Hex: The Full Story

Reviewed by Chris Sangwin

Hex: The Full Story
by Ryan B. Hayward
with Bjarne Toft

In *Hex: The Full Story*, Ryan B. Hayward and Bjarne Toft provide a comprehensive history of the game of Hex, up to and including an account of contemporary work. The book addresses three issues: (i) a history of Hex; (ii) a discussion of the theory behind Hex play; and (iii) recent work including the development of computer opponents.

You may be forgiven for never having come across Hex yourself. Indeed, I discovered it by accident in the search for a mathematical engagement activity for school students, and in common with many other mathematicians immediately fell in love with the game. It could well be the best strategy board game for two players.

Hex is played on a board made up of n rows and m columns of hexagonal cells arranged in a rhombus (normally $n = m$). There are two players who each own two opposite sides of the board. Counters can be placed in any unoccupied hexes. Players alternately place counters of their colour to occupy hexes. To win, connect your sides of the board with a continuous path made from counters of your colour. Note that normally counters cannot be

Figure 1. An example game of Hex.

moved, and that the four corner hexagons each belong to both adjacent sides.

Despite the simplicity of the rules, Hex is both very difficult to master and surprisingly fun to play. An example game is shown in Figure 1. In this game Red moved first, taking a very strong centre position. Blue was unable to block, but put up a spirited defence before Red finally won, connecting the two red sides. A 7×7 board is adequate for a couple of training games but very soon becomes too small, and 11×11 is the minimum size for a sensible game.

In Hex, it is best to play defensively: defence is also attack. By blocking your opponent you are also going to connect up your own sides. Also, since there is no hidden information or randomness, each player has *perfect knowledge*. A consequence is that if you can think of a strong response to your own move, then look for a better one!

The game also makes use of patterns, known as templates. The simplest such template, shown in Figure 2, is called a *bridge*. These two red counters are strongly connected. If Blue takes one empty hex, then Red immediately takes the other.

A further template is shown in Figure 3. The cells involved in the template are shaded in yellow, and Red has occupied one cell of the template. If Blue plays in the

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Figure 2. The bridge template.

Figure 3. A template connecting Red to Red's edge.

shaded area, then provided Red responds immediately, Red can always connect to the red edge.

As players gain experience, the game evolves into a joust between patterns such as templates which ultimately must overlap. There is no need to always place your pieces in adjacent hexes. Use bridges to make connections between your pieces *and* simultaneously to block your opponent. But, beware of empty cells which are part of multiple overlapping templates.

There are two remarkable theorems. The first is that games can never end in a tie, i.e., one player must always win. The second is that the first player should win. For this reason, going first is possibly unfair. Normally pieces cannot be moved but one important exception used by most players is called the *swap rule*. On their first move the second player may move normally, or choose to swap their piece with that placed by the first player. This encourages the first player to only choose a moderately strong first move and so reduces any advantage of going first. For new players the swap rule is confusing, and it is best to ignore the swap rule for the first few games. If you have not played Hex, then now would be a good time to do so.

Originally the game was played on disposable pads, with each player marking their move in pencil. Nestorgames (<https://nestorgames.com>) makes a convenient roll-up set in 11×11 and also a 14×14 set, but there are few commercial alternatives of which I'm aware. Otherwise I have found Hex sets surprisingly difficult to purchase. Somewhat irked at this I made my own boards, and a .pdf 11×11 board is available here, <https://www.maths.ed.ac.uk/~csangwin/hex/>, together with some basic notes. The computer opponent Hexy (vanshel.com/Hexy/) should be sufficient to thwart most beginners once their friends and relatives have become thoroughly frustrated.

Hex was invented by the Danish mathematician Piet Hein in the early 1940s, and he published a series of puzzles calling the game "Polygon" in the Danish newspaper *Politiken*. The same game was independently invented by John Nash in 1947 at Princeton University. Hex was popularized by Martin Gardner in his article [Gar59]. *Hex: The*

Full Story contains a comprehensive history of the discovery and early development of the game, and this historical account is the heart of the book. The authors have provided extensive quotations from original letters, and large numbers of figures, showing original manuscripts from Hein, Gardner, and others involved in the early development of Hex. Photographs of the protagonists and background information complete this story. They sensitively explain the potential controversy between Hein and Nash about the discovery, and the key diplomatic role played by Martin Gardner in the whole affair. I confess I was absolutely fascinated by their account, and thoroughly recommend the book to anyone interested in Hex, or for scholars looking for source material on independent reinvention in science and mathematics.

