Albert Baernstein II

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David Drasin

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Albert (Al) Baernstein made significant and original contributions to classical analysis. An outstanding and scrupulous scholar and teacher, he had a special talent for developing his own methods to settle long-standing problems. During his long service at Washington University, he trained a large cadre of PhD students, both from the US and abroad, and their contributions to complex and harmonic analysis continue.

Al's own research impact lies in his approach to symmetrization, now encapsulated by the term *Baernstein star function* (Baernstein $\star$ -function). Symmetrization is an ancient concept in mathematical analysis and geometry. Its most famous result is the *isoperimetric principle*: among all plane regions of fixed area, the one of least perimeter is the disk.

The general situation is concerned with a class of functions $\mathfrak{F}$ or domains $\mathfrak{D}$, and the issue is

to show that a given functional Φ is maximized, when restricted to either $\mathfrak{F}$ or $\mathfrak{D}$, by a function f or domain D which has additional regularity or symmetry properties. Al's inspiration from complex analysis arose in unexpected ways; but isoperimetry and symmetry are also important in statistics, probability, physics, potential theory, etc. Pólya-Szegő [MR 0043486] is the classical reference.

Al was raised in Tuscaloosa, Alabama, where his father owned a men's clothing store. Shortly after graduating from Cornell in 1962, he was married to Judy (Haynes) and had lost (nearly all of) his southern accent. A year as a cost analyst in the insurance industry convinced him to switch to graduate school. Thus, in 1963 he and Judy (and their new daughter) moved to Madison, Wisconsin. Al had broad mathematical interests but soon decided to concentrate on analysis, a discipline in which Wisconsin traditionally excelled. He consulted with Simon Hellerstein, whose course in entire and meromorphic functions had fascinated Al. Since Si had a full deck of students, he sent Al to a new colleague, Daniel Shea.

The thesis problem Dan proposed had little to do with symmetrization, but the star function arose from it in a natural manner a few years later. Al was already fascinated by Norbert Wiener's famous Tauberian theorem, so Dan suggested a type of regularity theorem (of Tauberian character) for what at first sight is an artificially narrow class of entire functions (an entire function is a function f analytic in the full complex plane). Functions in this class grow more slowly than the exponential function $w = e^z$ (i.e., f has order $\rho < 1$) and, assuming $f(0) = 1$, have the simple factorization

$$(1) \quad f(z) = \prod_n \left(1 - \frac{z}{a_n}\right)$$

(without the assumption of small order, this infinite product will not converge). Their special, defining

David Drasin is Emeritus Professor of mathematics at Purdue University. His email address is drasin@purdue.edu.

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property is that all zeros $\{a_n\}$ lie on the negative real axis.

Since the 1920s the standard method to study entire and meromorphic functions has been with Rolf Nevanlinna's value-distribution theory and characteristic function $T(r)$, $r > 0$; its most famous result is a deep and systematic generalization of E. Picard's theorem, expressed through Nevanlinna's notion of *deficient value*. In the situation here, very little preparation is needed; the *Nevanlinna characteristic* $T(r)$ may be defined as (2)

$$T(r) = \frac{1}{2\pi} \int_{re^{i\theta} \in P} \log |f(re^{i\theta})| d\theta \quad (0 < r < \infty),$$

where $P = \{z; \log |f(z)| \geq 0\}$. In fact, $T : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is an increasing function whose growth and behavior capture many asymptotic properties of f . It is elementary to see from the zeros $\{a_n\}$ in (1) being negative that $P \cap \{|z| = r\}$ is an arc $\{re^{i\theta}, |\theta| \leq \beta(r)\}$ and that ∂P is the arc $\Gamma = \{re^{i\beta(r)}; r > 0\}$ together with its reflection $\bar{\Gamma} = \{re^{-i\beta(r)}; r > 0\}$. The Tauberian nature of the problem is to obtain from an asymptotic formula for $T(r)$ a corresponding asymptotic formula for the number of zeros $\{a_n\}$ inside all large disks $\{|z| < r\}$; the converse relation was known already early in the last century. The location of the curves $\Gamma, \bar{\Gamma}$ is unspecified—only the values of f on them are given—but one consequence of this theorem is that Γ and $\bar{\Gamma}$ are asymptotic rays.

By late 1967 Al solved this problem [MR 0257358] with his “nonlinear Tauberian theorem,” combining Wiener's theorem with techniques from potential theory and harmonic analysis.

An assistant professorship at Syracuse followed in 1968, where Daniel Waterman was a leading harmonic analyst. In preparation, Al studied a survey article by Waterman and Casper Goffman [MR 0252940] and was able to settle several problems in harmonic analysis stated there [MR 0310523, MR 0304954, MR 0305044]. The genesis

of his star function came soon after, from a visit to Syracuse in 1971 by Allen Weitsman. Allen urged Al to work with Albert Edrei (also on the Syracuse faculty and thesis adviser to both Shea and Weitsman) to resolve a problem Edrei had championed for several years: to prove the *spread conjecture*. This refined problem in classical Nevanlinna theory suggested a concrete interpretation of a value $w = a$ being “deficient” in Nevanlinna's sense. Thus let f be meromorphic in the plane of order ρ with deficiency $\delta = \delta(a, f) > 0$ for some $a \in \mathbb{C} \cup \infty$. The conjecture was that there must be a sequence $r_n \rightarrow \infty$ and a network of circles $S_n = \{|z| = r_n\}$ so that $|f(r_ne^{i\theta}) - a|$ will be small on a θ -set of angular measure of at least

$$\min \left(\frac{4}{\rho} \sin^{-1} \sqrt{\frac{\delta}{2}}, 2\pi \right)$$

(this sequence $\{r_n\}$ is independent of a ; a slightly weaker form of this assertion was proposed far earlier by O. Teichmüller).

