Steklov Spectrum and Elliptic Problems with Nonlinear Boundary Conditions

Nsoki Mamie Mavinga

Problems with nonlinear boundary conditions arise naturally in many applications. For instance, in population dynamics where an impact of habitat-edges (boundary) on the dispersal pattern of species as they reach the boundary takes place in spatial ecology [CC06]. They occur when the biochemical reactions take place at or near the boundary, for example, in the limb bud development of a chick in which a chemical reaction produces outgrowth due to cell growth and division, and interactions between morphogens produced in several zones of the limb bud [DO99]. They also appear in noninvasive testing methods to locate defects in a medium by using boundary data measurements (see, e.g., [CCMM16]). In cryosurgery (a minimally invasive treatment used to treat some types of cancers and some conditions that may become cancer), a highly exothermic reaction takes place in a thin layer around the boundary in order to destroy abnormal tissue [LOS98]. These examples are not exhaustive.

Diffusion-type equations play a crucial role in these problems, and associated steady state problem and eigenvalue problems are critical in understanding the dynamics of diffusion-type equations. Hence, the qualitative (analytical) study of such equations is essential for better understanding and modeling nonlinear processes. The investigation of problems with nonlinear boundary conditions has therefore attracted a lot of attention in recent years; see for instance [Ama76, MN10, Mav12, MP17, LS18, CL15] and references therein.

In this article, we first introduce the spectral problem for elliptic equations with spectral-parameter dependent boundary conditions. We then discuss some recent results on the solvability of nonlinear diffusion problems when the nonlinearity on the boundary interacts in some sense with the spectrum, especially the effect of the first eigenvalue. We will present some of the results without proofs. References will be mentioned as appropriate.

In the following sections, we will first consider the linear Steklov problem in which the spectral parameter is in the boundary condition. Then, we discuss in-depth the properties of the first eigenvalue as well as briefly consider the one-dimensional case. In the last section, we take up the case of nonlinear perturbations of the linear Steklov problem, and set up the problem as a nonlinear first-order boundary-flux equation with a second-order elliptic partial differential equation "constraint" inside the domain. Considering asymptotic conditions on the boundary nonlinearity, we present existence, bifurcation and multiplicity results. A sketch of the bifurcation diagram is also provided.

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Steklov Spectrum and its Properties

In the seminal paper entitled "Sur les problèmes fondamentaux de la physique mathématique (suite et fin)," published in 1902 in the Annales Scientifiques de l'École Normale Supérieure [Ste02], W. Steklov considered the (spectral) problem of finding a harmonic function v inside a convex bounded region Ω in the plane $\mathbb{R}^2$ with smooth boundary surface $S = \partial\Omega$ which satisfies the boundary condition

$$\frac{\partial v}{\partial n} = \lambda \phi v \quad \text{on } S,$$

where $\partial/\partial n$ denotes the directional derivative in the direction of the (unit) outward normal vector to the boundary S, $\lambda \in \mathbb{R}$ is a spectral parameter, and ϕ is a given smooth positive (weight) function. The convexity condition on the region Ω and the smoothness of the surface S were relaxed by H. Poincaré, and the positive weight function ϕ was introduced by E. Le Roy. Soon after that S. Zaremba considered the more general (spectral) problem with a lower-order term

$$\begin{aligned} \Delta v + \xi v &= 0 \qquad \text{in } \Omega, \\ \frac{\partial v}{\partial n} &= \lambda \phi v \quad \text{on } S, \end{aligned}$$

where Δ is the Laplace operator and ξ is a (fixed) constant.

Later on in [Pay67], Payne presented a physical problem that describes the vibration of an elastic membrane with its whole mass uniformly distributed on the boundary with density ϕ leading to the problem

$$\begin{aligned} \Delta v &= 0 \qquad \text{in } \Omega, \\ \frac{\partial v}{\partial n} &= \lambda \phi v \quad \text{on } S, \end{aligned} \tag{1}$$

where Ω denotes an n-dimensional body with boundary $S = \partial\Omega$.

There have been many results and generalizations of the Steklov problem. We mention the book by Bandle [Ban80] and the papers by Auchmuty [Auc04] and Mavinga [Mav12] for higher dimensions with mild regularity conditions on the data. We refer also to Amann [Ama76] who discussed the existence of the first eigenvalue of the spectrum for this problem under somewhat strong regularity conditions on the data. Although a more general linear operator with lower order terms was considered in [Ama76], the techniques there used the theory of positive operators (Krein–Rutman theorem); which of course does not apply when trying to obtain higher eigenvalues. The arguments in [Auc04], which yielded higher eigenvalues as well, involved maximization of the boundary functional on bounded closed convex subsets of the Sobolev space $H^1(\Omega)$.

In this article, we present an approach used in [Mav12] where the minimization of the (differential) functional on an appropriate subspace of $H^1(\Omega)$ is used.

Steklov eigenproblem. Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with smooth boundary $\partial\Omega$. Consider the second-order elliptic equation

$$\begin{aligned} -\Delta v + c(x)v &= 0 \qquad \text{in } \Omega, \\ \frac{\partial v}{\partial n} &= \mu \sigma v \quad \text{on } \partial\Omega, \end{aligned} \tag{2}$$

where the (given) functions $c : \Omega \to \mathbb{R}$ and the weight $\sigma : \partial\Omega \to \mathbb{R}$ satisfy the following conditions.

(C) $c \in L^\infty(\Omega)$ and $\sigma \in L^\infty(\partial\Omega)$ are nonnegative functions such that $\int_\Omega c(x)\,dx > 0$ and $\int_{\partial\Omega} \sigma\, dS > 0$. Here, L^∞ denotes the real Lebesgue space of bounded functions.

Throughout this article, $H^1(\Omega)$ denotes the usual real Sobolev space of functions on Ω; which is a Hilbert space endowed with the c-inner product defined by

$$(u, v)_c = \int_\Omega \nabla u \nabla v + \int_\Omega c(x) uv, \tag{3}$$

with the associated norm denoted by $\|v\|_c$. This norm is equivalent to the standard $H^1(\Omega)$-norm.

