Remembering Carlos A. Berenstein (1944–2019)

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A Note on Carlos's Mathematical Work

Daniele C. Struppa

Carlos A. Berenstein passed away on August 24, 2019, after a long illness that did nothing to weaken the memory and love that many of us felt for him. I was his first doctoral student, taking a several complex variables class with him in the fall of 1978, as a new graduate student at the University of Maryland. Carlos was the dream advisor for a young, but passionate, aspiring mathematician. As his beloved daughter Nadia recounted in her beautiful eulogy, he used to answer my naive questions with variations of “this is actually not trivial,” something that encouraged my willingness to ask questions, and that betrayed his ability to discover depth in apparent simplicity. In the next few paragraphs I will offer a brief overview of Carlos's work, and the evolution of his wide array of interests. While it is impossible to fully convey the richness of his work, I hope that my notes, complemented by both mathematical and personal remembrances from his friends and colleagues,

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Carlos A. Berenstein, shown in late 2005 outside the Institute for Systems Research at the University of Maryland in College Park, Maryland.

may help us remember his vibrant intelligence and serve as a further stimulus for more work in the areas he loved.

Carlos was a student of Leon Ehrenpreis, and wrote his dissertation in 1970 on convolution operators and quasianalytic classes [15]. It was an exciting time for

Ehrenpreis, who had announced his famous Fundamental Principle only a few years earlier, and whose seminal book [17] appeared also in 1970 (Carlos being among the students whom Ehrenpreis thanks in the introduction to his book). The Fundamental Principle was a groundbreaking result that demonstrated that every solution of a system of linear constant coefficients partial differential equations could be represented as convergent integrals of exponentials on the algebraic variety associated to the zeros of the symbols of the differential operators. Naturally, in order to take into account the multiplicities, it was not enough to consider exponentials, but one had to apply some polynomial coefficients differential operators (known as the Noetherian operators) to those exponentials in order to obtain all possible solutions. The proof of this result relies on the ability to study the quotients of polynomial ideals in suitable spaces of entire functions with growth conditions.

Ehrenpreis spent a chapter in [17] indicating possible ways to extend his result to the case in which differential operators are replaced by convolution operators. In this case (which in the one-variable setting had been studied and understood by L. Schwartz), the situation becomes increasingly complex because the ideals to be considered are now generated by the Fourier transform of the convolutors. These Fourier transforms are entire functions satisfying suitable growth conditions established by some Paley–Wiener-like theorem, and thus one has to replace the algebraic variety of the zeros of polynomials with the analytic variety associated to the zeros of these functions. This introduces several difficulties, the first of which is the fact that the multiplicities of such varieties are unbounded.

Those were the kinds of problems that Carlos began to work on, and his dissertation begins the path that will eventually lead him to a formulation of the Fundamental Principle for convolution equations. In this period Carlos began a very fruitful collaboration with Milos Dostal, himself a student of Ehrenpreis, and together they wrote a series of papers on analytically uniform spaces (the spaces in which the Fundamental Principle could be proved) that culminated with [3], in which Carlos and Dostal prove a first extension of the Fundamental Principle to a very special case of convolution equations. This work (as well as those that preceded it) clarifies the kinds of techniques that will eventually be necessary to attack the problem in greater generality, and especially the role played by lower estimates, not just for polynomials (as Ehrenpreis had done), but for the Fourier transforms of convolutors, such as exponential sums, and exponential polynomials.

As Carlos continued to work on this project, a second important collaboration began to be established, the one with Alan Taylor, and this gave birth to several important papers, including [11], where Carlos and Taylor reimagine

interpolation for entire functions of one complex variable from a totally new viewpoint. This approach eventually allowed them to obtain a Fundamental Principle for convolution equations in [2]. Clearly, a full extension of the Fundamental Principle to solutions of convolution equations is not possible, and some restrictions have to be imposed on the convolutors for such a result to be conceivable. These restrictions (that allow division with bounds) gave rise to the notion of *slowly decreasing* convolutors, a notion whose name derives from some early work of Ehrenpreis. The key ingredient towards the integral representation of solutions of convolution equations is a new interpolation formula that Carlos and Taylor attribute, in its early form, to Jacobi, and that they exploit to great success. The extension of their result to the case of general rectangular systems of convolution equations constituted later the first part of my own dissertation.

This was a period of great creative productivity for Carlos, who, in addition to his collaboration with Taylor, also worked with Larry Zalcman on the Pompeiu problem, bringing to fruition several significant papers (e.g. [14]), as well as with me on some further generalizations of the Fundamental Principle, [10], and on some ideas first enunciated by Ehrenpreis on the Fabry gap theorem [9]. Both these themes, Pompeiu and Fabry, remained a constant in Carlos's production, and he continued to contribute to a more detailed understanding of these questions throughout the years.

