Nonlocal Diffusion Equations with Integrable Kernels

Julio D. Rossi

1. Local and Nonlocal Diffusions

The word diffusion derives from the Latin word *diffundere*, which means “to spread out.”

In applied sciences one phenomenon that is often studied is diffusion. Diffusion is understood as net movement of a species (for example, individuals, atoms, particles) from a region of higher concentration to a region of lower concentration. Diffusion is thus driven by a gradient in concentration. For example, if you spray a gas at one end of a room, eventually the gas particles will be all over the room.

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To the memory of Fuensanta Andreu.

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The concept of diffusion is widely used in many different and unrelated fields, including physics (particle diffusion), chemistry, biology, sociology, economics, and finance (diffusion of people, ideas, and price values). The central idea of diffusion, however, is common to all of these: an object (a particle, an idea, etc.) undergoing diffusion spreads out from points or locations at which there is a higher concentration of that object. There are two ways to introduce the notion of diffusion: either a phenomenological approach (Fick’s law), or a physical/atomistic one, considering the random walk of the diffusing particles. In the phenomenological approach, according to Fick’s law, the diffusion is proportional to the negative gradient of concentrations (it goes from regions of higher concentration to regions of lower concentration). From the atomistic point of view, diffusion is considered as a result of the random walk of the diffusing particles.

Here, by nonlocal diffusion we understand the evolution of a distribution (that we will call $u(x, t)$ emphasizing that the relevant quantity under study, u , depends on a

spacial variable $x \in \mathbb{R}^N$ and a time variable $t \in \mathbb{R}_+$) that takes place in such a way that the function u at some point (x, t) tries to equalize a weighted mean value of the function in a neighborhood of the point. That is, the function $u(x, t)$ evolves, aiming to increase in time at points where the mean value is larger than the value at the point and to decrease when the mean value is smaller. In this informal way of looking at diffusion we expect that the governing equations (that are given in terms of a partial differential equation of parabolic type) are such that the asymptotic behavior of the solutions is given by the convergence as $t \rightarrow \infty$ to a constant. We will not consider here external sources, hence focusing the analysis on the pure diffusive case.

Going into concrete examples, if you think about a linear diffusion equation, probably the first one that will come to your mind is the classical heat equation

$$\frac{\partial u}{\partial t}(x, t) = \Delta u(x, t). \quad (1)$$

The operator involved as the right-hand side of the heat equation is the usual Laplacian, a linear second-order differential operator, given by

$$\Delta u(x) = \sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2}(x).$$

This operator is closely related to a mean value formula; in fact, for a C^2 function, from a simple Taylor expansion we obtain

$$C\Delta u(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \int_{B_\varepsilon(x)} (u(y) - u(x)) dy. \quad (2)$$

Here C is a constant that depends only on the dimension and is given by $C = (\mathcal{f}_{B_1(0)} x_1^2 dx)^{-1}$.

The stationary counterpart to (1) is given by harmonic functions, that is, solutions to

$$\Delta u(x) = 0$$

(which is an elliptic equation). Harmonic functions are well known to satisfy a mean value property in balls; that is, for every $r > 0$, it holds that

$$u(x) = \int_{B_r(x)} u(y) dy.$$

The heat equation (1) is naturally associated with the energy

$$E(u) = \int \frac{|\nabla u|^2}{2}(x) dx$$

in the sense that (1) is the gradient flow associated to $E(u)$. That is, the heat equation can be written as $\frac{\partial u}{\partial t}(t) = -\partial E(u)(t)$; see [Eva10].

Notice that the heat equation can be written as

$$\frac{\partial u}{\partial t}(x, t) = -\operatorname{div}_x(F(x, t)), \quad F(x, t) = -\nabla_x u(x, t).$$

Hence, the heat equation can be viewed as the simplest case of Fick's law. In addition, concerning the physical/atomistic interpretation of diffusion, the heat equation is closely related to probability in the sense that if a particle is sorted at time $t = 0$ according to a probability density u_0 and it moves at random (Brownian motion), then at a positive time $t > 0$ the probability density of the particle is given by $u(x, t)$, the solution to the heat equation with initial datum u_0 [App09].

If you go one step further and consider nonlocal diffusion problems, one popular choice is

$$\frac{\partial u}{\partial t}(x, t) = \int_{\mathbb{R}^N} J(x - y)(u(y, t) - u(x, t)) dy, \quad (3)$$

where $J : \mathbb{R}^N \rightarrow \mathbb{R}$ is a nonnegative radial function with $\int_{\mathbb{R}^N} J = 1$. Here we will consider only integrable kernels. This rules out the fractional powers of the Laplacian for which $J(x - y) \sim |x - y|^{-N-2s}$. For singular kernels there is a very large number of recent references concerning evolution equations and their stationary counterparts, including free boundary problems, regularity results, etc.; see for example [RO16, CROS17]. The corresponding spaces are the well-known fractional Sobolev spaces; see [DNPV12].