Besides the Sobolev spaces, we make use, in what follows, of the real Lebesgue space $L^2(\partial\Omega)$ of square integrable functions, and the compactness of the trace operator $\Gamma : H^1(\Omega) \to L^2(\partial\Omega)$ (see, e.g., [Bre11] and references therein). Sometimes we will just use u in place of Γu when considering the trace of a function on $\partial\Omega$. We will denote the $L^2(\partial\Omega)$-inner product by $(u, v)_\partial = \int_{\partial\Omega} uv$ and the associated norm by $\|u\|_\partial$. We also set

$$(u, v)_\sigma = \int_{\partial\Omega} \sigma(x) uv \text{ and } \|u\|_\sigma^2 = \int_{\partial\Omega} \sigma(x) u^2, \tag{4}$$

for $u, v \in H^1(\Omega)$. Observe that because of condition (**C**), it can happen that $\sigma = 0$ in a subset of positive measure (on the boundary $\partial\Omega$) where $u \neq 0$. In this case $\|.\|_\sigma$ is only a semi-norm. Set $V_\sigma(\Omega) := \{u \in H^1(\Omega) : \|u\|_\sigma = 0\}$. It is readily seen that $V_\sigma(\Omega)$ is a closed linear subspace of $H^1(\Omega)$. Observe that if $\sigma(x) > 0$ on $\partial\Omega$, then the subspace $V_\sigma(\Omega)$ reduces simply to $H_0^1(\Omega)$. Let us denote the c-orthogonal complement of $V_\sigma(\Omega)$ by $H_\sigma(\Omega) = [V_\sigma(\Omega)]^\perp$. Therefore, one can split

$$H^1(\Omega) = V_\sigma(\Omega) \oplus_c H_\sigma(\Omega) \tag{5}$$

as a direct orthogonal sum (in the sense of H^1-c-inner product); that is, every $u \in H^1(\Omega)$ can be written in a unique way in the form $u = u_1 + u_2$, where $u_1 \in V_\sigma(\Omega)$ and $u_2 \in H_\sigma(\Omega)$ with $(u_1, u_2)_c = 0$.

Definition. The *Steklov eigenproblem* (in its variational form) is to find a pair $(\mu, \varphi) \in \mathbb{R} \times H^1(\Omega)$ with $\varphi \not\equiv 0$ such that

$$\int_\Omega \nabla\varphi \nabla\psi + \int_\Omega c(x)\varphi\psi = \mu \int_{\partial\Omega} \sigma(x)\varphi\psi$$

for every $\psi \in H^1(\Omega)$. The real number μ is called an *eigenvalue* of (2) and the function φ is said to be an *eigenfunction* associated to the eigenvalue μ.

Now, choosing $\psi = \varphi$, one sees immediately that if there is such an eigenpair, then $\mu > 0$ and $\int_{\partial\Omega} \sigma\varphi^2 > 0$. Otherwise, φ would be a constant function; which would contradict the assumption (**C**) imposed on $c(x)$. Therefore $\varphi \perp V_\sigma(\Omega)$ in the H^1-c-inner product defined in (3) above. Notice also that, if $u \in V_\sigma$ with $u \neq 0$, then $\|u\|_c > 0$ and the quotient $\|u\|_c/\|u\|_\sigma := \infty$.

The set Σ of $\mu \in \mathbb{R}$ such that (2) has a nontrivial solution, is called the *Steklov spectrum*.

In what follows, we present some results on the properties of the Steklov spectrum. Namely that Σ forms a countably infinite set $\{\mu_k : k \in \mathbb{N}\} \subset \mathbb{R}^+$ without finite accumulation point. Thus, its elements can be arranged in an increasing sequence. We omit the details and refer to [Mav12] for the proof.

Variational characterization of the Steklov spectrum. Assume that the condition (**C**) holds, then

(i) The Steklov eigenproblem (2) has a sequence of real eigenvalues

$$0 < \mu_1 < \mu_2 \leq \ldots \leq \mu_j \leq \ldots \to \infty, \text{ as } j \to \infty,$$

and these eigenvalues satisfy the variational characterizations

$$\mu_1 = \inf_{\substack{u \in H^1 \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^2 + \int_\Omega c(x)u^2}{\int_{\partial\Omega} \sigma(x)u^2} \tag{6}$$

and for $j = 1, 2, \ldots$

$$\mu_{j+1} = \inf_{\substack{u \in W_j \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^2 + \int_\Omega c(x)u^2}{\int_{\partial\Omega} \sigma(x)u^2}, \tag{7}$$

where $W_j = \{u \in H_\sigma(\Omega) : (u, \varphi_i)_\sigma = 0 \text{ for } i = 1, \ldots, j\}$ and φ_i are the eigenfunctions corresponding to μ_j. (Hence, each eigenvalue has a finite-dimensional eigenspace.)

(ii) The normalized eigenfunctions provide a complete c-orthonormal basis of $H_\sigma(\Omega)$. Moreover, each function $u \in H_\sigma(\Omega)$ has a unique representation of the form

$$u = \sum_{j=1}^{\infty} c_j \varphi_j \text{ with } c_j := \frac{1}{\mu_j}(u, \varphi_j)_c = (u, \varphi_j)_\sigma$$

$$\text{and } \|u\|_c^2 = \sum_{j=1}^{\infty} \mu_j |c_j|^2.$$

In addition

$$\|u\|_\sigma^2 = \sum_{j=1}^{\infty} |c_j|^2.$$

Observe that the variational characterization (6) gives the trace inequality

$$\mu_1 \int_{\partial\Omega} \sigma(x)u^2 \leq \int_\Omega |\nabla u|^2 + \int_\Omega c(x)u^2 \tag{8}$$

for all $u \in H^1(\Omega)$. Moreover if equality holds in (8), then u is a multiple of an eigenfunction of Eq.(2) corresponding to μ_1.

On the other hand, for every $v \in \oplus_{i \leq j} E(\mu_i)$ and $w \in \oplus_{i \geq j+1} E(\mu_i)$ we have that

$$\|v\|_c^2 \leq \mu_j \|v\|_\sigma^2 \quad \textit{and} \quad \|w\|_c^2 \geq \mu_{j+1} \|w\|_\sigma^2, \tag{9}$$

where $E(\mu_i)$ denotes the μ_i-eigenspace and $\oplus_{i \leq j} E(\mu_i)$ is the span of eigenfunctions associated to eigenvalues up to μ_j.

