Modeling Homotopy Theories

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Throughout mathematics, each field has fundamental objects of study, whether it be real-valued functions or vector spaces or Lie algebras or smooth manifolds. Furthermore, we are not only interested in what these objects are but also in appropriate kinds of functions between them, such as linear transformations of vector spaces or smooth functions between manifolds. Of particular interest are those functions that tell us that two specific objects are the same as each other in some critical way. For example, we typically do not distinguish between two different sets that each have three elements, since they are in bijection with one another. Similarly, we ignore the difference between groups that are isomorphic or topological spaces that are homeomorphic. In each of these examples, the criterion for “sameness” is quite rigid: there is a function of the appropriate flavor (function of sets, group homomorphism, or continuous map) that admits an inverse.

In other situations, we have notions of “sameness” that are not quite so strict. For instance, since it is generally extremely difficult to determine whether there is a homeomorphism between two topological spaces, one might ask instead whether there is a homotopy equivalence between them, namely, a map that may not have an inverse on the nose but only up to homotopy.

Many further examples can be described via invariants. One might consider weak homotopy equivalences of topological spaces, which induce isomorphisms on all homotopy groups, or quasi-isomorphisms of chain complexes,

which induce isomorphisms on all homology groups. In these examples, there is no reason to assume that such a function has any kind of meaningful function going in the reverse direction at all. These two situations provide classical examples of “homotopy theories.”

Our aim in this article is to give an overview of the development of the notion of homotopy theory, starting with these classical examples and leading up to more modern perspectives. As we discuss briefly at the end, we have chosen one of several possible entry points into this subject and in particular do not say much about the higher categorical aspects of the theory. We simply remark that the “homotopy theories” that we discuss here coincide with the “ $(\infty, 1)$ -categories” that are being used widely in a number of related fields.

Topological Spaces and Chain Complexes

We begin our investigations with the classical homotopy theory of topological spaces. Let us first recall the precise definition of homotopy equivalence.

Definition 1. A continuous map $f : X \rightarrow Y$ of topological spaces is a *homotopy equivalence* if there exists a continuous map $g : Y \rightarrow X$ such that $g \circ f$ is homotopic to the identity map of X and $f \circ g$ is homotopic to the identity map of Y .

To define the related notion of weak homotopy equivalence, we need to consider the homotopy groups of a topological space. The most familiar of these groups is the following.

Definition 2. Let X be a topological space with a chosen basepoint x_0 . Then its *fundamental group* is $\pi_1(X, x_0) = [S^1, X]_*$, the set of homotopy classes of basepoint-

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DOI: <https://doi.org/10.1090/noti1957>

Figure 1. A quasi-circle.

preserving maps $S^1 \rightarrow X$ which has a natural group operation.

While not as frequently presented in an introductory course, higher homotopy groups can be defined analogously.

Definition 3. Let X be a topological space with basepoint x_0 . Then for any $n \geq 1$, the n th homotopy group of X is $\pi_n(X, x_0) = [S^n, X]_*$.

This definition also works when $n = 0$, but in that case $\pi_0(X)$ is simply the set of path components of X and does not have a natural group structure.

When $n \geq 1$, the group $\pi_n(X, x_0)$ does not depend on the specific basepoint x_0 but only on the path component from which it is chosen. Hence, in what follows we implicitly assume that appropriate statements hold for all path components and simply write the homotopy groups as $\pi_n(X)$.

An important feature of homotopy groups is that they are *functorial*: given a continuous map of topological spaces $f: X \rightarrow Y$, there are induced group homomorphisms $f_*: \pi_n(X) \rightarrow \pi_n(Y)$.

Definition 4. A continuous map of topological spaces $X \rightarrow Y$ is a *weak homotopy equivalence* if, for every $n \geq 0$, the map

$$\pi_n(X) \rightarrow \pi_n(Y)$$

is an isomorphism.

One can check, using functoriality of homotopy groups, that any homotopy equivalence is a weak homotopy equivalence. However, the converse does not hold, as the following example illustrates.

Example 5. Let W be the quasi-circle in which an arc of S^1 is replaced by a $\sin(\frac{1}{x})$ curve and its limit interval, as depicted in Figure 1. Then the unique map $W \rightarrow *$ is a weak homotopy equivalence, since the homotopy groups of both spaces are all trivial. However, this map is not a homotopy equivalence, since there is no map $* \rightarrow W$ such

that the composite $W \rightarrow * \rightarrow W$ is homotopic to the identity on W .

However, we had to use a rather exotic example of a topological space to see the difference between our two notions of equivalence, so we might ask if they agree for maps between spaces that are sufficiently nice. A good answer comes from the following two theorems.

