Hochschild Cohomology

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Introduction

At the end of the nineteenth century, Poincaré created invariants to distinguish different topological spaces and their features, foreshadowing the homology and cohomology groups that would appear later. Towards the middle of the twentieth century, these notions were imported from topology to algebra: The subject of group homology and cohomology was founded by Eilenberg and Mac Lane, and the subject of Hochschild homology and cohomology by Hochschild. The uses of homological techniques continued to grow and spread, spilling out of algebra and topology into many other fields.

In this article we will focus on Hochschild cohomology, which now appears in the settings of algebraic geometry, category theory, functional analysis, topology, and beyond. There are strong connections to cyclic homology and K-theory. Many mathematicians use Hochschild cohomology in their research, and many continue to develop theoretical and computational techniques for better understanding. Hochschild cohomology is a broad and growing field, with connections to diverse parts of mathematics.

Our scope here is exclusively Hochschild cohomology for algebras, including Hochschild's original design [3] and just a few of the many important uses and recent developments in algebra. Some details and further references may be found in [5, 12, 13].

Gerhard Hochschild, 2003.

We will begin this story by setting the scene: We are interested here in a ring A that is also a vector space over a field k such as $\mathbb{R}$ or $\mathbb{C}$. We require the multiplication map on A to be bilinear; that is, the map $A \times A \rightarrow A$ given by $(a, b) \mapsto ab$ for a, b in A is bilinear (over k). A ring with this additional structure is called an *algebra* over k . Some examples are polynomial rings $A = k[x_1, \dots, x_n]$, which are commutative, and matrix rings $A = M_n(k)$, which are non-commutative when $n > 1$.

Our focus will be on multilinear maps

$$A \times \cdots \times A \rightarrow A$$

and what they can tell us about the structure and the representations, that is, modules, of A . Our story has two threads: One starts with particular functions on A called derivations. The other starts with the *center* $Z(A)$ of A , that is, the subalgebra of all elements commuting with every

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element of A . Both are then viewed in a wider, multilinear context. These two threads lead in the same direction and combine in the theory of Hochschild cohomology and in its two important binary operations, the Gerstenhaber bracket and the cup product, as we will see.

Derivations

Every calculus student has seen the Leibniz rule (also known as the product rule),

$$\frac{d}{dx}(fg) = \frac{d}{dx}(f)g + f\frac{d}{dx}(g),$$

for differentiable functions f, g . The set of all differentiable functions from $\mathbb{R}$ to $\mathbb{R}$ forms a ring A that is also a vector space over $\mathbb{R}$, and in fact is an algebra over $\mathbb{R}$.

Generalizing the Leibniz rule to other algebras over our field k , we define a *derivation* on an algebra A to be a k -linear function $\delta : A \rightarrow A$ such that

$$\delta(ab) = \delta(a)b + a\delta(b)$$

for all a, b in A . We will be interested in the set of all derivations,

$$\text{Der}(A) = \{\delta : A \rightarrow A \mid \delta \text{ is a derivation}\},$$

which is itself a vector space under addition and scalar multiplication of functions.

As a small example, let $A = k[x]$, that is, polynomials in one indeterminate x with coefficients in k . Define the derivation $\frac{d}{dx}$ as usual, that is,

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

for all $n \geq 1$ and $\frac{d}{dx}(1) = 0$, and extend linearly so that $\frac{d}{dx}(p)$ is defined for each polynomial $p = p(x)$. A calculation shows that $\text{Der}(A) = k[x]\frac{d}{dx}$; that is, the derivations are precisely the $k[x]$ -multiples of $\frac{d}{dx}$, where, for polynomials p and q , the function $q\frac{d}{dx}$ takes p to q times its derivative $\frac{dp}{dx}$.

Is the space $\text{Der}(A)$ of derivations closed under composition of functions? The answer is *no*, and a counterexample is given by composing $\frac{d}{dx}$ with itself as a function on $k[x]$, resulting in a function that is not itself a derivation. However, $\text{Der}(A)$ is closed under a commutator, or *bracket*. Let

$$[\delta, \delta'] = \delta \circ \delta' - \delta' \circ \delta \quad (1)$$

for all δ, δ' in $\text{Der}(A)$. Calculations show that $[\delta, \delta']$ is indeed a derivation, that this commutator is bilinear in its two arguments, and that it has the following additional properties: $[\delta, \delta] = 0$ and

$$[\delta, [\delta', \delta'']] + [\delta', [\delta'', \delta]] + [\delta'', [\delta, \delta']] = 0$$

for all $\delta, \delta', \delta''$ in $\text{Der}(A)$. That is, the bracket operation $[\ , \]$ is *alternating* and the *Jacobi identity* holds. Put another way, $\text{Der}(A)$ is a *Lie algebra*.

