The Geodesic Spring on the Euclidean Sphere with Parallel-Transport-Based Damping

Jason M. Osborne and Gregory P. Hicks

Figure 1. Identification of Curves on the Sphere with Orientations of a Robotic Arm. In this article, pointing a robotic arm is achieved using a generalized mass-spring-damper (MSD) control design on the sphere.

In this article differential geometric methods are applied to the design of a tracking algorithm which compels a particle that is constrained to the unit sphere $\mathbb{S}^2$ into a moving target that is also constrained to the sphere. By tracking we mean that both the position and velocity of the chasing particle are made to

Jason M. Osborne is a full-time lecturer at Boston University. His email address is josborne@bu.edu.

Gregory P. Hicks is a member of the senior professional staff at the Johns Hopkins University Applied Physics Laboratory. His email address is gregory.hicks@jhuapl.edu.

DOI: <http://dx.doi.org/10.1090/noti997>

match that of the target particle. A mechanical system analog of this mathematical scenario is the pointing of the end effector of a robotic arm whose range of motion is restricted to the surface of a sphere (see Figure 1). At the heart of this article is the 1D mass-spring-damper (MSD) system on the real line, which is reviewed and summarized in Sidebar 1 (see page 11), so as to be easily extensible to a unit sphere $\mathbb{S}^2$ -generalization. This spherical generalization, depicted by $(\checkmark \text{---} \text{---} \checkmark)$ in Figure 1, is the so-called *geodesic spring* tracking algorithm. Essentially, the spring $(\checkmark \text{---} \text{---} \checkmark)$ pulls the robotic arm close to its goal configuration, and the damper $(\text{---} \text{---})$ dissipates the kinetic energy of the arm so that it eventually reaches and stops at the goal configuration.

There are some new mathematical avenues explored in this article; however, with the belief that a good example can often be as useful as a good theorem, this article's main purpose is to instantiate the more research-oriented articles on which it is founded. Specifically, while the geodesic spring (with damping) design is theoretically formulated on a general Riemannian manifold in [1], in this article we actually implement this general result on a specific manifold, namely $\mathbb{S}^2$. In so doing, one is confronted with the *geometric*, *coordinate-free*, *nonlinear*, and *almost global* nature of the control design that, at least to these authors, provides an additional layer of understanding of the general theorem.

The authors' broader hope for this article is that, by providing an example which is completely expressed in closed form, some insight may be gained by the reader into the control and

stabilization of systems on manifolds. While this article is narrowly focused on the manifold $\mathbb{S}^2$, the potential usefulness of its results are limited only by the imagination of the reader to formulate a problem in terms of a specific Riemannian manifold of interest. For example, as exceptionally relayed and detailed by [2], in his investigations of surface curvature, Gauss invented a mapping of normal vectors on a surface $\mathbb{S}^2$. This map, now called the *Gauss map*, and some of the techniques of this article on $\mathbb{S}^2$ are used by [3] to devise a geometric-based guidance law for an interceptor problem. That is, unlike this article, where the goal is to match both the position and velocity of a reference trajectory $r(t) \in \mathbb{S}^2$, the goal in an interceptor problem is to match only position $r(t_f) \in \mathbb{R}^3$ for some intercept time t_f ; the velocity at which the interception is made is immaterial. An additional avenue for exploration of the techniques of this article is to a Riemannian manifold called the *Poincaré upper-half plane*, denoted $\mathbb{H}^2$, where, as in $\mathbb{S}^2$, the geodesic and parallel transport equations both have closed-form realizations. Perhaps then the control design of this article on $\mathbb{H}^2$ might find some application to electrical impedance tomography [4] and to microwave technology [5], where the geometry of $\mathbb{H}^2$ is shown to play a role.

Figure 2. Geodesic Distance on $\mathbb{S}^2$. The geodesic distance is given by $\theta = \arccos(\langle r, q \rangle)$ where θ is the angle subtended by the segment of the great circle (geodesic, black solid line) between q and r . Projecting the differential of $\frac{1}{2}\theta^2$ onto the tangent plane at q yields a vector F_p , which can be interpreted as a geodesic-spring force pushing q towards r .

the evolution of a *configuration error function* ϕ and the *transport map* $\mathcal{T}$ pairing. For the purposes

A broad outline of this article is as follows. In Sidebar 1 (page 11) we review the 1D MSD. Those readers already familiar with the material from this section and who want to learn about its generalization to the geodesic spring system on a Riemannian manifold could proceed to “History of the Geodesic Spring”, where a general theorem is outlined. In “MSD System on $\mathbb{S}^2$ ” we instantiate this general theorem on the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, and in “Implementation of MSD System on $\mathbb{S}^2$ ” we present several examples and simulations.

History of the Geodesic Spring

The evolution of the 1D MSD system to its general form on a Riemannian manifold can be traced to

of this article, we think of a Riemannian manifold Q simply as a surface $S \subset \mathbb{R}^3$ that is equipped with an inner product $\langle \cdot, \cdot \rangle$, which effectively induces a definition of distance between points $q, r \in Q$. A more formal definition of Riemannian manifold, though not necessary for this article, can be found in most books on differential geometry (for example, [6], [7]). In general, the value $\phi(q, r)$ is a measure of the difference (not necessarily distance) between q and r and satisfies $\phi(q, r) \rightarrow 0$ as $q \rightarrow r$. The transport map $\mathcal{T}$ is discussed shortly. The geometric setting of [8] is the Euclidean sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. One of the control problems solved in [8, Problem 2.2, p. 6] is to steer asymptotically a point particle q constrained to $\mathbb{S}^2$ to a *fixed* reference $r \in \mathbb{S}^2$. The authors use the function

$$(1) \quad \frac{1}{2} \left[\text{dist}_{\mathbb{S}^2}(q, r) \right]^2 = \frac{1}{2} \left[\arccos(\langle r, q \rangle) \right]^2 = \frac{1}{2} \theta^2$$

to obtain a *geodesic spring* force F_p on $\mathbb{S}^2$ (illustrated in Figure 2) that stabilizes a curve $q(t)$ to fixed r . Since $q, r \in \mathbb{S}^2$ and $\arccos(\langle r, q \rangle) \rightarrow 0$ as $q \rightarrow r$, then (1) is an example of a configuration error function $\phi(q, r)$ with sample plots shown in Figure 3. A comparison of this plot with Figure B1 along with the discussion in Sidebar 1, shows that (1) plays the role of a quadratic potential function on $\mathbb{S}^2$ centered at $r \in \mathbb{S}^2$. Extending the quadratic potential $V(q)$ on $\mathbb{R}$ to the quadratic potential $\phi(q, r)$ on $\mathbb{S}^2$ is the first step in generalizing the 1D MSD to $\mathbb{S}^2$.