Whitehead's Theorem. *Any weak homotopy equivalence between CW complexes is a homotopy equivalence.*

A CW complex is a topological space that, roughly speaking, is built by gluing together cells of different dimensions. Such spaces are considered well-behaved because they can be described so explicitly, but the following theorem also tells us that they are sufficiently general to capture much of the behavior of topological spaces.

CW Approximation Theorem. *Given any topological space X , there exists a weak homotopy equivalence $Y \rightarrow X$ with Y a CW complex.*

Taking these two theorems together, we can transition from the usual category of topological spaces and continuous maps to a new category in which weak homotopy equivalences become isomorphisms. This category still has all topological spaces as objects, but given two spaces X and Y , we define the set of maps $X \rightarrow Y$ to consist of homotopy classes of maps between CW replacements of X and Y . Since homotopy equivalences have inverses up to homotopy, their corresponding homotopy classes have inverses in this new category, which we call the *homotopy category of spaces*.

Now let us look at an algebraic example for which the same kind of phenomenon occurs. Let R be a ring, and consider nonnegatively graded chain complexes of R -modules

$$C_\bullet = (C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} C_2 \leftarrow \cdots).$$

Each C_i is an R -module, and each composite

$$C_{i-1} \xleftarrow{\partial_i} C_i \xleftarrow{\partial_{i+1}} C_{i+1}$$

is the zero map. Given another such chain complex $D_\bullet$, a *chain map* $f: C_\bullet \rightarrow D_\bullet$ consists of R -module homomorphisms $f_i: C_i \rightarrow D_i$ for each $i \geq 0$ so that the diagram

$$\begin{array}{ccccccc} C_0 & \xleftarrow{\quad} & C_1 & \xleftarrow{\quad} & C_2 & \xleftarrow{\quad} & \cdots \\ \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & & \\ D_0 & \xleftarrow{\quad} & D_1 & \xleftarrow{\quad} & D_2 & \xleftarrow{\quad} & \cdots \end{array}$$

commutes.

Just as for the example of topological spaces, there are two natural ways to think of equivalences between chain complexes. First, there is the notion of *chain homotopy equivalence*, which means having an inverse up to chain

homotopy. There is also the weaker notion of *quasi-isomorphism* or map that induces isomorphisms on all homology groups $H_i(C_\bullet) = \ker(\partial_i) / \text{im}(\partial_{i+1})$. So, again, we see two flavors of equivalence: one that produces isomorphisms after taking appropriate homotopy classes of maps, and one defined by an algebraic invariant.

So what is the comparable kind of object to CW complexes here? The following theorem provides an answer.

Theorem 6. *Any quasi-isomorphism between nonnegatively graded chain complexes made up of projective modules is a chain homotopy equivalence.*

But, as before, chain complexes of projective modules are also quite general.

Theorem 7. *Any nonnegatively graded chain complex has a projective resolution, namely, a quasi-isomorphic chain complex made up entirely of projective modules.*

Once again, we can define a category whose objects are nonnegatively graded chain complexes of R -modules and whose maps are chain homotopy classes of maps between projective resolutions. The resulting category is often called the *derived category* in homological algebra.

This second example suggests that the behavior of a homotopy theory is not isolated to the world of topological spaces but can occur in other areas of mathematics.

Model Categories

In his work on rational homotopy theory [16], Quillen observed further examples of a similar flavor, such as the category of topological spaces with rational equivalences and the category of differential graded Lie algebras with appropriate quasi-isomorphisms. He proceeded to specify axioms for a “homotopy theory” to capture this behavior; the resulting structure is known as a *model category*. We give a brief introduction to model categories here and refer the reader to [6] or [9] for more details.

Before stating the definition, let us summarize the ingredients that seemed to make the examples from the previous section work. First, we need a category together with some morphisms that we have designated as “weak equivalences,” which should capture the desired notion of “sameness” between objects. To construct the homotopy category, however, our examples suggest that we also need:

- some collection of objects that are sufficiently nice and such that any object in the category is weakly equivalent to one;
- a notion of homotopy of functions, at least between “nice” objects; and
- an agreement between “homotopy equivalence” and “weak equivalence” on the “nice” objects.

As Quillen realized, one can encode this information via the data of two additional classes of morphisms which,

following the primary example of topological spaces, are called *fibrations* and *cofibrations*, which are subject to several axioms. The following definition first appeared in [15], although we follow more closely the one given in [6].

Definition 8. A *model category* is a category $\mathcal{M}$ together with a choice of three distinguished classes of morphisms: *weak equivalences* ($\xrightarrow{\sim}$), *cofibrations* ($\hookrightarrow$), and *fibrations* ($\twoheadrightarrow$), each closed under composition and containing all isomorphisms and such that the following five axioms hold.