Notice that in (3) the diffusion of the density u at a point x and time t depends on the values of u at all points in the set $x + \operatorname{supp} J$, which is what makes the diffusion operator nonlocal. Evolution equations of this form and variations of it have been widely used to model diffusion processes; see for instance [AVMRTM10, CF05, Fif03]. As stated in [Fif03], if $u(x, t)$ is thought of as the density of a single population at the point x at time t , and $J(x - y)$ is regarded as the probability distribution of an individual of the species jumping from location y to location x (see section 4.4), then the rate at which individuals are arriving at position x from all other places is given by $\int_{\mathbb{R}^N} J(y - x)u(y, t) dy$, while the rate at which they are leaving location x to travel to all other sites is given by $-\int_{\mathbb{R}^N} J(y - x)u(x, t) dy = -u(x, t)$. Therefore, in the absence of external or internal sources, the density u satisfies the nonlocal equation (3). In this case there is also an energy that governs the evolution problem, namely,

$$E_J(u) = \frac{1}{4} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x - y)(u(y) - u(x))^2 dx dy.$$

That is, the nonlocal equation can also be seen as a gradient flow, $\frac{\partial u}{\partial t}(t) = -\partial E_J(u)(t)$; see [AVMRTM10].

Notice that the right-hand side of the nonlocal equation can be written as

$$\int_{\mathbb{R}^N} J(x - y)u(y, t) dy - u(x, t).$$

Taking the special kernel

$$J(z) = \frac{1}{|B_\varepsilon(0)|} \chi_{B_\varepsilon(0)}(z),$$

one can look at the usual Laplacian as an infinitesimal version of this mean value nonlocal operator in the sense that (2) holds.

The nonlocal equation (3) shares many properties with the classical heat equation $\frac{\partial u}{\partial t} = \Delta u$, such as: bounded stationary solutions are constant, a maximum principle holds for both of them, and even if J is compactly supported, perturbations propagate with infinite speed, [AVM-RTM10, Fif03]. However, in the nonlocal case there is no regularizing effect.

This kind of equation has been used to model very different applied situations, for example in biology, image processing, particle systems, coagulation models, etc., [HG06, HMMV03, KOJ05]. In particular, we want to mention a nonlocal model for elasticity called peridynamics, introduced in [Sil00], that has become quite popular nowadays. To analyze this model the authors of [DGLZ13] developed a complete nonlocal analogue to the usual vector calculus (including a nonlocal gradient, a nonlocal divergence, etc.). Here the kernel is assumed to be compactly supported, and the size of the support (called the horizon) plays a fundamental role in the model, measuring the range of the interactions inside the elastic material.

2. Boundary Conditions

Now we fix a bounded smooth domain $\Omega \subset \mathbb{R}^N$ and look at diffusion inside Ω . This has to be supplemented with extra conditions on the boundary for the local heat equation and in the complement of the domain for the nonlocal equation.

2.1. Dirichlet boundary conditions. When the heat equation is regarded in a bounded domain one has to impose boundary conditions to obtain a well-posed problem. In particular one can impose that the density/temperature $u(x, t)$ takes a prescribed value (we just take zero for simplicity) and consider

$$\begin{aligned}\frac{\partial u}{\partial t}(x, t) &= \Delta u(x, t), & \Omega \times \mathbb{R}_+, \\ u(x, t) &= 0, & \partial\Omega \times \mathbb{R}_+, \\ u(x, 0) &= u_0(x), & \Omega.\end{aligned}$$

Here we impose that the density vanishes on the boundary of the domain.

Concerning boundary conditions for nonlocal problems we consider

$$\begin{aligned}\frac{\partial u}{\partial t}(x, t) &= \int_{\mathbb{R}^N} J(x-y)u(y, t)dy - u(x, t), & \Omega \times \mathbb{R}_+, \\ u(x, t) &= 0, & \Omega^c \times \mathbb{R}_+, \\ u(x, 0) &= u_0(x), & \Omega.\end{aligned}$$

In this model we have that diffusion takes place in the whole $\mathbb{R}^N$, but we impose that u vanishes outside Ω . This

is the analogue of what is called Dirichlet boundary conditions for the heat equation. Notice that the boundary data needs to be imposed in the set $\{x \notin \Omega : \text{dist}(x, \partial\Omega) < \text{supp}(J)\}$. Here we assume that the individuals/particles that jump outside Ω are killed instantaneously (and therefore the density $u(x, t) \equiv 0$ in Ω^c).

2.2. Neumann boundary conditions. Now, we observe that one can impose that the flux across the boundary of $u(x, t)$ is prescribed (again we just take zero for simplicity) and consider

$$\begin{aligned}\frac{\partial u}{\partial t}(x, t) &= \Delta u(x, t), & \Omega \times \mathbb{R}_+, \\ \frac{\partial u}{\partial \eta}(x, t) &= 0, & \partial\Omega \times \mathbb{R}_+, \\ u(x, 0) &= u_0(x), & \Omega.\end{aligned}$$

Here we impose that the normal derivative of the density, $\frac{\partial u}{\partial \eta}$, vanishes on the boundary of the domain.