In an effort to generalize the damping force, [9, Chapter 11], [10], and [11, Chapter 4] work within the context of Riemannian manifolds, of which the surface $\mathbb{S}^2 \subset \mathbb{R}^3$ is but one example. Within this broader geometric setting, Bullo et al. introduce the general definitions of a pair of mappings called the *configuration error* and the *transport map*, which are denoted by ϕ and $\mathcal{T}$, respectively. Recall from Sidebar 1 that, when the tracker $q(t)$ is confined to the real line $\mathbb{R}$ and when the target $r \in \mathbb{R}$ is fixed, the difference $e(t) \triangleq q(t) - r$ (the position error) makes sense as a difference in $\mathbb{R}$ and further that the time derivative of the error (the velocity error) is simply $\dot{e} \triangleq \frac{d}{dt} e(t)$. In the transition to a Riemannian manifold Q (think of Q as a surface $S \subset \mathbb{R}^3$) with $q(t) \in S$ and a moving reference $r(t) \in S$, some care needs to be taken when differencing the velocity vectors $\dot{q}$ and $\dot{r}$, which lie in two separate tangent planes: one at $q(t)$ and the other at $r(t)$, respectively. This differencing is achieved by Bullo et al. through the introduction of a transport map $\mathcal{T}$, which transports $\dot{r}$ into the tangent plane at q , denoted $\mathcal{T}\dot{r}$, where the differencing with $\dot{q}$ makes sense using the vector space properties of the tangent plane $T_q S$. That is, as a result of the transport

Figure 3. Illustrations of the Potential Function $\phi(q, r) = \frac{1}{2} [\text{dist}_{S^2}(q, r)]^2 = \frac{1}{2} [\arccos(\langle r, q \rangle)]^2$.

map $\mathcal{T}$, a new definition of the velocity error on a Riemannian manifold is given by

$$\dot{e} \triangleq \dot{q} - \mathcal{T}\dot{r} \in T_q S.$$

With these general mappings ϕ and $\mathcal{T}$, Bullo et al. generalize the 1D MSD error system from Sidebar 1 as follows. Newton's equation for a free particle constrained to a Riemannian manifold is the geodesic equation

$$\frac{D\dot{q}}{dt} = 0,$$

where $\frac{D}{dt}$ denotes covariant differentiation along the curve $q(t)$. The general idea of covariant differentiation on a Riemannian manifold is addressed in most differential geometry books and can be illustrated in terms of, and indeed historically developed from, differentiation on a surface $S \subset \mathbb{R}^3$. Specifically, the covariant derivative at $q \in S$ is the standard derivative followed by a projection onto the tangent plane to S at q ,

$$(2) \quad \frac{D}{dt} = \text{proj}_q \circ \frac{d}{dt}.$$

Defining a proportional (spring) force F_P , a derivative (damping) force F_D , and an additional so-called feed-forward force $F_{\mathcal{F}\mathcal{F}}$ (which compensates for the nonzero accelerations of $r(t)$)

$$F_P \triangleq -k_P \text{proj}_q(d_q \phi) \triangleq -k_P e \in T_q S,$$

$$F_D \triangleq -k_D(\dot{q} - \mathcal{T}\dot{r}) \triangleq -k_D \dot{e} \in T_q S,$$

$$F_{\mathcal{F}\mathcal{F}} \triangleq \frac{D}{dt} [\mathcal{T}\dot{r}] \in T_q S$$

we find that Newton's equation with forcing ΣF is the closed-loop system.

$$(3) \quad \frac{D\dot{q}}{dt} = \Sigma F = F_P + F_D + F_{\mathcal{F}\mathcal{F}},$$

This simplifies to the error system

$$\frac{D\dot{e}}{dt} + k_D \dot{e} + k_P e = 0.$$

A word or two on notation is in order at this point. The vector $v_r = d_r \phi$ (the derivative of ϕ with respect to r) is in the tangent plane $T_r S$, but the vector $\text{proj}_r(v_r)$ is in the tangent plane (see Sidebar 2, page 12). From now on we denote the vectors $\text{proj}_r(d_r \phi)$ and $\text{proj}_q(d_q \phi)$ by $d_r \phi|_r$ and $d_q \phi|_q$, respectively. Assuming that ϕ and $\mathcal{T}$ satisfy a *compatibility condition*

$$(4) \quad \langle d_r \phi|_r, \dot{r} \rangle = -\langle d_q \phi|_q, \mathcal{T}\dot{r} \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the dot (inner) product on $\mathbb{R}^3$ applied to tangent vectors to S , the time derivative of the total energy $L = \frac{k_D}{2} \langle \dot{e}, \dot{e} \rangle + k_P \phi(q, r)$ along the solution to the forced dynamics reduces to

$$(5) \quad \begin{aligned} \frac{dL}{dt} &= k_P [\langle d_q \phi|_q, \dot{q} \rangle + \langle d_r \phi|_r, \dot{r} \rangle] + k_D \left\langle \frac{D\dot{e}}{dt}, \dot{e} \right\rangle \\ &= -k_D \langle \dot{e}, \dot{e} \rangle \leq 0 \quad [k_D > 0]. \end{aligned}$$

Equation (5) indicates that the energy function is decreasing and guarantees that $q = q(t)$ Lyapunov stabilizes to $r = r(t)$. Assuming technical conditions on the *uniformity* of ϕ [9, p. 536] and *boundedness* assumptions on, for example, k_P and k_D [9, pp. 540–541], it is proven in [9, Chapter 11] that $q = q(t)$ exponentially stabilizes to $r = r(t)$. This theorem (hereafter referred to as Bullo's Theorem) can be summarized as

Bullo's Theorem. *The solution e to the error system $\frac{D\dot{e}}{dt} + k_D \dot{e} + k_P e = 0$ satisfies $e \rightarrow 0$ as $t \rightarrow \infty$ so long as the mappings $(\phi, \mathcal{T})$ that define $e \triangleq d_q \phi|_q$ and $\dot{e} \triangleq \dot{q} - \mathcal{T}\dot{r}$ are compatible in the sense of $\langle d_r \phi|_r, \dot{r} \rangle = -\langle d_q \phi|_q, \mathcal{T}\dot{r} \rangle$.*

The objective now is to actually construct such a compatible pair of functions. In an attempt to implement Bullo's Theorem, [1] showed that the geodesic distance and parallel transport on a Riemannian manifold lead to such a compatible pair of mappings.

Figure 4. A Double Gimbal Mechanism (DGM).

A difficulty faced with the use of this pair is a reliance on numerical solutions to the geodesic equation (as a nonlinear, two-point boundary value problem) and to the parallel-transport equations (as a linear, possibly time-dependent, initial value problem). In [12], the geodesic distance and parallel transport pairing was utilized on a Riemannian manifold called the *double gimbal torus*,

which encapsulates the physics of the double gimbal mechanism (DGM) shown in Figure 4. For a special DGM (called the *flat DGM*), the parallel-transport and geodesic equations can be expressed in closed form, and the closed-loop equations (3) reduce to a system of uncoupled 1D MSD error systems. However, for a general DGM (that is, *nonflat*), the formulation of the closed-loop dynamics still requires numerical solutions.

The works cited here are by no means a complete list and represent only those most directly related to this article. For a broader range of references addressing control design on spheres and the use of geodesics in optimal control, see, for example, [13], [14], and the references and introductions of works cited in this section.

MSD System on $\mathbb{S}^2$

The goal of this section is to first give the details of the closed-form constructions of both the compatible configuration error and transport map pairing and the closed-loop dynamics (3). The configuration error function ϕ is recognizable as the quadratic potential well centered on $r(t) \in \mathbb{S}^2$ from (1) shown in Figure 3. As illustrated in Figure B1, this potential is sufficient to attract the curve $q(t) \in \mathbb{S}^2$ into the vicinity of $r(t)$. The main focus of “A Compatible Pair of Mappings on $\mathbb{S}^2$ ” below is the geometric construction of a transport map, shown to be parallel transport along the geodesics of $\mathbb{S}^2$. As illustrated in Figure B1, this transport map is used to define a source of friction on the potential surface that eventually slows the curve $q(t)$ to $r(t)$ located at the bottom of the quadratic potential well.

