Eigenvalues of Random Matrices in the General Linear Group in the Large- N Limit

Brian C. Hall

Random Matrices and the Circular Law

Random matrix theory was introduced by Eugene Wigner [3] in a 1955 paper modeling energy levels in atomic nuclei. The subject has now blossomed into a large industry with deep connections to physics. For my talk the most relevant sort of random matrix is the *Ginibre ensemble*. Here we consider $N \times N$ matrices with the entries chosen independently. Each entry X_{jk} is chosen to be a complex number, with the real and imaginary parts being independent normal random variables with mean zero and variance $1/(2N)$. We now consider the eigenvalues of this random matrix; these form a random collection of N points in the complex plane.

Eigenvalues of a random matrix from the Ginibre ensemble, with $N = 2,000$

Brian C. Hall is a professor of mathematics at the University of Notre Dame. His email address is bhall@nd.edu.

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Ginibre found that when N is large, these eigenvalues follow the *circular law*: with high probability, almost all of the eigenvalues lie in the unit disk and they are almost uniformly distributed there. This assertion exemplifies a general property of random matrices, that the bulk properties of the eigenvalues of randomly chosen $N \times N$ matrices tend to become nonrandom when N is large. Figure 1 shows a simulation of the Ginibre ensemble with $N = 2,000$.

Brownian Motion in the General Linear Group

I will discuss another random matrix model which is a deformation of the Ginibre ensemble. Let $GL(N; \mathbb{C})$ denote the general linear group, that is, the group of all $N \times N$ invertible matrices. Then consider Brownian motion b_t^N in $GL(N; \mathbb{C})$, starting from the identity. In a general Riemannian manifold, Brownian motion is a random path that can be obtained as the limit of random walks with very small step sizes. In the case of $GL(N; \mathbb{C})$, we should construct the approximating random walks *multiplicatively*; that is, at each stage, we *multiply* the current position by a matrix close to the identity.

A simulation of the eigenvalues of b_t^N with $N = 2,000$ and $t = 3.9$ shown on a plot of the domain Σ_t

The motivation for considering Brownian motion is this. If we consider Brownian motion in the space of all $N \times N$ matrices—but constructing the approximating random walks *additively*—the distribution at any time t is just the Ginibre ensemble scaled by a factor of $\sqrt{t}$. Thus, the distribution of Brownian motion in $GL(N; \mathbb{C})$ is a multiplicative analogue of the Ginibre ensemble.

In my talk, I will discuss results with Bruce Driver and Todd Kemp concerning the distribution of the eigenvalues of b_t^N in the large- N limit. Our first result [2] is that the eigenvalues cluster as $N \rightarrow \infty$ into a certain domain Σ_t in the plane identified by Biane [1]. We then have work in progress indicating a remarkably simple structure to the limiting distribution of eigenvalues within this domain. Figure 2 shows a simulation of the eigenvalues with $t = 3.9$ and $N = 2,000$.

References

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Brian C. Hall

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