Szemerédi's Theorem: Combinatorics, Ergodic Theory, and Algebra

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In 1930 Erdős and Turán conjectured that a set of natural numbers which is not too sparse must contain fairly long arithmetic progressions:

For any $\delta > 0$ and any integer $k \geq 1$ there is an $N_0(\delta, k)$ for which the following holds. If $N \geq N_0(\delta, k)$, and $E \subseteq \{1, 2, \dots, N\}$ contains at least δN points, then E contains k points in arithmetic progression: that is,

$$E \supseteq \{a, a + n, a + 2n, \dots, a + (k - 1)n\}$$

for some a and $n \geq 1$.

The case $k = 3$ was proved by Roth in 1953, and the general case was proved by Szemerédi in 1975. Roth's proof made a subtle use of Fourier analysis. Szemerédi's was purely combinatorial but required several deep new tools that have since become essential machinery in extremal combinatorics.

In 1977 Furstenberg gave a new proof of Szemerédi's Theorem by deducing it from an analogous result in ergodic theory called the Multiple Recurrence Theorem. A year later he and Katznelson were able to extend this approach to prove a "multi-dimensional" generalization of Szemerédi's Theorem: If N is large enough and E is a subset of $\{1, 2, \dots, N\}^d$ which is not too sparse, then

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$$E \supseteq \{a, a + ne_1, \dots, a + ne_d\}$$

for some integer vector a and $n \geq 1$,

where $e_1, \dots, e_d$ are the standard basis in $\mathbb{R}^d$.

Since these developments, Szemerédi's Theorem and various relatives have continued to attract attention. One motivation is to improve the known upper bounds on $N_0(\delta, k)$. At present, the best bound for the general case of Szemerédi's Theorem comes from Gowers's new approach from around 2000: he bounds $N_0(\delta, k)$ by a tower of nine iterated exponentials. (Better bounds are known when $k = 3$.)

Another motivation is the search for analogs of Szemerédi's Theorem in other settings. For instance, Szemerédi's Theorem itself does not imply that the primes contain arbitrarily long arithmetic progressions, because the number of primes in $\{1, \dots, N\}$ grows more slowly than any fixed multiple of N , but in 2006 Green and Tao proved this result by adapting ideas from several of the previous proofs of Szemerédi's Theorem.

Perhaps the deepest reason for continued interest in Szemerédi's Theorem is that it still generates so much interesting mathematics. Szemerédi's work launched a new field within combinatorics, and novel applications of his basic methods continue to appear every year. Furstenberg's approach did the same for ergodic theory: the basis of his argument is a certain classification of dynamical systems with invariant measures, and more recent work by Host, Kra, and others has focused on what roles are played by all the different species. Gowers's approach required a whole new generalization of Fourier analysis, now sometimes called "higher-order Fourier analysis." Other recent works have also uncovered some deep features that these approaches have in common.

This talk will start with a rough sketch of some of these directions and will then focus on one recent project to emerge from this area. Although Gowers's general approach gives the most efficient proof of Szemerédi's Theorem and it has now been greatly enhanced by Green, Tao, Ziegler, and Szegedy, it has not been

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extended to prove the multi-dimensional theorem of Furstenberg and Katznelson. The known bounds in that theorem are correspondingly much worse. It turns out that the search for a multidimensional version of Gowers's work quickly runs into purely algebraic difficulties, before one even starts on the analysis or combinatorics. I will finish with a sketch of an elementary but surprisingly rich class of linear equations for functions on Abelian groups which arise in this search and some preliminary results about them.

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