A Compatible Pair of Mappings on $\mathbb{S}^2$

Define a configuration error function ϕ and transport map $\mathcal{T}$ pairing on $\mathbb{S}^2$ by

$$(6) \quad (\phi, \mathcal{T}_{r \rightarrow q}) = \left[\frac{1}{2} (\arccos(\beta))^2, \beta I_3 + \left(\frac{1 - \beta}{\langle \omega, \omega \rangle} \right) \omega \otimes \omega + \hat{\omega} \right],$$

where $\beta \triangleq \langle q, r \rangle = \cos(\theta)$, $\omega \triangleq r \times q$, and $\omega \otimes \omega$ is defined by its action on a vector v as $(\omega \otimes \omega)(v) \triangleq \langle \omega, v \rangle \omega$. Furthermore, the mapping $\hat{\cdot}$ is defined by

$$(7) \quad \hat{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$$

and has the property $\hat{v} * w = v \times w$, where $*$ is matrix-vector multiplication. The geometric interpretation of the transport map $\mathcal{T}_{r \rightarrow q}$ is revealed as parallel transport by considering the Frenet-Serret frame along the segment of the great circle between r and q . The vectors r and q lead to the orthonormal set $\{r, \text{vers}(q^{\perp r})\}$, which can then be used to define the curve

$$(8) \quad \sigma(\tau) \triangleq \cos(\tau) r + \sin(\tau) \text{vers}(q^{\perp r}).$$

Geometrically, $\sigma(\tau)$, $\tau \in [0, 2\pi)$, is the great circle in the plane defined by the orthonormal vectors $\{r, \text{vers}(q^{\perp r})\}$ with $\sigma(0) = r$ and $\sigma(\theta) = \beta r + \sin(\theta) \text{vers}(q^{\perp r}) = q$. In general, the legs of the Frenet-Serret frame $F(\tau)$ along an arbitrary curve $\sigma(\tau)$ are defined by the vectors

$$T(\tau) = \text{vers} \left(\frac{d}{d\tau} \sigma(\tau) \right)$$

and

$$N(\tau) = \text{vers} \left(\frac{d}{d\tau} T(\tau) \right),$$

and $B(\tau) = T(\tau) \times N(\tau)$. Specifically along the great circle $\sigma(\tau)$ defined in (8), the Frenet-Serret frames at $r = \sigma(0)$ and $q = \sigma(\theta)$ are computed to be

$$(9) \quad F_r = \{T_r, N_r, B_r\} = \{\text{vers}(q^{\perp r}), -r, \text{vers}(r \times q)\},$$

$$(10) \quad F_q = \{\sin(\theta)N_r + \cos(\theta)T_r, \cos(\theta)N_r - \sin(\theta)T_r, B_r\}.$$

Comparing (9) and (10), the two frames F_r and F_q are related by the equation $F_q = T * F_r$ where T is a rotation matrix given by

$$(11) \quad T = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Consequently, the equation $F_q = T * F_r$ is interpreted geometrically to mean that F_q (the frame at q) is obtained by an anticlockwise rotation of F_r (the frame at r) around the $r \times q$ axis through the angle subtended by the great circle between r and q . With the means now to transport the Frenet-Serret frame field from r to q , the transport of v_r (a vector at r) to v_q (a vector at q) is a two-step process:

Step 1. Compute the components of v_r relative to the frame F_r by finding the coefficients c_i of $v_r = \sum_{i=1}^3 c_i F_r^i$. These components are readily

Figure 5. Parallel Transport on $\mathbb{S}^2$. Shown are the transport of Frenet-Serret frame F_r to F_q and the transport of the vector v_r to v_q . Geometrically, the parallel transport of the vector v_r to v_q on $\mathbb{S}^2$ is an anticlockwise rotation around the $r \times q$ axis through the angle subtended by the great circle between r and q .

from (9), the matrix of the transport map $\mathcal{T} = \mathcal{T}_{r \rightarrow q}$ relative to $\mathcal{B}$ is found by applying the transport map to each of the basis vectors. A straightforward computation using (B8)–(B12) determines that the matrix of the transport map $\mathcal{T}$ is equal to the matrix T from (11) and the transport map $\mathcal{T}_{r \rightarrow q}$ is in fact the coordinate-free expression for parallel transport on $\mathbb{S}^2$.

The mappings ϕ and $\mathcal{T}$ from (6) are now shown to satisfy the compatibility condition (4) where $d_q \phi|_q = -\hat{q}^2(d_q \phi)$ and $d_r \phi|_r = -\hat{r}^2(d_r \phi)$ are the vectors on $\mathbb{S}^2$ obtained by projecting the vectors $d_q \phi$ and $d_r \phi$ in $\mathbb{R}^3$ onto the tangent planes $T_q \mathbb{S}^2$ and $T_r \mathbb{S}^2$, respectively (cf. Sidebar 2). A straightforward computation determines the derivative of ϕ with respect to q and its projection to be

$$d_q \phi = -(\theta / \sin(\theta)) r, d_q \phi|_q = -\theta \text{vers}(r^{\perp q}).$$

Similarly, the projection of the derivative of ϕ with respect to r is $d_r \phi|_r = -\theta \text{vers}(q^{\perp r})$. Therefore the right-hand side of (4) is computed to be

$$\begin{aligned} (12) \quad & -\langle d_q \phi|_q, \mathcal{T}_{r \rightarrow q} v_r \rangle \\ &= \theta \beta \langle \text{vers}(r^{\perp q}), v_r \rangle + \theta \langle \text{vers}(r^{\perp q}), \omega \times v_r \rangle \\ &\stackrel{(B8)}{=} \langle \theta \beta \text{vers}(r^{\perp q}) - \theta(\omega \times \text{vers}(q^{\perp r})), v_r \rangle \\ &\stackrel{(B12)}{=} \langle \theta \beta \text{vers}(r^{\perp q}) - \theta \text{vers}(q^{\perp r}) - \theta \beta \text{vers}(r^{\perp q}), v_r \rangle \\ &= \langle d_r \phi|_r, v_r \rangle. \end{aligned}$$

computed to be $c_i = \langle v_r, F_r^i \rangle$, since F_r is an orthonormal frame field.

Step 2. Define the vector v_q with the components from Step 1 and the frame F_q to obtain $v_q \triangleq \sum_{i=1}^3 c_i F_q^i$. Taken together these two steps define a transport mapping called *parallel transport* along the great circle between r and q .

Figure 5 illustrates both the parallel transport of the frame field F_r to F_q and a specific vector v_r to v_q . While this definition of parallel transport is geometrically intuitive, it is a coordinate-based definition. In contrast, the transport mapping $\mathcal{T}_{r \rightarrow q}$ from (6) is a coordinate-free expression. In a coordinate system given by the ordered, orthonormal basis $\mathcal{B} = \{T_r, N_r, B_r\}$

Since the equality in (12) holds for an arbitrary vector $v_r \in T_r \mathbb{S}^2$, ϕ and $\mathcal{T}_{r \rightarrow q}$ form a compatible configuration error and transport map pair, and, consequently, they can be used to implement Bullo's Theorem on $\mathbb{S}^2$.

Recall from Sidebar 1 that the potential function $V(q) = \frac{1}{2}(\text{dist}_{\mathbb{R}}(q, r))^2$ generated the spring force $F_P = -k_P \nabla V = \pm k_P \text{dist}_{\mathbb{R}}(q, r)$, where the direction of the force $\pm$ is chosen so that F_P points from position q to fixed reference r . Unlike the 1D case, where there are only two directions $\pm$ (left, right) to select from, on the sphere there are an infinite number of directions to choose from, namely, one for every vector $v_q \in T_q \mathbb{S}^2$. Strategically selecting

(a) Geodesic Distance θ is large away from the goal, so the strength (length) of spring force F_P is also large.

(b) Geodesic Distance θ is small near the goal, so the strength (length) of the spring force F_P is also small.