- (MC1) The category $\mathcal{M}$ has all small limits and colimits.
- (MC2) All three classes of morphisms are closed under retracts.
- (MC3) The class of weak equivalences satisfies the *two-out-of-three property*: if $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are morphisms in $\mathcal{M}$ such that two of f , g , and gf are weak equivalences, then so is the third.
- (MC4) Consider any commutative square

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow i & \nearrow & \downarrow p \\ B & \longrightarrow & Y. \end{array}$$

If either i or p is a weak equivalence, then a dotted arrow lift exists, making both triangular diagrams commute.

- (MC5) Any morphism $f: X \rightarrow Y$ can be factored in two ways: as

$$X \xrightarrow{\sim} X' \twoheadrightarrow Y$$

and as

$$X \hookrightarrow Y' \xrightarrow{\sim} Y.$$

It is not immediately clear that these axioms provide the desired structure that we specified above. Let us first address the question of identifying suitably “nice” objects. The fact that $\mathcal{M}$ has limits and colimits implies, in particular, that $\mathcal{M}$ has both an initial object, which we denote by $\emptyset$, and a terminal object, which we denote by $*$. Recall that an object $\emptyset$ of a category is *initial* if there is a unique morphism $\emptyset \rightarrow X$ for any object X of $\mathcal{M}$. Dually, an object $*$ is *terminal* if there is a unique map $X \rightarrow *$ for any X .

Definition 9. An object X of a model category $\mathcal{M}$ is *cofibrant* if the unique map $\emptyset \rightarrow X$ is a cofibration. It is *fibrant* if the unique map $X \rightarrow *$ is a fibration.

We thus propose that the “nice” objects in a model category are those that are both cofibrant and fibrant. To show that any object is weakly equivalent to such, we can apply axiom (MC5) as follows. Given any object X , we can factor the map $\emptyset \rightarrow X$ as

$$\emptyset \hookrightarrow X^c \xrightarrow{\sim} X,$$

and X^c is a *cofibrant replacement* of X . Dually, X has a *fibrant replacement* X^f , coming from the factorization

$$X \xrightarrow{\sim} X^f \rightarrow *.$$

One can check that taking a fibrant replacement of a cofibrant object, denoted by X^{cf} , produces an object that is still cofibrant, and similarly if we do the replacements in the opposite order.

A more sophisticated argument can be used to show that this structure allows for a well-defined “homotopy” equivalence relation between morphisms $f: X \rightarrow Y$ in $\mathcal{M}$ for which X and Y are both cofibrant and fibrant. We refer the reader to [6, §4] for a good treatment of this construction.

Definition 10. Let $\mathcal{M}$ be a model category. Define its *homotopy category* $\mathrm{Ho}(\mathcal{M})$ to have the same objects as $\mathcal{M}$ and to have morphisms between objects X and Y defined by

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(X, Y) = \mathrm{Hom}_{\mathcal{M}}(X^{cf}, Y^{cf}) / \sim,$$

where we identify maps in the same homotopy class.

In other words, maps $X \rightarrow Y$ in the homotopy category are defined by the homotopy classes of maps between cofibrant and fibrant replacements of X and Y .

There is a natural functor $\mathcal{M} \rightarrow \mathrm{Ho}(\mathcal{M})$ that takes weak equivalences of $\mathcal{M}$ to isomorphisms in $\mathrm{Ho}(\mathcal{M})$. Thus, the homotopy category is the context in which weak equivalences from $\mathcal{M}$ become isomorphisms.

Now that we have a definition of model categories, let us see how the examples we have considered fit into this framework.

Example 11. The category of topological spaces with weak homotopy equivalences has a model structure. The fibrations are the *Serre fibrations*, or continuous maps $X \rightarrow Y$, such that, for any CW complex A , a lift exists in any diagram of the form

$$\begin{array}{ccc} A \times \{0\} & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ A \times [0, 1] & \longrightarrow & Y. \end{array}$$

The cofibrations are the retracts of inclusions given by cell attachments. With this structure, all objects are fibrant, and the cofibrant objects are the retracts of CW complexes.

Example 12. The category of nonnegatively graded chain complexes with quasi-isomorphisms has a model structure in which the fibrations are levelwise epimorphisms and the cofibrations are levelwise monomorphisms that each has projective cokernel. Again, all objects are fibrant, and the cofibrant objects are precisely the chain complexes whose component modules are all projective.

There is also a dual model structure with the same weak equivalences in which the cofibrations are levelwise monomorphisms and the fibrations are levelwise epimorphisms that each has injective cokernel. In this structure, all objects are cofibrant, and the fibrant objects are the chain complexes made up of injective modules.