For example, $\text{Der}(M_2(k))$ may be identified with the Lie algebra $\mathfrak{sl}_2(k)$ of 2×2 matrices of trace 0, called the *special linear algebra*, provided the characteristic is not 2. Specifically, each matrix X of trace 0 determines a derivation δ_X given by a commutator: For all Y in $M_2(k)$, set $\delta_X(Y) = XY - YX$. All derivations are of the form δ_X for some X in $\mathfrak{sl}_2(k)$, and distinct matrices X, X' in $\mathfrak{sl}_2(k)$ determine distinct derivations $\delta_X, \delta_{X'}$.

From another vantage point, we see Lie algebras acting on an algebra A by derivations, that is, through Lie algebra homomorphisms to $\text{Der}(A)$. Such actions are ubiquitous and uncover important structural information about A .

Multilinear Functions

We wish to consider more generally multilinear functions

$$\delta : A^n \rightarrow A$$

where $A^n = A \times \cdots \times A$ (n factors). Are there useful multilinear analogs of derivations on A ? The answer is *yes*, and this is where we start to see the ideas of Hochschild from the 1940s.

From now on we will use some homological terminology: A derivation is also called a *Hochschild 1-cocycle*. Now suppose $\delta : A^2 \rightarrow A$ is a bilinear function. It is a *Hochschild 2-cocycle* if

$$\delta(ab, c) - \delta(a, bc) = a\delta(b, c) - \delta(a, b)c$$

for all a, b, c in A . More generally, suppose $\delta : A^n \rightarrow A$ is an n -linear function. It is a *Hochschild n -cocycle* if

$$\begin{aligned} & \sum_{i=1}^n (-1)^i \delta(a_1, \dots, a_i a_{i+1}, \dots, a_{n+1}) \\ &= -a_1 \delta(a_2, \dots, a_{n+1}) + (-1)^n \delta(a_1, \dots, a_n) a_{n+1} \end{aligned}$$

for all $a_1, \dots, a_{n+1} \in A$. Let $C^n(A)$ be the vector space of all multilinear functions $\delta : A^n \rightarrow A$, and set

$$Z^n(A) = \{\delta \in C^n(A) \mid \delta \text{ is a Hochschild } n\text{-cocycle}\},$$

so that $Z^1(A) = \text{Der}(A)$ as before. These historical definitions contain remnants of a template from algebraic topology. We will see that these definitions are meaningful in algebra in a number of ways.

We saw that $Z^1(A) = \text{Der}(A)$ is a Lie algebra. More generally, is $Z^n(A)$ a Lie algebra? The answer is *no*, at least not in a meaningful way. (It is always possible to define the bracket of two vector space elements to be 0, but we are looking for more interesting answers.) Gerstenhaber [2] had an insightful idea in the 1960s.

Murray Gerstenhaber, 2017.

His idea was this: Consider all of the vector spaces $Z^n(A)$ together, setting

$$Z^*(A) = \bigoplus_{n \geq 0} Z^n(A),$$

with $Z^0(A) = Z(A)$ the center of the algebra A . Then $Z^*(A)$ is in fact a graded version of a Lie algebra for which the bracket of two elements in $Z^1(A)$ is as before. If δ is in $Z^m(A)$ and δ' is in $Z^n(A)$, their bracket $[\delta, \delta']$ is defined similarly to the case $m = 1, n = 1$ as in equation (1), but we replace composition of functions with a generalized version of composition. We illustrate for small values of m and n : If δ is in $Z^1(A)$ and δ' is in $Z^2(A)$, define $[\delta, \delta']$ in $Z^2(A)$ by

$$[\delta, \delta'](a, b) = \delta(\delta'(a, b)) - \delta'(\delta(a), b) + \delta'(a, \delta(b)),$$

and if δ, δ' are both in $Z^2(A)$, define $[\delta, \delta']$ in $Z^3(A)$ by

$$(a, b, c) = \delta(\delta'(a, b), c) - \delta(a, \delta'(b, c)) + \delta'(\delta(a, b), c) - \delta'(a, \delta(b, c))$$

for all a, b, c in A . These bracket operations can be visualized with the aid of tree diagrams. For example, the four terms on the right side of the last equation above correspond to the four trees

