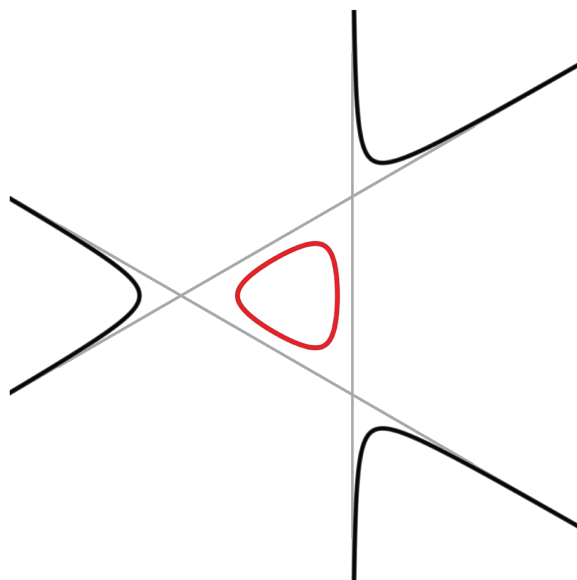# Enumerative Geometry Meets Statistics, Combinatorics, and Topology

Mateusz Michałek

1. Introduction

We explain connections among several, a priori unrelated, areas of mathematics: combinatorics, algebraic statistics, topology, and enumerative algebraic geometry. Our focus is on discrete invariants, strongly related to the theory of Lorentzian polynomials. The main concept joining these fields is a linear space of matrices.

The following questions arise prominently in different branches of mathematics.

1. Given a graph G, how many proper vertex colorings with k colors exist?
2. What is the degree of a general linear concentration model? What is its maximum likelihood degree?

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3. What is the Euler characteristic of a hypersurface defined by the determinant of a matrix with linear entries?
4. How many degree two hypersurfaces pass through a general points and are tangent to b general hyperplanes?

We will start by explaining the meaning of the above questions. Our main aim is to show that the central objects we encounter are in fact shadows of one construction and all of the above questions are in fact one (or more precisely two related) question(s). It turns out that the unifying setting is surprisingly simple: we will always start from a linear space L of square matrices. To such a space we canonically associate a polynomial with integral coefficients. This is a special instance of the so-called volume polynomial and an example of a *Lorentzian polynomial*. Roughly speaking, the coefficients of this polynomial form the multidegree of the graph of the gradient of the determinant restricted to L. We also present powerful, modern geometric tools to study these basic invariants from a new perspective. This will be achieved by performing intersection theory on

smooth, projective, and beautiful varieties. We will work over the field $\mathbb{C}$.

2. Main Players

Multidegree. For our purposes a variety $V(f_1, \ldots, f_n)$ is the set of zeros of a polynomial system $f_i = 0$. Given homogeneous polynomials on a vector space V, it is natural to consider their zero set as a subset of the projective space $\mathbb{P}(V)$. The corresponding variety is called *a projective variety* [MS21, Chapters 1–3].

Example 2.1. Let $f = xy - zt$. We may regard $V(f)$ as a two-dimensional quadratic surface in $\mathbb{P}^3$.

The two most important invariants of a variety in a projective space are its dimension and degree. One of a few equivalent ways to define the degree of a variety $X \subset \mathbb{P}^n$ is as the number of points one obtains after intersecting it with $\dim X$-many general hyperplanes. A variety X is *irreducible* if, whenever it is a union of two varieties $X = X_1 \cup X_2$ then $X = X_1$ or $X = X_2$. To the projective space $\mathbb{P}^n$ one associates a ring $H^*(\mathbb{P}^n)$, which in this case coincides with the Chow ring and the cohomology ring. Elements of this ring are formal linear combinations of classes of irreducible subvarieties of $\mathbb{P}^n$, called cohomology classes. A variety is equivalent to a formal sum of its irreducible components. Further, two, possibly reducible, varieties are equivalent, if they have the same degree and all components are of the same dimension. The multiplication in $H^*(\mathbb{P}^n)$ corresponds to intersection of varieties[1]. Note that all hyperplanes define the same cohomology class. In fact, the cohomology ring with rational coefficients of $\mathbb{P}^n$ is $\mathbb{Q}[H]/(H^{n+1})$, where H is the class of the hyperplane. Then the class of an irreducible variety X equals $(\deg X)H^{n-\dim X}$.

Example 2.2. Continuing the example of $f = xy - zt$, one can check that $V(f)$ has degree two. In general, the notion of the degree of a variety generalizes the notion of the degree of a polynomial.

When X is a subvariety of a product $\mathbb{P}^n \times \mathbb{P}^m$ of projective spaces, the analogue of the degree is the *multidegree*. Indeed, we have two different (families of) "hyperplanes" in $\mathbb{P}^n \times \mathbb{P}^m$. Namely, the product H_1 of a hyperplane in $\mathbb{P}^n$ with $\mathbb{P}^m$, and the product H_2 of $\mathbb{P}^n$ with a hyperplane in $\mathbb{P}^m$. Thus, instead of one number, we obtain a sequence of $(\dim X + 1)$-many numbers, by intersecting X with a general hyperplanes of type H_1 and b general hyperplanes of type H_2, where $a + b = \dim X$. Analogously to the previous case, the cohomology ring of $\mathbb{P}^n \times \mathbb{P}^m$ is $\mathbb{Q}[H_1, H_2]/(H_1^{n+1}, H_2^{m+1})$. The multidegree of an irreducible variety X tells us the cohomology class of

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[1] *Assuming we choose representatives of classes that intersect in a nice way — formally we have to assume they intersect transversally.*

X. For more information about the multidegree we refer to the book [MS05, Chapter 8].

Graphs. Let $G = (V, E)$ be a loopless graph. A *proper vertex coloring* using $k \in \mathbb{Z}_{\geq 0}$ colors is a function $f : V \to \{1, \ldots, k\}$, such that whenever two vertices v_1, v_2 are connected by an edge, we have $f(v_1) \neq f(v_2)$. The function $\chi_G : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ that to k assigns the number of proper vertex colorings using k colors is known as the *chromatic polynomial* of G. It is indeed a polynomial, as one may prove by induction on the number of edges, by contracting and deleting a given edge. The same proof shows that χ_G is a polynomial of degree $|V|$ with integral coefficients, whose signs are alternating. If G has at least one edge then χ_G has a root at one. We thus define the *reduced chromatic polynomial* $\overline{\chi}_G(k) := \chi_G(k)/(k-1)$. From now on, for simplicity, we will assume that G is connected.