Let us turn our attention to the nonlocal case with Neumann boundary conditions. We have the following model for $(x, t) \in \Omega \times \mathbb{R}_+$:

$$\begin{aligned}\frac{\partial u}{\partial t}(x, t) &= \int_{\Omega} J(x-y)(u(y, t) - u(x, t))dy, \\ u(x, 0) &= u_0(x).\end{aligned}$$

In this model we have that the integral terms take into account the diffusion only inside Ω . In fact, as we have explained before, the integral $\int J(x-y)(u(y, t) - u(x, t))dy$ takes into account the individuals arriving at or leaving position x from other places. Since we are integrating in Ω , we are imposing that diffusion take place only in Ω . The individuals may not enter or leave Ω . This is the analogue of what is called homogeneous Neumann boundary conditions in the literature, since there is no flux of individuals to/from Ω from/to the exterior.

3. Analysis of the Nonlocal Problems

The main goal of this section is to prove existence and uniqueness of solutions and to describe their asymptotic behavior. Since some of the arguments are easy to follow and illustrative of what one can expect to do when analyzing a nonlocal problem, we include some ideas of the proofs.

3.1. Cauchy problem. We consider the linear nonlocal diffusion problem in the whole $\mathbb{R}^N$,

$$\begin{aligned}\frac{\partial u}{\partial t}(x, t) &= \int_{\mathbb{R}^N} J(x-y)u(y, t)dy - u(x, t), \\ u(x, 0) &= u_0(x).\end{aligned}\tag{4}$$

In what follows we denote by $\hat{f}$ the Fourier transform of f that is given by

$$\hat{f}(\xi) = \int_{\mathbb{R}^N} f(x) e^{-2\pi i \langle x, \xi \rangle} dx.$$

We will make extensive use of the Fourier transform in order to obtain explicit solutions in the frequency variable, ξ . Let us recall that if $f \in L^1(\mathbb{R}^N)$, then $\widehat{f}$ and $\check{f}$ are bounded and continuous, where $\widehat{f}$ is the Fourier transform of f and $\check{f}$ is its inverse Fourier transform. Moreover,

$$\lim_{|\xi| \rightarrow \infty} \widehat{f}(\xi) = 0 \quad \text{and} \quad \lim_{|x| \rightarrow \infty} \check{f}(x) = 0.$$

We begin by collecting some properties of the function J (recall that J is a radial probability density).

(i) $|\widehat{J}(\xi)| \leq 1$, $\widehat{J}(0) = 1$.

(ii) If $\int_{\mathbb{R}^N} J(x)|x| dx < +\infty$, then

$$(\nabla_{\xi} \widehat{J})(0) = -i \int_{\mathbb{R}^N} x_i J(x) dx = 0;$$

and if $\int_{\mathbb{R}^N} J(x)|x|^2 dx < +\infty$, then

$$(D^2 \widehat{J})_{ij}(0) = - \int_{\mathbb{R}^N} x_i x_j J(x) dx.$$

Therefore $(D^2 \widehat{J})_{ij}(0) = 0$ when $i \neq j$ and $(D^2 \widehat{J})_{ii}(0) \neq 0$.

Hence the Hessian matrix of $\widehat{J}$ at the origin is given by

$$D^2 \widehat{J}(0) = - \left(\frac{1}{N} \int_{\mathbb{R}^N} |x|^2 J(x) dx \right) \cdot \text{Id}.$$

(iii) If $\widehat{J}(\xi) = 1 - A|\xi|^\alpha + o(|\xi|^\alpha)$ for $\xi \sim 0$, then necessarily $\alpha \in (0, 2]$ (otherwise we will have $\int_{\mathbb{R}^N} |x|^2 J(x) dx = 0$, contradicting that J is nonnegative and nontrivial); and if J has a first momentum, then $\alpha \neq 1$.

Now, we prove existence and uniqueness of solutions using the Fourier transform.

Theorem 1. *Given $u_0 \in L^1(\mathbb{R}^N)$ with $\widehat{u_0} \in L^1(\mathbb{R}^N)$, there exists a unique solution $u \in C^0([0, \infty); L^1(\mathbb{R}^N))$ of (4). Moreover, in the Fourier space the solution is given by*

$$\widehat{u}(\xi, t) = e^{(\widehat{J}(\xi)-1)t} \widehat{u_0}(\xi).$$

Proof. Our nonlocal equation can be written as

$$\frac{\partial u}{\partial t}(x, t) = J * u(x, t) - u(x, t).$$

Applying the Fourier transform to this equation we obtain

$$\frac{\partial \widehat{u}}{\partial t}(\xi, t) = \widehat{u}(\xi, t)(\widehat{J}(\xi) - 1).$$

Hence,

$$\widehat{u}(\xi, t) = e^{(\widehat{J}(\xi)-1)t} \widehat{u_0}(\xi).$$

Since $\widehat{u_0} \in L^1(\mathbb{R}^N)$ and $e^{(\widehat{J}(\xi)-1)t}$ is continuous and bounded, the result follows by taking the inverse of the Fourier transform. $\square$

Now we prove a lemma concerning the fundamental solution of (4).