Hence, this gives a splitting of the space $H_\sigma(\Omega)$ (and hence of $H^1(\Omega)$).

Remark. Note that if $c \equiv 0$ (i.e., the harmonic equation case) then $\mu = 0$ is the first eigenvalue of (2) with eigenfunction $\varphi \equiv 1$ on $\overline{\Omega}$. Let us mention here that the eigenvalues and eigenfunctions of the harmonic operator are used in fluid mechanics, heat transmission, electromagnetism, and material design (see, e.g., [Lip98, CCMM16]). They play an important role in the study of isoperimetric inequalities (see, e.g., [Pay67]).

Properties of the first eigenvalue. The first eigenvalue μ_1 is principal; that is, it is simple (i.e., its associated eigenfunctions are each a constant multiple of one another) and the associated eigenfunction φ_1 doesn't change sign in Ω (i.e., it is either strictly positive or strictly negative).

We first show that φ_1 does not change sign in Ω. Indeed, suppose it does, and let $\varphi_1^+ = \max\{\varphi_1, 0\}$ and $\varphi_1^- = \min\{\varphi_1, 0\}$, we know that φ_1^+ and $\varphi_1^- \in H^1(\Omega)$.

By the characterization of μ_1 it follows that $(\varphi_1, \varphi_1)_c = \mu_1(\varphi_1, \varphi_1)_\sigma$. Therefore,

$$\begin{aligned} 0 &\leq (\varphi_1^+, \varphi_1^+)_c + (\varphi_1^-, \varphi_1^-)_c \\ &\quad - \mu_1(\varphi_1^+, \varphi_1^+)_\sigma - \mu_1(\varphi_1^-, \varphi_1^-)_\sigma \\ &= (\varphi_1, \varphi_1)_c - \mu_1(\varphi_1, \varphi_1)_\sigma = 0. \end{aligned}$$

It follows immediately that φ_1^+ and φ_1^- are also eigenfunctions corresponding to μ_1. From [Bre11], we get that $\varphi_1^+ > 0$ a.e in Ω and $\varphi_1^- < 0$ a.e in Ω, which is impossible. Thus φ_1 does not change sign in Ω.

Next, we claim that μ_1 is simple if and only if φ_1 does not change sign. Indeed, If φ_1 changes sign then φ_1^+ and φ_1^- are also eigenfunctions corresponding to μ_1 and they are linearly independent. Hence, μ_1 is not simple. On the other hand, suppose that μ_1 is not simple and let φ and ψ be two eigenfunctions corresponding to μ_1; they are linearly independent. If φ or ψ changes sign then the claim is proved. Otherwise suppose without loss of generality that φ and ψ are positive then we will prove that there exists $\alpha \in \mathbb{R}$ such that the eigenfunction (corresponding to μ_1)

$\varphi + a\psi$ changes sign. Indeed, suppose that for all $\alpha \in \mathbb{R}$, $\varphi + \alpha\psi$ does not change. Let the function $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(\alpha) = \int \varphi + \alpha \int \psi$. Since h is continuous there exists $a \in \mathbb{R}$ such that $h(a) = \int \varphi + a \int \psi = 0$. Hence, $\varphi = -a\psi$ which contradicts the fact that φ and ψ are linearly independent. Thus, $\varphi + a\psi$ changes sign.

Remark. If the boundary $\partial\Omega$ is smooth and the functions c and σ are Hölder continuous, then by the regularity arguments for elliptic equations (see, e.g., [Bre11, Theorem 9.26 and Theorem 9.34]) it follows that $\varphi_1 \in C^{2,\gamma}(\Omega) \cap C^{0,\gamma}(\overline{\Omega})$ where $0 < \gamma < 1$. The Hopf Lemma and the subsequent strong maximum principle (or Boundary Point Lemma) shows that the outer normal derivative $\frac{\partial \varphi_1}{\partial n}(x) < 0$ whenever $\varphi_1(x) = 0$ with $x \in \partial\Omega$. Hence, one has that $\varphi_1 > 0$ on $\overline{\Omega}$.

Remark. As shown above, we have completely described the spectrum of the Laplace operator. We would like to mention that in the case of the p-Laplacian operator, the Steklov spectrum is not completely known, although one may still obtain an infinite sequence of eigenvalues. Moreover, if σ changes sign appropriately on $\partial\Omega$, then problem (2) possesses an infinite sequence of positive eigenvalues and an infinite sequence of negative eigenvalues

$$-\infty \leftarrow \ldots \leq \mu_{-2} < \mu_{-1} < 0 < \mu_1 < \mu_2 \leq \ldots \to \infty,$$

as $j \to \infty$. In addition, $\mu_{\pm 1}$ are both principal eigenvalues. We refer to [CL15] for details.

So far, we have presented the Steklov spectrum for the N-dimensional equation with $N \geq 2$. For sake of completeness, let us make some comments on the one-dimensional case.

The one-dimensional case. Consider the one-dimensional domain $\Omega = (0, 1)$ with $c \equiv 1$ and $\sigma \equiv 1$. The spectral problem (2) can be rewritten as a second order ordinary differential equation

$$\begin{aligned} -v'' + v &= 0 \qquad \text{in } (0,1), \\ -v'(0) &= \mu v(0) \\ v'(0) &= \mu v(1) \end{aligned} \tag{10}$$

In this case, the differential equation can be solved explicitly by using the characteristic polynomial technique, and the general solution of (10) is of the form $u(x) = c_1 e^x + c_2 e^{-x}$, where c_1 and c_2 are constants. Taking into account the boundary conditions, we obtain only two (simple) eigenvalues

$$\mu_1 = \frac{e-1}{e+1} < \mu_2 = \frac{1}{\mu_1} = \frac{e+1}{e-1}.$$

The eigenfunctions associated to μ_1 and μ_2 are given by $\varphi_1(x) = e^x + e^{1-x}$ and $\varphi_2(x) = e^x - e^{1-x}$, respectively. Observe that $\varphi_1(0) = \varphi_1(1) = 1 + e$ and $\varphi_1(x) > 0$ for all $x \in [0, 1]$, and $\varphi_2(0) = 1 - e = -\varphi_2(1)$. We see that φ_2 changes sign and it is orthogonal to φ_1 with respect to the $L^2((0, 1))$-inner product as well as the $H^1((0, 1))$-inner product.