Example 2.3. Let P_n be the path with n vertices. Given k colors we may assign to the first vertex any of the colors. Then to each consecutive vertex we may assign $(k-1)$ colors. We obtain:

$$\chi_{P_n}(k) = k(k-1)^{n-1}$$
$$\overline{\chi}_{P_n}(k) = k(k-1)^{n-2}.$$

Let C_3 be the 3-cycle, i.e., a triangle. If we remove one edge, we obtain a path P_3 with three vertices. A proper coloring of P_3 is not a proper coloring of C_3 if and only if the two end vertices have the same color. Thus, there are exactly $\chi_{P_2}(k) = k(k-1)$ such non-proper colorings. We obtain:

$$\chi_{C_3}(k) = \chi_{P_3}(k) - \chi_{P_2}(k) = k(k-1)(k-2)$$
$$\overline{\chi}_{C_3}(k) = k(k-2).$$

Example 2.4. Let C_4 be the 4-cycle and let P_4 be the path obtained by removing one edge from C_4. A proper coloring of P_4 is not a proper coloring of C_4 if and only if the two end vertices have the same color. Thus, there are exactly $\chi_{C_3}(k) = k(k-1)(k-2)$ such colorings. We obtain:

$$\chi_{C_4}(k) = k(k-1)\big((k-1)^2 - (k-2)\big) =$$
$$k(k-1)(k^2 - 3k + 3)$$
$$\overline{\chi}_{C_4}(k) = k^3 - 3k^2 + 3k.$$

Let $n := |E|$. Below, we describe a classical construction of associating to G a subspace of $\mathbb{C}^n$, which is a special case of a representation of a matroid. Let $\mathbb{C}^n$ be the vector space with basis f_e for $e \in E$. We orient the edges of G in an arbitrary way. Let L_G be the subspace of $\mathbb{C}^n$ spanned by all vectors w_v indexed by vertices $v \in V$, where the coordinate of w_v corresponding to the edge e is given by:

$$f_e^*(w_v) = \begin{cases} 1 & \text{when } e = (v, v') \\ -1 & \text{when } e = (v', v) \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.5. When C_4 is the oriented 4-cycle we see that L_G is a codimension one subspace of $\mathbb{C}^4$ given by the linear equation: sum of coordinates equal to zero.

A magnificent connection of invariants of L_G with the chromatic polynomial χ_G was discovered by Huh [Huh12]. This was a cornerstone to solutions of several long-standing conjectures allowing applications of powerful theorems from algebraic geometry to combinatorics. We describe this connection below using a language different from [Huh12], however better suited to exhibit connections with other topics. First, the ambient space $\mathbb{C}^n$ will be identified with the space M_n^D of $n \times n$ diagonal matrices. On the projective $(n-1)$-dimensional space $\mathbb{P}(M_n^D)$ we have a rational[2] map:

$$F : \mathbb{P}(M_n^D) \dashrightarrow \mathbb{P}^{n-1}$$

given by the (nonzero) $(n-1)\times(n-1)$ minors. At this point the reader should see that this is simply the classical Cremona transformation that inverts coordinates. One could also say this is the gradient of the determinant. We restrict F to $\mathbb{P}(L_G)$ and look at the graph:

$$\Gamma_G := \overline{\{(x, y) \in \mathbb{P}(L_G) \times \mathbb{P}^{n-1} : y = F(x)\}}.$$

The dimension of Γ_G is the dimension of $\mathbb{P}(L_G)$. For connected graphs it simply equals the cardinality of the vertex set minus two[3]. The multidegree turns out to be utterly interesting!

Theorem 2.1 ([Huh12]). *The multidegree sequence of Γ_G equals the sequence of absolute values of coefficients of the reduced chromatic polynomial $\overline{\chi}_G$.*

Example 2.6. Continuing the example when C_4 is the 4-cycle we obtain the restriction of the map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^3$:

$$(x : y : z : t) \mapsto (yzt : xzt : xyt : xyz)$$

to $\mathbb{P}^2 \subset \mathbb{P}^3$ given by $x + y + z + t = 0$. The image of the restriction is the hypersurface defined by the degree three elementary symmetric polynomial. This degree three is also one entry of the multidegree; it is given by the product $[\Gamma_{C_4}][H_2]^2 = 3[pt]$ in the cohomology ring $H^*(\mathbb{P}^3 \times \mathbb{P}^3)$, where $[pt]$ is the class of a point. It also corresponds to the lowest degree term $3k$ in $\overline{\chi_{C_4}}$ in Example 2.4. The computation of the multidegree can be achieved, e.g., through the `Cremona` package in Macaulay2 [GS93].

```
R=QQ[a,b,c,d], S=QQ[x,y,z,t]
f=map(S/ideal(x+y+z+t),R,
{y*z*t,x*z*t,x*y*t,x*y*z})
loadPackage"Cremona"
projectiveDegrees f
```

The output, consistent with Example 2.4, is:

```
o4 = {1,3,3}
```

---

[2] *i.e., a map that is not defined everywhere*

[3] *Experts may recognize that this is in fact the rank of the matroid minus one.*