Lemma 2. *Let $J \in \mathcal{S}(\mathbb{R}^N)$, the space of rapidly decreasing functions. The fundamental solution of (4), that is, the solution of (4) with initial condition $u_0 = \delta_0$, can be decomposed as*

$$w(x, t) = e^{-t} \delta_0(x) + v(x, t), \quad (5)$$

with $v(x, t)$ smooth. Moreover, if u is a solution of (4) it can be written as

$$u(x, t) = (w * u_0)(x, t) = \int_{\mathbb{R}^N} w(x - z, t) u_0(z) dz.$$

Proof. By the previous result we have

$$\frac{\partial \widehat{w}}{\partial t}(\xi, t) = \widehat{w}(\xi, t)(\widehat{J}(\xi) - 1).$$

Hence, as the initial datum verifies $\widehat{u_0} = \widehat{\delta_0} = 1$,

$$\widehat{w}(\xi, t) = e^{(\widehat{J}(\xi)-1)t} = e^{-t} + e^{-t}(e^{\widehat{J}(\xi)t} - 1).$$

The first part of the lemma follows by applying the inverse Fourier transform in $\mathcal{S}(\mathbb{R}^N)$.

To finish the proof we just observe that $w * u_0$ is a solution of (4) (using Fubini's theorem) with $(w * u_0)(x, 0) = u_0(x)$. $\square$

Remark 3. The above proof together with the fact that $\widehat{J}(\xi) \rightarrow 0$ (since $J \in L^1(\mathbb{R}^N)$) shows that if $\widehat{J} \in L^1(\mathbb{R}^N)$, then the same decomposition (5) holds and the result also applies.

Moreover, this result shows that the solution can be written as

$$u(x, t) = e^{-t} u_0(x) + \int_{\mathbb{R}^N} v(x - z, t) u_0(z) dz.$$

Since v is smooth, we obtain that there is no regularizing effect ($u(\cdot, t)$ is as smooth as $u_0(\cdot)$ is).

Now we state without proof what is the asymptotic behavior of the solutions as $t \rightarrow \infty$; see [CCR06] for the proof.

Theorem 4. *Let u be a solution of (4) with $u_0, \widehat{u_0} \in L^1(\mathbb{R}^N)$. If there exist $A > 0$ and $0 < \alpha \leq 2$ such that*

$$\widehat{J}(\xi) = 1 - A|\xi|^\alpha + o(|\xi|^\alpha), \quad \xi \rightarrow 0,$$

then the asymptotic behavior of $u(x, t)$ is given by

$$\lim_{t \rightarrow +\infty} t^{N/\alpha} \max_x |u(x, t) - v(x, t)| = 0,$$

where v is the solution to

$$v_t(x, t) = -A(-\Delta)^{\alpha/2} v(x, t),$$

$$v(x, 0) = u_0(x).$$

Moreover, we have

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq C t^{-N/\alpha},$$

and the asymptotic profile is given by

$$\lim_{t \rightarrow +\infty} \max_y |t^{N/\alpha} u(y t^{1/\alpha}, t) - \|u_0\|_{L^1} G_A(y)| = 0,$$

where $G_A(y)$ satisfies $\widehat{G_A}(\xi) = e^{-A|\xi|^\alpha}$.

In the special case $\alpha = 2$, the decay rate is $t^{-N/2}$, and the asymptotic profile is a Gaussian,

$$G_A(y) = (4\pi A)^{N/2} \exp(-A|y|^2/4),$$

with $A \cdot \text{Id} = -(1/2)D^2\hat{J}(0)$. Note that in this case (which occurs, for example, when J is compactly supported) the asymptotic decay is the same as the one for solutions of the heat equation and, as happens for the heat equation, the asymptotic profile is a Gaussian.

The decay in L^∞ of the solutions together with the conservation of mass give the decay of the L^p -norms by interpolation. As a consequence of the previous theorem, we find that this decay is analogous to the decay of the evolution given by the fractional Laplacian, that is,

$$\|u(\cdot, t)\|_{L^p(\mathbb{R}^N)} \leq C t^{-\frac{N}{\alpha}(1-\frac{1}{p})}.$$

3.2. Dirichlet problem. Next we consider a bounded smooth domain $\Omega \subset \mathbb{R}^N$ and impose boundary conditions on our model. From now on we assume that J is continuous.

Consider the nonlocal problem for $(x, t) \in \Omega \times \mathbb{R}_+$:

$$\begin{aligned} \frac{\partial u}{\partial t}(x, t) &= \int_{\mathbb{R}^N} J(x-y)u(y, t)dy - u(x, t), \\ u(x, t) &= 0, \quad \Omega^c \times \mathbb{R}_+, \\ u(x, 0) &= u_0(x). \end{aligned} \quad (6)$$

Before studying the asymptotic behavior, we shall first derive existence and uniqueness of solutions, which are a consequence of Banach's fixed point theorem.