More generally, when $c \in L^1(0, 1)$ is nonnegative and $\int_0^1 c > 0$, and the weight $\sigma : \{0, 1\} \to \mathbb{R}$ is a nonnegative function with $\sigma(0) + \sigma(1) \neq 0$, then by using variational characterizations discussed above, we also obtain exactly two positive eigenvalues

$$\mu_1 = \inf_{\substack{u \in H^1 \\ u \neq 0}} \frac{\int_0^1 |(u')^2 + c(x)u^2}{\sigma(0)u^2(0) + \sigma(1)u^2(1)}$$

and

$$\mu_2 = \inf_{\substack{u \in W \\ u \neq 0}} \frac{\int_0^1 |(u')^2 + c(x)u^2}{\sigma(0)u^2(0) + \sigma(1)u^2(1)},$$

where $W = \{u \in H_\sigma(\Omega) : (u, \varphi_1)_\sigma = \sigma(1)\varphi_1(1)u(1) + \sigma(0)\varphi_1(0)u(0) = 0\}$ with φ_1 an eigenfunction corresponding to μ_1, and $\|u\|_\sigma = [\sigma(0)u^2(0) + \sigma(1)u^2(1)]^{1/2}$. Moreover, μ_1 is principal and μ_2 is simple with eigenfunction changing sign.

We notice that in the one-dimensional case the Steklov spectrum has only two elements whereas in N-dimensional case with $N \geq 2$, the Steklov spectrum is unbounded, infinite and discrete.

Linear nonhomogeneous Steklov problem. Consider the linear nonhomogeneous problem

$$\begin{aligned} -\Delta u + c(x)u = 0 \quad &\text{a.e. in } \Omega, \\ \frac{\partial u}{\partial \eta} = (\mu_k + \lambda)\sigma(x)u + h(x) \quad &\text{on } \partial\Omega, \end{aligned} \tag{11}$$

where $h \in L^2(\partial\Omega)$, μ_k is a Steklov eigenvalue of (2), and $\lambda \in \mathbb{R}$. From the Fredholm alternative theorem [Bre11], we have that

(i) If $\mu_k + \lambda$ is not an eigenvalue of (2), then Equation (11) has a unique solution for every h.
(ii) If $\mu_k + \lambda$ is an eigenvalue of (2), then Equation (11) has a solution if and only if h is orthogonal to the eigenspace associated to $(\mu_k + \lambda)$.

Because of the properties of the first eigenvalue μ_1, our analysis will only be focused on the case where $\mu_k = \mu_1$. From the above discussion, the Fredholm alternative-type arguments describe completely the structure of the solution set of (11).

In what follows, we will be concerned with nonlinear pertubations of equation (2). We will also analyze the structure of the solution set in the framework of bifurcation from infinity.

Problems with Nonlinear Boundary Conditions

Consider a nonlinear pertubation of equation (2) given by

$$\begin{aligned} -\Delta u + c(x)u = 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \eta} = (\mu_1 + \lambda)\sigma(x)u + g(x,u) + h(x) \quad \text{on } \partial\Omega. \end{aligned} \tag{12}$$

Here Ω is a smooth bounded domain in $\mathbb{R}^N$, where $N \geq 2$. Although much weaker regularity conditions may be considered on the data as seen in the previous sections, we assume for the sake of simplicity and clarity of the presentation that the coefficient-functions c and σ, as well as the nonhomogenous term h and the nonlinearity g, are smooth on their domains of definition, and that $g(x,u)$ is asymptotically sublinear at infinity in u uniformly in x (see below). Moreover, we assume that c and σ are nonnegative with $\int_{\partial\Omega} \sigma\, dS > 0$. In addition, μ_1 is the first Steklov eigenvalue of equation (2) and the (real) parameter λ varies in a neighborhood of zero.

When $g \not\equiv 0$ is a (genuine) nonlinearity, the structure of the solution set may be quite different from that of the nonhomogeneous linear equation (11). Therefore, we will present some results on the solution set structure; namely, the location and behavior of the solution set for the nonlinear problem (12) for λ in a neighborhood of zero (i.e., $\mu_1+\lambda$ in a neighborhood of μ_1), and the nonlinearity g satisfies some asymptotic conditions. In particular, the existence of multiple solutions with (potentially) large norms.

By a solution to Eq.(12) we mean a function $u \in W_p^2(\Omega)$, $p > N$, which satisfies (12). (For the definitions and properties of the Sobolev spaces $W_p^k(\Omega)$, (Sobolev) *trace*-spaces $W_p^{k-p}(\partial\Omega)$ and Hölder spaces $C^{0,\gamma}(\partial\Omega)$, we refer for instance to [Bre11].)

The nonlinear problem (12) has received much attention in recent years. A few results on a disk ($N = 2$) were obtained in the case of linear elliptic equations where the nonlinearity on the boundary was compared with the first Steklov eigenvalue. We refer to Klingelhöfer [Kli68]. The results in [Kli68] were significantly generalized to higher dimensions in [Ama76] in the framework of the sub- and super-solutions method.

Let us mention here that in [MP17], the authors proved multiplicity results for weak solutions (in $H^1(\Omega)$) for problems somewhat similar to (12) by using a priori estimates and bifurcation theory. Their results considered the case $c \equiv 1$. The harmonic function situation, i.e., $c \equiv 0$, was not included. In [Mav12, MN10] the authors proved the existence of weak solutions for elliptic equations with nonlinear boundary conditions using variational arguments.

To obtain existence, multiplicity, and bifurcation from infinity results for equation (12), we impose the following general conditions on the (boundary) nonlinearity g and the nonhomogeneous term h, and appropriately cast the problem in an abstract setting.

Conditions on the nonlinearity g.