But possibly the most important scientific encounter in Carlos's career was with a young French mathematician, Alain Yger, who became his long-term collaborator on a series of truly groundbreaking papers. Yger was a student of Yves Meyer, and I remember Carlos's showing me some of Yger's early papers on difference-differential equations. In the early 1980s Carlos and Yger began a collaboration on the problem of deconvolution (described in more detail in one of the contributions below). The problem has important applications in image reconstruction, and immediately leads to the study of Bezout-like equations in spaces of entire functions with growth conditions. Carlos and Yger were interested in finding effective and constructive ways to solve these Bezout equations. This area of research led Carlos to begin significant collaborations with engineers (such as J. Baras and P. Krishnaprasad) and more applied mathematicians (something that he will continue to do until the end of his career), but his most important results remain those connected with effective Bezout identities and residue calculus. While we are not able to offer in this article a more detailed discussion, we will at least refer the reader to [12] and [13], among Carlos's finest articles.

In 2003 Ehrenpreis published a long-awaited new work, his major contribution to the theory of the Radon

transform. This book had an extremely long gestation, but when it appeared it rekindled Carlos's interest in this subject, and the study of the Radon transform and its applications became one of the major themes in the last part of his career, as recounted below by Peter Kuchment.

Carlos was not only a very prolific and successful researcher. He also appreciated the importance of transferring his research knowledge into original textbooks. Examples of this additional contribution he has given the mathematical community are the two volumes he coauthored with Roger Gay, [5], [6], as well as the work that brought him again together with Yger, Gay, and another student of his, Alekos Vidras, [7].

Carlos's mathematics spanned a variety of different arenas, and in so doing he touched and worked with many mathematicians whose careers he helped define, many of them participating in this memorial article. Carlos's generosity and impact on so many of us have been inestimable; his career has not only advanced mathematics, but allowed many younger mathematicians to come of age, and to learn from him in so many ways.

Daniele C. Struppa

Mark Agranovsky

Before meeting in person, Carlos Berenstein and I knew each other via publications. I was impressed by his works, especially by his very beautiful article on the Pompeiu problem [16], where he proved that the infinite Pompeiu spectrum implies that the domain is a ball.

The first time we met in Moscow in 1989, I got a feeling like we had already been friends for a long time. Carlos emanated a human warmth and openness. He invited me to visit the University of Maryland. It was my first trip from the Soviet Union to the West and Carlos was my first guide to the open mathematical world. This is how our friendship and collaboration began. I loved to work with him, and to talk to him about life. I liked his funny stories about his young years in Argentina, and his humor,

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decorated by his special smile. I remember Carlos permanently busy with supporting or helping somebody.

He did beautiful mathematics. His mathematical interests were broad: from several complex variables and harmonic analysis, to inverse problems and integral geometry.

Doing joint work in mathematics was for Carlos a form of human friendship. Working with him had no limitations in space and time. Often, while I visited him in Maryland, we started working in his office in the morning, continued at his home, and the workday ended at night, after having dinner at his hospitable home with his nice family.

Carlos was a hard worker, but sometimes his eyes looked too tired, probably a sign of the future illness from which he suffered the last years of his life. He and his wife Elsa visited me in Israel; their present, a painting, has been since then on the wall in our living room and reminds me of the beautiful years of our friendship. It was a nice and important chapter of my life. Carlos was a brilliant mathematician, a most gentle and generous person, a good friend. I was lucky to have met Carlos on the path of my life. "Do not say with sadness: they passed away, but say with gratitude: they were!" (Russian poet V. Zhukovsky, nineteenth century). I will remember Carlos with tenderness and gratitude.

Mark Agranovsky

John Baras

I first met Carlos Berenstein in 1972 when I was a PhD student in applied mathematics and he was an assistant professor of mathematics, at Harvard. I was working on my PhD thesis with Roger Brockett, studying the infinite-dimensional "realization problem" for linear time-invariant distributed parameter systems (i.e., those governed by delay differential equations, partial differential equations, etc.). This was an ambitious undertaking to generalize the then recently developed finite-dimensional

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realization theory by Rudy Kalman. The input-output description of the systems I was investigating was given via convolution operators whose kernels had nonrational Laplace (or Fourier) transforms, which included meromorphic functions, functions with branch cuts, etc. Carlos was an expert in several complex variables, so several of his colleagues at Harvard highly recommended that I meet him and discuss these problems with him. My acquaintance with Carlos quickly developed into a close friendship, as he helped me considerably with guidance and most importantly with constant encouragement through some very difficult moments in my PhD research. He introduced me to both Lars Ahlfors at Harvard and to Kenneth Hoffman at MIT, as I started exploring mathematics that used the Hardy function algebras on the disk and on half planes, the related invariant subspace theory of the shift, ideals in these function algebras, inner and outer functions, and associated factorizations. I still remember very gratefully Carlos's encouragement to continue my ambitious research program despite recommendations from both Ahlfors and Hoffman to change the objectives of my PhD research. Carlos helped me through the proof and understanding of the Carleson Corona Theorem and guided my ideas in trying to connect the spectra of the partial differential operators resulting from my theory with the singularities of the Laplace transforms of the convolution kernels in the input-output description of the systems I was studying. I graduated in 1973 and came to the University of Maryland as an assistant professor in the Electrical and Computer Engineering Department and the Applied Mathematics and Scientific Computation Interdisciplinary Program.