Fix $t_0 > 0$ and consider the Banach space

$$X_{t_0} = \{w \in C([0, t_0]; L^1(\Omega))\}$$

with the norm

$$\|w\| = \max_{0 \leq t \leq t_0} \|w(\cdot, t)\|_{L^1(\Omega)}.$$

We can obtain the existence and uniqueness of solutions as a fixed point of the operator $\mathcal{T} : X_{t_0} \rightarrow X_{t_0}$ defined by

$$\begin{aligned} \mathcal{T}_{w_0}(w)(x, t) \\ = w_0(x) + \int_0^t \int_{\mathbb{R}^N} J(x-y)(w(y, s) - w(x, s))dyds, \end{aligned}$$

for $x \in \Omega$, $t \in [0, T]$, and

$$\mathcal{T}_{w_0}(w)(x, t) = 0, \quad x \notin \Omega.$$

Lemma 5. *Let $w_0, z_0 \in L^1(\Omega)$ and $w, z \in X_{t_0}$. Then there exists a constant C depending on J and Ω such that*

$$\|\mathcal{T}_{w_0}(w) - \mathcal{T}_{z_0}(z)\| \leq Ct_0\|w - z\| + \|w_0 - z_0\|_{L^1(\Omega)}.$$

Proof. We have

$$\begin{aligned} & \int_{\Omega} |\mathcal{T}_{w_0}(w)(x, t) - \mathcal{T}_{z_0}(z)(x, t)| dx \\ & \leq \int_{\Omega} |w_0 - z_0|(x) dx \\ & \quad + \int_{\Omega} \left| \int_0^t \int_{\mathbb{R}^N} J(x-y) [(w(y, s) - z(y, s)) \right. \\ & \quad \left. - (w(x, s) - z(x, s))] dy ds \right| dx. \end{aligned}$$

Hence, taking into account that w and z vanish outside Ω , we obtain

$$\|\mathcal{T}_{w_0}(w) - \mathcal{T}_{z_0}(z)\| \leq \|w_0 - z_0\|_{L^1(\Omega)} + Ct_0\|w - z\|,$$

as we wanted to prove. $\square$

This fixed point argument allows us to obtain existence and uniqueness of solutions.

Theorem 6. *For every $u_0 \in L^1(\Omega)$ there exists a unique solution u , such that $u \in C([0, \infty); L^1(\Omega))$.*

Proof. We check first that $\mathcal{T}_{u_0}$ maps X_{t_0} into X_{t_0} . Taking $z_0, z \equiv 0$ in Lemma 5 we get that $\mathcal{T}(w) \in C([0, t_0]; L^1(\Omega))$.

Choose t_0 such that $Ct_0 < 1$. Now taking $z_0 \equiv w_0 \equiv u_0$ in Lemma 5 we get that $\mathcal{T}_{u_0}$ is a strict contraction in X_{t_0} and the existence and uniqueness part of the theorem follows from Banach's fixed point theorem in the interval $[0, t_0]$. To extend the solution to $[0, \infty)$ we may take as initial data $u(x, t_0) \in L^1(\Omega)$ and obtain a solution up to $[0, 2t_0]$. Iterating this procedure we get a solution defined in $[0, \infty)$. $\square$

Next we look for steady states of (6).

Proposition 7. *$u \equiv 0$ is the unique stationary solution of (6).*

Proof. Let u be a stationary solution of (6). Then

$$0 = \int_{\mathbb{R}^N} J(x-y)(u(y) - u(x)) dy, \quad x \in \Omega,$$

and $u(x) = 0$ for $x \notin \Omega$. Hence, using that $\int J = 1$ we obtain that for every $x \in \Omega$ it holds that

$$u(x) = \int_{\mathbb{R}^N} J(x-y)u(y) dy.$$

Since we assumed that J is continuous, we get that u is continuous in Ω and globally bounded. Set $K = \max_{x \in \bar{\Omega}} u(x)$ and assume that $K > 0$ (the argument assuming that $\min_{x \in \bar{\Omega}} u(x) < 0$ is analogous). Consider the set $\mathcal{A} = \{x \in \Omega \mid u(x) = K\}$. The set $\mathcal{A}$ is clearly closed and nonempty. Now, for $x_0 \in \mathcal{A}$, we have

$$K = u(x_0) = \int_{\Omega} J(x_0 - y)u(y) dy \leq K.$$

This implies that $u(y) = u(x_0) = K$ for all $y \in \Omega \cap B(x_0, d)$, and hence $\mathcal{A}$ is open. Consequently, as Ω is connected, $\mathcal{A} = \Omega$ and $u \equiv K$ is constant. Then the equation, together

with $u(x) = 0$ for $x \notin \Omega$, gives a contradiction that implies that $u \equiv 0$. $\square$

Now, let us analyze the asymptotic behavior of the solutions. As there exists a unique stationary solution, it is expected that solutions converge to zero as $t \rightarrow \infty$. Our main concern will be the rate of convergence. In this case we find an exponential decay given by the first eigenvalue of an associated problem, and the asymptotic behavior of solutions is described by the unique (up to a constant) associated eigenfunction. Let $\lambda_1 = \lambda_1(\Omega)$ be given by

$$\lambda_1 = \inf_A \frac{\frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x-y)(u(x) - u(y))^2 dx dy}{\int_{\Omega} (u(x))^2 dx}$$

with $A = \{u : u|_{\Omega^c} \equiv 0 \text{ and } u \in L^2(\Omega)\}$. We denote by ϕ_1 an associated eigenfunction (a function where the infimum is attained) that is unique up to scalar multiples (the first eigenvalue is simple).