An important feature of model categories is that their homotopy categories do not depend on the model structure used but only on the weak equivalences. Thus, if we have an underlying category and choice of weak equivalences but two model structures with different fibrations and cofibrations, then the resulting homotopy categories are suitably equivalent to one another.

However, different model categories, even with different underlying categories, can still have equivalent homotopy categories. We would like to specify such an equivalence, and to do so we first consider what kind of functor between model categories preserves the relevant structure. Since this structure appears in two dual flavors, namely, that of the cofibrations and that of the fibrations, it is convenient to talk about an adjoint pair of functors rather than a single functor between two model categories. Recall that $F: \mathcal{M} \rightarrow \mathcal{N}$ and $G: \mathcal{N} \rightarrow \mathcal{M}$ are *adjoint functors*, with F the *left adjoint* and G the *right adjoint*, if there is an isomorphism

$$\mathrm{Hom}_{\mathcal{N}}(FX, Y) \cong \mathrm{Hom}_{\mathcal{M}}(X, GY)$$

that is natural for objects X of $\mathcal{M}$ and objects Y of $\mathcal{N}$. Let us now incorporate model structures.

Definition 13. Let $\mathcal{M}$ and $\mathcal{N}$ be model categories. An adjoint pair of functors

$$F: \mathcal{M} \rightleftarrows \mathcal{N}: G$$

is a *Quillen pair* if the left adjoint F preserves cofibrations and the right adjoint G preserves fibrations.

To get an appropriate notion of equivalence between model categories, we incorporate weak equivalence data.

Definition 14. A Quillen pair $F: \mathcal{M} \rightleftarrows \mathcal{N}: G$ is a *Quillen equivalence* when a morphism $FX \rightarrow Y$, with X cofibrant and Y fibrant, is a weak equivalence in $\mathcal{N}$ precisely when its corresponding adjoint morphism $X \rightarrow GY$ is a weak equivalence in $\mathcal{M}$.

The following result establishes that this definition produces the desired effect on the homotopy category.

Proposition 15. If $F: \mathcal{M} \rightleftarrows \mathcal{N}: G$ is a Quillen equivalence, then the induced adjoint pair

$$F: \mathrm{Ho}(\mathcal{M}) \rightleftarrows \mathrm{Ho}(\mathcal{N}): G$$

defines an equivalence of categories.

Having a Quillen equivalence between model categories is stronger than simply having an equivalence between the two homotopy categories, however. After giving an example, we will look more deeply into this structure.

Simplicial Sets

Let us consider a classical example of two model categories that are Quillen equivalent to one another: the model structure on topological spaces that we have already seen and a corresponding model structure on simplicial sets. The latter are combinatorial objects of a similar flavor to, but more general than, simplicial complexes. Aside from this relationship with topological spaces, simplicial sets play a central role in the study of general homotopy theories.

Definition 16. The category Δ has as objects the finite ordered sets $[n] = \{0 \leq 1 \leq \dots \leq n\}$ and as morphisms the weakly order-preserving functions.

Depicting only the generating morphisms for this category, we can visualize Δ as

$$[0] \rightleftarrows [1] \rightleftarrows [2] \dots$$

Given any category C , we can take its *opposite category* C^{op} , which has the same objects as C but in which the direction of the morphisms is reversed. We thus consider the category Δ^{op} .

Definition 17. A *simplicial set* is a functor $K: \Delta^{\text{op}} \rightarrow \text{Sets}$.

We can thus think of a simplicial set as a diagram of sets

$$K_0 \rightleftarrows K_1 \rightleftarrows K_2 \dots$$

satisfying certain relations. The maps $K_n \rightarrow K_{n-1}$ are called *face maps* and behave analogously to face maps between simplicial complexes. The main difference between simplicial sets and simplicial complexes is that n -simplices, here the elements of the set K_n , are not required to be generalized triangles in terms of how their faces fit together. The maps $K_n \rightarrow K_{n+1}$ are called *degeneracy maps* and provide a specific way for an n -simplex to be thought of as a “degenerate” simplex in each higher dimension.

Given a simplicial set K , it can be geometrically realized to a topological space $|K|$. The idea is to take a geometric n -simplex Δ^n for each element of K_n , for all $n \geq 0$, and glue together simplices as specified by the face maps. Thus, although they are defined combinatorially, we can think of simplicial sets geometrically. Let us look at some examples.

Example 18. For any $n \geq 0$, define the *standard n -simplex* to be the simplicial set $\Delta[n] = \text{Hom}_{\Delta}(-, [n])$. As its name suggests, its geometric realization is the usual topological n -simplex Δ^n . If we remove the n -simplex given by the identity map in $\text{Hom}_{\Delta}([n], [n])$, we obtain its *boundary* $\partial\Delta[n]$.