I cannot describe easily my pleasant surprise and joy when I ran into Carlos again, as he had joined the Mathematics Department at the University of Maryland the same year. We became very close friends and developed research collaborations in several different areas, and coadvised several students. Our families became close, and we frequently spent happy times at each other's home, as we were living very near to each other. Our two sons, Alex and Aris, grew up with his daughter Nadia and son Ariel. My wife Mary and Elsa became very close friends as well. Quite often we had interesting discussions and debates on various social and political issues. But none of our discussions matched the intensity of Carlos's political discussions about the situation in Argentina when he was visited by colleagues and friends of the same national origin. Carlos was an optimistic and joyful family man, a great and caring father, who participated fully and enthusiastically in various social, cultural, and family events and celebrations. He was very proud of his children Nadia and Ariel.

My friendship and collaboration with Carlos continued throughout his life. I worked further to develop frequency domain methods for the design of controllers for these infinite-dimensional systems, which quickly led me to Bezout (aka Diophantine) equations describing the solution (this is the same as the deconvolution problem) involving H^∞ functions in one or several complex variables, an area where Carlos was a world expert [12], [13]. With Carlos's encouragement I developed a module theory for such infinite-dimensional systems, generalizing significantly the module theory of Kalman for finite-dimensional systems, a complexity and dimensionality theory that employed the division of inner functions associated with some of these function ideals that emerged from the realization constructions.

With Carlos and my PhD student Nikos Sidiropoulos we worked on morphological signal processing, a discrete random set approach to signal processing and filtering, with applications to image processing, computer vision, medical image diagnostics, object recognition, deconvolution and multisensor fusion. We coauthored many papers on this fascinating subject and successfully applied our resulting algorithms to many practical problems. We also worked with Carlos and my PhD student Ziv Berman on wavelet transforms and in particular on the parametric wavelet maxima representation, which led us to prove a version of the famous Marr Conjecture regarding the reconstruction of images from its multiscale edges.

Later on, we worked with Carlos and David MacEnany on algorithms for electrical impedance tomography (EIT) (various versions of the Calderon problem), one of Carlos's favorite areas, where we developed new algorithms exploiting Radon transform on symmetric spaces. We implemented the resulting new algorithms in various medical diagnostic problems under funding from the NIH, and showed that these new algorithms outperformed the then state of the art. Carlos had immense knowledge in several complex variables and hyperbolic spaces. He transferred his enthusiasm for these mathematics and associated computations to me and several students. With Carlos we co-supervised the PhD thesis of Franklin Gavilanez, and investigated the extension and application of the EIT mathematics and algorithms to general network tomography problems, including communication and sensor network problems. We coauthored several invited papers on these mathematics and algorithms. The influence and inspiration on my work from Carlos's enthusiasm for hyperbolic spaces, and various graph embeddings in hyperbolic spaces, led me more recently to the development of efficient and fast routing algorithms that employ greedy hyperbolic embedding for mobile (wireless) ad hoc networks.

I could go on for a long time describing our close collaborations and friendship. Carlos was a brilliant mathematician, curious as a child to apply novel mathematics to challenging problems, always enthusiastic about such works and investigations, with great humor, an engaging, almost permanent smile, and an unmatched collaborative spirit. He worked with several colleagues in engineering and applied mathematics at Maryland and he was heavily involved in the effort that established the Institute for Systems Research (ISR), initially as one of the first six NSF Engineering Research Centers (ERCs), and later as a permanent institute at Maryland. For his contributions to the development and research of ISR Carlos was named a Permanent Joint Appointment of the ISR in 1995 (one of only five such appointments in the 35-year history of the ISR). Carlos's brilliance, positive attitude, and open collaborative spirit will be missed by many.

John Baras

Der-Chen Chang and Bao Qin Li

We both first met Carlos at the University of Maryland at College Park around the beginning of the 1990s, Bao Qin as his PhD student and Der-Chen as an assistant professor. At that time, Carlos had several other students, and also visitors, among whom were L. Aizenberg, C. T. Chuang, R. Gay, M. Morimoto, A. Yger, and others. When Carlos walked the hallways of the department, he was always among a group of several people, leading an animated conversation. He would stop at our offices and chat with us often; sometimes we met in the departmental lounge, which had blackboards surrounding the wall of the room so we could eat and talk and write things on the board. He also stopped by Der-Chen's house on his way home occasionally to discuss mathematics. Carlos was so knowledgeable and insightful that he could talk to and collaborate with many mathematicians from different areas of pure and

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applied mathematics, and with his profound breadth and depth of understanding, he could connect innumerable ideas and discover new problems. During the period of time all three of us were in Maryland, we wrote six papers together, and at the same time Carlos worked with Bao Qin separately on interpolation problems in several complex variables, and with Der-Chen on Pompeiu problems in the Heisenberg group.