(G1) g is asymptotically sublinear at infinity in u, uniformly in x; that is, $\lim_{|u|\to\infty} \frac{g(x,u)}{u} = 0$ uniformly in x in the sense that for every $\varepsilon > 0$ there is a constant $r_\varepsilon > 0$ such that

$$|g(x,u)| \leq \varepsilon |u|$$

for all $x \in \partial\Omega$ and all $u \in \mathbb{R}$ with $|u| \geq r_\varepsilon$.

(G2) g satisfies a sign-like condition, i.e., there are functions $A \in C(\partial\Omega)$ and $B \in C(\partial\Omega)$ and constants r, R with $r < 0 < R$ such that

$$g(x,u) \geq A(x) \quad \text{for all } x \in \partial\Omega$$

and all $u \in \mathbb{R}$ with $u \geq R$;

$$g(x,u) \leq B(x) \quad \text{for all } x \in \partial\Omega$$

and all $u \in \mathbb{R}$ with $u \leq r$.

Conditions on the nonhomogeneous function h. The nonhomogeneous function h satisfies the orthogonality-like conditions

(H)

$$\int_{\partial\Omega} B(x)\varphi_1 \leq -\int_{\partial\Omega} h(x)\varphi_1 \leq \int_{\partial\Omega} A(x)\varphi_1, \tag{13}$$

We would like to mention that the sublinearity condition **(G1)** guarantees the existence of unbounded branches of solutions when the parameter λ approaches zero. These branches bifurcate from infinity in the sense of Rabinowitz; see [Rab73]. Conditions **(G2)** and **(H)** are used in connection with the so-called Landesman–Lazer resonance conditions.

Problem framework. We set up problem (12) in terms of the normal derivative trace equation on the boundary, and Nemytskĭi operators on trace-spaces. More specifically, we cast the problem as a nonlinear first-order differential equation "through" the boundary sub-manifold $\partial\Omega$ (i.e., a normal derivative trace equation) along with homogeneous linear second-order partial differential equations (diffusion-type) "constraint" inside the domain Ω. Since the regularity conditions on the data may be significantly weakened as aforementioned, we indicate how we set up the problem in terms of Sobolev spaces.

We define the linear (Steklov) boundary operator

$$\mathcal{B} : \operatorname{Dom}(\mathcal{B}) \subset W_p^{1-1/p}(\partial\Omega) \to W_p^{1-1/p}(\partial\Omega) \text{ by}$$

$$\mathcal{B}u := \frac{\partial u}{\partial \nu} - \mu_1\sigma(x)u,$$

where $X := \operatorname{Dom}(\mathcal{B}) = \{u \in W_p^2(\Omega) : -\Delta u + c(x)u = 0 \text{ a.e. in } \Omega\}$.

Since $X \subset W_p^2(\Omega)$, we write symbolically $W_p^2(\Omega) \overset{c}{\hookrightarrow} W_p^{1-1/p}(\partial\Omega)$ to simply mean that the trace-extension operator $W_p^2(\Omega) \hookrightarrow W_p^{2-1/p}(\partial\Omega) \Subset W_p^{1-1/p}(\partial\Omega)$ is a compact linear operator from $W_p^{2-1/p}(\partial\Omega)$ into $W_p^{1-1/p}(\partial\Omega)$ (see, e.g., [Bre11] and references therein). Notice also that the second-order differential equation defines (or more precisely is included as a "constraint" in) the domain of the linear (boundary) operator $\mathcal{B}$, and that X is a closed subspace of $W_p^2(\Omega)$.

Now, we define the nonlinear (Nemytskiĭ) superposition-operator

$$\mathcal{N} : X \subset W_p^{1-1/p}(\partial\Omega) \to W_p^{1-1/p}(\partial\Omega) \quad \text{by}$$

$$\mathcal{N}u = \mathrm{g}(\cdot, u(\cdot)).$$

Eq.(12) is then equivalent to

$$\mathcal{B}u = \lambda\sigma(\cdot)u + \mathcal{N}u + h, u \in X. \tag{14}$$

This abstract set up on the trace-spaces together with a combination of degree theory (see, e.g., [Maw79]), continuation methods, and Rabinowitz bifurcation from infinity arguments [Rab73] are used to establish the existence and multiplicity of solutions and to provide the location and the behavior of the solution sets.

In order to apply degree theory, one should establish at least an a priori bound for all possible solutions to a homotopy associated with Eq. (12); see below.

Proposition 1 (*a priori estimate*). *Assume that the assumptions* ***(G1)–(G2)*** *and* ***(H)*** *hold true. Let $\lambda_0 \in \mathbb{R}$ be a fixed constant such that $0 < \lambda_0 < \mu_2 - \mu_1$. Then, there is a constant $R_0 := R_0(\lambda_0) > 0$ such that all possible solutions of Eq.(12), with $0 < \lambda \leq \lambda_0$, satisfy*

$$|u|_{W_p^2(\Omega)} \leq R_0.$$

That is, all possible solutions of Eq.(12) are (uniformly) bounded in $W_p^2(\Omega)$ independently of λ, provided $0 < \lambda \leq \lambda_0$.

Let us mention that a similar result holds for all λ negative (and bounded away from zero). More precisely, we have the following uniform a priori bound.

Proposition 2 (*a priori estimate*). *Let $\lambda_0, \lambda_1 \in \mathbb{R}$ be (fixed negative) constants such that $-\infty < \lambda_0 < \lambda_1 < 0$. Suppose that the assumptions* ***(G1)*** *holds. Then there exists a constant $R_0 := R_0(\lambda_0, \lambda_1) > 0$ such that all possible solutions of Eq.(12), with $\lambda_0 \leq \lambda \leq \lambda_1$, satisfy*

$$|u|_{W_p^2(\Omega)} \leq R_0.$$

That is, all possible solutions of Eq.(2) are (uniformly) bounded in $W_p^2(\Omega)$ independently of λ, provided that $\lambda_0 \leq \lambda \leq \lambda_1 < 0$.

Existence of solutions.

Theorem 1 (Existence). *Assume that the assumption* ***(G1)–(G2)*** *and* ***(H)*** *hold, then Eq.(12) has at least one solution for every $\lambda < \mu_2 - \mu_1$.*

Moreover, for $0 < \lambda \leq \lambda_0$, with $\lambda_0 < \mu_2 - \mu_1$, all solutions are uniformly bounded in $W_p^2(\Omega)$, independently of λ.