Figure 6. Illustration of the Spring Force

$F_P = -k_P \text{vers}(r^{\perp q}) \theta$ of the Proportional-Derivative Force $F_{PD} = F_P + F_D$. Notice that the force F_P is directed along the segment of the geodesic (great circle) connecting the blue configuration curve to the red reference curve (in this case a fixed point). The strength (length) of the spring force is proportional to the distance between the current and reference configuration.

the direction vector for F_P on $\mathbb{S}^2$ is at the heart of the geodesic-spring design. For the potential function $\phi(q, r) = \frac{1}{2}(\arccos(\langle q, r \rangle))^2$, the generated geodesic-spring force is $F_P = -k_P d_q \phi|_q = k_P \text{vers}(r^{\perp q}) \theta$. That is, $\text{vers}(r^{\perp q})$ is the direction

Figure 7. Closed-Loop System Behaves as a Geodesic Spring (Example 1 and 2 Illustrated). When the reference $R(t)$ (by choice) and $Q(t)$ (by appropriate choice of initial conditions) are restricted to a geodesic (the equator in this case), the exact solution $Q(t) = [\theta(t), \phi(t)]$ to the closed-loop system behaves as a 1D MSD about the equilibrium $R(t)$. These plots should be compared to Figure B2.

of the force F_P with strength given by the geodesic distance θ between q and r (see Figure 6).

Control Forces on $\mathbb{S}^2$

Having shown that ϕ and $\mathcal{T}$ from (6) are a compatible pair on $\mathbb{S}^2$, we can now theoretically apply Bullo's Theorem on $\mathbb{S}^2$. To actually implement this theorem, we must compute the resulting proportional derivative and feedforward forces

Figure 8. Illustration of Critically Damped Motion in Example 1 along the Equator from the $\theta = 3$ Longitude to the $\theta = 2$ Longitude.

($F_{PD} = F_P + F_D$ and F_{FF} , respectively) which stabilize the configuration curve q to the reference curve r in the closed loop. Unlike the general Riemannian manifold case in which the geodesic equations (as a nonlinear, two-point boundary value problem) and the parallel transport equations (as a linear, possibly time-dependent, initial value problem) have at best only numerical solutions, closed form

expressions of these forces on $\mathbb{S}^2 \subset \mathbb{R}^3$ lead to a result that generalizes the MSD error system to $\mathbb{S}^2$.

Result 2.1. (MSD System on $\mathbb{S}^2$) Let $q = q(t) \in \mathbb{S}^2$ be the solution to the nonlinear dynamic equations for a free particle

$$\frac{D\dot{q}}{dt} = \text{proj}_q \left(\frac{d\dot{q}}{dt} \right) = -\hat{q}^2 \left(\frac{d\dot{q}}{dt} \right) = \ddot{q} - \langle \ddot{q}, q \rangle q = 0$$

and let $r = r(t) \in \mathbb{S}^2$ be a known reference curve. The compatible pair of functions $(\phi, \mathcal{T})$ defined by (6) generates closed-form expressions of the external forces $\Sigma F = F_{PD} + F_{FF}$ [$k_P > 0, k_D > 0$] given by

$$\begin{aligned} F_{PD} &= F_P + F_D = -k_P d_q \phi|_q - k_D (\dot{q} - \mathcal{T}_{r \rightarrow q} \dot{r}), \\ &= k_P \left(\frac{-\arccos(\beta)}{\sqrt{1-\beta^2}} \right) (r - \beta q) \\ &\quad - k_D [\dot{q} - q \times (\dot{r} \times r) - \alpha \omega], [\beta = \langle r, q \rangle], \end{aligned}$$

where $\alpha \triangleq (1 - \beta) \frac{\langle \omega, \dot{r} \rangle}{\langle \omega, \omega \rangle}$ and

$$\begin{aligned} F_{FF} &= \frac{D}{dt} [\mathcal{T}_{r \rightarrow q} \dot{r}] = q \times (\dot{r} \times r) + \langle q, \dot{r} \times r \rangle (\dot{q} \times q) \\ &\quad + \dot{\alpha}_q \omega + \alpha (\dot{r} \times q) + \dot{\alpha}_r \omega - \beta \alpha (\dot{q} \times q), \end{aligned}$$

with

$$\begin{aligned} \dot{\alpha}_q &\triangleq \alpha + (1 - \beta) \left[\frac{\langle \dot{r} \times q, \dot{r} \rangle + \langle \omega, \dot{r} \rangle}{\langle \omega, \omega \rangle} - 2 \frac{\langle \omega, \dot{r} \rangle \langle \dot{r} \times q, \omega \rangle}{\langle \omega, \omega \rangle^2} \right], \\ \dot{\alpha}_r &\triangleq \alpha + (1 - \beta) \left[\frac{\langle r \times \dot{q}, \dot{r} \rangle}{\langle \omega, \omega \rangle} - 2 \frac{\langle \omega, \dot{r} \rangle \langle r \times \dot{q}, \omega \rangle}{\langle \omega, \omega \rangle^2} \right], \end{aligned}$$

Figure 9. Tracking on the Euclidean Sphere (3D view of Figure 10 (a, b, c), respectively). In all three cases, red is the reference curve $r(t)$, black the solution to $D\dot{q}/dt = 0$, and blue the configuration curve $q(t)$ solution to $D\dot{q}/dt = \Sigma F = F_{pD} + F_{fF}$. Notice that when the forces ΣF are applied, the blue solution turns away from the black curve and tracks the red curve. Therefore, the chasing objective is achieved. The control algorithm is coordinate free, meaning that it can be easily adapted to any coordinate system. The algorithm allows for near-global, large-angle maneuvers of the chasing particle.

and $\% = -\frac{\langle q, \dot{r} \rangle \langle \omega, \dot{r} \rangle}{\langle \omega, \omega \rangle}$. The closed-loop dynamics of the particle q subjected to these forces are $\frac{D\dot{q}}{dt} = \Sigma F$. Since ϕ and $\mathcal{T}$ are a compatible pair, then in the closed loop and by Bullo's Theorem, $q(t) \rightarrow r(t)$ as $t \rightarrow \infty$ (provided the uniformity

and bounding assumptions [9, pp. 536, 540–541] are met and provided that q and r never become antipodally positioned).

The proof of this result is purely computational and requires only repeated use of the straightforward fact that differentiation of both the inner product and the cross product satisfy the “product” rules

$$\begin{aligned} \frac{d}{dt} [v(t) \times w(t)] &= \dot{v}(t) \times w(t) + v(t) \times \dot{w}(t), \\ \frac{d}{dt} \langle v(t), w(t) \rangle &= \langle \dot{v}(t), w(t) \rangle + \langle v(t), \dot{w}(t) \rangle. \end{aligned}$$

However, the derivative of the transport map presents some subtleties. Since the configuration $q(t)$ is time-varying and the reference $r(t)$ can be, in general, time-varying, the derivative of $\mathcal{T}\dot{r}$ is a total derivative

$$\frac{d}{dt} [\mathcal{T}_{r(t)-q(t)} \dot{r}(t)] \triangleq \frac{d}{dt} [\mathcal{T}_{r(t)-q} \dot{r}(t)] + \frac{d}{dt} [\mathcal{T}_{r-q(t)} \dot{r}],$$

where

$$\begin{aligned} \mathcal{T}_{r(t)-q(t)} \dot{r}(t) &= q(t) \times (\dot{r}(t) \times r(t)) \\ &\quad + [1 - \langle q(t), r(t) \rangle] \text{proj}_{r(t) \times q(t)} \dot{r}(t) \end{aligned}$$

and where

$$\begin{aligned} \frac{d}{dt} [\mathcal{T}_{r(t)-q} \dot{r}(t)] &= q \times (\ddot{r}(t) \times r(t)) + \dot{\alpha}_q(r(t) \times q) \\ &\quad + \alpha_q(\dot{r}(t) \times q), \end{aligned} \tag{13}$$

$$\begin{aligned} \frac{d}{dt} [\mathcal{T}_{r-q(t)} \dot{r}] &= \dot{q}(t) \times (\dot{r} \times r) + \dot{\alpha}_r(r \times q(t)) \\ &\quad + \alpha_r(r \times \dot{q}(t)), \end{aligned} \tag{14}$$