The interpolation problem Carlos was interested in can be phrased in a very broad fashion. Roughly speaking, characterizations for interpolating varieties are sought so that every holomorphic function with certain growth constraints on a given analytic variety has a global extension to an entire function in the whole space satisfying the same kind of growth constraints. Being one of the major problems in several complex variables, it has very rich connections to other problems in mathematics, such as harmonic analysis, number theory, and systems theory. For example, the problem (in special cases) is equivalent to some questions in harmonic analysis about solutions of the (overdetermined) system of convolution equations $u_i * f = g_i$ ($1 \leq i \leq N$), where u_i (compactly supported distributions) and g_i are known, and f is the unknown. These questions arise in problems like robust filtering, imaging processing, etc. The system can be solved if one could find deconvolvers v_i such that $\sum v_i * u_i = \delta$ (deconvolution problem). The equivalent formulation of the deconvolution problem, via the Fourier transformations, is to find v_i such that $\sum \hat{v}_i \hat{u}_i = 1$ (division problem). By a theorem of Hörmander, the division problem is solvable if a Lojasiewicz-type inequality holds for $\sum |\hat{u}_j|$ (strong coprimeness), which also plays a role in transcendental number theory. Carlos did seminal work in these areas, besides his many other achievements, which are reflected in his over one hundred publications. An example was his series of important work with B. A. Taylor, which resulted in elegant characterizations of interpolating varieties and led to fruitful connections to other problems such as the seemingly unrelated transcendental Bézout problem, originally due to P. A. Griffiths, on entire analytic mappings.

The Pompeiu problem Carlos introduced to Der-Chen can be stated as follows. Let X be a locally compact topological space, let μ be a nonnegative invariant Radon measure on X , let $\{\gamma_j\}_{j=1}^N$ be a finite collection of compact subsets in X , and let G be a topological group acting on X and leaving the measure μ invariant. Consider the *Pompeiu transform*: $P : C(X) \rightarrow (C(G))^N$ defined by $(Pf)(\mathbf{g}) := \int_{\mathbf{g}\gamma_j} f d\mu$, where P_j is the j th component of the transform P and $\mathbf{g}\gamma = \{\mathbf{g} \circ x : x \in \gamma\}$.

We say the family $\{\gamma_j\}_{j=1}^N$ has the *Pompeiu property* if P is injective. Note that for this definition to be nontrivial

one essentially needs the action of G to be transitive. The *Pompeiu problem* is to decide *as explicitly as possible* whether the family has the Pompeiu property. For example, let μ_r be the normalized surface measure on the sphere $C_r(0) = \{x \in \mathbb{R}^n : |x| = r\}$. Then we want to know whether the spherical means of f , $f * \mu_r = 0$ imply that $f \equiv 0$. If $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$, by taking the Fourier transform,

$$\widehat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx.$$

This implies that $\widehat{f * \mu_r} = \widehat{f} \cdot \widehat{\mu_r} = 0$ and hence $f \equiv 0$. This is usually called a *one radius theorem*. Or, the sphere $C_r(0)$ satisfies the Pompeiu property. However, if we take

$$f(x) = \phi_\lambda(x) = c_n \frac{J_{\frac{n}{2}-1}(\lambda|x|)}{(\lambda|x|)^{\frac{n}{2}-1}},$$

where J_α is the Bessel function of order α , then $f * \mu_r(x) = \phi_\lambda(r) \phi_\lambda(x)$, where $\phi_\lambda(r)$ stands for $\phi_\lambda(y)$ with $|y| = r$. Therefore, if we choose $r \in \mathbb{R}$ such that $J_{\frac{n}{2}-1}(\lambda r) = 0$, then $f * \mu_r = 0$. Note that $\phi_\lambda \in L^p(\mathbb{R}^n)$ for $p > \frac{2n}{n-1}$. Hence, we have:

- (1) If $p > \frac{2n}{n-1}$, a one radius theorem is not true for $L^p(\mathbb{R}^n)$;
- (2) If $1 \leq p \leq \frac{2n}{n-1}$, $C_r(0)$ indeed satisfies the Pompeiu property for any $r > 0$ by a theorem of S. Thangavelu.

