Wild Wild Whitehead

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Take a knot. Take a tube around the knot. Put a new knot in the tube, twisted around and clasping itself as in Figure 1. The new knot goes once around the tube, and then “doubles back” and clasps itself. The new knot is the *Whitehead double* of the old.

Figure 1. The green knot is the Whitehead double of the black knot.

Let’s call the first knot K_0 and its Whitehead double K_1 , and let’s call the tube around K_0 (actually a solid torus) N_0 . It makes sense to take the Whitehead double of any knot, but in Figure 1, K_0 is a trivial knot; i.e., it bounds an embedded disk in S^3 . In this case, K_1 is trivial too: it bounds an embedded disk in S^3 . But K_1 is *knotted* in N_0 . Any embedded disk that K_1 bounds must go outside N_0 .

On the other hand, K_1 is *homotopically* trivial in N_0 ; i.e., it bounds an *immersed* disk, one that crosses itself, but does not cross N_0 . To see this, just push one of the clasps of K_1 through the other one; this undoes the knotting, and the

result can be shrunk down to a point. The track of the knot K_1 during this process sweeps out an immersed disk.

In terms of the fundamental group, a knot K in a space X determines a conjugacy class $[K]$ in the fundamental group $\pi_1(X)$. Now, $\pi_1(N_0) = \mathbb{Z}$, and since K_1 bounds an immersed disk, $[K_1]$ is trivial in $\pi_1(N_0)$.

We can keep going. Let N_1 be a tube around K_1 , thin enough to fit in N_0 , and let $K_2 \subset N_1$ be the Whitehead double of K_1 . And so on. Each K_i bounds an embedded disk in S^3 , but each of these disks must go (many times!) all the way outside N_0 .

The tubes get thinner and thinner as we go, and longer and longer. Consequently, the knots must get longer and longer too: Each K_n must wind back and forth at least 2^n times around N_0 , clasping itself in a complicated way at the end. The infinite intersection $\bigcap_i N_i$ is called the *Whitehead continuum*, which we write Wh , see Figure 2. The Whitehead continuum is connected but not path connected. It has an entangled dyadic Cantor set of “strands” that wind around N_0 .

The complement $S^3 - Wh$ is an open 3-manifold called the *Whitehead manifold*. It turns out that $S^3 - Wh$ is contractible but not homeomorphic to a 3-ball. Let’s see why.

The outside of N_0 in S^3 is another solid torus N'_0 , whose core is a knot K'_0 linking K_0 in a Hopf link. $[K'_0]$ is the generator of $\pi_1(S^3 - N_0) = \pi_1(N'_0) = \mathbb{Z}$. The knots K'_0 and K_1 together form a 2-component link called the *Whitehead link*. This link is symmetric: we can isotope it around and interchange K'_0 and K_1 ; see Figure 3.

Since $[K_1]$ is trivial in $\pi_1(N_0) = \pi_1(S^3 - N'_0)$, it follows by symmetry that $[K'_0]$ is trivial in $\pi_1(S^3 - N_1)$. Consequently the inclusion map $S^3 - N_0 \rightarrow S^3 - N_1$

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Figure 2. Successive approximations converge to Wh .

Figure 3. The Whitehead link is symmetric.

induces the zero map on π_1 . Each N_i is unknotted in S^3 , and each N_{i+1} sits in N_i the same way that N_1 sits in N_0 . So each $\pi_1(S^3 - N_i) = \mathbb{Z}$, and each inclusion $S^3 - N_n \rightarrow S^3 - N_{n+1}$ induces the zero map on π_1 . Taking a direct limit, $\pi_1(S^3 - Wh)$ itself is trivial. A similar argument shows that all the homotopy groups of $S^3 - Wh$ vanish, and it is contractible.

On the other hand, the complement $S^3 - Wh - N'_0 = N_0 - Wh$ has an *infinitely generated* fundamental group; each $\pi_1(N_i - N_{i+1})$ is complicated (it contains free groups of every rank), and all of them include as subgroups of $\pi_1(N_0 - Wh)$. This shows that $S^3 - Wh$ is not a ball.

Now let's compactify $S^3 - Wh$ by adding a single point at infinity. This compact space can also be thought of as the quotient space S^3/Wh that we get by crushing Wh to a single point. Because $S^3 - Wh$ is not a ball, S^3/Wh is not a manifold. However—remarkably—it is a *manifold factor*: the product $(S^3/Wh) \times \mathbb{R}$ is homeomorphic to $S^3 \times \mathbb{R}$!

How can this possibly be???