with $\alpha, \dot{\alpha}_q, \dot{\alpha}_r$ defined as in the statement of the result. Notice that a distinction is being made between the time-dependent and the fixed quantities. For example, $\frac{d}{dt} [\mathcal{T}_{r(t)-q} \dot{r}(t)]$ indicates that the regular time derivative is taken of $\mathcal{T}_{r(t)-q} \dot{r}(t)$ with q fixed and $r(t)$ time varying. Now, recall that while $v_q = \mathcal{T}_{r-q} \dot{r} \in T_q \mathbb{S}^2$, the derivative (or acceleration) $\frac{d}{dt} (v_q)$ does not generally lie in $T_q \mathbb{S}^2$. To ensure that the derivative of v_q lies in $T_q \mathbb{S}^2$, one must follow the derivative by a projection. That is, covariant differentiation must be used rather than regular differentiation. Computing the projections $-\hat{q}^2(\cdot)$ and $-\widehat{q(t)}^2(\cdot)$ of (13) and (14), respectively, and adding the results complete the computation of the feedforward force F_{fF} .

Implementation of MSD System on $\mathbb{S}^2$

With the external forces $\Sigma F = F_{pD} + F_{fF}$ now defined in terms of the intrinsic geometry of $\mathbb{S}^2$, Result 2.1 is *coordinate free* and can be implemented using any set of coordinates on $\mathbb{S}^2$ (see Sidebar 3). This distinction is important to note. It is only *after* we have formulated the

Figure 10. Simulations of Result 2.1 in Spherical and Projection Coordinates. In the projection coordinates simulations, the reference curve (red) is the coordinate form of a “short” circle. Since this reference curve is *not* a geodesic, the closed-loop dynamics do not present a closed-form solution. Instead, numerical solutions must be considered. The geodesic distance is computed by $\arccos(\langle r(t), q(t) \rangle)$ where, in projection coordinates, for example, $r(t) = \Phi_{sp}([u_r(t), v_r(t)])$ and $q(t) = \Phi_{sp}([u(t), v(t)])$. See Figure 9 for the trajectories on the sphere.

control law that coordinates are introduced. This is in contrast to *first* introducing coordinates, say spherical coordinates, to formulate the free dynamics $\frac{D\dot{q}}{dt} = 0$ and *then* devising the control forces $\Sigma F = F_{PD} + F_{FF}$ to achieve the tracking objective. This method would be a *coordinate-based* approach and works for the chosen coordinate system and no other.

In this section we give two examples that, in spherical coordinates, have closed-form solutions due to the fact that the reference curve $r(t)$ and the configuration curve $q(t)$ are restricted to the equator (a geodesic). We also present simulations (shown in both stereographic and projection

coordinates in Figure 10) where the reference curve is *not* the equator (i.e., *not* a geodesic). Exact solutions to this case are unknown to this author and so numerical solutions are considered. The goal of each example is to demonstrate that no matter the coordinate system, the geodesic spring design with parallel-transport-based damping exhibits the behavior of a mass-spring-damper error system in the geodesic distance.

Examples

Two examples of Result 2.1 with closed-form solutions are presented. These two examples are informed by a general theorem [1, p. 5] which proves that for the configuration and transport

map pairing $(\phi, \mathcal{T})$ given by (6), the geodesic distance exhibits the behavior seen from the 1D MSD in Figure B2. The first example is the simplest example of the closed-loop error system from Result 2.1, namely, the fixed reference case.

Example 1 ($\dot{r} = 0$). In spherical coordinates, let r be a fixed reference with coordinate description $R = [\theta_r, \phi_r] = [2, \pi/2]$, and assume the initial conditions $\{\theta(0) = 3, \phi(0) = \pi/2, \dot{\theta}(0) = 0, \dot{\phi}(0) = 0\}$ for the configuration curve $q(t)$ with coordinate description $Q(t) = [\theta(t), \phi(t)]$. In other words, both $q(0)$ and r start on the equator and have zero initial velocity. Because the reference is fixed, the transport map of $\dot{r}$ and its derivatives is 0, and the closed-loop error system of Result 2.1 reduces to $\frac{D\dot{q}}{dt} = F_{PD}$ with

$$F_{PD} = F_P + F_D = -k_P d_q \phi|_q - k_D \dot{q} [k_P > 0, k_D > 0]$$

$$= k_P \left(\frac{-\arccos(\beta)}{\sqrt{1-\beta^2}} \right) (r - \beta q) - k_D \dot{q} [\beta = \langle r, q \rangle].$$

The *first* and *third* equations of the closed-loop error system are

$$(15) \quad \ddot{\theta} + 2\mathfrak{c}_\theta \dot{\phi} \dot{\theta} + k_D \dot{\theta} + k_P \mathfrak{s}_\theta = 0,$$

$$(16) \quad \ddot{\phi} - \mathfrak{c}_\phi \mathfrak{s}_\phi \dot{\theta}^2 + k_D \dot{\phi} + k_P \mathfrak{s}_\phi = 0,$$

where

$$\mathfrak{s}_\theta = \frac{\arccos(\mathfrak{c}_\phi \mathfrak{c}_{\phi_r} + \mathfrak{c}_{\theta-\theta_r} \mathfrak{s}_\phi \mathfrak{s}_{\phi_r})}{\mathfrak{s}_\phi \sqrt{1 - (\mathfrak{c}_\phi \mathfrak{c}_{\phi_r} + \mathfrak{c}_{\theta-\theta_r} \mathfrak{s}_\phi \mathfrak{s}_{\phi_r})^2}}$$

$$\mathfrak{s}_\phi = \frac{\arccos(\mathfrak{c}_\phi \mathfrak{c}_{\phi_r} + \mathfrak{c}_{\theta-\theta_r} \mathfrak{s}_\phi \mathfrak{s}_{\phi_r})}{\sqrt{1 - (\mathfrak{c}_\phi \mathfrak{c}_{\phi_r} + \mathfrak{c}_{\theta-\theta_r} \mathfrak{s}_\phi \mathfrak{s}_{\phi_r})^2}}$$

and where, for example, $\mathfrak{c}_{\theta_r-\theta} = \cos(\theta_r - \theta)$, $\mathfrak{s}_\phi = \sin(\phi)$, and $\mathfrak{c}_\theta = \cot \phi$.

Why the first and the third equations? Note that computing the free particle dynamics on $\mathbb{S}^2$ by $\frac{D\dot{q}}{dt} = \ddot{q} - \langle \ddot{q}, q \rangle q = 0$, where $q(t) = \Phi_{sc}([\theta(t), \phi(t)])$ yields three equations:

$$(17) \quad \ddot{\phi} - \mathfrak{c}_\phi \mathfrak{s}_\phi \dot{\theta}^2 = 0,$$

$$(18) \quad \mathfrak{c}_\theta \mathfrak{s}_\phi \ddot{\theta} + \mathfrak{c}_\phi \mathfrak{s}_\theta \ddot{\phi} - \mathfrak{c}_\phi^2 \mathfrak{s}_\theta \mathfrak{s}_\phi \dot{\theta}^2 + 2\mathfrak{c}_\theta \mathfrak{c}_\phi \dot{\theta} \dot{\phi} = 0,$$

$$(19) \quad \mathfrak{c}_\theta \mathfrak{c}_\phi \ddot{\phi} - \ddot{\theta} \mathfrak{s}_\theta \mathfrak{s}_\phi + \mathfrak{c}_\theta \mathfrak{c}_\phi^2 \mathfrak{s}_\phi \dot{\theta}^2 - 2\mathfrak{c}_\phi \mathfrak{s}_\theta \dot{\theta} \mathfrak{s}_\phi = 0.$$

Solving (17) for $\ddot{\phi}$ and then substituting the result into (18) and (19) yields the same result:

$$\ddot{\theta} + 2\mathfrak{c}_\theta \dot{\phi} \dot{\theta} = 0.$$

Consequently, only the *first* and *third* (or *second*) of the free particle equations are distinct. The equations

$$\ddot{\phi} - \mathfrak{c}_\phi \mathfrak{s}_\phi \dot{\theta}^2 = 0,$$

$$\ddot{\theta} + 2\mathfrak{c}_\theta \dot{\phi} \dot{\theta} = 0$$

are the typical form (in spherical coordinates) of the geodesic equations on $\mathbb{S}^2 \subset \mathbb{R}^3$. With the addition of the external forces ΣF , the closed-loop dynamics $\frac{D\dot{q}}{dt} = \Sigma F$ also have redundancies. As in

the free particle case, the first and third equations are distinct.