Statistical models. We start with the following thought experiment. We are given a coin with probability of heads p and tails $1 - p$. Say we know that p was chosen from the interval $(\frac{1}{3}, \frac{2}{3})$. Such an assumption corresponds to choosing a *statistical model*. How to find the value of p? We could start throwing the coin many times. Say, after throwing 1000 times we had 400 heads. Intuitively we estimate $p = 0.4$. A rigorous way is to compute the probability of our experiment — i.e., 400 heads — as a function of p. This is referred to as the likelihood function, which is $L(p) := p^{400}(1-p)^{600}$. The p we want to find *maximizes the likelihood function*. Indeed, it may be checked that $p = 0.4$ is the correct one. This method is called *maximum likelihood estimation*. In practice, one often maximizes the log-likelihood function $\log L(p)$. An efficient way is to check when the derivative is equal to zero. In the described case:

$$\frac{d(\log L)}{dp} = \frac{200(2 - 5p)}{p(1 - p)}$$

equals zero precisely when $p = 0.4$. For more complicated statistical models we may get several critical values, one of which is the desired maximum (assuming it is achieved). The number of complex critical points is known as the *maximum likelihood degree* (ML-degree) and is one of the basic algebraic measures of the complexity of the model [SU10].

Our main interest will be *Gaussian models*. As we will soon see the degree and the ML-degree of linear multivariate Gaussian models will be governed by the multidegree of a graph of a map given by the gradient of the determinant. The classical Gaussian distribution on $\mathbb{R}$ has density given by:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$

Here $\mu \in \mathbb{R}$ is the mean and $\sigma^2 \in \mathbb{R}_+$ is the variance. A slight generalization is a multivariate Gaussian distribution, i.e., distribution on $\mathbb{R}^n$. In particular, the mean $\mu \in \mathbb{R}^n$ is now a vector. How should we generalize the variance? We note that the exponent in $f(x)$ is actually an evaluation of a quadratic form on $x - \mu$. In the one-dimensional case we only had to choose one coefficient for the quadratic form. For $\mathbb{R}^n$ the right generalization is a positive-definite symmetric $n \times n$ matrix Σ, known as the covariance matrix. We obtain:

$$f_{\Sigma,\mu}(x) =$$
$$\det(2\pi\Sigma)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(\mathbf{x} - \mu)^\mathsf{T}\Sigma^{-1}(\mathbf{x} - \mu)\right).$$

As Σ is positive-definite the function is integrable. The coefficient $\det(2\pi\Sigma)^{-\frac{1}{2}}$ in front of the exponential is chosen so that the integral of $f_{\Sigma,\mu}(x)$ over $\mathbb{R}^n$ equals one. Thus indeed we obtain a probability distribution.

In the examples above, we were not specifying one probability distribution, but a set of those. This motivates the following definition.

Definition 2.1. A *statistical model* is a family of probability distributions.

Such a family often comes with an additional structure, e.g., as a subset of $\mathbb{R}^m$. For the general multivariate Gaussian model we would take $m = \binom{n+1}{2} + n$ and identify the probability distribution with a point $(\Sigma, \mu) \in \mathbb{R}^m$. Even when dealing with Gaussian distributions we often make further assumptions on Σ and μ. First, we note that if we have data that we believe come from a multivariate Gaussian distribution, we may estimate the mean μ by taking the mean of the data. By shifting the data we will assume from now on that $\mu = 0 \in \mathbb{R}^n$. Thus, the model is specified by determining a set $S \subset \mathbb{R}^{\binom{n+1}{2}}$ to which Σ may belong. Of particular interest for us will be the *linear concentration model* described below. The matrix $K := \Sigma^{-1}$ is called the *concentration matrix*. Let us fix a linear space L of symmetric $n \times n$ matrices that contains a positive-definite matrix. A linear concentration model is given by:

$$S_L := \{\Sigma : K = \Sigma^{-1} \in L\}.$$

Formally we still require Σ, and hence also K, to be positive definite.

We have two important invariants of S_L. One is the *degree of the model*, which is simply the degree of the variety $L^{-1} := \overline{S_L}$ that is the Zariski closure of S_L in the ambient space of $n \times n$ matrices. We note that $\overline{S_L}$ is also the Zariski or Euclidean closure of the locus of inverses of all invertible matrices in L. The other invariant is the ML-degree that we define below in analogy to the case of a coin. Suppose we are given a data vector $d_1 \in \mathbb{R}^n$. It does not make sense to ask what is the probability of observing d_1, as this is zero. Still, the value of the density function on d_1 is the correct measure of how likely it is to make such an observation. Hence, for general data $d_1, \ldots, d_k \in \mathbb{R}^n$ with mean zero the *ML-degree* is the number of (complex) critical points of the log-likelihood function:

$$\Sigma \mapsto \log\left(\prod_{i=1}^{k} f_{\Sigma,0}(d_i)\right).$$

As our probability distribution is no longer discrete, as it was in case of the coin, instead of maximizing the probability we maximize the product of values of density functions. If no condition on Σ is required, i.e., L is the whole ambient space, one can check [DSS09, pp. 43 and 44] that the optimal Σ is given by:

$$\hat{\Sigma} := \frac{1}{k}\sum_{i=1}^{k} d_i d_i^t$$

and is called *the sample covariance matrix*.