When $n = 2$, then by Green's theorem,

$$f * \mu_r(\zeta) = \int_{C_r(\zeta)} f(z) dz = 2i \int \int_{D_r(\zeta)} \frac{\partial f}{\partial \bar{z}} dV(z)$$

for $f \in C^1(\mathbb{R}^2)$. Here $C_r(\zeta) = \{z \in \mathbb{C} : |z - \zeta| = r\}$ and $D_r(\zeta) = \{z \in \mathbb{C} : |z - \zeta| \leq r\}$. Therefore, in $\mathbb{C}^1$, the Pompeiu property implies the analyticity of the function f . Morera's theorem is a well-known theorem in complex analysis: Let $f \in C(\mathbb{C})$. Then f is an entire function in $\mathbb{C}$ if $\int_\gamma f(z) dz = 0$ for all piecewise C^1 Jordan curves γ in $\mathbb{C}$. In fact, the condition for Morera's theorem is too strong. Of course, there is no L^p entire function in $\mathbb{C}$. Obviously, one radius is definitely not enough. However, L. Zalcman first had obtained a so-called *two radii theorem*: Let $f \in C(\mathbb{R}^2)$ and suppose there exist positive numbers r_1, r_2 such that $\int_{C_{r_j}(\zeta)} f(z) dz = 0$, $j = 1, 2$ for every $\zeta \in \mathbb{C}$. Then f is holomorphic as long as r_1/r_2 is not an element of $\mathcal{Q}(J_1)$, the set of quotients of nontrivial zeros of Bessel functions of order 1. In case $r_1/r_2 \in \mathcal{Q}(J_1)$ then f need not be holomorphic anywhere. Since then, Carlos and his collaborators obtained many good results in (real and complex) Euclidean spaces along this research direction. Carlos told Der-Chen maybe one can consider similar problems on the boundary of a "reasonable" domain in $\mathbb{C}^n$. The most natural example will be the Heisenberg group, which can

be considered as a good local approximation to the boundary of a strongly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. That was the first time we worked with M. Agranovsky and D. Pascuas. We finished a series of papers on a characterization of L^p , $1 \leq p \leq \infty$, CR functions in the Heisenberg group by studying Gelfand pairs. The main ingredients to prove those theorems were based on Laguerre calculus (first studied by P. Greiner), and the Hulanicki–Ricci–Wiener–Tauberian theorem.

Carlos was a kind person, a great advisor, and a wonderful friend. He shared his mathematical ideas with his friends, students, and collaborators generously. He had a profound impact on us and many others throughout his life. We will forever remember his contributions to our field, the Christmas Eve's we spent together with his family at his home, his warm voice, his big-hearted smiles.... To us, his passing was a great loss of a teacher, mentor, and friend. We will miss him deeply.

Der-Chen Chang

Bao Qin Li

Roger Gay

Having heard M. Dostal in Strasbourg talk about L. Ehrenpreis, I already knew when arriving in 1977 at Bordeaux the proximity and importance of Carlos to his thesis supervisor. I met Carlos for the first time when he responded favorably to my invitation to teach the last course of a year of study devoted to his joint work with B. A. Taylor [11]. My vivid memory of this day remains the certainty of having been in the presence of an influential guide with extensive knowledge eager to share his enthusiasm and passion.

This day initiated the adventure (parallel to many others around the world) common to the mathematics department of the University of Bordeaux and its counterpart at Maryland (College Park). The Bordeaux protagonists, besides me, were my former student and very dear friend A. Sebbar, whom Carlos invited to spend a year at College Park, and with whom he wrote a very nice paper, and A. Yger, with whom he developed many important works.

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I thank providence for having granted me the presence of a kind man with a brilliant brain, a generous heart, constantly happy, with a bright smile, who made me enjoy the benevolent welcome of his family during our years of work together. We took advantage of these awesome circumstances to develop, among other things, some works dealing with inverse problems of the *Pompeiu problem type*, [4], as well as the writing of a complex analysis textbook mainly based, as [11] was, on the existence of solutions for the inhomogeneous problem of the Cauchy–Riemann equations.

Our cooperation lasted a few years after I retired. This decision was unfortunately followed, after a few years, by the sad news of the illness and recent death of Carlos. I am one of the many devastated people for whom Carlos will be sorely missed. I will endeavor to keep in mind the precious and joyous days that were one of the great gifts of my life.

Roger Gay

My Dear Friend Carlos Berenstein

Peter Kuchment

It is hard to believe that I am taking part in writing an obituary for my dear friend and colleague Carlos Berenstein, with whom I had communicated, met, and worked since the end of the 1970s. As I explain below, I am indebted to him not only for the pleasures of great friendship and collaboration, but also for his and his wife Elsa's encouragement and crucial help in my settling in the US.

From the end of the 1970s to the beginning of the 1980s, I started studying properties of solutions of periodic elliptic and hypoelliptic PDEs. This required, in particular, handling various difficult issues of the theory of several complex variables. Among these were intricate

interpolation theorems for entire functions of some classes, which I found in the joint works by Carlos with B. A. Taylor and Carlos's Lecture Notes in Mathematics book with M. Dostal. Thus an active, and extremely helpful for me, letter exchange with Carlos ensued. The resulting theory that was significantly helped by communicating with Carlos was published in several major articles and surveys, as well as in my 1993 book on Floquet theory for PDEs. The communication with Carlos intensified sometime later, when a chain of associations led me to harmonic analysis on symmetric spaces. By a strange turn of fate, which would repeat itself later, Carlos had also started working in this area. Due to this direction, I have had very significant exchanges of letters with Carlos and with another hero of mine, Sigurdur Helgason.