Given the geodesic spring motivation of the closed-loop error system, (15) and (16) have a closed-form solution. For the chosen initial conditions, the solution must stay on the equator $\phi = \phi_r = \pi/2$ (since $\dot{\phi}(0) = 0$) and behave like a 1D MSD measured from θ_r . The claim is $Q(t) = [\theta(t), \phi(t)] = [\theta_r + \mathfrak{g}(t), \pi/2]$, with $\mathfrak{g}(t)$ the solution to the 1D MSD initial value problem $\ddot{\mathfrak{g}} + k_D \dot{\mathfrak{g}} + k_P \mathfrak{g} = 0$ with $\mathfrak{g}(0) = \theta(0) - \theta_r = 1$, and $\dot{\mathfrak{g}}(0) = \dot{\theta}(0) = 0$ is the exact solution to (15) and (16). Since $\dot{\phi} = \ddot{\phi} = 0$, $\mathfrak{c}_\phi = 0$, and $\mathfrak{s}_\phi = 1$, (16) is trivially satisfied. Under the same assumptions and since $\dot{\theta} = \dot{\mathfrak{g}}$, $\ddot{\theta} = \ddot{\mathfrak{g}}$, $\theta_r - \theta = \mathfrak{g}$, (15) simplifies to an equation in $\mathfrak{g}$ as

$$\ddot{\theta} + k_D \dot{\theta} + k_P (\theta_r - \theta) \rightarrow \ddot{\mathfrak{g}} + k_D \dot{\mathfrak{g}} + k_P \mathfrak{g} = 0.$$

For example, when $k_D = 2$ and $k_P = 1$, $\mathfrak{g}(t) = e^{-t}(1+t)$ is the critically damped solution to $\ddot{\mathfrak{g}} + k_D \dot{\mathfrak{g}} + k_P \mathfrak{g} = 0$ with $\mathfrak{g}(t) \rightarrow 0$ as $t \rightarrow \infty$. It follows that $\theta(t) = \theta_r + \mathfrak{g}(t)$ is the critically damped solution to (15) where $\theta \rightarrow \theta_r$ as $t \rightarrow \infty$. Figure 8 shows the solution (blue curve) $q(t) = \Phi_{sc}[\phi_r + \mathfrak{g}(t), \pi/2]$ on $\mathbb{S}^2$. Figures 7(a) and 7(c) corroborate the underdamped and critically damped solutions.

Example 2 ($\dot{r} \neq 0$). This second example extends the first example to consider a moving reference $r(t)$ along the equator. In spherical coordinates, let $r(t)$ be the reference with coordinate description $R(t) = [\theta_r(t), \phi_r] = [2 - \frac{t}{5}, \pi/2]$, and assume the initial conditions $\{\theta(0) = 3, \phi(0) = \pi/2, \dot{\theta}(0) = 0, \dot{\phi}(0) = 0\}$ for the configuration curve $q(t)$ with coordinate description $Q(t) = [\theta(t), \phi(t)]$. Since the reference is no longer fixed and yet lies on the geodesic (equator), it follows that $\langle \omega, \dot{r} \rangle = 0$; therefore, the middle term of the transport map is 0. Consequently, the closed-loop error system of Result 2.1 reduces to $\frac{D\dot{q}}{dt} = F_{PD} + F_{FF}$ with

$$F_{PD} = F_P + F_D = -k_P d_q \phi|_q - k_D (\dot{q} - \mathcal{T}_{r \rightarrow q} \dot{r})$$

$$= k_P \left(\frac{-\arccos(\beta)}{\sqrt{1-\beta^2}} \right) (r - \beta q) - k_D [\dot{q} - q \times (\dot{r} \times r)],$$

$$F_{FF} = \frac{D}{dt} [\mathcal{T}_{r \rightarrow q} \dot{r}] = q \times (\ddot{r} \times r).$$

The third equation of the closed-loop system is trivially satisfied along the equator $\phi = \phi_r = \pi/2$. In terms of the new variable $\mathfrak{g} = \theta_r - \theta$, which defines $\ddot{\theta} = \ddot{\mathfrak{g}}$, $\dot{\theta} = \dot{\mathfrak{g}} - \frac{1}{5}$, the first equation simplifies to

$$\frac{k_D}{5} + k_P (\theta_r - \theta) + k_D \dot{\theta} + \ddot{\theta} = 0 \rightarrow k_P \mathfrak{g} + k_D \dot{\mathfrak{g}} + \ddot{\mathfrak{g}} = 0.$$

Therefore, as in the first example, the solution to the first and third equations of the closed-loop system is given by $Q(t) = [\theta(t), \phi(t)] = [\theta_r(t) + \mathfrak{g}(t), \pi/2]$, where $\mathfrak{g}(t)$ is a solution to

$\ddot{\vartheta} + k_D \dot{\vartheta} + k_P \vartheta = 0$, $\vartheta(0) = 1$, $\dot{\vartheta}(0) = 1/5$. Figures 7(b) and 7(d) corroborate the underdamped and critically damped solutions in this example.

Simulations

Simulations of Result 2.1 are performed in spherical and stereographic coordinates using Mathematica's `NDsolve` command.

Sidebar 1. 1D Mass-Spring-Damper (MSD) System

Consider a particle of mass m constrained to move on a frictionless rod (analogous to the real line $\mathbb{R}$). Assume that the particle, initially at position q_0 , is set into motion by an initial force that imparts an initial velocity v_0 to the particle but after which is subject to no other outside forces. That is, the particle is *free* with external forces $\Sigma F = 0$. As a differential equation, the free dynamics for the particle are given by Newton's equation $\Sigma F = ma$, specifically,

$$(B1) \quad \frac{dq}{dt} = 0$$

with solution $q(t) = v_0 t + q_0$. Depending on the direction (left, $-$) or (right, $+$) of the initial velocity, the particle's position $q(t) \rightarrow \pm \infty$ as $t \rightarrow \infty$. Forcing the particle to approach a *fixed* reference position r (rather than $\pm \infty$) can be achieved by connecting the particle to the rod with a spring-damper where the linear spring has a stiffness modeled by the constant $k_P > 0$ and the damper provides a resistance modeled by the constant $k_D > 0$. Define the proportional (spring) force F_P and the derivative (damping) force F_D by

$$(B2) \quad F_P \triangleq -k_P e(t) \quad \text{and} \quad F_D \triangleq -k_D \frac{d}{dt} [e(t)],$$

where $e(t) \triangleq q(t) - r$ is the error between the particle's position $q(t)$ and the fixed reference position r . It follows that Newton's equation $m \frac{dq}{dt} = \Sigma F$ with external forces $\Sigma F = F_P + F_D$ simplifies to the 1D mass-spring-damper (MSD) error system

$$(B3) \quad \ddot{e} + 2\delta \dot{e} + \omega^2 e = 0,$$

where $\delta = k_D/2m$, $\omega = \sqrt{k_P/m}$, and $\dot{e}, \ddot{e}$ denote in this case the first and second time derivatives of $e(t)$. The solution to this error system is a standard result in introductory ordinary differential equations. For any initial conditions q_0, v_0 , the error $e(t) \rightarrow 0$ as $t \rightarrow \infty$. In other words, the position of the forced particle $q(t)$ approaches the reference position r as $t \rightarrow \infty$. Consequently, depending on the selections for δ and ω , the control objective $q(t) \rightarrow r$ is achieved in an underdamped, overdamped, or critically damped manner (see Figure B2).