Each tree indicates on which two arguments to evaluate δ' (respectively, δ) first, then δ (respectively, δ') is evaluated on that output together with the remaining argument. In

general, the bracket is a function

$$[\ , \] : Z^m(A) \times Z^n(A) \rightarrow Z^{m+n-1}(A)$$

for all m, n , called the *Gerstenhaber bracket*, named in honor of Gerstenhaber [2], who first defined it. It can be shown that $Z^*(A)$ is indeed a graded Lie algebra under $[\ , \]$. That is, it satisfies modified versions of the alternating property and the Jacobi identity that include signs depending on the degrees of elements.

Hochschild Cohomology Vector Space

We will define the Hochschild cohomology vector space in degree n as a quotient of the vector space $Z^n(A)$ of Hochschild n -cocycles as follows. Let $\delta' : A^{n-1} \rightarrow A$ be any $(n-1)$ -linear function. Define $\delta : A^n \rightarrow A$ by

$$\begin{aligned} \delta(a_1, \dots, a_n) &= a_1 \delta'(a_2, \dots, a_n) \\ &+ \sum_{i=1}^{n-1} (-1)^i \delta'(a_1, \dots, a_i a_{i+1}, \dots, a_n) \\ &+ (-1)^n \delta'(a_1, \dots, a_{n-1}) a_n \end{aligned}$$

for $a_1, \dots, a_n$ in A . Call such a function δ a *Hochschild n -coboundary*. Recall that $C^n(A)$ is the vector space of all multilinear functions from A^n to A . Let

$$\begin{aligned} B^n(A) &= \{ \delta \in C^n(A) \mid \delta \text{ is a Hochschild } n\text{-coboundary} \} \end{aligned}$$

for $n > 0$ and $B^0(A) = 0$. A calculation shows that $B^n(A)$ is a subspace of $Z^n(A)$ for each n . Now let

$$\begin{aligned} \text{HH}^n(A) &= Z^n(A)/B^n(A) \\ \text{and } \text{HH}^*(A) &= \bigoplus_{n \geq 0} \text{HH}^n(A). \end{aligned}$$

The latter is named the *Hochschild cohomology vector space* of A in honor of Hochschild [3], who first defined it. It can be shown that the Gerstenhaber bracket $[\ , \]$ as defined on $Z^*(A)$ passes to this quotient, making $\text{HH}^*(A)$ a graded Lie algebra.

There is an important dual notion, *Hochschild homology*: Vector spaces $\text{HH}_n(A)$ are defined analogously as quotients of subspaces of some spaces constructed from n copies of A . There are also more general versions of both Hochschild cohomology and homology, defined for A -bimodules M , denoted $\text{HH}^*(A, M)$ and $\text{HH}_*(A, M)$. We will not need the details of these constructions in this article.

As a small Hochschild cohomology example, consider $\text{HH}^n(k[x])$. This space turns out to be nonzero only when $n = 0$ and $n = 1$. When $n = 0$, it is simply the algebra $k[x]$. When $n = 1$, it is essentially the space of derivations $k[x] \frac{d}{dx}$ that we saw before since $B^1(k[x])$ turns out to be 0. Thus

$$\text{HH}^*(k[x]) = k[x] \oplus k[x] \frac{d}{dx}.$$

As another small example, take a polynomial ring in two indeterminates, $A = k[x, y]$. Then

$$\mathrm{HH}^*(A) = A \oplus \left(A \frac{\partial}{\partial x} \oplus A \frac{\partial}{\partial y} \right) \oplus A \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}.$$

The two middle summands above constitute $\mathrm{HH}^1(A)$ and together correspond to all A -linear combinations of the standard derivations $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. The first summand above is $\mathrm{HH}^0(A)$, corresponding to the algebra A itself. The last summand above is $\mathrm{HH}^2(A)$, corresponding to all A -multiples of the bilinear map $\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ given by applying $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ to the first and second arguments, respectively, then multiplying. The dual notion of Hochschild homology, $\mathrm{HH}_n(A)$, turns out to be the same vector space as Hochschild cohomology $\mathrm{HH}^n(A)$ in each degree n for this example as well as for the previous one.

