Differential Operators and the Geometry of Domains in Euclidean Space

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Figure 1. Picture Lake and Mount Shuksan, North Cascades National Park, WA

If asked to use the adjectives *rough* and *smooth* to describe the objects in the landscape in Figure 1, most of us would say that the mountains are rough and that the water is smooth. What would you say if I asked you to explain your choices? In this talk, I will provide one explanation. I will describe the deep connection between the temperature distribution in the mountains and in the lake, and the geometry of the peaks and the edge, bottom, and surface of the lake. Although somewhat allegorical, this analogy encompasses the content of an area of mathematics that aims to describe the geometry of domains in Euclidean space in terms of the properties of solutions to canonical partial differential operators defined on these domains. The reader should think of the lake as a domain in $\mathbb{R}^3$ whose boundary includes the edge and the bottom of the lake as well as the interface between the water and the air. The temperature in steady state is a solution to Laplace's equation in this domain. Solutions to Laplace's equation are harmonic functions. My goal is to illustrate the relationship between the boundary behavior of harmonic functions on a domain and the geometry of the boundary of that domain.

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The Laplacian has different expressions in homogeneous media (e.g., the lake) versus inhomogeneous media (e.g., the mountain). Nevertheless, under mild assumptions on the domains, in both cases there exists a canonical measure supported on the boundary of the domain where the operators are studied. These canonical measures (harmonic measure in the homogeneous case and elliptic measure in the inhomogeneous case) are associated in a unique way to the domain and the operator and constitute the building blocks for all solutions to Laplace's equation. In the domains considered in this talk there is another canonical measure supported on the boundary, namely, the surface measure. (In Figure 1 this measure corresponds to the area. It can tell us the size of the glacier on top of Mount Shuksan.) From the mathematical point of view my objective is to discuss how the relationship between the harmonic measure and the surface measure characterizes the geometry of a large class of domains. I will mention the challenges that arise in the inhomogeneous case and conclude with a complete characterization of the inhomogeneous media that behave like the homogeneous one under this lens. The relationship between two measures is described using notions that come from harmonic analysis. The geometry of the domain is described using notions arising in quantitative geometric measure theory.

Early on, the work of F. and M. Riesz [RR] put in evidence the deep connection that exists between the properties of the harmonic measure of a domain and the regularity of the boundary expressed in terms of its differentiability, that is, in terms of the existence of tangent planes almost everywhere with respect to surface measure. Recently, the relationship between the properties of the harmonic measure of a domain with respect to the surface measure of its boundary and the weakly differentiable properties of it has been an area of active inquiry. Here the weakly differentiable properties of the boundary refer to the existence of tangent planes almost everywhere (rectifiability) and a quantitative analogous notion (uniform rectifiability).

For instance, in the scale-invariant sense and under some quantitative topological assumptions, one can show that the A_∞ property (quantitative mutual absolute continuity) of harmonic measure with respect to surface measure characterizes the uniform rectifiability of the boundary or even to the nontangentially approximability of the exterior domain. This characterization is a team effort achieved over time by different groups: [Sem], [DJ], [HMUT], and [AHMNT].

One of the main motivations of my recent work has been to understand whether the elliptic measure of a variable-coefficient divergence form elliptic operator (the Laplacian in an inhomogeneous medium) distinguishes between a rectifiable and a purely unrectifiable boundary. Together with S. Hofmann, J. M. Martell, S. Mayboroda, and Z. Zhao, we embarked on this project a few years ago. We have found

the optimal solution, namely, a large class of variable-coefficient elliptic operators for which the A_∞ property of the associated elliptic measure with respect to surface measure guarantees that the boundary of the domain is uniformly rectifiable ([HMMTZ1] and [HMMTZ2]).

In this talk I will share some of the highlights of this journey. I hope to convey the excitement of obtaining an optimal result after working for several years on a challenging project.

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Credits

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