Theorem 8. *For every $u_0 \in L^1(\Omega)$ there exists a unique solution u of (6) such that $u \in C([0, \infty); L^1(\Omega))$. Moreover, if $u_0 \in L^2(\Omega)$, solutions decay to zero as $t \rightarrow \infty$ with an exponential rate*

$$\|u(\cdot, t)\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)} e^{-\lambda_1 t}.$$

If u_0 is continuous, positive, and bounded, then there exist positive constants C and C^* such that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C e^{-\lambda_1 t}$$

and

$$\lim_{t \rightarrow \infty} \max_x |e^{\lambda_1 t} u(x, t) - C^* \phi_1(x)| = 0.$$

Remark 9. Note that the first eigenfunction ϕ_1 is strictly positive in Ω (with positive continuous extension to $\overline{\Omega}$) and vanishes outside Ω . Therefore a discontinuity occurs on $\partial\Omega$, and the boundary value is not taken in the usual "classical" continuous sense.

3.3. Neumann problem. Let us turn our attention to Neumann boundary conditions. We study

$$\begin{aligned} \frac{\partial u}{\partial t}(x, t) &= \int_{\Omega} J(x-y)(u(y, t) - u(x, t)) dy, \\ u(x, 0) &= u_0(x), \end{aligned} \quad (7)$$

with $(x, t) \in \Omega \times \mathbb{R}_+$. Again solutions are to be understood in an integral sense; see Theorem 10.

As happens for the Dirichlet model, existence and uniqueness will be a consequence of Banach's fixed point theorem. The main arguments are basically the same.

Fix $t_0 > 0$ and consider the Banach space

$$X_{t_0} = C([0, t_0]; L^1(\Omega))$$

with the norm

$$\|w\| = \max_{0 \leq t \leq t_0} \|w(\cdot, t)\|_{L^1(\Omega)}.$$

We will obtain the solution as a fixed point of the operator $\mathcal{T} : X_{t_0} \rightarrow X_{t_0}$ defined by

$$\begin{aligned} \mathcal{T}_{w_0}(w)(x, t) \\ = w_0(x) + \int_0^t \int_{\Omega} J(x-y)(w(y, s) - w(x, s)) dy ds. \end{aligned}$$

Again one can show that, given $w_0, z_0 \in L^1(\Omega)$ and $w, z \in X_{t_0}$, there exists a constant C depending only on Ω and J such that

$$\|\mathcal{T}_{w_0}(w) - \mathcal{T}_{z_0}(z)\| \leq C t_0 (\|w - z\| + \|w_0 - z_0\|_{L^1(\Omega)}).$$

With this estimate one can easily prove the following result.

Theorem 10. *For every $u_0 \in L^1(\Omega)$ there exists a unique solution u of (7) such that $u \in C([0, \infty); L^1(\Omega))$. Moreover, the total mass in Ω is preserved along the evolution,*

$$\int_{\Omega} u(y, t) dy = \int_{\Omega} u_0(y) dy.$$

Now we study the asymptotic behavior as $t \rightarrow \infty$. We start by analyzing the corresponding stationary problem, so we consider the equation

$$0 = \int_{\Omega} J(x-y)(\varphi(y) - \varphi(x)) dy. \quad (8)$$

The only solutions are constants. In fact, in particular, (8) implies that φ is a continuous function. Now, we argue as for the Dirichlet case. Set

$$K = \max_{x \in \overline{\Omega}} \varphi(x)$$

and consider the set $\mathcal{A} = \{x \in \overline{\Omega} \mid \varphi(x) = K\}$. The set $\mathcal{A}$ is clearly closed and nonempty. We claim that it is also open in $\overline{\Omega}$. Let $x_0 \in \mathcal{A}$. We then have

$$\varphi(x_0) = \left(\int_{\Omega} J(x_0 - y) dy \right)^{-1} \int_{\Omega} J(x_0 - y) \varphi(y) dy,$$

and $\varphi(y) \leq \varphi(x_0)$. This implies that $\varphi(y) = \varphi(x_0)$ for all $y \in \Omega \cap B(x_0, d)$, and hence $\mathcal{A}$ is open as claimed. Consequently, as Ω is connected, $\mathcal{A} = \overline{\Omega}$ and φ is constant.

We have proved the following proposition:

Proposition 11. *Every stationary solution of (7) is constant in Ω .*

Now we are ready to look at the asymptotic behavior. In this case it is given by an exponential decay to the mean value of the initial condition.

Let β_1 be the first nontrivial eigenvalue associated with our equation. It is given by

$$\beta_1 = \inf_B \frac{\frac{1}{2} \int_{\Omega} \int_{\Omega} J(x-y)(u(y) - u(x))^2 dy dx}{\int_{\Omega} (u(x))^2 dx}.$$

Here $B = \{u \in L^2(\Omega) : \int_{\Omega} u = 0\}$. We remark that β_1 is strictly positive; see [AVMRTM10].