In an n -simplex, or its boundary, observe that the 0-simplices have a natural ordering, so we can label them by $0, 1, \dots, n$. We thus define the *k -horn* $V[n, k]$ by removing the face opposite the vertex labeled by k from $\partial\Delta[n]$ for any $0 \leq k \leq n$. When $n = 2$, we can visualize $\partial\Delta[2]$ as

Then the horns $V[2, 0]$, $V[2, 1]$, and $V[2, 2]$ can be depicted respectively as

Since simplicial sets are defined as functors, we can take natural transformations between them to obtain a category of simplicial sets, which we denote by SSets . More concretely, a map of simplicial sets $K \rightarrow L$ consists of maps $K_n \rightarrow L_n$ for all $n \geq 0$ that commute with the face and degeneracy maps appropriately.

Geometric realization defines a functor

$$|-|: \text{SSets} \rightarrow \text{Top},$$

where Top denotes the category of topological spaces and continuous maps. This functor has a right adjoint, the singular functor

$$\text{Sing}: \text{Top} \rightarrow \text{SSets},$$

defined by $\text{Sing}(X)_n = \text{Hom}(\Delta^n, X)$.

We can use the geometric realization functor to define a model structure on the category of simplicial sets.

Definition 19. A map $K \rightarrow L$ of simplicial sets is a *weak equivalence* if its geometric realization $|K| \rightarrow |L|$ is a weak homotopy equivalence of topological spaces.

We define the cofibrations simply to be monomorphisms of simplicial sets. To define the fibrations, we use the horn inclusions $V[n, k] \rightarrow \Delta[n]$ from the above example.

Definition 20. A map $K \rightarrow L$ of simplicial sets is a *Kan fibration* if, for any $n \geq 1$ and $0 \leq k \leq n$, a dotted arrow lift exists in any commutative diagram of the form

$$\begin{array}{ccc} V[n, k] & \longrightarrow & K \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ \Delta[n] & \longrightarrow & L. \end{array}$$

Theorem 21. There is a model structure on the category of simplicial sets with weak equivalences, fibrations, and cofibrations defined as above.

Using these definitions, observe that every object is cofibrant and that the fibrant objects are the *Kan complexes* or simplicial sets K for which the unique map $K \rightarrow \Delta[0]$ is a Kan fibration.

The following theorem of Quillen [15], which is proved concurrently with the one above, justifies our claim that simplicial sets are combinatorial models for topological spaces.

Theorem 22. *The adjoint pair $(|-|, \text{Sing})$ defines a Quillen equivalence between the model structures on simplicial sets and topological spaces.*

This result reinforces the idea that simplicial sets provide a combinatorial model for topological spaces in that they have the same essential homotopy theory.

Categories with Weak Equivalences and Localization

Unfortunately, not every category with a good notion of weak equivalence actually has a model structure. Sometimes the problem is categorical, in that the underlying category does not have enough limits or colimits. In other situations it is difficult or even impossible to find appropriate classes of fibrations and cofibrations that work well. So one might ask, can we get by without a model structure?

Formally, the answer is yes. Given any category C and choice of class $\mathcal{W}$ of morphisms in that category, we can take a *localization* of C by formally adjoining inverses to all morphisms of $\mathcal{W}$, as well as all composites involving those new morphisms, to again obtain a category. Indeed, we can take the resulting category to be the homotopy category of the pair $(C, \mathcal{W})$. This homotopy category is a *localization* of $(C, \mathcal{W})$, in that any functor from C to another category $\mathcal{D}$ that takes weak equivalences of C to isomorphisms in $\mathcal{D}$ must factor through the homotopy category.

Since that description seems so much more simple, why did we need to go to all the trouble of a model structure? One reason is that the formal localization that we described here is often not set-theoretically well-behaved. Typically, we do not assume that a category has a *set* of objects, but can have a proper class, as we see in many common examples, such as the category of sets or the category of topological spaces. Consequently, the collection of morphisms often also forms a proper class. However, it is usually assumed, given two objects X and Y of a category, that $\text{Hom}(X, Y)$ is a set rather than a class. The localization process that we have just described does not necessarily preserve this property; the output could be a structure with proper hom classes rather than hom sets. There are ways to work around this issue, for example, invoking passage to a higher set-theoretic universe. When we apply the homotopy category construction to a model category, we avoid these issues and get a category in the preferred sense.

Nonetheless, let us now turn to a more general theory

of categories with weak equivalences and no assumed additional structure, sometimes referred to as *relative categories*.

Definition 23. *A category with weak equivalences is a category C together with a subcategory $\mathcal{W}$ whose morphisms are called *weak equivalences* and satisfy the two-out-of-three property.*

Because the homotopy category does not really depend on a given model structure and because we do not need a model structure to have the data of a category with weak equivalences anyway, we consider a category with weak equivalences to be the essential data of what we might call a “homotopy theory.”

