickness[0.003], Dashed,
  Line[{{AAP, BBP, CCP, DDP, AAP}, {AA, AAP}, {BB, BBP}, {CC, CCP}}]}, Boxed -> False];
cubeA = Show[gantaA, gantbA]; santNbA =
  Line[{{AAN, BBN, CCN, DDN, AAN}, {AAN, AAPN}, {BBN, BBPN}, {CCN, CCPN}, {DDN, DDPN}}];
santNbB = Line[{{AAPN, BBPN, CCPN, DDPN, AAPN}}];
gantNaB = Graphics3D[{Thickness[0.003], Dashed, santNbB}, Boxed -> False];
gantNbB = Graphics3D[{Thickness[0.003], Dashed, santNbB}, Boxed -> False];
cubeB = Show[gantNaB, gantNbB];
sup = Line[{{AAN[[1]], AAN[[2]], DDP[[3]]}, {BB[[1]], AAN[[2]], DDP[[3]]}}];
rgt = Line[{{BB[[1]], AAN[[2]], DDP[[3]]}, {BB[[1]], AAN[[2]], BBPN[[3]]}}];
suplft = Line[{{AAN[[1]], AAN[[2]], DD[[3]]}, {AAN[[1]], DDP[[2]], DD[[3]]}}];
bcklft = Line[{{AAN[[1]], DDP[[2]], DD[[3]]}, {AAN[[1]], DDP[[2]], AAPN[[3]]}}];
bckdwn = Line[{{AAN[[1]], DDP[[2]], AAPN[[3]]}, {BBP[[1]], BBP[[2]], AAPN[[3]]}}];
brgtdwn = Line[{{BBP[[1]], BBP[[2]], AAPN[[3]]}, {BBP[[1]], BBN[[2]], AAPN[[3]]}}];
intersct = Graphics3D[{{Thickness[0.006], Red, sup, rgt},
  {Dashing[{0.03, 0.02}], Thickness[0.006], Red, {suplft, bcklft, bckdwn, brgtdwn}}},
  Axes -> True, AxesLabel -> {"X", "Y", "Z"}, 
  PlotRange -> {{0, 2.1}, {0, 1.5}, {0, 1.5}}, Boxed -> False]; CCNsh = {1, 1/8, 1/7};

DDNsh = {1, 1/8, 1};
DDPsh = {1/15, 1, 1};
BBSH = {1, 1/4.5, 0};
agg = Graphics3D[{Thickness[0.003],
  Line[{{DDN, CCN, CCPN, DDPN, DDN}, {CCN, BBN, BBPN, CCPN, CCN, BBN, CCNsh}, {CC, CCSHO, CC}
    (DDN, DDNsh, DDN), {DDP, DDPsh, DDP}, {BB, BBSH, BB}}}], Boxed -> False];
Fig8C = Show[intersct, cubeA, cubeB, agg, labels8C]

```



```
Export["Fig8C.eps", Fig8C];
```