We are now ready to state the result concerning the asymptotic behavior of solutions of (7). For the proof we refer to [CCR06].

Theorem 12. *Let $k = \frac{1}{|\Omega|} \int_{\Omega} u_0$ and let $\beta_1 > 0$ as above. Then the asymptotic behavior of solutions of (7) is described as follows: if $u_0 \in L^2(\Omega)$, then*

$$\|u(\cdot, t) - k\|_{L^2(\Omega)} \leq e^{-\beta_1 t} \|u_0 - k\|_{L^2(\Omega)},$$

and if u_0 is continuous and bounded there exists a positive constant C such that

$$\|u(\cdot, t) - k\|_{L^\infty(\Omega)} \leq C e^{-\beta_1 t}.$$

4. Extra Notes

In this final section we collect some extra results concerning extension of the results to the nonlinear case, non-symmetric kernels, rescales of the equation, and at the end we add something a little more technical concerning the probabilistic interpretation of nonlocal diffusion equations and related geometric objects.

4.1. Nonlinear diffusion. One can consider nonlinear diffusion equations, for example,

$$\frac{\partial u}{\partial t}(x, t) = \Delta_p u(x, t).$$

The operator involved here is the p -Laplacian, given by

$$\Delta_p u(x) = \operatorname{div}(|\nabla u|^{p-2} \nabla u(x))$$

with $p \geq 1$ and associated with the energy

$$E(u) = \int \frac{|\nabla u|^p}{p}(x) dx.$$

If we want a similar nonlocal nonlinear diffusion problem, we obtain

$$\frac{\partial u}{\partial t}(x, t) = \int_{\mathbb{R}^N} J(x-y) |u(y, t) - u(x, t)|^{p-2} \times (u(y, t) - u(x, t)) dy,$$

that is, the gradient flow of the energy

$$E(u) = \frac{1}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x-y) |u(y) - u(x)|^p dx dy.$$

For results concerning this kind of equation, see [AVMRTM10]. There are striking differences between the local and the nonlocal models due to the lack of compactness. For example, for the local problem with $p > 2$ there is finite speed of propagation (for a compactly supported initial condition the solutions are compactly supported for every positive time, with a support that expands with time); however solutions to the nonlocal model with a nonnegative and nontrivial initial datum become positive

everywhere instantaneously. In addition, it is known that the quotient associated with the first Dirichlet eigenvalue

$$\lambda_1 = \inf_A \frac{\frac{1}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x-y) |u(x) - u(y)|^p dx dy}{\int_{\Omega} |u(x)|^p dx}$$

with $A = \{u : u|_{\Omega^c} \equiv 0 \text{ and } u \in L^p(\Omega)\}$ is strictly positive, but it is not known if it is attained (except for $p = 2$, as mentioned previously). Remark that when we take a minimizing sequence, normalized by $\int_{\Omega} |u(x)|^p dx = 1$, we can only extract a subsequence that is weakly convergent in L^p and hence we cannot assert that $\int_{\Omega} |u(x)|^p dx = 1$ in the limit. There is a lack of compactness that is due to the integrability of the kernel.

Another well-studied nonlinear diffusion equation is the porous medium equation

$$\frac{\partial u}{\partial t}(x, t) = \Delta u^m(x, t).$$

There are also nonlocal analogues to this problem such as

$$\frac{\partial u}{\partial t}(x, t) = \int_{\mathbb{R}^N} J(x-y) (u^m(y, t) - u^m(x, t)) dy.$$

For this model with an integrable kernel we refer to [AVMRTM10] and with a singular one to [Váz14].

4.2. Equations that are not in convolution form. One can also consider equations of the form

$$\frac{\partial u}{\partial t}(x, t) = \int_{\mathbb{R}^N} J(x, y) (u(y, t) - u(x, t)) dy,$$

where the kernel is not in convolution form (even if we assume that the kernel is symmetric, i.e., $J(x, y) = J(y, x)$). This makes the Fourier transform useless, and new methods have been developed, inspired by energy considerations; we refer to [IR09].

When the kernel is not symmetric, $J(x, y) \neq J(y, x)$, we obtain possible analogues to a convection-diffusion equation, $u_t = \Delta u + \langle \nabla u, \xi \rangle$; see [AVMRTM10].

4.3. Rescaling the kernel. The main goal now is to show that local evolution problems for the heat equation (in the whole $\mathbb{R}^N$ or with Dirichlet or Neumann boundary conditions) can be approximated by suitable nonlocal problems when the kernel is rescaled. More precisely, for a given J and a given $\varepsilon > 0$ we consider the rescaled kernel

$$J_\varepsilon(\xi) = C_1 \frac{1}{\varepsilon^N} J\left(\frac{\xi}{\varepsilon}\right),$$

with

$$C_1^{-1} = \frac{1}{2} \int_{B(0,d)} J(z) z_N^2 dz.$$

This C_1 is a normalizing constant in order to obtain the Laplacian in the limit instead of a multiple of it. Let $u^\varepsilon(x, t)$

be the solution of

$$\frac{\partial u^\varepsilon}{\partial t}(x, t) = \int_{\Omega} \frac{J_\varepsilon(x - y)}{\varepsilon^2} (u^\varepsilon(y, t) - u^\varepsilon(x, t)) dy.$$

Our main result now reads as follows.

