Alexander Smith Wins the First David Goss Prize in Number Theory

Dorian Goldfeld and Simone Munao

David Goss

David Goss, who was the former editor-in-chief of the *Journal of Number Theory* (JNT), passed away in 2017. With David's passing, the mathematical community lost one of its most distinguished members, whose outstanding work had a profound influence on the way we see number theory today.

David received his PhD in 1977 from Harvard University under the supervision of Barry Mazur. He held positions at Princeton, Berkeley, and Brandeis before joining The Ohio State University in 1982, from where he retired in 2013 and became professor emeritus. David was the advisor for three doctoral students. Generous with his time and knowledge and full of enthusiasm, David was always available with support for his colleagues, especially junior researchers in number theory. He was viewed by many as a mentor and a friend: his broad and deep knowledge, as well as his kindness and warmth, will be missed by many.

David's research was centered on algebraic number theory and algebraic geometry, where he made extensive contributions to the arithmetic theory of function fields, publishing some forty papers. He also authored the book *Basic Structures of Function Field Arithmetic* [Gos96], which is widely viewed as one of the standard and most accessible references in the field.

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Figure 1. David Goss.

Starting with his Harvard thesis, David introduced his novel and revolutionary theory of modular forms and Eisenstein series in the function field setting. As all the functions here have characteristic p values, the theory is distinct from the ones developed by Weil, Harder, and Langlands. In 1979, while at Princeton, he achieved another great milestone by defining what are now known as Goss zeta functions [Gos79]. In this seminal work, he also introduced the character group, and the analogs of analytic continuation, the values at negative integers, the trivial zeroes, and other important structures which have inspired so many mathematicians. He later continued developing

his theory of zeta and L-functions, following the path of Euler and Riemann. With unlimited energy, he inspired and encouraged research in these and other topics in number theory.

David Goss became an editor of the *Journal of Number Theory* in 1988 and then served as its editor-in-chief for over fifteen years. He led the journal through difficult times with his strength and his devotion and was deeply appreciated by all the editors and the Elsevier staff.

The David Goss Prize

Elsevier has decided to establish the Goss prize (US\$10K), to be awarded every two years to celebrate and recognize the efforts of David Goss on behalf of the journal. The prize, to be awarded to a mathematician under the age of thirty-five for outstanding contributions to number theory, was established in 2018 and paired with the new series of JNT Biennial Conferences, whose aim is to disseminate recent advances and breakthroughs in all areas of number theory.

The first David Goss Prize in Number Theory was awarded in July 2019 in Cetraro, Italy, at the first JNT Biennial Conference. The enthusiasm and support shown by the Goss family is very much appreciated.

David's daughter Alyssa, son Noah, and wife Rita Eppler-Goss attended the conference and the ceremony for the Goss Prize, which was awarded unanimously to Alexander Smith by the selection committee, consisting of Dorian Goldfeld (Columbia University), Philippe Michel (École Polytechnique, Lausanne), Dipendra Prasad (Tata Institute), Emmanuel Ullmo (IHES), Umberto Zannier (Scuola Normale de Pisa), and Shou-Wu Zhang (Princeton University).

Alexander Smith

Alexander Smith was awarded the 2019 David Goss Prize for his stunning work [Smi17] on class groups of number fields and Selmer groups of elliptic curves. He showed that the 2^∞ -class groups of imaginary quadratic fields are distributed according to the Cohen–Lenstra heuristic [CL84]. Smith also proved that the 2^∞ -Selmer groups of quadratic twists of elliptic curves $E/\mathbb{Q}$, with full rational 2-torsion and no rational cyclic subgroup of order four, satisfy the Delaunay heuristic [Del01] and Goldfeld's conjecture [Gol79]. The prize also recognizes his outstanding undergraduate thesis on the congruent number problem (see the revised version [Smi16]) following work of Heath–Brown [HB94], Monsky [Mon96], and Tian–Yuan–Zhang [TYZ17].

Alexander Smith was born on March 4, 1993, in Massachusetts, USA, and is currently a fourth-year graduate student at Harvard University. Smith's amazing research

Figure 2. Alexander Smith.

achievements began with his undergraduate senior thesis at Princeton written under the supervision of Shou-Wu Zhang. A congruent number is a positive integer which is the area of a right triangle, all of whose sides have rational number length. The congruent number problem, dating back to antiquity, asks if there is an algorithm to determine if a positive integer n is a congruent number. Smith [Smi16] proved that at least 55.9% of all square-free positive integers $n \equiv 5, 6, 7 \pmod{8}$ are congruent numbers. This was the first such positive density result ever obtained.

