Quantizations, Representations, and Morse Theory

Thomas Nevins

Suppose X is a nonsingular complex algebraic variety, and let $\mathbb{C}[X]$ denote its coordinate ring (ring of regular functions). If $X \subset \mathbb{C}^n$, then $\mathbb{C}[X]$ is the algebra of functions that are restrictions of polynomial functions on $\mathbb{C}^n$. For example, if $X = \{(x, y) \in \mathbb{C}^2 \mid y = x^2\}$, two polynomials $f_1(x, y)$ and $f_2(x, y)$ have the same restriction to X if and only if $y - x^2$ divides $f_1 - f_2$ in the polynomial ring $\mathbb{C}[x, y]$; thus, $\mathbb{C}[X]$ is identified with the quotient ring $\mathbb{C}[x, y]/(y - x^2)$.

Similarly, let $\mathcal{D} = \mathcal{D}(X)$ denote the algebra of linear differential operators (with regular function coefficients) on X . For example, if $X = \mathbb{C}^n$, then

$$(1) \quad \mathcal{D}(X) = \mathbb{C}[x_1, \dots, x_n, \partial/\partial x_1, \dots, \partial/\partial x_n],$$

the n th Weyl algebra. In general, $\mathcal{D}(X)$ is defined via operators acting on $\mathbb{C}[X]$.

Enhancing a $\mathbb{C}[X]$ -module M to a left $\mathcal{D}$ -module amounts to choosing a flat (algebraic) connection on M . Via the latter description and the Riemann-Hilbert correspondence, $\mathcal{D}$ -modules have close connections to topology and specifically to the Hodge theory of complex varieties.

On the other hand, the algebra $\mathcal{D}(X)$ is a deformation of the algebra $\mathbb{C}[T^*X]$ of functions on the cotangent bundle, as one can imagine when $X = \mathbb{C}^n$ from formula (1). That deformation is closely connected to the canonical symplectic structure on the cotangent bundle T^*X , and thus $\mathcal{D}$ -modules connect closely to the symplectic geometry of T^*X , the relationship at the heart of microlocal analysis.

Thomas Nevins is professor of mathematics at the University of Illinois at Urbana-Champaign. His email address is nevins@illinois.edu.

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$\mathcal{D}$ -modules on algebraic varieties provide paradigmatic tools for realizing representations of interesting algebras. For example, if a complex Lie group G acts on X , then differentiating the action yields a homomorphism $U(\mathfrak{g}) \rightarrow \mathcal{D}(X)$ from the enveloping algebra of the Lie algebra $\mathfrak{g}$ of G , and thus every $\mathcal{D}$ -module on X yields a representation of $U(\mathfrak{g})$. In the case when $X = G/B$ is the flag variety of a complex semisimple group G , this construction mediates between topology (and Hodge theory) of the projective variety G/B and representations of $\mathfrak{g}$ in the proof (by Beilinson-Bernstein and Brylinski-Kashiwara) of the “Kazhdan-Lusztig conjecture” of Lie theory.

Replacing T^*X by a more general symplectic algebraic variety (or algebraic stack/equivariant space) Y and $\mathcal{D}$ by a quantization of the algebra of functions on Y leads to many more algebras of importance: for example, Cherednik algebras, W -algebras, and more. The space Y yields a rich interplay of symplectic algebraic geometry, quantum algebra, and representation theory that is the subject of intense current research.

One recent new research direction concerns Morse theory. Morse theory provides structural principles governing how the space Y is built from simpler pieces. Furthermore, just as Morse theory controls the cohomology of Y , it also appears “one categorical level higher,” controlling how categories of representations, realized via quantum geometry of Y , are assembled from basic constituents. The *categorical Morse theory* for quantum symplectic varieties that thus emerges has new consequences not only for representation theory but also “at lower categorical level” for the topology of Y .

Credit

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THE AUTHOR

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