A Homotopy Theory of Homotopy Theories

We are now in a position to investigate homotopy theories as objects of study. Returning to the ideas from the beginning of this article, we could then ask, what are the appropriate functions between homotopy theories, and what might a natural choice of equivalences be? We have seen the answers to these questions in the setting of model categories, but what about for more general categories with weak equivalences?

For functions $(C, \mathcal{W}) \rightarrow (C', \mathcal{W}')$, a natural choice is to take functors $F: C \rightarrow \mathcal{D}$ such that $F(\mathcal{W}) \subseteq \mathcal{W}'$, so that weak equivalences are preserved. To answer the question of what equivalences should be, let us first recall the definition of equivalence of categories.

Definition 24. *A functor $F: C \rightarrow \mathcal{D}$ is an *equivalence of categories* if:*

- (1) *F is fully faithful*, in that, for any objects x and y of C , the map

$$\text{Hom}_C(x, y) \rightarrow \text{Hom}_{\mathcal{D}}(Fx, Fy)$$

is a bijection of sets; and

- (2) *F is essentially surjective*, in that, for any object z of $\mathcal{D}$, there is an object x of C such that $Fx \cong z$.

The question of incorporating weak equivalence data into this definition turns out to be surprisingly delicate. For example, a natural generalization of essential surjectivity would be that any object of $\mathcal{D}$ is weakly equivalent to an object in the image of F . However, if “weakly equivalent” is to be an equivalence relation, two objects can be weakly equivalent without having a single weak equivalence between them, but instead have a zig-zag such as

$$\bullet \xrightarrow{\sim} \bullet \xleftarrow{\sim} \bullet \xrightarrow{\sim} \bullet$$

connecting them.

This difficulty is not insurmountable, as shown in work of Barwick and Kan [1], but here we instead look at an earlier approach to this difficulty implemented by Dwyer and Kan. Using the methods of simplicial localization, they

repackage the data of a homotopy theory into that of a simplicial category [5].

Definition 25. A *simplicial category* is a category C such that the morphisms between objects X and Y form a simplicial set $\text{Map}_C(x, y)$ in a way compatible with composition and identity morphisms. A *simplicial functor* $C \rightarrow \mathcal{D}$ preserves this simplicial structure.

For example, the category of simplicial sets forms a simplicial category, where

$$\text{Map}(K, L)_n = \text{Hom}(K \times \Delta[n], L).$$

The idea behind Dwyer and Kan's simplicial localization is that we can build a simplicial category $L(C, W)$ from a category with weak equivalences (C, W) by defining $\text{Map}(x, y)$ to have 0-simplices given by zig-zags such as

$$x \rightarrow \bullet \xleftarrow{\sim} \bullet \rightarrow y$$

in which all left-going arrows are weak equivalences but which can be any length. Higher n -simplices are given by *hammocks* of length n ; an example when $n = 1$ is

One can recover the homotopy category (or ordinary localization) by taking the sets of components of each of these mapping spaces. Here, additional structure of these mapping spaces captures how much more information is present in a homotopy theory than in its associated homotopy category.

In the setting of simplicial categories, we have the following natural generalization of the notion of equivalence of categories.

Definition 26. A simplicial functor $f: C \rightarrow \mathcal{D}$ is a *Dwyer–Kan equivalence* if:

- (1) for any pair of objects x, y of C , the induced map of simplicial sets

$$\text{Map}_C(x, y) \rightarrow \text{Map}_{\mathcal{D}}(fx, fy)$$

is a weak equivalence of simplicial sets; and

- (2) the functor $\pi_0 C \rightarrow \pi_0 \mathcal{D}$ is essentially surjective, where $\pi_0 C$ denotes the category with the same objects as C and $\text{Hom}_{\pi_0 C}(x, y) = \pi_0 \text{Map}_C(x, y)$.

The following result of Dwyer and Kan from [4] tells us that simplicial categories always arise from simplicial localization.

Theorem 27. Up to Dwyer–Kan equivalence, every simplicial category can be obtained as the simplicial localization of a category with weak equivalences.

Thus, we can sensibly regard a simplicial category as a “homotopy theory” and the collection of all small simplicial categories, together with Dwyer–Kan equivalences, as the “homotopy theory of homotopy theories.”

In fact, we can say even more.

Theorem 28. There is a model structure on the category of small simplicial categories in which the weak equivalences are the Dwyer–Kan equivalences.

We omit the details of the fibrations and cofibrations here but refer the reader to [2].