These polynomial ring examples can be viewed as special cases of the Hochschild–Kostant–Rosenberg theorem [4]. The theorem states that for a smooth finitely generated commutative algebra A , the Hochschild cohomology vector space $\mathrm{HH}^*(A)$ is isomorphic to the space of polyvector fields, and there is a dual version for Hochschild homology. The hypothesis that A is smooth and finitely generated translates in a precise way to the statement that A is a ring of functions on a space that is smooth, that is, has no cusps or other singularities. The cohomology isomorphism of the Hochschild–Kostant–Rosenberg theorem is in fact an isomorphism of graded Lie algebras, and even further of Gerstenhaber algebras.

Gerstenhaber Algebra Structure

The bracket operation on the Hochschild cohomology space $\mathrm{HH}^*(A)$ was Gerstenhaber’s idea, as described above. It extends the Lie algebra structure on derivations of A and on $\mathrm{HH}^1(A)$, as we have seen. There is another, simpler, binary operation on the Hochschild cohomology vector space $\mathrm{HH}^*(A)$ that instead extends the associative algebra structure on the center $Z(A) = Z^0(A) = \mathrm{HH}^0(A)$ of the algebra A . We turn to this structure, given by a *cup product*, next. Letting δ be a Hochschild m -cocycle and δ' a Hochschild n -cocycle, define

$$\begin{aligned} (\delta \smile \delta')(a_1, \dots, a_{m+n}) \\ = (-1)^{mn} \delta(a_1, \dots, a_m) \delta'(a_{m+1}, \dots, a_{m+n}) \end{aligned}$$

for all $a_1, \dots, a_{m+n}$ in A . Then $\delta \smile \delta'$ is a Hochschild $(m+n)$ -cocycle. This product can be visualized with a tree

diagram. For $m = 2, n = 2$, the tree is

where μ is the multiplication map. This cup product on cocycles induces an associative product

$$\smile : \mathrm{HH}^m(A) \times \mathrm{HH}^n(A) \rightarrow \mathrm{HH}^{m+n}(A).$$

It can be shown that $\mathrm{HH}^*(A)$, under cup product, is a graded commutative algebra, specifically,

$$\bar{\delta} \smile \bar{\delta}' = (-1)^{mn} \bar{\delta}' \smile \bar{\delta},$$

where $\bar{\delta}, \bar{\delta}'$ are the images of δ, δ' in $\mathrm{HH}^m(A), \mathrm{HH}^n(A)$ under the respective quotient maps.

The two binary operations on $\mathrm{HH}^*(A)$, namely $[\ , \]$ and $\smile$, can be seen to work well together in the sense that the functions $[\bar{\delta}, -]$ from $\mathrm{HH}^*(A)$ to $\mathrm{HH}^*(A)$ are themselves graded derivations with respect to cup product, specifically,

$$[\bar{\delta}, \bar{\delta}' \smile \bar{\delta}''] = [\bar{\delta}, \bar{\delta}'] \smile \bar{\delta}'' + (-1)^{m(p-1)} \bar{\delta}' \smile [\bar{\delta}, \bar{\delta}'']$$

for all $\bar{\delta}, \bar{\delta}', \bar{\delta}''$ in $\mathrm{HH}^m(A), \mathrm{HH}^n(A), \mathrm{HH}^p(A)$, respectively. We thus say that $\mathrm{HH}^*(A)$ is a *Gerstenhaber algebra* in honor of Gerstenhaber, who first discovered this structure.

Take as an example $A = k[x, y]$: The cup product is the product on A combined with that on the exterior algebra on the derivations $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$, for example,

$$\left(p \frac{\partial}{\partial x} \right) \cdot \left(q \frac{\partial}{\partial y} \right) = pq \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y},$$

and the Gerstenhaber bracket is the Schouten bracket on polyvector fields, for example,

$$\left[p \frac{\partial}{\partial x}, q \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \right] = \left(p \frac{\partial}{\partial x} (q) - q \frac{\partial}{\partial x} (p) \right) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$$

for all p, q in $k[x, y]$. These structures extend to any polynomial ring in finitely many indeterminates.