Which $\Sigma \in S_L$ maximizes the log-likelihood function when $\hat{\Sigma} \notin S_L$? There is a beautiful geometric answer to this question. This is the unique positive definite matrix $\hat{\Sigma}_L \in S_L$ such that for all $K \in L$ we have $\langle K, \hat{\Sigma}_L \rangle = \langle K, \hat{\Sigma} \rangle$, where the pairing is the trace of the product of matrices. In other words $\hat{\Sigma}_L - \hat{\Sigma} \in L^\perp$ [SU10, p. 604]. For this reason, one considers the projection $\pi : \mathbb{P}(S^2\mathbb{C}^n) \dashrightarrow \mathbb{P}(S^2\mathbb{C}^n/L^\perp)$, where S^2 is the second symmetric power, i.e., $S^2\mathbb{C}^n$ is identified with the space of symmetric $n \times n$ matrices. The ML-maximization problem then turns out to be related to finding the fiber of the generically finite map $\pi_{|\overline{S_L}}$ over $\pi(\hat{\Sigma})$. Indeed, there is a unique positive definite matrix in S_L in the fiber and it is the maximizer of the likelihood function. What about other, possibly complex, points in the fiber? The cardinality of the fiber, for general $\hat{\Sigma}$, is precisely the ML-degree [AGK$^+$21]. In particular, we see that the ML-degree is the degree of the map $\pi : \mathbb{P}(\overline{S_L}) \dashrightarrow \mathbb{P}(S^2\mathbb{C}^n/L^\perp)$. Equivalently, it is the cardinality of the intersection $|(W \cap \mathbb{P}(\overline{S_L})) \setminus \mathbb{P}(L^\perp)|$, where W is a general projective subspace of dimension equal to the codimension of $\mathbb{P}(\overline{S_L})$ that contains $\mathbb{P}(L^\perp)$. This is always upper bounded by the degree of the model and equality holds when $\mathbb{P}(L^\perp) \cap \mathbb{P}(S_L) = \emptyset$. By a theorem of Teissier [Tei82, II.2.1.3], [JKW21, Lemma 3.1], this happens when L is general, however often fails for special L.

In algebraic statistics both cases are interesting: general and special L. Of particular interest are *graphical Gaussian models*, where a graph G with n vertices, indexed by integers $1, \ldots, n$, encodes the space L^G. Precisely, L^G is the subspace of symmetric matrices that have zeros on off-diagonal entries (a, b) whenever there is no edge between a and b in G. Note that we always allow arbitrary diagonal entries, thus one can imagine loops at every vertex of G. We emphasise that this is a very different construction of a subspace L of matrices from the one described in the previous section and denoted by L_G. Before we had a subspace of diagonal matrices, while now we obtain a space of symmetric matrices that contains all diagonal matrices. We thus have the following interesting invariants:

- the degree of the model for general L of codimension a, which is the degree of the variety $\overline{S_L}$. This degree also equals the ML-degree and is denoted by $\phi(n, a)$,
- the degree of the model L^G,
- the ML-degree of L^G.

The last two of course depend on G. It turns out that even for relatively simple G, computing those invariants is highly nontrivial.

Conjecture 2.1. [DSS09, 7.4] *If G is an n-cycle then the ML-degree equals:*

$$(n-3) \cdot 2^{n-2} + 1.$$

The following theorem was proved using the methods described at the end of the article and confirms a conjecture of Sturmfels and Uhler [SU10].

Theorem 2.2. [DMV21, Theorem 1.4] *If G is an n-cycle then the degree of the model equals:*

$$\frac{n+2}{4}\binom{2n}{n} - 3 \cdot 2^{2n-3}.$$

Recall that the map $\mathbb{P}(S^2\mathbb{C}^n) \dashrightarrow \mathbb{P}(S^2\mathbb{C}^n)$ that is given by inverting the matrix, is also the gradient of the determinant, i.e., the map given by all partial derivatives of the determinant. We have the following commutative diagram [DMS21, p.6]:

$$
\begin{array}{ccc}
\mathbb{P}(S^2\mathbb{C}^n) & \xdashrightarrow{\nabla\det} & \mathbb{P}(S^2\mathbb{C}^n) \\
\uparrow {\scriptstyle(\nabla\det)_{|\mathbb{P}(L)}} \nearrow & & \downarrow \pi \\
\mathbb{P}(L) & \xdashrightarrow{\nabla(\det_{|\mathbb{P}(L)})} & \mathbb{P}(L^*) = \mathbb{P}(S^2\mathbb{C}^n/L^\perp)
\end{array}
\tag{1}
$$

The degree of the model is the last integer in the multidegree of the graph of the diagonal map $(\nabla\det)_{|\mathbb{P}(L)}$, while the ML-degree is the last integer in the multidegree of graph of the lower map $\nabla(\det_{|\mathbb{P}(L)})$.

We would like to point out that there are other interesting questions regarding the geometry of the model, like the generators of the ideal of S_{L^G}, however this lies outside the scope of this article.

Euler characteristic. The Euler characteristic $\chi(X)$ is one of the most important discrete invariants of a topological space X. Assuming we may triangulate the space, we could define it as the number of vertices in the triangulation minus the number of edges plus the number of triangles etc. For complex algebraic constructible sets Euler characteristic is additive:

$$\chi(X) = \chi(X \setminus Y) + \chi(Y). \tag{2}$$

One may thus often compute it by breaking an algebraic set into pieces that are easier to understand. Another useful trick is to equip a projective variety X with an action of an algebraic torus $T = (\mathbb{C}^*)^k$. Then $\chi(X) = \chi(X^T)$, where X^T is the locus of torus fixed points. In particular, if there are finitely many T-fixed points, then the Euler characteristic equals the number of those points. For $X = \mathbb{P}^n$ with the standard multiplicative action of the torus consisting of points with all coordinates nonzero, we get $\chi(\mathbb{P}^n) = n+1$.