In 1989, Carlos and his wonderful wife Elsa visited Moscow. Notified about this trip by Carlos, I could not possibly miss the opportunity to finally meet him in person. I thus took the "lifeline" overnight train from my city Voronezh to Moscow ("lifeline," since it was used to bring meat, cheese, and other food unavailable in Voronezh back to our city, where much of it was in fact produced). We met with Carlos and Elsa and spent wonderful hours walking around Moscow. I realized what a friendly and cheerful person Carlos was. At some moment, discussing mathematics, I mentioned, "You know what? I have stumbled upon another great area!" "What is it?" asked Carlos. "It is tomography and related issues of integral geometry." And here Carlos started laughing and told me that in fact he had also just moved to tomography. What can one say? It was fate! The fate has not stopped at this, and years later we both independently started working on problems on graphs.¹

This meeting was also fateful for me in a more serious way. Elsa, getting acquainted with Moscow, said, "How can you live in this depressing country? You need to leave" (and she had not yet set her foot outside the capital!). However, during those Gorbachev times I still hoped for the better future of the Soviet Union, and thus did not plan to emigrate (although my parents and siblings with their families had left a decade before). Soon, however, things took a sharp turn for the worse. I was planning to visit in June–July 1989 for the first time my mother and sisters, who lived in New York City. A few days before my departure a pogrom of Turks in Fergana Valley in Central Asia happened, with many casualties. It became known that the authorities had known of the planned massacre in advance, but did not try to protect anyone. This is when I understood that Elsa was right, and I had to take care of

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¹The same kind of independent parallel turns in the areas of research we have also experienced with another great friend of mine and mathematician, Michael Shubin.

the safety of my family, since the government apparently could not care less. It was clear that bloodshed was ahead. And here my next story about Carlos started.

When I arrived in the US, I could speak only very limited English, to put it mildly, and had almost no idea what to expect. My first impression was that Americans were incomparably nicer to others than Russians normally were. I am not talking just about mathematicians, who accepted me extremely well (although I was not a prominent and well-known figure), but complete strangers everywhere (and I have traveled extensively to other states, visiting universities and participating in conferences). In particular, Carlos Berenstein invited me after my arrival to the University of Maryland to give a lecture. I had the privilege to stay in his home and enjoy the company of his wonderful family. I told Carlos about my plan to emigrate as soon as possible, due to the aggravating situation in the USSR. I knew practically nothing about what one should do to apply for jobs, what is a c.v., etc. Carlos explained all this to me, provided samples, and helped with materials preparation. Fortunately, I could get letters of reference from several prominent mathematicians in the US and abroad, but the job situation was very tough, in particular due to the huge influx of highly qualified mathematicians from the USSR and Eastern Europe. Thus I had to apply to hundreds of places (essentially to anything available). The problem was that I was about to return to the USSR and had no chance (as well as no money) to print and mail hundreds of packages. Carlos did something that triggered my enormous admiration and gratitude, which have not subsided since: he offered to do all this for me himself, while I would be back in the USSR. And he did complete that enormous task!

When I finally arrived in the US, due to visa delays it was way too late for the interviews to which I was invited. The future looked uncertain and I was preparing for all eventualities: working at a school, computer programming, taking actuarial exams, etc. All this time Carlos was offering me his unwavering support, help, and encouragement. Finally, I found a university position and was able to continue my academic career. All this time, since 1989 and till Carlos's serious illness, we were in close contact, visiting each other, discussing a lot of mathematics of common interest (inverse problems, complex analysis, integral geometry, graph theory, harmonic analysis), and doing joint research. Carlos's enthusiasm and cheerfulness were contagious.

I will cherish the fond memories of my wonderful friend and colleague Carlos Berenstein until my last days. I am honored to have been his friend.

Peter Kuchment

Remembering Carlos

Domenico Napoletani

You knew that there was nothing that could change the depth of Carlos's affection and his belief in your potential, once you were taken under his tutelage. Others will describe his accomplishments as a mathematician. I would like to focus on his lasting impact as a mentor to so many students and colleagues, and on the unique way he achieved such an impact.

Throughout the last year of Carlos's career, I had the opportunity to be close to him. The thing that stood out most to me was the way Carlos lifted you up to his level of scholarship and knowledge, until you realized that you had a place there, and that naturally you would manage to express your own understanding of mathematics.

I remember vividly how Carlos would share his delight for integral geometry or for the connection of Nevanlinna theory and Diophantine approximation as clarified by Paul Vojta. The latter was a topic to which Carlos would come back repeatedly in his last years. His own work made Carlos constantly fascinated by the deep connections of number theory and complex analysis, and he believed that more refined growth conditions on the class of meromorphic functions in Nevanlinna theory would allow for even stronger results in Diophantine approximation.