Appearances of Hochschild Cohomology

Hochschild cohomology has found many uses since its inception. We mention here only a few places where Hochschild cohomology appears in order to show how widespread and diverse its uses can be.

Noncommutative geometry. There are many notions from classical geometry that can be defined homologically, and Hochschild cohomology plays a role. In classical geometry, we glean information about geometric spaces from commutative rings of functions on the spaces. For example, the polynomial ring $\mathbb{R}[x, y]$ may be considered to

be a ring of functions from the plane $\mathbb{R}^2$ to $\mathbb{R}$, with x, y corresponding to the coordinate functions. In noncommutative geometry, we replace such commutative rings of functions by noncommutative rings and ask analogous questions about their structure. For example, the *Weyl algebra* is an algebra that looks very much like a polynomial ring, but we replace commutativity of indeterminates x and y with the relation $xy = yx - 1$. Notions such as dimension, smoothness, and differential forms can be defined in the setting of noncommutative geometry via Hochschild cohomology of noncommutative algebras. Such definitions may be viewed as generalizing those in commutative algebra.

Algebraic deformation theory. Some algebras are regarded as deformations of simpler ones. For example, the Weyl algebra is a deformation of a polynomial ring. A deformation has associated to it a Hochschild 2-cocycle, technically by first routing through formal power series over the original algebra. For example, the cocycle for the Weyl algebra is $-\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$: Applying $-\frac{\partial}{\partial x}$ to x , $\frac{\partial}{\partial y}$ to y , and multiplying yields -1 , the scalar in the defining relation $xy = yx - 1$ of the Weyl algebra that distinguishes it from the relation $xy = yx$ of the polynomial ring. Conversely, given a Hochschild 2-cocycle, in order for it to be associated to a deformation, other cocycles defined via the Gerstenhaber bracket $[\ , \]$ must be coboundaries. This connection to Hochschild cohomology can be profitable. For example, the classical Poincaré–Birkhoff–Witt theorem for Lie algebras in this language states that the universal enveloping algebra of a Lie algebra is a deformation of a polynomial ring, and one proof passes through Hochschild cohomology. A notable special case of algebraic deformation is deformation quantization, that is, deformation of a commutative algebra for which the Hochschild 2-cocycle is a Poisson bracket.

Representation theory. Some techniques in representation theory rely on the graded commutative property of the Hochschild cohomology ring under cup product that we saw earlier. This property opens the door to commutative algebraic and algebraic geometric techniques. Specifically, each A -module M is associated to a module for the graded commutative Hochschild cohomology ring $\mathrm{HH}^*(A)$, and this module itself is a cohomology ring $H^*(M)$ defined by M . Under some finiteness conditions, this yields an assignment of an algebraic variety, called a *support variety*, to each A -module M , that is, the spectrum of the annihilator in $\mathrm{HH}^*(A)$ of $H^*(M)$. This assignment grants access to geometric information encoding representation-theoretic properties. For example, in some settings, the dimension of the support variety of M is equal to a measure of complexity of M . This theory is both parallel to and more

James Stasheff, 2016.

general than the classical theory of support varieties for modules of finite groups.

Towards Structural Understanding

Due to the many appearances of Hochschild cohomology in mathematics, it is important to understand Hochschild cohomology spaces well. One approach is to understand first the precursor spaces $C^n(A)$ of multilinear functions from A^n to A , with the goal of understanding better their subspaces $Z^n(A)$ of Hochschild n -cocycles and the quotients $Z^n(A)/B^n(A) = \mathrm{HH}^n(A)$.

Identify the vector space $C^n(A)$ with that of all linear functions from $A^{\otimes n} = A \otimes_k \cdots \otimes_k A$ (n tensor factors) to A . Stasheff [8] considered some additional structure on the graded vector space

$$T(A) = \bigoplus_{n \geq 0} A^{\otimes n},$$

namely that of a coalgebra. This structure is given by deconcatenating tensor products in all possible ways and summing. Stasheff showed that Hochschild cohomology comes from the subspace of all coderivations on this coalgebra. He also initiated the study of higher multiplications and comultiplications on

$$C^*(A) = \bigoplus_{n \geq 0} C^n(A)$$

and on other spaces, thus founding the subject of A_∞ -algebras and related structures. Deligne [1] conjectured further structure on $C^*(A)$, that it is an algebra over an operad of little discs, one version of which can be viewed as existence of a particular infinity structure. The conjecture was proven in different ways by different groups of mathematicians around the turn of the millenium; see [6] for a

discussion and references. Operads underlie other important properties of $C^*(A)$ and thus of Hochschild cohomology. See [6, 9] and references therein for an introduction to operads.