While the control objective has been achieved, the result is rather unsatisfactory in the sense that we are not left with much understanding about how this result was achieved. What is the big picture, and how might knowing it aid in extending this spring design to the sphere? To begin answering this question, consider more closely the forces F_D and F_P . The control objective is partially achieved by the addition of the spring force F_P , which can be interpreted in terms of the quadratic potential function

$$(B4) \quad V(q) = \frac{1}{2} \text{dist}_{\mathbb{R}}(q, r)^2 \triangleq \frac{1}{2} |q - r|^2.$$

This potential can be viewed as a quadratic well centered at the fixed reference r , as shown in Figure B1, that generates a (restoring) force $F_P = -k_P \nabla V$ on the particle directed toward the reference r with a strength proportional to the distance between the particle's position q and reference r . In other words,

$$(B5) \quad F_P = \pm k_P \text{dist}_{\mathbb{R}}(q, r) = -k_P(q - r) = -k_P e,$$

where the direction $+$ (right) or $-$ (left) of the force is given by the expression $-(q - r)/|q - r|$. Due to energy considerations, the spring force F_P alone is not enough to achieve the control objective. Specifically, for zero damping $\delta = 0 \Rightarrow F_D = 0$, the total energy L (kinetic energy $\frac{1}{2} \dot{q}^2$ plus potential energy $\frac{1}{2} \omega^2 (q - r)^2$) of the particle along the solution to $m\ddot{e} = F_P$ is a constant function of the initial position and velocity of the particle, and therefore the spring force F_P alone cannot dissipate the energy. Introducing the external damping force F_D , the total derivative of the total energy of the particle along the solutions to $m\ddot{e} = F_D + F_P$ (equivalently, $\ddot{e} = -2\delta \dot{e} - \omega^2 e$) is

$$(B6) \quad \begin{aligned} \frac{dL}{dt} &= \dot{q} \ddot{q} + \omega^2 (q - r) \dot{q} = \dot{e} \ddot{e} + \omega^2 e \dot{e} \\ &= \dot{e}(-2\delta \dot{e} - \omega^2 e) + \omega^2 e \dot{e} \\ &= -2\delta \dot{e}^2 \leq 0. \end{aligned}$$

For the cases $\delta^2 > \omega^2$ and $\delta^2 = \omega^2$, Figure B2 (bottom, middle, right) illustrates that $\dot{e} \neq 0$, in which case $\frac{dL}{dt}$ is strictly negative, and therefore the total energy of the particle must dissipate to 0. Even when $\dot{e} = 0$ ($\delta^2 < \omega^2$), Figure B2 (bottom, left) illustrates that the total energy of the particle eventually dissipates to 0. In all three cases, the control objective $q(t) \rightarrow r$ is achieved.

In this article, the forces F_D and F_P are extended to the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ in order to stabilize a particle $q \in \mathbb{S}^2$ to a time-dependent reference curve $r(t) \in \mathbb{S}^2$. The overarching control methodology is to create a potential well, this time on the sphere, centered on the moving reference $r(t)$. The resulting potential force along with a geometrically defined damping force achieves the control objective $q(t) \rightarrow r(t)$.

(a) (left) The potential function $V(q)$ can be viewed as a quadratic potential well centered at $r \in \mathbb{R}$ in which the particle q is constrained to roll about. As q is released from its position of maximum potential V_0 , its energy is converted to purely kinetic energy at the bottom of the well and again to maximum potential at $V_1 = V_0$. The particle q will continue to roll around the point r but never settle on it. (b) (right) With the inclusion of a damping force, thought of as a source friction on the potential surface, the potential energy of particle q dissipates from V_1 to V_3 , and eventually $V = 0$ where the reference position r is located. Consequently, in the presence of damping, q settles on r . The overarching control methodology in this article is to create a quadratic potential surface (well), this time centered on a moving reference $r(t) \in \mathbb{S}^2$ (see Figure 2). The resulting potential force, along with a geometrically defined damping force, which defines a source of friction on the potential surface will achieve the control objective $q(t) \rightarrow r(t)$ on the sphere.

Figure B1. Illustration of the Potential Function $V(q) = \frac{1}{2} \left(\text{dist}_{\mathbb{R}}(q, r) \right)^2 = \frac{1}{2} (q - r)^2$ and Damping.

(a) The damped solution (heavy solid line) to the MSD error system (B3) is plotted for various values of δ^2 and ω^2 and overlaid with the free solution (the straight line) and the nondamped solution (dot-dashed line). In each case, the damped solution approaches the equilibrium position r .

(b) The total energy of the damped solution (heavy solid line) is plotted for various values of δ^2 and ω^2 and overlaid with the total energy of the free solution and the energy of the nondamped solution (the constant dot-dashed line), both of which are equal to L_0 . In each case, the energy of the damped solution approaches 0.

Figure B2. 1D Mass-Spring-Damper (MSD) Summary.

Sidebar 2. Visualization of Notation

The following equations illustrate the interaction between the inner product, cross-product, and projection and are a collection of the computations needed to prove the results of this article:

$$(B7) \quad v \times (w \times z) = \langle v, z \rangle w - \langle v, w \rangle z,$$

$$(B8) \quad \langle y, (w \times z) \times v \rangle = -\langle v, (w \times z) \times y \rangle,$$

$$(B9) \quad -\hat{r}^2 q = -(q \times r) \times r = q - \beta r \triangleq q^{\perp r},$$

$$(B10) \quad ||q^{\perp r}|| = ||r^{\perp q}|| = \sin(\theta),$$

$$(B11) \quad (r \times q) \times r^{\perp q} = q^{\perp r} + \beta r^{\perp q} = \sin^2(\theta) q,$$

$$(B12) \quad (r \times q) \times \text{vers}(q^{\perp r}) = -\sin(\theta) r,$$

Figure B3. Visualization of Notation. Points $r, q, v \in \mathbb{S}^2 \subset \mathbb{R}^3$ are identified with the vectors $\vec{r}, \vec{q}, \vec{v}$ at the origin $O \in \mathbb{R}^3$. Often the vector symbol $\vec{\cdot}$ is suppressed. The vectors v, r, q can be translated (without stretching or tilting) to the points $r, q \in \mathbb{S}^2$ to obtain the vectors q_r, v_r, q_q . Often the point of attachment is suppressed.

where $\beta \triangleq \langle q, r \rangle = \cos(\theta)$ and where $\text{vers}(\cdot)$ is the length-normalization mapping on vectors v defined by $\text{vers}(v) = v/||v||$. The mapping $\hat{\cdot}$ is defined in (7). Equation (B7), the standard triple cross-product formula, is the main equation from which the other equations are derived. In general, the vectors q_r and v_r are not in the tangent plane to $\mathbb{S}^2$ at r (denoted $T_r \mathbb{S}^2$). According to (B9) and as shown in Figure B3, $q^{\perp r} = -\hat{r}^2 q \in T_r \mathbb{S}^2$ and $-\hat{r}^2 v \in T_q \mathbb{S}^2$. That is, the operators $-\hat{r}^2(\cdot)$ and $-\hat{q}^2(\cdot)$ act as projections onto the tangent planes to $\mathbb{S}^2$ at r and q , respectively. Therefore the covariant derivative operator at $q \in \mathbb{S}^2$ from (2) can be explicitly given by

$$(B13) \quad \frac{D}{dt}|_q = -\hat{q}^2 \circ \frac{d}{dt}.$$

Note that $r^{\perp q} \triangleq -\hat{r}^2(q)$ is the vector at q pointing tangent to the geodesic segment (black solid curve) connecting q to r . In this article, we construct a force $F_p \propto r^{\perp q}$ that compels a particle $q(t) \in \mathbb{S}^2$ towards a moving target $r(t) \in \mathbb{S}^2$. It is important that the applied forces, like F_p for example, be tangent to the sphere. Otherwise, the curve $q(t)$ will leave the sphere, and, using the robotic arm analogy, the tool tip of the robotic arm will no longer be constrained to the sphere.