Theorem 13. *Let u^ε be the solution to the nonlocal Cauchy, Dirichlet, or Neumann problem with kernel J_ε and a fixed initial datum u_0 . Under smoothness of the domain Ω and the data, there exist positive constants C and α such that*

$$\sup_{t \in [0, T]} \|v - u^\varepsilon\|_{L^\infty(\Omega)} \leq C\varepsilon^\alpha \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0,$$

where v is the solution to the Cauchy, Dirichlet, or Neumann problem for the heat equation with the same initial condition.

4.4. A probabilistic interpretation. The fundamental works by Doob, Feller, Hunt, Kakutani, Kolmogorov, and many others show the deep connection between classical potential theory and probability theory. The main idea behind this relation is that harmonic functions and martingales have something in common: the mean value formulas. This relation is also quite fruitful in the nonlinear case, and the tug-of-war games are a clear evidence of this fact.

We have to mention the close relation between this kind of evolution problem and probability theory. In fact, when one looks at a Levy process ([Ber96]) the nonlocal operator that appears naturally is a fractional power of the Laplacian. This approach is out of the scope of this note, and we refer to [App09] for a reference concerning the interplay between nonlocal partial differential equations and probability.

Although we are not dealing with probability issues in this work, let us explain briefly why the concrete problem

$$\frac{\partial u}{\partial t}(x, t) = \int_{\mathbb{R}^N} J(x - y)(u(y, t) - u(x, t)) dy \quad (9)$$

has a probabilistic interpretation. Assume that a particle or an individual moves according to the following rules: the particle carries a clock that rings at a random time with exponential distribution (a Poisson clock). When the clock rings, the particle jumps from its actual position in space, x , to a different position, y , that is sorted with probability density $J(x - y)$. Assume that the initial position of the particle is sorted according to a fixed distribution given by $u_0(x)$, and take $u(x, t)$ as the probability density for the particle at time $t > 0$, that is, $\mathbb{P}(\text{particle} \in E \text{ at time } t) = \int_E u(x, t) dx$. Then, we have that $u(x, t)$ solves (9) with initial condition $u(x, 0) = u_0(x)$.

In more abstract terms we can also put all our previous models into a general probabilistic framework. Let $(E, \mathcal{E})$ be a measurable space and let $P : E \times \mathcal{E} \rightarrow [0, 1]$ be a probability transition on E . Then we define a Markovian

transition function as follows. For any $x \in E$, $\mathcal{A} \in \mathcal{E}$, let

$$P_t(x, \mathcal{A}) = e^{-t} \sum_{n=0}^{+\infty} \frac{t^n}{n!} P^{(n)}(x, \mathcal{A}) \quad t \in \mathbb{R}_+,$$

where $P^{(n)}$ denotes the n th iterate of P . The associated family of Markovian operators, $P_t f(x) = \int f(y) P_t(x, dy)$, satisfies

$$\frac{\partial}{\partial t} P_t f(x) = \int P_t f(y) P(x, dy) - P_t f(x).$$

If we consider a Markov process $(Z_t)_{t \geq 0}$ associated to the transition function $(P_t)_{t \geq 0}$, and if we denote by μ_t the distribution of Z_t , then the family $(\mu_t)_{t \geq 0}$ satisfies also a linear equation of the form

$$\frac{\partial}{\partial t} \mu_t = \int P(y, \cdot) \mu_t(dy) - \mu_t.$$

In particular, for $E = \mathbb{R}^N$, if the probability transition $P(x, dy)$ has a density $y \mapsto J(x, y)$, and μ_t has a density $y \mapsto u(y, t)$, then the following equation is satisfied:

$$\frac{\partial}{\partial t} u(x, t) = \int J(x, y) u(y, t) d\lambda(y) - u(x, t). \quad (10)$$

With different particular choices of P we recover the equation studied in the Cauchy, the Dirichlet, and the Neumann cases. For example, if

$$P(x, dy) = J(y - x) dy$$

is the transition probability of a random walk, equation (10) is just our nonlocal equation (9).

4.5. Nonlocal geometry. One can define a notion of nonlocal perimeter of a set K as

$$\text{Per}_J(K) = \int_K \int_{K^c} J(x - y) dx dy.$$

Associated with the variations of this notion of perimeter one has a nonlocal curvature for a point $x \in \mathbb{R}^N$. Just consider

$$\kappa_J(K)(x) = \int_{\mathbb{R}^N} J(x - y) \text{sign}(\chi_K(x) - \chi_K(y)) dx dy.$$

Associated to this curvature one can study a geometric evolution of a set (a nonlocal curvature flow). See [CF17] for singular kernels and [MRT19] for integrable kernels.

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