First, each N_1 slice can be unknotted by a tiny perturbation in $N_0 \times \mathbb{R}$. To distinguish the $\mathbb{R}$ factor and, for the sake of brevity, we refer to it as the “time” coordinate (this is purely a notational convenience). In this language, we unclasp N_1 from itself by nudging one clasp very slightly forward into the future, and the other very slightly back into the past. After the nudge, N_1 will not clasp itself, but it will clasp a “future” N_1 on one side, and a “past” N_1 on the other. Instead of N_1 clasp[ing] itself in a circle, we get a chain of successive N_1 's, each clasping the next, in a slowly ascending *spiral*. Let's let $\epsilon/4$ be the size of the perturbations of each clasp in the time direction, so that the projection of each N_1 to the time coordinate after it's been nudged has total length $\epsilon/2$.

Nudging adjusts points in $N_1 \times \mathbb{R}$ by sliding each point $\times \mathbb{R}$ slightly backward or forward in time. Nudging extends to a self-homeomorphism ν of $N_0 \times \mathbb{R}$, fixed on the boundary.

Figure 4. Folding the clasps of each N_1 back and forth in time nudges the union of all N_1 's into a collection of spirals.

By the way, there's not just one spiral—there's a circle's worth of them, filling the whole of $N_1 \times \mathbb{R}$. Two slices $\nu(N_1 \times t)$, $\nu(N_1 \times s)$ are in the same spiral if and only if $t - s$ is an integer multiple of $\epsilon/2$.

After nudging, the next move will straighten out this and every other spiral so that its projection to the S^3 factor is small (let's say for concreteness it has diameter $< \epsilon/2$) *without* affecting the projection to the $\mathbb{R}$ factor.

The cylinder $K_0 \times \mathbb{R} \subset S^3 \times \mathbb{R}$ has polar coordinates (θ, t) where $\theta \in \mathbb{R}/\mathbb{Z}$. Extend these polar coordinates to a small tubular neighborhood of $K_0 \times \mathbb{R}$ containing $N_1 \times \mathbb{R}$, with closure contained in the interior of $N_0 \times \mathbb{R}$.

We can “untwist” every spiral simultaneously by the map

$$(\theta, t) \rightarrow (\theta - 2t/\epsilon, t)$$

on our small tubular neighborhood. Twisting extends to a self-homeomorphism τ of $N_0 \times \mathbb{R}$, once again fixed on the boundary.

In summary, first we nudge, then we twist. After doing this, every $\tau\nu(N_1)$ slice projects to subsets of diameter at most $\epsilon/2$ in *both* the $\mathbb{R}$ and the S^3 directions. So $\tau\nu(N_1)$ has diameter at most ϵ .

In other words, $h_1 := \tau\nu$ simultaneously *shrinks* all the N_1 slices in $S^3 \times \mathbb{R}$ as small as we like, while keeping $(S^3 - N_0) \times \mathbb{R}$ fixed pointwise.

Figure 5. Screw the top and bottom in opposite directions like you're taking the lid off a pickle jar.

Take a sequence $\epsilon_i \rightarrow 0$, and repeat this operation for each $i > 1$ in place of 1 with ϵ_i in place of ϵ . We get a sequence of self-homeomorphisms $h_i : S^3 \times \mathbb{R} \rightarrow S^3 \times \mathbb{R}$, each supported in $N_{i-1} \times \mathbb{R}$, as a composition of a nudge-and-twist $h_i := \tau_i \nu_i$. Each N_i slice gets smaller and smaller in diameter as we apply consecutive h_i 's. Under application of successive h_i 's, the orbit of every point is a Cauchy sequence, and the infinite composition

$$h := \lim_{i \rightarrow \infty} h_i h_{i-1} \cdots h_1 : S^3 \times \mathbb{R} \rightarrow S^3 \times \mathbb{R}$$

is well-defined and continuous.

For any compact subset X of $S^3 - Wh$ the restriction of h to X is the composition of finitely many homeomorphisms, so the restriction of h to $(S^3 - Wh) \times \mathbb{R}$ is a homeomorphism. On the other hand, each Wh slice is successively shrunk smaller and smaller by successive h_i , so in the end h crushes each Wh slice to a point, and h factors as $h = g\pi$,

$$S^3 \times \mathbb{R} \xrightarrow{\pi} (S^3/Wh) \times \mathbb{R} \xrightarrow{g} S^3 \times \mathbb{R},$$

where $\pi : S^3 \times \mathbb{R} \rightarrow (S^3/Wh) \times \mathbb{R}$ is the quotient map, and g is the homeomorphism we've been looking for.

AUTHOR'S NOTE. The main theorem in this article and its proof are both well-known, and not due to me! They are due to J. Andrews and L. Rubin, *Bull. Amer. Math. Soc.* **71**(1965), 675-677.

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