Consider a projective hypersurface $X = V(f) \subset \mathbb{P}^n$. We will be mostly interested in the case when $\mathbb{P}^n$ is a subspace of the space of matrices and f is the restriction of the determinant. The bottom row in Diagram (1) is thus a special case of the map:

$$\mathbb{P}^n \xdashrightarrow{\nabla f} \mathbb{P}^n.$$

Huh [Huh12, p.912], following [DP03], observed a beautiful relation among the Euler characteristic and the multidegrees μ_i of the graph of ∇f:

$$\chi(\mathbb{P}^n \setminus X) = \sum_i (-1)^i \mu_i. \tag{3}$$

A well-known special case is when X is smooth. Then ∇f is defined everywhere. To compute μ_i we may use the isomorphism of the domain $\mathbb{P}^n$ with the graph. We then have to intersect i general hyperplanes, with $n-i$ polynomials, that are general linear combinations of partial derivatives of f. These partial derivatives do not have common zeros, thus by a theorem due to Bertini the intersection is smooth, and if $\deg f = d$, we have $\mu_i = (d-1)^{n-i}$ points, which is the expected Bézout bound. By (2) and (3), we obtain:

$$\chi(X) = \chi(\mathbb{P}^n) - \chi(\mathbb{P}^n \setminus X)$$
$$= (n+1) - \frac{1 - (1-d)^{n+1}}{d}.$$

When X has isolated singularities, then all but one μ_i's remain the same as in the smooth case. Namely as long as we intersect with at least one hyperplane in the domain, we will not see the singularities and we may still apply Bertini theorem obtaining the Bézout bound. However, for μ_n, the derivatives of f intersect in μ_n many simple points and in the singular points of X. Thus, from $(d-1)^n$ we have to subtract the sum of multiplicities of the isolated singularities of X. These multiplicities are known as *Milnor numbers* and in our case are nothing else than the dimension, as a vector space, of the algebra of the singular locus. We emphasise that Equation (3) works for arbitrary singularities.

This theory may be applied to answer Question 3 from the introduction. Indeed, a matrix with linear entries, may be identified with a space L of matrices. We may compute $\chi(\mathbb{P}(L) \cap V(\det))$ from the multidegree of the graph of the map $\mathbb{P}(L) \dashrightarrow \mathbb{P}(L^*)$ given by the gradient of the determinant composed with the projection from $L^\perp$.

Example 2.7. Consider the space L of 4×4 diagonal, traceless matrices, which we obtained from the 4-cycle. We identify $\mathbb{P}(L) = \mathbb{P}^2 \subset \mathbb{P}^3$. On $\mathbb{P}^3$ the determinant is the product of the four linear forms corresponding to the coordinates. Thus, the hypersurface $X = V(\det)$ is the hyperplane arrangement given by four planes. When restricted to $\mathbb{P}(L)$ we obtain four lines, which pairwise intersect. The six intersection points correspond to the $6 = \binom{4}{2}$, unique up to scalar multiplication, rank two, diagonal, traceless matrices. Denoting a point by pt, we have:

$$\chi(\mathbb{P}(L) \setminus X) = \chi(\mathbb{P}^2) - 4 \cdot \chi(\mathbb{P}^1) + 6 \cdot \chi(pt)$$
$$= 3 - 4 \cdot 2 + 6 = 1.$$

This, as expected, coincides with the signed sum of multidegrees, computed in Examples 2.4 and 2.6:

$$1 - 3 + 3 = 1.$$

Example 2.8. Let L be the space of 2×2 matrices. We have $\mathbb{P}(L) = \mathbb{P}^3$ and the gradient of the determinant is a linear isomorphism. In particular, all μ_i are equal to one and their signed sum equals 0. The determinantal hypersurface X is the quadric from Example 2.1. We obtain:

$$0 = \chi(\mathbb{P}^3 \setminus X) = 4 - \chi(X).$$

Thus, $\chi(X) = 4$. Indeed, readers with experience in algebraic geometry may recognize that X is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, via the Segre embedding.

Quadric hypersurfaces. Every homogeneous degree two polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ is uniquely represented by a symmetric $n \times n$ matrix M_f:

$$f(x) = x^t M_f x.$$

The matrix M_f has on the diagonal coefficients of x_i^2 in f and on (i, j)-th off-diagonal entry, half of the coefficient of $x_i x_j$. Clearly the properties of f and the variety $V(f)$ are related to the properties of M_f. We first learned this relation in school, where, for $n = 2$ and $f = ax^2 + bxy + cy^2$ we compute $\Delta = b^2 - 4ac$, which is nothing else than $4 \det M_f$. We know that $f = 0$ has two distinct complex solutions if and only if $\Delta \neq 0$, i.e., M_f has rank two.

From now on we identify the space of degree two homogeneous polynomials in n variables with $S^2\mathbb{C}^n$. We will always assume $f \neq 0$. In general, the quadric $V(f)$ is smooth[4] if and only if $\det M_f \neq 0$ and f is a square of a linear form if and only if rank of M_f equals one. Given a smooth quadric $V(f)$ we may consider all hyperplanes $H \subset \mathbb{P}^{n-1}$ that are tangent to $V(f)$ at some point. Each such H corresponds to a point $P_H \in (\mathbb{P}^{n-1})^*$ in the dual projective space. It turns out that the locus of all P_H for which H is tangent to $V(f)$, is also a quadratic hypersurface known as the *dual quadric*. The matrix associated to the dual quadric is, up to scaling, M_f^{-1}.

Our aim is to head towards Question 4 from the introduction. First, fix a point $P_1 \in \mathbb{P}^{n-1}$. Which quadrics pass through P_1? Note that $f(P_1) = 0$ is a *linear* equation in the coefficients of f. Thus we obtain a hyperplane $H_{P_1} \subset \mathbb{P}(S^2\mathbb{C}^n)$ of polynomials f such that $P_1 \in V(f)$. When we have more points we simply have to intersect $H_{P_1} \cap \cdots \cap H_{P_k}$ to obtain the locus of quadrics that pass through $P_1, \dots, P_k$. If P_i's are general we see that:

- as long as $k < \dim \mathbb{P}(S^2\mathbb{C}^n) = \binom{n+1}{2} - 1$ we have infinitely many quadrics,
- if $k = \binom{n+1}{2} - 1$ there is precisely one quadric, up to scaling,
- if $k > \binom{n+1}{2} - 1$ there are no quadrics

[4]*Here, we consider $V(f)$ as a scheme, thus if f is not reduced, i.e., a square of a linear form, we say it is not smooth.*

that pass through all P_i's. This is slightly less obvious than one may think, as general P_i's do *not* give general hyperplanes H_{P_i}. The formal proof could for example rely on the fact that H_P's do not have base locus, i.e., $\bigcap_{P \in \mathbb{P}^n} H_P = \emptyset$, which is equivalent to the fact that no quadric passes through all points, plus simple linear algebra. Just to point out what may go wrong we present the following example.