For example, in notes for a lecture on 12 December 1942 (only rediscovered in 2004), Hein said:

What I intend to provide tonight is merely a sketch of a thought to an introduction to a game. The idea is to look at mathematics as a game, and the game is a simple example of looking at games as mathematics. [HT19, p. 21]

This sentiment particularly interested me because my original motivation was to find a mathematical game as an engagement activity for school students. The idea of mathematics itself as a formal game is well established, but little did I know that this idea was shared by Hein in the context of Hex. I should add that my use of Hex was not entirely satisfactory: some young people got too consumed by the competitive element of the play, and this robbed others of the pleasure of seeing the mathematical structure I hoped they would enjoy. Nevertheless I still think Hex can act as a model of mathematics itself, with formal rules, structure, patterns of play, and heuristic thinking guiding strategy and thought.

The book also provides the story of the development of Hex puzzles. Figure 4 shows four Hex puzzles taken from [Gar59]. In each case Red is to move next and win. Right from the start, Hex was popularised by setting puzzles for the general public. It was natural to assume that Hein had authored these, but Chapter 4 suggests that Jens Lindhard had a leading role. Indeed the authors go as far as saying, "*the Polygon project was a two-man partnership, with Hein the expositor and Lindhard the game expert*" [HT19, p. 65]. Lindhard's papers and archive, examined in 2017, contain puzzles and drafts for puzzles.

Hex is a connection strategy game, and there are many equivalent and related games. For example, in Chapter 6 the authors explain how the game of Brig-it is really a special case of Hex, where the Hex board used has a regular arrangement of counters at the outset. Another example is the "Game of Y," which uses a triangular arrangement of hexagonal cells. In Y, each player now has the

Figure 4. Hex puzzles: Red to move and win.

same goal: connect all three sides with a continuous path of their colour. While Hex has players connecting opposite sides, in Y both players share the same goal. In this case Y generalizes Hex, since a partially filled-in Y board is identical to a Hex board. In fact, boards can be made in a wide, bewildering even, variety of other ways, using polygons which are not necessarily all hexagonal. Underlying all these games is a connectivity graph, and viewed in this way such games can be recast as the *Shannon switching game*. Criteria on the connectivity graph are used to establish the two, previously mentioned, important theorems: (i) games do not end in a tie and (ii) first player to win.

This book contains a very interesting account of early attempts to create automatic machines to play such games, including Shannon's machine to play Hex. The attempts to create automatic Hex opponents span all epochs in the development of the computer, including analogue computers, digital computers, and contemporary neural networks. Shannon's original device for playing Hex used current through resistors, effectively solving the discrete Hex problem by using an analogue machine.

Lastly, the authors give an account of very recent work to create computer opponents. They introduce, and explain, the idea of *virtual connections*, which allows a computer to represent and evaluate board positions in a more efficient manner, vastly reducing the search space of possible moves. Given my somewhat mixed experience of students' competitiveness when using Hex for an engagement activity, I find the positive role of competitive play in driving the research and development of computer opponents somewhat ironic. The desire to create better computer opponents has been a driving force in some areas of computer science, and the last few chapters of the book provide an informative and interesting account as applied to Hex. The authors have their own significant part to play in this research, which is ongoing and tied to similar efforts with the game of Go. Linking history to contemporary mathematics is an unusual genre of writing, but I found it refreshing and welcome. The book itself is not long, since

the generous use of illustrations and figures reduces the number of words.

I would not recommend this as a first book on how to play Hex. Instead, I'd recommend [Bro00] (and see also [Bro05] for comments on related games). I feel rather mean-spirited writing this in a review. Instead this book is a highly valuable record of original source material, including a comprehensive reproduction of original Hex puzzles in the appendix which might be otherwise very difficult to obtain. I do think most Hex players will want a copy of this book, both for the puzzles and the historical background.

I'm saddened that so few people know about Hex, and that it has slipped out of consciousness. While Chess, Go, and other classic games are well established in our cultural consciousness, Hex has yet to make this transition. So please, if you have not yet played Hex, do so, and then read this fascinating account of its invention, history, and contemporary research to develop automatic opponents.

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