For many questions involving Hochschild cohomology, it can be helpful first to refine the vector space $C^*(A)$ itself, since this space is quite large and can conceal important information. We look next at such an alternative approach.

Resolutions

We wish to place the Hochschild cohomology vector space $HH^*(A)$ into a bigger picture by giving an equivalent definition via an arbitrary resolution next. In fact, results and examples mentioned earlier depend heavily on this approach.

The algebra A acts on itself by left and right multiplication, making it into an A -bimodule. Equivalently, it is a left module over the *enveloping algebra* $A^e = A \otimes_k A^{op}$ where A^{op} is the opposite algebra (that is, A with the opposite multiplication, $a \cdot_{op} b = ba$ for all a and b in A). The left A^e -module structure on A is given by

$$(a \otimes_k b) \cdot c = acb$$

for all a and c in A and b in A^{op} . Considering A as an A^e -module in this way, we ask for a sequence of free A^e -modules F_n (that is, each F_n must be a direct sum of copies of A^e),

$$\cdots \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} A \longrightarrow 0 \quad (2)$$

with A^e -module homomorphisms $\varepsilon : F_0 \rightarrow A$ and $d_n : F_n \rightarrow F_{n-1}$ for all $n \geq 1$. We require $\text{im}(d_1) = \ker(\varepsilon)$ and $\text{im}(d_n) = \ker(d_{n-1})$ for all $n > 1$; that is, the sequence (2) is *exact*. Put another way, we say that the sequence (2), or rather the collection of bimodules F_n and maps d_n , is a *free resolution* of the A^e -module A .¹ An example is the *bar resolution* of A for which $F_n = A \otimes_k \cdots \otimes_k A$ ($n+2$ factors), $\varepsilon(a \otimes_k b) = ab$, and

$$\begin{aligned} d_n(a_0 \otimes_k \cdots \otimes_k a_{n+1}) \\ = \sum_{i=0}^n (-1)^i a_0 \otimes_k \cdots \otimes_k a_i a_{i+1} \otimes_k \cdots \otimes_k a_{n+1} \end{aligned}$$

for all $a_0, \dots, a_{n+1}$ in A . A resemblance to simplices and boundary maps in the theory of singular homology recalls roots in algebraic topology.

In general, consider the vector space of all A^e -module homomorphisms from F_n to A , denoted $\text{Hom}_{A^e}(F_n, A)$ for each n . Define the map

$$d_n^* : \text{Hom}_{A^e}(F_{n-1}, A) \rightarrow \text{Hom}_{A^e}(F_n, A)$$

¹More generally, we may take each F_n to be a projective A^e -module for a projective resolution.

to be a composition with d_n , that is, $d_n^*(\delta) = \delta \circ d_n$, for all $n > 1$, and define d_0^* to be the zero map. It turns out that there is a vector space isomorphism for each n ,

$$HH^n(A) \cong \ker(d_{n+1}^*)/\text{im}(d_n^*),$$

where $HH^n(A)$ is the Hochschild cohomology space of A in degree n , as defined earlier. The latter space above, $\ker(d_{n+1}^*)/\text{im}(d_n^*)$, is also denoted $\text{Ext}_{A^e}^n(A, A)$, and is independent of the choice of resolution (2) as a consequence of existence of comparison maps between resolutions. If we choose the bar resolution as described above, we return to the definition of Hochschild cohomology we gave earlier. If we take any other choice of free resolution of A , we obtain a new (equivalent) definition of the Hochschild cohomology spaces $HH^n(A)$. The resulting zoo of choices of definitions of $HH^n(A)$ vastly expands our toolbox for handling it.

In comparison, the dual notion of Hochschild homology $HH_n(A)$ is given by $\text{Tor}_n^{A^e}(A, A)$, and generally for any A -bimodule M and $n \geq 0$,

$$HH^n(A, M) = \text{Ext}_{A^e}^n(A, M),$$

$$HH_n(A, M) = \text{Tor}_n^{A^e}(M, A)$$

are the Hochschild cohomology and homology vector spaces with coefficients in M . These spaces also have further structure and uses that we will not discuss here. See [5, 12, 13] for more information.