I share this particular example to stress the aspect of Carlos's training that I found very unique. Unlike many mathematicians, he did not reach for a blackboard the moment he would think about mathematics. Instead, he would love to chat at length about ideas and methods, and about the likes and dislikes of his many distinguished friends.

These extended, leisurely conversations allowed students to develop something that is not easy to share, and that most people do not care to share: a taste for ideas. Eventually you would realize that the most technical work

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did not appear daunting anymore, because you understood the deep motivations for its development. This same kind, affirming attitude and gracious sharing of his intellect was also extended to junior faculty and Carlos's peers. He strongly rejected the way most departments evaluated their own faculty, through a relentless focus on constant publishing and grant applications, because Carlos did not need the proxy of money and statistical data on published papers to judge the contribution of a mathematician.

On this, I recall his appreciation for unpublished manuscripts and for papers that were not widely circulated. For example, he often stated the need to seriously study the theory of resurgent functions of Jean Écalle, which in his view was not as well known in the US as it deserved.

In an academic world that quickly discounts what you have done in the past, that flattens the worth of scholars on their latest papers, Carlos would never forget a collaborator, no matter how far in the past his or her active work was. I had seen him actively look for his previous students (and this is before the internet age) just to make sure they were doing fine or to know that their talent was being put to good use, even if they did not pursue an academic career.

Each of us needs concrete examples of individuals who show us how it is possible to keep our humanity in our work life, how to care for others, and how to grow throughout one's professional career. Carlos was for me, and for many others, such an example.

Domenico
Napolitano

Irene Sabadini

I first met Carlos Berenstein in the winter of 1994, while studying for my PhD thesis, when I was visiting George Mason University. My then advisor Daniele Struppa and I went to see him at College Park, and I was rather worried, since I did not know what to expect. Carlos was very friendly, and we immediately started to discuss what was our common interest at that time, namely, quaternionic

analysis. We visited him several times. The main focus was on regular functions, i.e., functions in the kernel of the so-called Cauchy–Fueter operator $D = \partial/\partial x_0 + i\partial/\partial x_1 + j\partial/\partial x_2 + k\partial/\partial x_3$, in several variables. The topic is difficult, and the noncommutativity plays a crucial role in preventing use of most of the classical techniques available for holomorphic functions in several variables.

One of Carlos's main interests was also the study of residues, and we started also discussing this topic and the few available results in hypercomplex analysis.

For me it was impressive to see how Carlos was quick in understanding links between different topics and to pass from one to another, his knowledge being very deep and wide. I was also impressed by how, after long discussions, he could go back to the original topic to close the loop. We discussed these topics a lot and eventually the idea of using an algebraic approach emerged. This was a turning point, since it gave us the opportunity to start studying the Cauchy–Fueter system and similar ones, using the techniques introduced by Ehrenpreis and Palamodov in the late sixties. The main idea was to rewrite the Cauchy–Fueter regularity condition $Df = 0$ as a system in $4n$ real variables, and, similarly, the regularity condition in n variables was expressed by a system of $4n$ real equations in $4n$ real variables. One then considers the (formal) Fourier transform to get a matrix whose entries are monomials in $4n$ real variables and constructs the module generated by its rows. This module, and the minimal free resolution constructed from it, yield many interesting analytical consequences. The construction of the minimal free resolution in this case and similar cases was the object of various studies and required competencies from commutative algebra, which were given at that time by W. W. Adams and P. Lousaunau. As a by-product, this hypercomplex case was also a concrete application of the tools introduced by Ehrenpreis and Palamodov. Our early results in this direction eventually resulted in [1].

I visited George Mason University various times after my PhD degree, also spending a semester there, and every time I visited Carlos he was always eager to discuss new problems. He was very generous in sharing ideas and giving advice and suggestions.

I recall a somewhat tender moment when in 2004 there was a conference at GMU in honor of Carlos's sixtieth birthday and Carlos was protesting that he was too young to deserve such a celebration! Some of his youth does indeed transpire from the pictures that were taken at that conference. For me it was a great honor to have the opportunity to edit, with Daniele Struppa and David Walnut, a Festschrift in his honor.

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I met Carlos one last time at the end of August 2005 when he invited me to give a lecture at College Park. I see this as the last act of generosity towards me.

Irene Sabadini

Carlos Berenstein as I Met and Remember Him

Ahmed Sebbar

Carlos, with his unequaled smile and permanent brilliance, visited quite often the Mathematics Institute of Bordeaux, mainly to work with my former advisor Roger Gay. At that time they were working, if I remember correctly, on some inverse spectral problems and their relation to the Pompiou problem, which eventually got published as [4]. They also planned to write their monumental treatise on complex analysis, published a little later [5], [6]. That is how I met him and how he offered me a one-year stay at the University of Maryland, after I had just defended my thesis. Everything was new for the young man I was, even those lunches that were happening with scrapbook and pencil, where meals and mathematics were mixed up, especially the discovery of a question by A. Hurwitz concerning certain germs of analytic functions (other than the exponential function) whose analytical continuation along certain closed paths gives the derivative. Apart from an example of H. Lewy, an article by A. Naftalevich, and the MS thesis by P. Gilden, nothing was known about this question of Hurwitz. The few years that the Hurwitz question demanded for its complete resolution [8] have been years of learning old and still contemporary mathematics, the works of Abel, Jacobi, Riemann, Kronecker, Appell, Hurwitz, and more recently of Artin and Weil, to give only a few names.