Example 2.9. We would like to answer the question: how may, up to scaling, homogeneous degree two polynomials $f = ax^2 + bxy + cy^2$

- have a double root and
- vanish at a given point $P \in \mathbb{P}^1$?

We consider the $\mathbb{P}^2$ of all polynomials. In this $\mathbb{P}^2$ the locus of polynomials that have a double root is the quadratic hypersurface $\Delta = 0$. We intersect this hypersurface with the line H_P. We expect two solutions. This is clearly wrong! If we fix a root P and require that a degree two polynomial has a double root, then P must be the double root. Every child knows that there is only one, up to scaling, such a polynomial. The geometry here is that the line H_P will be always tangent to $V(\Delta)$ and thus it will intersect it in a single point.

How about the locus of quadrics that are tangent to a given hyperplane $H \subset \mathbb{P}^{n-1}$? We already know that a smooth quadric represented by M_f is tangent to H if and only if the dual quadric represented by M_f^{-1} passes through the point $P_H \in (\mathbb{P}^{n-1})^*$. As we are working up to scaling we may replace M_f^{-1} by the adjugate matrix M_f^{adj}. The entries of M_f^{adj} are degree $(n-1)$ polynomials in the original coordinates, i.e., the entries of M_f. The condition $(P_H)^t M_f^{adj}(P_H) = 0$ is thus a degree $(n-1)$ polynomial defining a hypersurface $T_H \subset \mathbb{P}(S^2\mathbb{C}^n)$. Very explicitly this hypersurface is a linear combination of $(n-1) \times (n-1)$ minors of M_f. It thus makes sense to ask: how many quadrics are tangent to a given hyperplane and pass through $\binom{n+1}{2} - 2$ fixed general points? Geometrically we intersect T_H with hyperplanes and get $\deg T_H = n - 1$ many points[5].

We could be tempted to continue this game. If we consider a general points and $\binom{n+1}{2} - a - 1$ general hyperplanes, how many quadrics pass through the given points and are tangent to the given hyperplanes? Geometrically we intersect a hyperplanes and $\binom{n+1}{2} - a - 1$ hypersurfaces of degree $(n-1)$. Thus we expect that the answer is equal to the Bézout bound: $(n-1)^{\binom{n+1}{2} - a - 1}$. This, however, is wrong in general!

[5]*For experts: as our choices are not entirely generic, we should refer here to Kleiman's transversality theorem, assuring that we indeed obtain $(n-1)$ many points corresponding to smooth quadrics.*

The easiest way to see this is to consider $n = 3$ and five general lines in $\mathbb{P}^2$. If we ask for quadrics that are tangent to all five, we may equivalently ask for dual quadrics passing through five points. But here we already know the answer is one, not 32. What goes wrong? We recall that our hypersurfaces T_H are defined by linear combinations of minors of size $(n - 1)$. In particular, each T_H contains all matrices of rank at most $(n - 2)$. This is a very different situation than for the hyperplanes H_P, which had no base locus. The codimension of the locus of symmetric $n \times n$ matrices of rank $(n-2)$ equals three. Thus, if we take $b \geq 3$ general hyperplanes and intersect the corresponding hypersurfaces T_H we will get:

- a big codimension three component of matrices of rank at most $(n - 2)$ and
- a small codimension b component.

The meaningful geometric counting problem is to ask for *smooth* quadrics that pass through the given points and are tangent to the given hyperplanes. Thus, we would like to intersect the small codimension b component with hyperplanes H_{P_i} and count the number of points. However, the contribution of the big component makes such computations quite hard.

Let us state the enumerative problem in a way more suitable for this article. By fixing points $P_1, \dots, P_a$, we fix a linear space $L \subset \mathbb{P}(S^2\mathbb{C}^n)$ of quadrics that pass through those points. We may now consider the rational map:

$$\mathbb{P}(S^2\mathbb{C}^n) \dashrightarrow \mathbb{P}(S^2\mathbb{C}^n)^*$$

that is the gradient of the determinant, or equivalently, taking adjugate or inverse of a matrix[6]. We restrict the map to $\mathbb{P}(L)$, obtaining the diagonal map in Diagram (1). Each tangency to a hyperplane condition is in fact intersection with a hyperplane in the dual space $\mathbb{P}(S^2\mathbb{C}^n)^*$. We thus see that the number of quadrics that pass through the given points and are tangent to the correct number of general hyperplanes is in fact one entry in the multidegree of the graph of $(\nabla \det)_{|\mathbb{P}(L)}$. Note that here the problem of base locus and low rank matrices disappears. Indeed in the product of projective spaces both: hyperplanes in the domain and hyperplanes in the codomain do not have the base locus. Also as the graph is by definition the closure of the locus corresponding to full rank matrices, we do not have the additional large component. Note that for general points $P_1, \dots, P_a$ we get:

the number of quadrics that pass through all P_i's and are tangent to general $\binom{n+1}{2} - a - 1$ hyperplanes equals precisely the degree $\phi(n, a)$ of the general linear concentration model.

In many cases in enumerative problems the way to obtain the correct answer in a nice way is to change the

ambient space where intersection is performed. This usually requires very good, clever ideas. Passing to the graph of the map, instead of $\mathbb{P}^n$ or $\mathbb{P}(L)$ seems to be one. However, it is only the first step, as such graph is in general not smooth. Fortunately, in the cases interesting to us, great mathematicians before us already had the right ideas. We may present these in the next subsection.