An anonymous Arabic manuscript, written before the year 972, asks the following question (original version of the congruent number problem). Given an integer n , find a rational square α^2 such that $\alpha^2 \pm n$ are also squares of rational numbers (see [Dic66], p. 462). Tunnell [Tun83] showed that the congruent number problem can be solved for a positive integer n if the Birch–Swinnerton-Dyer (BSD) conjecture holds for the CM elliptic curve $y^2 = x^3 - n^2x$. It was known before Tunnell's work [Ste75] that if a weak form of BSD were proved for elliptic curves $y^2 = x^3 - n^2x$, then all positive integers $n \equiv 5, 6, 7 \pmod{8}$ would be congruent numbers. In [Smi16] Smith also proves that the BSD conjecture holds for at least 55.9% of such elliptic curves with square-free positive integers $n \equiv 5, 6, 7 \pmod{8}$.

Goldfeld's fifty-fifty conjecture [Gol79] states that given an elliptic curve E over $\mathbb{Q}$, 50% of the quadratic twists of E have analytic rank 0 and 50% have analytic rank 1, while 0% have higher analytic rank. Smith [Smi17] proves the following theorem and corollary.

Theorem. *Let $E/\mathbb{Q}$ be an elliptic curve with full rational 2-torsion. Assume that E has no rational cyclic subgroup of order four. Then, if the Birch and Swinnerton-Dyer conjecture holds for the set of twists of E , Goldfeld's conjecture holds for E .*

Smith actually proves that the number of quadratic twists $d \leq x$ of E (in the above theorem) which have rank > 1 is at most $x/(\log \log \log x)^c$ for some constant $c > 0$ as $x \rightarrow \infty$. This is stronger than proving that the number of quadratic twists of E that have rank > 1 is 0%.

Corollary. *Among positive square-free integers $n \equiv 1, 2, 3 \pmod{8}$, 100% are noncongruent numbers.*

We quote from Smith's paper [Smi17]: "From global root number calculations, we know that 50% of the twists will have even $\mathbf{Z}_2$ -Selmer corank, and 50% have odd $\mathbf{Z}_2$ corank. In light of this we have the above corollary."

The fact that 1 is not a congruent number has a very interesting history. Leonardo Pisano (better known as Fibonacci) stated in *Liber Quadratorum* (published in the year 1225) that 1 is not a congruent number. The proof was first found by Fermat more than four hundred years later. In fact, Fermat proved (by his method of infinite descent) that there is no right triangle with rational length sides whose area is a square. Andre Weil [Wei07] quotes from Fermat's letter to Huygens in 1659: "If the area of such a triangle were a square, then there would also be a smaller one with the same property, and so on, which is impossible," and adds that "to explain why would make his discourse too long, as the whole mystery of his method lies there."

Weil tops it off by saying, "Fortunately, just this once, Fermat had found room for this mystery in the margin of the very last proposition of Diophantus."

In our modern formulation, Fermat's descent argument establishes that the elliptic curve $y^2 = x^3 - x$ has rank zero over $\mathbb{Q}$. In the early twentieth century, Mordell generalized this argument to apply to other elliptic curves, proving that every elliptic curve over the rationals has finite rank. This would later lead to the development of Selmer groups of elliptic curves, effectively computable abelian groups whose rank gives an upper bound for the rank of the elliptic curve.

The 2^∞ Selmer rank of E is defined as the limit (as $k \rightarrow \infty$) of

$$r_k(E) = \text{rank}_{\mathbf{F}_2} \left(2^{k-1} \text{Sel}_{2^k}(E) / \left(2^{k-1} E[2] \right) \right),$$

where $\text{Sel}(E)$ denotes the Selmer group of E . Note that $r_1(E) \geq r_2(E) \geq \dots \geq \text{rank}(E)$. Smith proved that the distribution of r_k follows a Markov chain behavior: given r_k for $k = 1, 2, \dots, m$, the distribution of r_{m+1} depends only on r_m and is predicted by the Delaunay heuristic [Del01]. Smith finds that the distribution of the groups $2\text{Cl}(\mathbb{Q}(\sqrt{-d})) [2^\infty]$ is as predicted by Gerth [Ger84],

following Cohen-Lenstra [CL84].

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