Cup Product Revisited

Cup products can be found on an arbitrary resolution by using an algorithm analogous to a slide rule. In relation to the sequence (2), let δ be an element of $\ker(d_{m+1}^*)$ and let δ' be an element of $\ker(d_{n+1}^*)$, representing elements of $HH^m(A)$ and $HH^n(A)$, respectively. That is, δ is an A^e -module homomorphism from F_m to A for which $\delta \circ d_{m+1} = 0$ and δ' is an A^e -module homomorphism from F_n to A for which $\delta' \circ d_{n+1} = 0$. Consider two copies of the sequence (2), one above the other:

$$\cdots \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A$$

$$\cdots \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A$$

Slide the top copy n places to the right so that F_n is now above F_0 , called *shifting* the resolution. By virtue of being free, the A^e -module F_n may be mapped to F_0 via an A^e -module homomorphism δ'_0 in such a way that $\delta' = \varepsilon \circ \delta'_0$, as in the diagram below. Similarly, there are A^e -module homomorphisms $\delta'_i : F_{n+i} \rightarrow F_i$ for $1 \leq i \leq m$ for which the diagram below commutes; that is, composing maps along any two paths with the same start and end produces the same result. We call the collection of such maps δ'_i a

chain map:

$$\begin{array}{ccccccc}
 F_{m+n} & \xrightarrow{d_{m+n}} & \cdots & \xrightarrow{d_{n+2}} & F_{n+1} & \xrightarrow{d_{n+1}} & F_n \\
 \delta'_m \downarrow & & & & \delta'_1 \downarrow & & \delta'_0 \downarrow \\
 F_m & \xrightarrow{d_m} & \cdots & \xrightarrow{d_2} & F_1 & \xrightarrow{d_1} & F_0 \xrightarrow{\varepsilon} A
 \end{array}$$

Now define

$$\delta \smile \delta' = \delta \circ \delta'_m,$$

an A^e -module homomorphism from F_{m+n} to A :

$$\begin{array}{c}
 F_{m+n} \\
 \delta'_m \downarrow \\
 F_m \\
 \delta \downarrow \\
 A
 \end{array}
 \quad \delta \circ \delta'_m$$

Then $\delta \smile \delta'$ indeed represents the cup product in the Hochschild cohomology space $\mathrm{HH}^{m+n}(A)$. If we take the bar resolution as described earlier, one choice of maps δ'_i leads to precisely the formula for cup product that we had before.

This alternate definition of cup product, on an arbitrary resolution, increases the versatility and therefore the usefulness of this operation. Often we choose a resolution tailored specifically to a given type of algebra, aiding both computation and theoretical understanding. The above method of defining cup products is constructive and is used in practice to compute them. There are yet other equivalent definitions of cup product, also useful, that we do not discuss here.

Gerstenhaber Bracket Revisited

Gerstenhaber brackets may be found on an arbitrary resolution by a similar technique to that described above for cup products. Again, in relation to the sequence (2), let δ be an element of $\ker(d_{m+1}^*)$ and let δ' be an element of $\ker(d_{n+1}^*)$, representing elements of $\mathrm{HH}^m(A)$ and $\mathrm{HH}^n(A)$, respectively. Again we line up two copies of the sequence (2), but this time we slide the top copy $n-1$ places to the right, so that F_{n-1} is above F_0 . We are interested in A^e -module homomorphisms $\delta'_{[i]} : F_{i+n-1} \rightarrow F_i$ as in the diagram below:

$$\begin{array}{ccccccc}
 F_{m+n-1} & \longrightarrow & \cdots & \longrightarrow & F_n & \longrightarrow & F_{n-1} \\
 \delta'_{[m]} \downarrow & & & & \delta'_{[1]} \downarrow & & \delta'_{[0]} \downarrow \\
 F_m & \longrightarrow & \cdots & \longrightarrow & F_1 & \longrightarrow & F_0
 \end{array}$$

However instead of requiring that the diagram commute, we require the homomorphisms $\delta'_{[i]}$ to comprise a homotopy between two maps defined in terms of δ' and a chain map from the sequence (2) to its tensor square over A .