To prove Theorem 1, we first consider the case when $\lambda \geq 0$ is fixed. Picking $\delta \in \mathbb{R}$ such that $0 < \delta < \mu_2 - \mu_1$, and following the notation of the previous section, we consider the homotopy

$$\mathcal{B}u - \delta\sigma(\cdot)u = \theta[(\lambda - \delta)\sigma(\cdot)u + \mathcal{N}u + h], u \in X, \tag{15}$$

where $\theta \in [0,1)$; which, when $\theta = 0$, reduces to the homogeneous linear problem $\mathcal{B}u - \delta\sigma(\cdot)u = 0$ that has only the trivial solution. (It would reduce to our original nonlinear problem (12) if θ were equal to 1.) Since the linear operator $\mathcal{B} - \delta\sigma(\cdot)I$ defined by $\mathcal{B} - \delta\sigma(\cdot)I : X \to W_p^{1-1/p}(\partial\Omega)$ is bounded, one-to-one and onto (by the continuity of the trace operator and the Fredholm alternative), it follows that (15) is equivalent to the fixed point homotopy

$$u = \theta(\mathcal{B} - \delta\sigma(\cdot)I)^{-1}\left((\lambda - \delta)\sigma(\cdot)Iu + \mathcal{N}u + h\right), \tag{16}$$

$u \in X := \mathrm{Dom}(\mathcal{B})$. Therefore, by the compactness of the trace operator $W_p^2(\Omega) \overset{c}{\hookrightarrow} W_p^{1-1/p}(\partial\Omega)$ and the topological degree theory (see, e.g., [Maw79]), it suffices to show that all possible solutions of the homotopy (16) are bounded in $W_p^2(\Omega)$, independently of $\theta \in [0,1)$, in order to conclude that Eq.(16) has at least one solution for $\theta = 1$ as well.

Indeed, observing that $0 < (1-\theta)\delta + \theta\lambda \leq \max\{\lambda, \delta\} := \lambda_0 < \mu_2 - \mu_1$ for $0 \leq \theta < 1$, it follows from Proposition 1 that all possible solutions of Eq.(15) (or equivalently Eq.(16)) are (uniformly) bounded in $W_p^2(\Omega)$ independently of $\theta \in [0,1)$. This proves the first part of Theorem 1. The second part of Theorem 1 follows readily from Proposition 1.

Now, to prove the existence of at least one solution for $\lambda < 0$ (fixed), we consider the homotopy (15) where $\delta < 0$ and now $\theta \in [0,1]$. (Notice that $\theta = 1$ is included here.) Observing that $\lambda_0 := \min\{\lambda, \delta\} \leq (1-\theta)\delta + \theta\lambda \leq \max\{\lambda, \delta\} := \lambda_1 < 0$ for $0 \leq \theta \leq 1$, it follows Proposition 2 that all possible solutions of Eq.(15) are (uniformly) bounded in $W_p^2(\Omega)$ independently of $\theta \in [0,1]$. The existence of at least one solution for each $\theta \in [0,1]$ follows from topological degree arguments as above. (It should be noted that Assumptions (**G2**)–(**H**) do not matter when $\lambda < 0$, at least as far as existence of at least one solution is concerned.) □

Recall that no multiplicity results occur when $\mathrm{g} \equiv 0$ and either $\lambda < 0$ or $0 < \lambda < \mu_2 - \mu_1$, since the Fredholm alternative guarantees uniqueness in this case! We claim that, by strengthening somewhat either (**G2**) or (**H**), we obtain multiplicity results and more importantly we describe the behavior of the solution set. The first result is motivated by the fact that one may allow the equality $A(x) = B(x)$ for $x \in \partial\Omega$ in (**G2**). We would like to point out that, in this instance, multiplicity may occur only for one value of λ; more precisely at $\lambda = 0$ (even if $g \not\equiv 0$), with the bifurcation branches in the $(\lambda, |u|_{C^{0,\alpha}(\partial\Omega)})$-plane being only (semi-infinite) straight line rays located on the vertical $|u|_{C^{0,\alpha}(\partial\Omega)}$-axis, as illustrated in the following remark.

Remark. Consider any (nonlinearity) g such that $g(x, u) = 0$ for all $x \in \partial\Omega$ and $u \in \mathbb{R}$ with $|u| \geq R$, where $R > 0$ is a fixed number; i.e., the function g vanishes outside a "cylindrical shell" $\partial\Omega \times [-R, R]$. For $\lambda = 0$, it is easily seen that the function defined by $u_t := t\varphi_1$ is a solution to Eq.(12) for every $t \in \mathbb{R}$ that is such that $|t| \min_{\partial\Omega}\{\varphi_1(x)\} \geq R$; provided the nonhomogeneous term $h \equiv 0$ of course. An analysis of the proof of the above existence result (or the multiplicity results below) will indicate that, provided h is $L^2(\partial\Omega)$-orthogonal to φ_1, $\lambda = 0$ is the only parameter-value for which large solutions exist, and the bifurcation from infinity branches are (semi-infinite) straight line rays on the $|u|_{C^{0,\alpha}(\partial\Omega)}$-axis in the $(\lambda, |u|_{C^{0,\alpha}(\partial\Omega)})$-plane, as described above. Therefore, the bifurcation from infinity parameter-interval collapses to just one-point interval $\{\lambda\} = \{0\}$.

Therefore, for the rest of the article, we will be interested in nonlinearities g that satisfy a sign-like condition and that are not identically null outside a compact u-interval in $\mathbb{R}$.

Bifurcation from infinity. In addition to a (fairly) general existence result (see Theorem 1 above), our multiplicity results state that as long as the nonlinearity g satisfies a condition asymptotically, then when λ is in an appropriate interval on one side of zero, Eq.(12) has at least two (large-norm) solutions, provided h is in an appropriate range. Moreover, all solutions with λ on the other side (of zero) are uniformly bounded. In this way, we locate the solution set and describe its behavior in terms of bifurcation from infinity as the parameter λ varies. Our asymptotic conditions include the so-called "very strong resonance" (see Theorem 2 below); i.e., $g \to 0$ as $|u| \to \infty$ at $\lambda = 0$, and no "decay-rate" at infinity is required; "standard resonance" (see Theorem 3) such as the so-called Landesman–Lazer-type conditions (i.e., $g \nrightarrow 0$ as $|u| \to \infty$).