In addition to being a friend and collaborator, Carlos was also a teacher. This is how I will remember him.

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Ahmed Sebbar

Alain Yger

It was in January 1979 while I was preparing my doctoral thesis under the supervision of Yves Meyer that I received a letter from the University of Maryland. "It was a very happy coincidence that you sent me your paper since I had in my desk the first part of it for review ..." were the first words of Carlos that I read that day. This phrase started an uninterrupted collaboration with Carlos that lasted more than twenty-five years. Is "collaboration" really the appropriate term? I remain deeply indebted to Carlos for all my research in mathematics and indeed for much more than that. How to express in a few lines his tremendous generosity, his communicative enthusiasm, his unlimited passion to share ideas and intuitions across all scientific fields, from mathematics to engineering, from physics to computer science, and certainly far beyond the artificial borderline between theory and applications? It is so hard to find the words to express how lucky I was to participate in such an adventure.

From the beginning, Carlos introduced me to fascinating questions. Can distinct zeros of exponential sums with purely imaginary algebraic frequencies and algebraic coefficients be arbitrarily close to each other? How effective can Ax-Schanuel's result about algebraic independence of polynomial and exponential functions be when facing questions such as the spectral synthesis problem with respect to systems of delayed-differential equations? How could one profit from rephrasing division questions in terms of integration by parts based on the use of global or at least semilocal Bernstein-Sato identities? These questions remained constantly present through our collaboration as permanent guidelines. Most of them remain with a question mark. I cannot imagine thinking about them today without my friend Carlos's presence and comments.

At the beginning of everything there was some encounter (most of the time quite unexpected). In front of it, as always, was Carlos's generosity, accompanied by

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his communicative passion to share ideas and, I should add, his fantastic mathematical intuition. Some of those encounters come back to my memory. I remember the discussions with Bernard Teissier and Patrice Philippon in Palaiseau about Briançon–Skoda’s theorem. I also recall the first time we discovered (attending Orsay’s complex analysis seminar around 1982) the construction by Mats Andersson and Bo Berndtsson of weighted Bochner–Martinelli interpolation-division formulae and how Carlos almost immediately guessed that such constructions could lead to explicit analytic Bézout identities, together with applications to signal analysis or optics through the process of deconvolution. How could I forget the unexpected discussion we had at the beginning of 1986 with Dale Brownawell after his colloquium at Johns Hopkins about Nullstellen inequalities? It is indeed from it that grew our common interest with Carlos in questions in computational polynomial geometry, involving later on multivariate residue calculus.

Speaking about multidimensional residue calculus, there is an image I keep present in my mind associating it with Carlos. It was during a conference held in 1992 in the old building of the Banach Center in Warsaw, which I was attending with Carlos, Lev Aizenberg, Mikael Passare, and August Tsikh. One morning, the windows of each of our rooms opened almost exactly (pure coincidence) at the same moment. From this time, I really understood what collaboration means and the pleasure it may bring. We were participating all together in the multidimensional residue adventure, windows opening at the same time towards the future. I understand better now why I associate this image with Carlos: I learned from him the pleasure of collaborating and sharing ideas. Carlos was much more than a collaborator. He was my friend. How could I forget the hours spent in Paris, Bordeaux, or most often in Bethesda, in front of the computer or a piece of paper, writing and then rewriting notes with him.

Time was running sometimes so fast, as it was during the genesis of our 1993 Birkhäuser “Green book” [7] about Bézout identities and multidimensional residue theory, with our common friend Roger Gay and Carlos’s former student Alekos Vidras. Our efforts with Alekos, since 2010, were to update this book, incorporating all the new developments of analytic multidimensional residue theory occurring for the last three decades (almost). This work would not make sense without the constant presence of Carlos beside us, despite his so cruel, untimely illness. I feel impatient for this long-term project to be carried out and finalized.

My words here were about the mathematician, but I would have had so much to say about Carlos beyond mathematics: Carlos and Elsa, Nadia and Ariel, our

children growing up together; Carlos and Argentina, speaking about his “mathematical father” Alberto Calderón, about Miguel Herrera or Micha Cotlar, discussing with Alicia Dickenstein or Joos Heintz; Carlos and his roots, that evening I spent walking with him in Kazimierz, while he was searching for his past. I would like to conclude with a funny little story. I remember one day around 1983 our walk (commissioned by my wife to buy some bread) and our return hours later with some mathematical question solved, but no bread! Carlos, we miss you so much but you remain so present.

Alain Yger

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