Complete varieties. In a quite non-standard way, we do *not* refer to complete varieties, as a synonym for proper. The complete varieties as described below are very special projective varieties, that may be regarded as particularly nice compactifications of the locus of nondegenerate matrices.

Let W be the space of diagonal or symmetric or general $n \times n$ matrices. Let W° be the subset of full rank matrices. We could see $\mathbb{P}(W^\circ) \subset \mathbb{P}(W)$ as a natural compactification. However, as we have seen in the previous section the low rank matrices in $\mathbb{P}(W)$ often turn out to be problematic. There exists a well-known procedure in algebraic geometry of replacing a small set by a divisor, i.e., a codimension one set, known as the *blow-up*. Say we have a subset $S \subset \mathbb{P}^n$, where $S = V(g_0, \dots, g_s)$ and all g_i's are polynomials of fixed degree d. For our purposes, we define the blow-up of $\mathbb{P}^n$ at S as the graph of the rational map given by g_i's:

$$\mathbb{P}^n \dashrightarrow \mathbb{P}^s.$$

This procedure is particularly nice when S is smooth. In our case $\mathbb{P}^n = \mathbb{P}(W)$ and S is the set of rank at most $n - 2$ matrices. Thus, the graph we considered, is the blow-up construction, where g_i's are the minors. However, the set S is not smooth, it is singular along matrices of rank at most $n - 3$, which further is a set singular along matrices of rank at most $n - 4$ etc. until rank one matrices, which is a smooth locus in $\mathbb{P}(W)$.

The idea is to first blow-up rank one matrices, then rank two matrices, etc. until finally we blow-up rank $n - 2$ matrices. This may be realized at once as follows. Consider the rational map:

$$\psi : \mathbb{P}(W) \dashrightarrow \mathbb{P}(W_2) \times \cdots \times \mathbb{P}(W_{n-1})$$

where the map $\mathbb{P}(W) \dashrightarrow \mathbb{P}(W_i)$ is given by $i \times i$ minors. The dimension of $\mathbb{P}(W_i)$ depends on the case we are in, e.g., for diagonal matrices $\dim W_i = \binom{n}{i}$, while for general matrices $\dim W_i = \binom{n}{i}^2$, however otherwise the construction remains the same. The (closure of the) graph of ψ is known as:

- permutohedral variety in the diagonal case,
- variety of complete quadrics in the symmetric case,
- variety of complete collineations in the case of general square matrices[7].

It is equal to the iterative blow-up of rank i matrices for $i = 1, \dots, n - 2$ anticipated above. This variety is smooth. By projecting to $\mathbb{P}(W) \times \mathbb{P}(W_{n-1})$ it becomes a resolution of singularities of the graph of the map inverting the matrix. In this article, to emphasise the common features of the above varieties, we refer to all of them as *complete varieties*.

As an example let us show how one may think about points of the variety X of complete quadrics. First, there is an open locus, corresponding to quadrics of rank at least $(n - 1)$, which is isomorphic to an open subset of $\mathbb{P}(W)$, via the projection $X \to \mathbb{P}(W)$. However, there are many points $x \in X$ mapping to a given quadric $Q \in \mathbb{P}(W)$ if the rank of Q is at most $n - 2$. Fix such a quadric $Q \in \mathbb{P}(W)$. Let $V \subset \mathbb{C}^n$ be the image of the symmetric matrix A_Q. It turns out that the fiber over Q is the variety of complete quadrics over the vector space $\mathbb{C}^n/V$. We see that the points of the variety of complete quadrics may be identified with the data consisting of:

- flags $V_1 \subset \cdots \subset V_k = \mathbb{C}^n$ and
- full rank quadrics on each V_{i+1}/V_i, i.e., elements of $\mathbb{P}(S^2(V_{i+1}/V_i)^\circ)$.

A particularly interesting case is when the flag is full, as then the data of quadrics is trivial — there is only one up to scaling nonzero, degree two homogeneous polynomial in one variable. Thus we see that the variety of complete quadrics naturally contains the full flag variety F. For experts: the inclusion $F \to X$ induces the inclusion of Picard groups $\mathrm{Pic}(X) \to \mathrm{Pic}(F)$. The right-hand side is understood in terms of the root system of type A. This allows us to have a good understanding of $\mathrm{Pic}(X)$.

3. Intersection Theory

Classical algebraic intersection theory associates to an algebraic variety X, the graded Chow ring $\mathrm{CH}(X)$. We have already seen this on the example of $X = \mathbb{P}^n$. In general, the degree k part of $\mathrm{CH}(X)$ is spanned by classes of codimension k subvarieties, modulo rational equivalence. For example for $k = 1$ we obtain the divisor class group, which for smooth X coincides with the Picard group $\mathrm{Pic}(X)$. The multiplication in $\mathrm{CH}(X)$ corresponds to intersection. Precisely, if $Y, Z \subset X$ intersect transversally, then $[Y] \cdot [Z] = [Y \cap Z]$. For all the varieties that we consider in this article the Chow ring is isomorphic to the cohomology ring. We refer the reader interested in intersection theory to the book [EH16].

Volume polynomials. Given a k-dimensional subvariety $Y \subset X$ we may define a polynomial function on the Picard group $\mathrm{Pic}(X)$:

$$\mathrm{Pic}(X) \ni D \mapsto \deg([Y][D]^k) \in \mathbb{Z},$$

where deg is the degree function that to a zero-dimensional (class of a) scheme associates its degree. If we prefer to work in coordinates and obtain a polynomial we fix

divisors $D_1, \dots, D_n \in \mathrm{Pic}(X)$. We define the *volume polynomial* of Y on $\mathbb{Q}^n$ by:

$$(t_1, \dots, t_n) \mapsto \deg\left((\textstyle\sum t_i[D_i])^k[Y]\right).$$

A divisor D is called nef (numerically effective) if for every curve $C \subset X$ the number $\deg([C] \cdot [D])$ is nonnegative. A particularly nice case is that of a volume polynomial when $D_1, \dots, D_n$ are nef. Then the volume polynomial is *Lorentzian* [BH20, Theorem 4.6], exhibiting a lot of nice properties.