Sidebar 3. Spherical and Projection Coordinates

Spherical and stereographic projection coordinates are the two coordinate systems employed in this article. The general concept of coordinates on $\mathbb{S}^2$ can be illustrated using the example of stereographic projection. Since we are viewing the unit sphere $\mathbb{S}^2$ as a subset of $\mathbb{R}^3$, one description of the sphere is the algebraic relation $x^2 + y^2 + z^2 = 1$ between the *three* variables (x, y, z) . Imagine now that a light

Figure B4. Stereographic Projection Illustration. Because the geodesic spring control design is coordinate independent, it can be implemented in any coordinate system. The coordinates defined by stereographic projection are ideal because of the large domain of definition, $\mathbb{S}^2$ – north or south pole.

source and planar screen are placed at the north and south poles of a translucent sphere, respectively. Any region on the sphere will then cast a shadow onto the screen to create a two-dimensional representation of that region (see Figure B4). The *two* variables which describe the position of the shadow are the (stereographic) projection coordinates of $\mathbb{S}^2$. The spherical coordinate (θ, ϕ) and stereographic (south pole) projection coordinate (u, v) representations of the sphere $\Phi_{sc} : \mathcal{D}_{sc} \subset \mathbb{R}^2 \rightarrow \mathbb{S}_{sc}^2 \subset \mathbb{S}^2$ and $\Phi_{sp} : \mathcal{D}_{sp} \subset \mathbb{R}^2 \rightarrow \mathbb{S}_{sp}^2 \subset \mathbb{S}^2$ are given by

$$(B14) \quad \Phi_{sc}(\theta, \phi) = [\cos(\theta) \sin(\phi), \sin(\phi) \sin(\theta), \cos(\phi)],$$

$$(B15)$$

$$\Phi_{sp}(u, v) = \left[\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{1-u^2-v^2}{1+u^2+v^2} \right],$$

where $\mathcal{D}_{sc} = (0, 2\pi) \times (0, \pi)$ and $\mathcal{D}_{sp} = (-\infty, \infty) \times (-\infty, \infty)$ are the domains chosen so that Φ_{sc} and Φ_{sp} are one-to-one mappings. For completeness, the mapping which assigns to each point $(x, y, z) \in \mathbb{R}^3$ its projection coordinates (u, v) is

$$(B16) \quad \Phi_{sp}^{-1}(x, y, z) = \left[\frac{2x}{1+z}, \frac{2y}{1+z} \right].$$

Acknowledgments

The first author would like to thank Robert Fuentes and Gregory Hicks for bringing his attention to the geodesic spring control design during a Space Scholars internship at the Air Force Research Labs (AFRL). The general results on which this article are founded, namely, using both parallel transport and geodesic distance on a Riemannian manifold [1] as a compatible pair of

mappings, originated with them. The first author further appreciates the efforts made by AFRL in offering the Space Scholars program and for the valuable opportunities it provided. The first author would like to also formally express a sincere gratitude to his coauthor (Greg) for the many conversations related to geometry and control and for helping to make this article possible. We wish to sincerely thank the reviewers and the editors whose comments and suggestions have made this article more accessible as an expository article and more transparent to the big picture. Finally, and certainly not least, the first author wishes to thank Jill Reese (wife, software engineer, and copy editor extraordinaire) for her ever-present support of and belief in me. Without her and her infinite patience this article would not exist (at a maximum) and be riddled with missed and misapplied apostrophes and hyphens (at a minimum). Any errors that may lurk within are, of course, the author's own.

References

- [1] R. FUENTES, G. HICKS, and J. OSBORNE, The Spring Paradigm in Tracking Control of Simple Mechanical Systems, *Automatica*, 47(5):993-1000, 2011.
- [2] M. SPIVAK, *A Comprehensive Introduction to Differential Geometry (Vol. II)*, Publish or Perish, Inc., Houston, Texas, 1999.
- [3] G. HICKS and R. FUENTES, A Geometric Treatment of Directed Guidance Regulation. In *Proceedings of AIAA*, 2010.
- [4] C. A. BERENSTEIN and E. C. TARABUS, Integral Geometry in Hyperbolic Spaces and Electrical Impedance Tomography, *SIAM J. Appl. Math.*, 56(3):755-764, 1996.
- [5] A. TERRAS, *Harmonic Analysis on Symmetric Spaces and Applications*, Springer-Verlag, 1985.
- [6] M. P. DOCARMO, *Riemannian Geometry*, Birkhäuser, Boston, 1992.
- [7] M. BERGER, *A Panoramic View of Riemannian Geometry*, Springer, New York, 2003.
- [8] F. BULLO, R. M. MURRAY, and A. SARTI, Control on the Sphere and Reduced Attitude Stabilization. In *Symposium on Nonlinear Control Systems*, volume 2, pages 495-501, Tahoe City, CA, June 1995.
- [9] F. BULLO and A. D. LEWIS, *Geometric Control of Mechanical Systems*, Springer, New York, 2005.
- [10] F. BULLO and R. M. MURRAY, Tracking for Fully Actuated Mechanical Systems: A Geometric Framework, *Automatica*, 35:17-34, 1999.
- [11] FRANCESCO BULLO, *Nonlinear Control of Mechanical Systems: A Riemannian Geometry Approach*, Ph.D. dissertation, California Institute of Technology, Pasadena, CA, 1999.
- [12] J. OSBORNE, G. HICKS, and R. FUENTES, Global Analysis of the Double Gimbal Mechanism, *IEEE Contr. Sys. Mag.*, 28(4):44-64, 2008.
- [13] R. W. BROCKETT, Lie Theory and Control Systems Defined on Spheres, *SIAM J. Appl. Math.*, 25(2), 1973.
- [14] A. M. BLOCH, P. E. CROUCH, J. E. MARSDEN, and A. K. SANYAL, Optimal Control and Geodesics on Quadratic Matrix Lie Groups, *Found. Comput. Math.*, 8:469-500, 2008.

Far from Ordinary

The U.S. is under cyber attack, every minute of every day. That's why cyberspace has become today's new front line. What you know can make a difference at the National Security Agency. Whether it's collecting foreign intelligence or preventing foreign adversaries from accessing U.S. secrets, you can protect the nation by putting your intelligence to work. Explore technology that's years ahead of the private sector. Plus exciting career fields, paid internships, co-op and scholarship opportunities. See how you can be a part of our tradition of excellence and help the nation stay a step ahead of the cyber threat.

KNOWINGMATTERS

Excellent Career Opportunities for Experts in the Following:

- Number Theory
- Probability Theory
- Group Theory
- Finite Field Theory
- Combinatorics
- Linear Algebra

>> Plus other opportunities

NSA
NSA.gov/Careers

APPLY TODAY

WHERE INTELLIGENCE GOES TO WORK®

U.S. citizenship is required. NSA is an Equal Opportunity Employer.