Thinking of a homotopy theory as being modeled by a simplicial category, we can take the following alternative perspective. Taking the components of mapping spaces in a simplicial category via the functor π_0 results in an ordinary category. We can thus think of any morphism in a given mapping space as representing an equivalence class in that category and any two such equivalent morphisms as being “homotopic” to one another. In this way, we can think of a simplicial category as representing a “category up to homotopy,” in the sense that morphisms can be deformed in a manner analogous to homotopies of maps of topological spaces.

For many applications, it is useful to push this idea still further and ask that composition of morphisms be defined only up to homotopy or that associativity of composition be required to hold only up to homotopy. It is this idea that we wish to explore in the next two sections.

What Is a Category Up to Homotopy?

Now that we have the concept of a category up to homotopy, let us develop it from a different angle, starting with the following way to think of an ordinary category as a simplicial set.

Definition 29. Let C be a small category. Its *nerve* is the simplicial set $\text{nerve}(C)$ given by

$$\text{nerve}(C)_n = \text{Hom}([n], C).$$

Observe that, in this definition, we are considering the ordered set $[n]$, which we previously regarded as an object of Δ , as a category with $n + 1$ objects, depicted as

$$0 \rightarrow 1 \rightarrow 2 \rightarrow \cdots \rightarrow n.$$

Thus, the 0-simplices of the nerve correspond to the objects of C and the 1-simplices to the morphisms. For $n \geq 2$, the set of n -simplices of the nerve of C is precisely the set of chains of n composable arrows in the category C .

A natural question, then, is whether the nerve functor takes equivalences of categories to weak equivalences of categories, which is indeed the case.

Proposition 30. If $F: \mathcal{C} \rightarrow \mathcal{D}$ is an equivalence of categories, then $\text{nerve}(F): \text{nerve}(\mathcal{C}) \rightarrow \text{nerve}(\mathcal{D})$ is a weak equivalence of simplicial sets.

However, the converse statement is false, as the following example illustrates.

Example 31. Consider the category $[0]$, which consists of one object and no nonidentity morphisms, and the category $[1]$, which consists of two objects and a single nonidentity morphism between them. The natural inclusion $[0] \hookrightarrow [1]$ is not an equivalence of categories, since the object 1 of $[1]$ is not isomorphic to an object in the image of this functor. However, its nerve is an inclusion $\Delta[0] \rightarrow \Delta[1]$ that after geometric realization is simply the inclusion of the endpoint of an interval. Since both of these spaces are contractible, this map is a weak equivalence.

The problem here arises because weak equivalences between simplicial sets are defined in terms of geometric realization, which forgets the directionality of 1-simplices in a simplicial set. More importantly for nerves of categories, we lose information about whether a 1-simplex arises from an isomorphism or from a noninvertible morphism. This data, however, is critical for verifying essential surjectivity. We would like to have a more refined notion of weak equivalence that does not identify nerves of nonequivalent categories.

We are not going to describe these new weak equivalences explicitly, since it is quite technical to do so, but rather use elements of the previously described model structure on simplicial sets to point toward them. In particular, we look at the relationship between Kan complexes and nerves of *groupoids* or categories with all morphisms invertible.

Recall from above that a *Kan complex* is a simplicial set K such that a dotted arrow lift exists in any diagram of the form

$$\begin{array}{ccc} V[n, k] & \longrightarrow & K \\ \downarrow & \nearrow \text{dotted} & \\ \Delta[n] & & \end{array}$$

where $n \geq 1$ and $0 \leq k \leq n$, making the diagram commute.

Proposition 32. A simplicial set is the nerve of a groupoid if and only if it is a strict Kan complex, in that the lifts along the horn inclusions are unique.

One can thus think of a Kan complex in which the lifts are not unique as a groupoid up to homotopy. Since every object in a model category is weakly equivalent to one that is fibrant, we see our difficulty from another angle: weak equivalences cannot tell nerves of groupoids apart from

other simplicial sets and in particular from nerves of more general categories.

Let us return to our example of horns in a 2-simplex. What does being able to extend them in a 2-simplex tell us? In the case of $V[2, 1]$, a unique lifting corresponds to the existence of a composite 1-simplex, as we would expect in the nerve of a category. However, lifting along the horns $V[2, 0]$ and $V[2, 2]$ does not give us composition data but rather the existence of one-sided inverses. This example suggests the following result.

Proposition 33. A simplicial set is the nerve of a category if and only if it has unique liftings along the inner horn inclusions, namely, $V[n, k] \rightarrow \Delta[n]$ for $0 < k < n$.

We thus have good candidates for categories up to homotopy by removing the uniqueness assumption.

Definition 34. A *quasi-category* is a simplicial set K such that a lift exists in any diagram of the form

$$\begin{array}{ccc} V[n, k] & \longrightarrow & K \\ \downarrow & \nearrow \text{dotted} & \\ \Delta[n] & & \end{array}$$

for every $n \geq 2$ and $0 < k < n$.