Specifically, write F_* for the truncation of sequence (2) in which we have deleted the rightmost nonzero term A . Let Δ be any chain map from F_* to the tensor product $F_* \otimes_A F_*$ that extends the canonical isomorphism $A \xrightarrow{\sim} A \otimes_A A$. We require the map $d_{i+1}\delta'_{[i+1]} + (-1)^n \delta'_{[i]} d_{i+n}$ to be equal to the difference between $(\delta' \otimes_A 1)\Delta_{i+n}$ and $(1 \otimes_A \delta')\Delta_{i+n}$ for all i , as well as requiring $\delta'_{[0]}$ to satisfy an initial condition. (Here we take the value of δ' to be 0 on all F_j with $j \neq n$.) The collection of such maps $\delta'_{[i]}$ is called a *homotopy lifting* of δ' [11, 13]. Now define

$$[\delta, \delta'] = \delta \circ \delta'_{[m]} - (-1)^{(m-1)(n-1)} \delta' \circ \delta_{[n]},$$

an A^e -module homomorphism from F_{m+n-1} to A . Then $[\delta, \delta']$ represents the Gerstenhaber bracket in Hochschild cohomology $\mathrm{HH}^{m+n-1}(A)$. If we take the bar resolution as described earlier, one choice of maps $\delta'_{[i]}$ produces exactly the formula for the Gerstenhaber bracket that we had before.

This alternate definition of Gerstenhaber bracket, on an arbitrary resolution $F_{*,*}$ is due to Volkov [11] in this general form. The notion of homotopy lifting is directly related to Stasheff's coderivations [8] in case F_* is the bar resolution. A method for finding homotopy liftings under some additional conditions is in an earlier paper with Negron [7]. In case F_* is the bar resolution, homotopy liftings of 1-cocycles are related to earlier, more general results of Suárez-Álvarez [10]. All of these new techniques have been used in further computational and theoretical work on Gerstenhaber brackets for some classes of algebras, and they promise to lead to yet further advances.

References

- [1] A. A. Voronov and M. Gerstenhaber, *Higher operations and the Hochschild complex*, *Funct. Anal. Appl.* **29** (1995), no. 1, 1–5. MR1328534
- [2] M. Gerstenhaber, *The cohomology structure of an associative ring*, *Ann. Math.* **78** (1963), no. 2, 267–288. MR161898
- [3] G. Hochschild, *On the cohomology groups of an associative algebra*, *Ann. Math.* **46** (1945), no. 2, 58–67. MR11076
- [4] G. Hochschild, B. Kostant, and A. Rosenberg, *Differential forms on regular affine algebras*, *Trans. Amer. Math. Soc.* **102** (1962), 383–408. MR142598
- [5] J.-L. Loday, *Cyclic Homology*, *Grundlehren der Mathematischen Wissenschaften*, vol. 301, Springer-Verlag, 1998. MR1600246
- [6] J. E. McClure and J. H. Smith, *Operads and cosimplicial objects: an introduction*, *Axiomatic, enriched and motivic homotopy theory*, 133–171, *NATO Sci. Ser. II Math. Phys. Chem.* **131**, Kluwer Acad. Publ., Dordrecht, 2004. MR2061854
- [7] C. Negron and S. Witherspoon, *An alternate approach to the Lie bracket on Hochschild cohomology*, *Homology, Homotopy and Applications* **18** (2016), no. 1, 265–285. MR3498646

- [8] J. Stasheff, *The intrinsic bracket on the deformation complex of an associative algebra*, J. Pure Appl. Algebra **89** (1993), 231–235. MR1239562
- [9] J. Stasheff, *What is an operad?* Notices Amer. Math. Soc. **51** (2004), no. 6, 630–631. MR2064150
- [10] M. Suárez-Álvarez, *A little bit of extra functoriality for Ext and the computation of the Gerstenhaber bracket*, J. Pure Appl. Algebra **221** (2017), no. 8, 1981–1998. MR3623179
- [11] Y. Volkov, *Gerstenhaber bracket on the Hochschild cohomology via an arbitrary resolution*, Proc. Edinburgh Math. Soc., 2019, 20 pp. MR3974969
- [12] C. A. Weibel, *An Introduction to Homological Algebra*, Cambridge University Press, 1994. MR1269324
- [13] S. Witherspoon, *Hochschild Cohomology for Algebras*, Graduate Studies in Mathematics, 204, American Mathematical Society, Providence, RI, 2019. MR3971234

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