Definition. We say that (λ_∞, ∞) is a bifurcation point from infinity (on the the boundary) if there exists a sequence of solutions (λ_n, u_n) such that $\lambda_n \to \lambda_\infty$ and $|u_n|_{C^{0,\alpha}(\partial\Omega)} \to \infty$ as $n \to \infty$.

Theorem 2 (Bifurcation from infinity). *Assume that condition **(G1)** is met, and that **(G2)** holds on $\partial\Omega$ with strict inequalities; that is, there are functions $A, B \in C(\partial\Omega)$ and constants $r < 0 < R$ such that*

$$\textbf{(SS)}\quad \begin{aligned} &g(x,u) > A(x) \quad \textit{for all } x \in \partial\Omega \\ &\textit{and all } u \in \mathbb{R} \textit{ with } u \geq R; \\ &g(x,u) < B(x) \quad \textit{for all } x \in \partial\Omega \\ &\textit{and all } u \in \mathbb{R} \textit{ with } u \leq r. \end{aligned}$$

*Provided **(H)** holds, $(0, \infty)$ is a bifurcation point from infinity; that is, there is a constant $\lambda_- < 0$ such that for every $\varepsilon \in (0, |\lambda_-|)$ Eq.(12) has at least two solutions, denoted $(\lambda_\varepsilon^+, u_\varepsilon)$ and $(\lambda_\varepsilon^-, v_\varepsilon)$, with $-\varepsilon < \lambda_\varepsilon^\pm < 0$ such that for some $0 < \alpha < 1$,*

$$\lim_{\varepsilon \to 0^+} \min\left\{|u_\varepsilon|_{C^{0,\alpha}(\partial\Omega)}, |v_\varepsilon|_{C^{0,\alpha}(\partial\Omega)}\right\} = \infty;$$

that is, they bifurcate from infinity since $\lambda_\varepsilon^\pm \to 0$ as $\varepsilon \to 0^+$.

Moreover, for $0 \leq \lambda \leq \lambda_0$ with $\lambda_0 < \mu_2 - \mu_1$, all solutions (which exist by Theorem 1) are uniformly bounded, independently of λ. Therefore, bifurcation from infinity occurs only (strictly) to the left of the eigenvalue μ_1. (In some sense, the "strong resonance" conditions "bend" the bifurcation branches; see Figure 1 below.)

A simple example to keep in mind here is the (continuous) function g given by $g(x, u) := \eta_+(x)(1 + u^2)^{-1}$ for $u \geq R > 0$ and $g(x, u) := -\eta_-(x)(1+u^2)^{-1}$ for $u \leq -r < 0$, where $\eta_\pm$ are smooth positive functions on $\partial\Omega$, or a nonbounded counterpart $g(x, u) := \sqrt[3]{u}\sin^2(u) \pm \eta_\pm(x)(1 + u^2)^{-1}$. Notice that here, $A(x) = B(x) = 0$; which by **(H)** requires h to be $L^2(\partial\Omega)$-orthogonal to φ_1. Observe that in either case $\liminf_{u\to\infty} g(x, u) = 0 = \limsup_{u\to-\infty} g(x, u)$ and $\liminf_{u\to\infty} ug(x, u) = 0 = \limsup_{u\to-\infty} ug(x, u)$; that is, no (linear) "decay rate" at infinity is required (see, e.g., [AA95] and references therein). Thus, the terminology (asymptotic) strong resonance used here! We also point out that the so-called Landesman–Lazer condition **(LL)** (see below) is not satisfied for these nonlinearities since one has equalities in **(H)**, but we are still able to "locate" and "describe" the solution-branches.

Note that the "stronger" condition (SS) may be used to establish that all possible solutions of Eq.(14) are (uniformly) bounded in $W_p^2(\Omega)$ when $\lambda = 0$ as well; that is, the conclusion of Theorem 1 actually holds true for all $\lambda \in [0, \lambda_0]$.

To prove Theorem 2, we consider the fixed point equation

$$u = \theta(\mathcal{B} - \delta\sigma(\cdot)I)^{-1}\left((\lambda - \delta)\sigma(\cdot)Iu + \mathcal{N}u + h\right). \qquad (17)$$

Setting

$$\mu := \lambda + \delta, Lu := [(\mathcal{B} + \delta\sigma(\cdot)I)^{-1}\sigma(\cdot)I]u$$

and

$$Ku := (\mathcal{B} + \delta\sigma(\cdot)I)^{-1}\left(\mathcal{N}u + h\right),$$

it follows that the above fixed point equation is equivalent to the nonlinear "normal derivative trace" equation

$$u = \mu Lu + K(u), u \in C^{0,\alpha}(\partial\Omega), 0 < \alpha < 1. \qquad (18)$$

From this setup, it follows that $\mu^{-1} = \delta^{-1}$, i.e., $\lambda = 0$, is the principal eigenvalue of L and that, by the compactness of the trace operator, the solution-map (through the use of the "bootstrap" regularity argument as above)

$$L : C^{0,\alpha}(\partial\Omega) \to C^{1,\alpha\gamma}(\overline{\Omega}) \overset{c}{\hookrightarrow} C^{0,\alpha}(\partial\Omega)$$

is a compact linear operator when considered as an operator from $C^{0,\alpha}(\partial\Omega)$ into $C^{0,\alpha}(\partial\Omega)$. Using the regularity of g and h and a "bootstrap" argument again one shows that

$$K : C^{0,\alpha}(\partial\Omega) \overset{c}{\hookrightarrow} C^{0,\alpha}(\partial\Omega)$$