We then describe our desired equivalences as the weak equivalences in a model structure characterized by its nice objects as follows [11].

Theorem 35. There is a model structure on the category of simplicial sets such that the fibrant and cofibrant objects are precisely the quasi-categories. This model structure is Quillen equivalent to the one for simplicial categories.

This theorem serves to give us another model for the homotopy theory of homotopy theories, but we also have that the weak equivalences (often called *Joyal equivalences*) between nerves of categories correspond exactly to equivalences of categories.

Another Approach to Categories Up to Homotopy

Rezk developed a different strategy for defining categories up to homotopy [17]. The idea is to keep the same notion of weak equivalence between simplicial sets but to separate out isomorphism data in the nerve of a category. We do this separation via an extra simplicial direction.

Definition 36. A *bisimplicial set* is a functor $\Delta^{\text{op}} \rightarrow \mathbf{SSets}$.

In other words, a bisimplicial set consists of a simplicial diagram of simplicial sets. To describe how to obtain one from a category, let us first give some notation.

Let $\mathcal{C}$ be a small category and let $n \geq 0$. Then the category of functors $\mathcal{C}^{[n]}$ has as objects the functors $[n] \rightarrow \mathcal{C}$ and as morphisms the natural transformations between

such functors. Given a small category $\mathcal{D}$, denote by $\text{iso}(\mathcal{D})$ the subcategory of $\mathcal{D}$ consisting only of the isomorphisms of $\mathcal{D}$.

Definition 37. Let C be a small category. Its *classifying diagram* NC is the bisimplicial set given by

$$(NC)_n = \text{nerve}(\text{iso}(C^{[n]})).$$

Observe that $(NC)_0$ is essentially the nerve of $\text{iso}(C)$, so contains only information about the isomorphisms of C . Data about other morphisms on C appears in $(NC)_1$.

We refer the reader to [17] for a more explicit version of the following result.

Proposition 38.

(1) For every $n \geq 2$, there is a natural isomorphism

$$(NC)_n \cong \underbrace{(NC)_1 \times_{(NC)_0} \cdots \times_{(NC)_0} (NC)_1}_n.$$

(2) The image of the degeneracy map $(NC)_0 \rightarrow (NC)_1$ consists precisely of the information about isomorphisms in C .

(3) The simplicial set $(NC)_{*,0}$ is precisely the nerve of C .

It follows from this result that the classifying diagram does what we want: a functor $C \rightarrow \mathcal{D}$ is an equivalence of categories if and only if, for every $n \geq 0$, the map $(NC)_n \rightarrow (N\mathcal{D})_n$ is a weak equivalence of simplicial sets. Thus, we have a way to describe categories in terms of bisimplicial sets. To produce a model for categories up to homotopy, we weaken the first two properties of the classifying diagram.

Definition 39.

(1) A bisimplicial set X is a *Segal space* if, for each $n \geq 2$, the map

$$X \rightarrow \underbrace{X_1 \times_{X_0}^h \cdots \times_{X_0}^h X_1}_n$$

is a weak equivalence of simplicial sets, where the right-hand side denotes an iterated homotopy pull-back.

(2) It is a *complete Segal space* if, additionally, the image of the degeneracy map $X_0 \rightarrow X_1$ is weakly equivalent to the subsimplicial set of homotopy equivalences.

We again have an associated model structure that we describe in terms of the nice objects [17].

Theorem 40. *There is a model structure on the category of bisimplicial sets in which all objects are cofibrant and the fibrant objects are complete Segal spaces.*

Furthermore, property (3) from Proposition 38 above generalizes as follows.

Theorem 41. *If X is a complete Segal space, then the simplicial set $X_{*,0}$ is a quasi-category. Indeed, this assignment defines a Quillen equivalence between the two model structures.*

This result is proved in [10]; a comparison with simplicial categories can be found in [3].

Other Perspectives

There are also other ways to model such structures, including Segal categories [7], [14]; an appropriate notion of A_∞ -categories [8]; the 1-complicial sets of Verity [13], [20]; and, going back to the origins of the subject, a model structure on the category of small categories with weak equivalences [1]. An axiomatic treatment for such models was given by Toën [19], and a very general approach using the framework of 2-categories is being developed by Riehl and Verity [18].

A further point of view that we have not considered here, but which is nonetheless very important, is that homotopy theories, or categories up to homotopy, are a certain flavor of higher categories called $(\infty, 1)$ -categories. An introductory treatment to this point of view was presented in [12]. Whether regarded as a homotopy theory or as a higher category, these structures have become widely used, not only in homotopy theory but in a range of subjects including mathematical physics and algebraic geometry.

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