In case of complete varieties there is a distinguished set of nef divisors. By definition, the complete variety is a subvariety of the product $\mathbb{P}(W) \times \mathbb{P}(W_2) \times \cdots \times \mathbb{P}(W_{n-1})$, the hyperplane $H_i \subset \mathbb{P}(W_i)$ (times the remaining $\mathbb{P}(W_j)$'s) gives a nef divisor L_i on X.

For example, when X is the variety of complete quadrics, the L_i's are the extremal rays generating the nef cone — all nef divisors are nonnegative linear combinations of L_i's. Via the containment of $\mathrm{Pic}(X)$ in the root system of type A the L_i's correspond to (twice) the fundamental roots. For the permutohedral variety the L_i's do not generate the nef cone, however they generate the S_n (i.e., permutation) invariant part.

To sum up: consider an a-dimensional linear space L of $n \times n$ matrices. Assume L contains an invertible matrix. The closure of invertible matrices in L in a complete variety X (which depends on the type of matrices L consists of) gives a subvariety Y_L. We obtain *the Lorentzian polynomial* given by $\deg(\sum t_i L_i)^{a-1}[Y_L]$. In [CM21] the coefficients of this polynomial were introduced and called the *characteristic numbers*.

By setting $t_2 = \cdots = t_{n-1} = 0$ we recover the *chromatic polynomial* of a tensor, defined also in [CM21]. Its coefficients (up to binomial factors) are precisely the multidegrees of the graph of the map inverting matrices[8] from L.

In a very related construction, replacing the restriction to L of the gradient of the determinant, by the gradient of the restriction to L, we obtain the *relative chromatic polynomial*. These two coincide, e.g., when L consists of diagonal matrices, or when L is general. These two invariants seem very important and, in particular, provide answers to all the questions from the introduction.

Methods in a nutshell. Here, we would like to briefly mention a few methods that rely on complete varieties and allow to compute the intersection numbers. For more details we refer to [MMM+20, MMW21, DMS21]. For simplicity, consider the variety X of compete quadrics. Say we want to compute $[L_1]^a[L_{n-1}]^{\binom{n+1}{2}-1-a}$. The first trick is due to Schubert. One notices that on X we have another set of divisors. Indeed, let S_i be the exceptional divisor coming from the blow-up of rank i matrices. It turns out that S_i

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[8]*Equivalently gradient of the determinant*

correspond to (twice) the simple positive roots. We thus have relations in Pic(X) that allow us to translate from L_i's to S_i's. We obtain:

$$[L_1] = \frac{1}{n} \sum_{j=1}^{n-1} (n-j)[S_j].$$

This reduces the problem to computing all $[S_j][L_1]^{a-1}[L_{n-1}]^{\binom{n+1}{2}-1-a}$. At first this looks like a more complicated problem. The trick is to realize that one can replace $[S_j]$ by a simpler variety. First, we have a rational map $f_j : S_j \dashrightarrow G(j,n)$, from S_j to the Grassmannian parameterizing j-dimensional subspaces of an n-dimensional vector space. To a point $s \in S_j$ mapping to a rank j symmetric matrix M_s the map f_j associates the image of M_s. Fixing a point $V \in G(j,n)$ we see that the fiber of f_j is birational with: $\mathbb{P}(S^2 V)$ for the choice of M_s times $\mathbb{P}(S^2(\mathbb{C}^n/V)^*)$, as after fixing M_s we said that the fiber of X over M_s is isomorphic to the variety of complete quadrics on $\mathbb{C}^n/V$. Hence, S_j is birational to the product bundle $\mathbb{P}(S^2\mathcal{U}) \times \mathbb{P}(S^2\mathcal{Q}^*)$ over $G(j,n)$, where $\mathcal{U}$ and $\mathcal{Q}$ are respectively the universal and quotient bundle. This birational morphism is good enough to switch from intersecting with $[S_j]$ on X to intersecting on the product bundle. The classes $[L_1]$ and $[L_{n-1}]$ also translate nicely and may be expressed in terms of characteristic classes of $S^2\mathcal{U}$ and $S^2\mathcal{Q}$. These have been investigated in detail, cf. [Pra88] and references therein.

4. Future

The unified perspective on the presented problems provides not only new tools, but may also be used to try to extend theories from one field to another. For example, when L is general, in the diagonal case, the associated matroid is uniform and characteristic numbers are simply binomial coefficients $\binom{n}{a}$. For fixed a, this is a polynomial in n with very nice reciprocity properties. In the symmetric case we obtain $\phi(n,a)$, which is also a polynomial in n. However, here the reciprocity results are much less understood, with first, very recent results presented in [Gał22]. For positive n and a the function $\phi(n,a)$ counts the number of quadrics satisfying given conditions. Is there a nice interpretation for fixed positive a and negative n?

Another interesting perspective is to consider the given constructions as invariants of tensors. Indeed, each 3-way tensor naturally gives a space of matrices. What does the associated Lorentzian polynomial tell us about the tensor? When do two tensors have the same Lorentzian polynomial? Can we obtain more invariants coming from algebraic geometry from the variety L^{-1}? The study of tensors from this perspective has recently been initiated in [CK22].

The *Bodensee program* [DMS21] studies how discrete invariants, like $\phi(n,a)$, change, as the variety, here n, changes.

We already mentioned the polynomiality result of $\phi(n,a)$, however the coefficients of this polynomial remain quite mysterious. Their absolute values seem to be log-concave. At present, we do not know the theory that could explain such phenomena.

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