is a completely continuous mapping when viewed as a nonlinear operator from $C^{0,\alpha}(\partial\Omega)$ into $C^{0,\alpha}(\partial\Omega)$. Then using the sublinear growth condition **(G1)**, one can show that $K(u) = o(|u|_{C^{0,\alpha}(\partial\Omega)})$ as $|u|_{C^{0,\alpha}(\partial\Omega)} \to \infty$. Notice that Eq.(18) has now an abstract form considered, e.g., in [Rab73] for bifurcation from infinity purposes. Therefore, $\lambda = 0$ is a bifurcation point from infinity since all assumptions of the bifurcation from infinity result are fulfilled (see, e.g., [Rab73, p. 465, Theorem 1.6 and Corollary 1.8]; that is, there exist two connected sets of solutions $\mathcal{C}^+, \mathcal{C}^- \subset \mathbb{R}\times C^{0,\alpha}(\partial\Omega)$ with $\mathcal{C}^+\cap\mathcal{C}^- = \emptyset$ which are such that for every (sufficiently) small $\varepsilon > 0$, $\mathcal{C}^+ \cap U_\varepsilon \neq \emptyset$, $\mathcal{C}^- \cap U_\varepsilon \neq \emptyset$ where $U_\varepsilon := \{(\lambda, u) \in \mathbb{R} \times C^{0,\alpha}(\partial\Omega) : |\lambda| < \varepsilon, |u|_{C^{0,\alpha}(\partial\Omega)} > 1/\varepsilon\}$. (Observe that, by the regularity of solutions, $u \in C^{0,\alpha}(\partial\Omega) \cap X$ since it is a solution of the fixed point equation (18).) Since all solutions are uniformly bounded in $W^2_p(\Omega)$ for all $\lambda \in [0, \lambda_0]$ with $\lambda_0 < \mu_2 - \mu_1$ (see Proposition 1 and the bound in the case $\lambda = 0$) and for all $\lambda \in [\lambda_0, \lambda_1]$ with $\lambda_1 < 0$, there therefore exists a deleted left-neighborhood of 0 in $\mathbb{R}$; i.e., there is $\lambda_- < 0$, such that for every $\varepsilon > 0$ with $\varepsilon < |\lambda_-|$, there are two distinct solutions $(\lambda^+_\varepsilon, u_\varepsilon) \in C^+$ and $(\lambda^-_\varepsilon, v_\varepsilon) \in C^-$ with $-\varepsilon < \lambda^\pm_\varepsilon < 0$, $u_\varepsilon \neq v_\varepsilon$, and $\min\{|u_\varepsilon|_{C^{0,\alpha}(\partial\Omega)}, |v_\varepsilon|_{C^{0,\alpha}(\partial\Omega)}\} > 1/\varepsilon$. It follows that $\lambda^\pm_\varepsilon \to 0$ and $\min\{|u_\varepsilon|_{C^{0,\alpha}(\partial\Omega)}, |v_\varepsilon|_{C^{0,\alpha}(\partial\Omega)}\} \to \infty$ as $\varepsilon \to 0^+$.

Theorem 3 (Bifurcation from infinity). *Assume that **(G1)**–**(G2)** hold and that*

$$\textbf{(LL)} \int_{\partial\Omega} g_-(x)\varphi_1 < -\int_{\partial\Omega} h(x)\varphi_1 < \int_{\partial\Omega} g_+(x)\varphi_1, \quad (19)$$

where $g_+(x) := \liminf_{u\to\infty} g(x,u)$ *and* $g_-(x) := \limsup_{u\to-\infty} g(x,u)$.

Then $(0,\infty)$ *is a bifurcation point from infinity; that is, the conclusion of Theorem 2 holds.*

In the above result we strengthen "a little bit" the condition **(H)** by requiring strict inequalities while keeping **(G2)** as it is given. This is the so-called Landesman–Lazer-type conditions; which was considered in the literature in some other setting. To show that all possible solutions of Eq.(14) are (uniformly) bounded in $W^2_p(\Omega)$ when $\lambda = 0$, we use the Landesman–Lazer condition **(LL)** and Fatou's lemma. Then proceed as in the proof of Theorem 1.

A simple example to keep in mind here is the (smooth) function g (independent of x) given by $g(u) := \eta_\pm \tanh(u)$ for $|u| \geq R > 0$ with $\eta_+ > 0$ applying when $u > R$ and $\eta_- > 0$ applying when $u < -R$, or a nonbounded counterpart $g(u) := \sqrt[3]{u}\sin^2(u) + \eta_\pm \tanh(u)$ for $|u| \geq R > 0$.

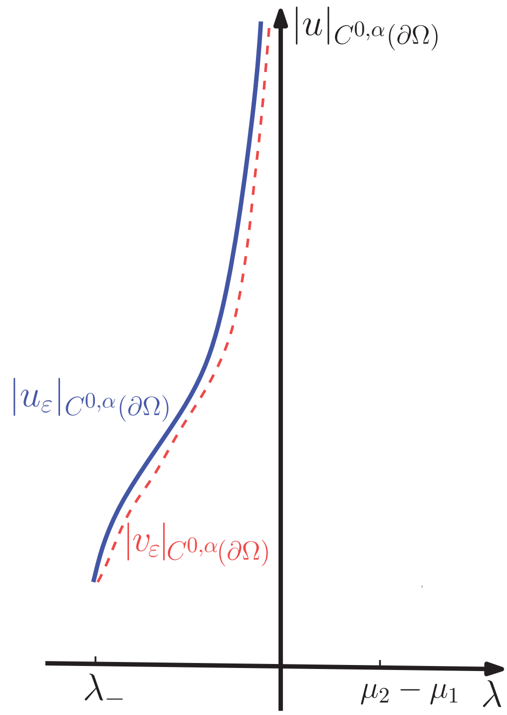

Figure 1. Bifurcation diagram in the case of a "strong resonance" nonlinearity.

Notice that in either case $\liminf_{u\to\infty} g(u) = \eta_+$ and $\limsup_{u\to-\infty} g(u) = -\eta_-$. Therefore the nonhomogeneous term h has to satisfy the strict inequalities

$$-\eta_- \int_{\partial\Omega} \varphi_1 < -\int_{\partial\Omega} h(x)\varphi_1 < \eta_+ \int_{\partial\Omega} \varphi_1.$$

Another aspect that we have not considered here, due in part to space limitation, but which is nonetheless important is the numerical analysis and simulation for these problems.

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