Within six months Al obtained the spread relation [MR 0372429]. Most significantly, its realization required his creating the star function $T^*(re^{i\theta})$, $(0 \leq \theta \leq \pi)$. The contrast between $T(r)$ (recall (2)) and T^* is evident from the definition

$$T^*(re^{i\theta}) = \max_{|E|=2\theta} \int_E \log |f(re^{i\psi})| d\psi,$$

concealing the remarkable property that T^* is subharmonic in the upper half-plane $H : \{re^{i\theta}, 0 < \theta < \pi\}$. When applied to a general entire function f (not of the form (1)), T^* converts the values $\{f(re^{i\theta}), |\theta| \leq \pi\}$ into their *symmetric decreasing rearrangement*, a procedure standard in symmetrization, and then integrates back. The spread relation then followed from analysis of the subharmonic function T^* with known boundary values on ∂H . (The definitions of T and T^* are slightly more complicated when f has poles.)

While Al and many authors later applied the star function to other questions about meromorphic functions, several classical problems in which these functions grow more rapidly than the exponential function (i.e., functions of order $\rho > 1$) seem to require different methods and remain open.

Al's proof of the spread relation led to prompt international recognition and an offer in 1971 from Washington University in St. Louis. Since this breakthrough occurred early in his career, he, along with his students, coauthors and the full mathematical world, had many opportunities, which led to an invitation to address the 1978 International Congress in Helsinki [MR 562666], but made him a frequent and popular speaker at many mathematical conferences and symposia. The results of many of these presentations remain in the literature as surveys of and orientations on a wide variety of topics.

Walter Hayman recalls the contemporary impact of T^* : “I remember well when I first met Al

Baernstein at the conference in Canterbury in 1973. He told us about his invention of the star function and how you could use it to prove Edrei's spread conjecture and to solve a number of other problems. So he became the star of the conference. He has been one of the brightest stars in the sky of function theory ever since."

For the rest of his life, the center of Al's activities was at Washington University, where he and Judy raised their daughters, Prudence Renée and Amy, and hosted legions of visitors and students. The Baernsteins loved the spirit of the mathematics department and especially the cultural life in St. Louis, with its rich traditions in music and opera. Al's passion for the drama of the libretti matched his love of the music, and he and Judy were proud supporters of St. Louis opera companies. Indeed, his memory for libretti as well as for dates, events, talks, and on their many mathematical memories seemed almost beyond human capacity.

The Baernsteins thrived on outdoor activities, from matters as routine as Al's daily walks to his office to family float trips on Missouri rivers and long hikes both near home and during their many trips. Judy hailed from Maine, and nearly annual summer pilgrimages to their cabin in Maine and hikes in Acadia National Park were cherished family rituals. Al had serious cardiac issues all his life, but superb medical care allowed him to remain active until his last years, when a neurological disease made both mathematics and intense physical activity impossible.

Shortly after arriving in St. Louis, Al propelled his theory far beyond meromorphic functions in his most-cited paper (*Acta Mathematica* [MR 0417406]). One famous result dealt with univalent (one-one) functions $S : f(z) = z + \sum_{j \geq 2} a_j z^j$ in the unit disk $D := \{|z| < 1\}$. The presumed extremal function $k(z) = z/(1-z)^2 = \sum_{j \geq 1} jz^j$ (Koebe function) maps D onto the plane with the radial segment $(-\infty, -1/4]$ deleted, so that $k(D)$ has maximal symmetry. Bieberbach's famous conjecture was that if $f \in S$, $f \neq k$, then $|a_j| < j$ for every $j \geq 2$ (finally proved in 1985 by de Branges). That k is extremal in other respects is also evident from Al's main result, which stands independently of de Branges's: *If Φ is convex on $(-\infty, \infty)$ and $r > 0$, then the integral means $\int_{-\pi}^{\pi} \Phi(\pm \log |f(re^{i\theta})|) d\theta$ are extremal when $f(z) = k(z)$ (if Φ is strictly convex, k is the unique extremal).* The paper contains many other applications, for example, that symmetrization increases the L_p -means of Green's functions. Good accounts now appear in standard monographs, e.g., Duren [MR 0708494] and volume 2 of Hayman [MR 1049148]. Relatively recently, Al published a major main chapter in Kühnau's encyclopedia [MR 1966196], with his own survey of the star function's impact in complex analysis. Some years earlier, he began to prepare his own research monograph,

Symmetrization in Analysis, one purpose being to present a full account of the star function in the universe of rearrangements, harmonic analysis, and geometric measure theory. Although about 20 percent of the project remained incomplete at his death, enough instructions and outlines were left to enable a group of colleagues and former students to prepare a full version for publication.

When Al arrived, Washington was already a powerhouse in classical mathematical analysis, primarily related to the Zygmund school of harmonic analysis. Al flourished in this environment while developing his own research agenda. A strong cadre of graduate students interested in analysis was arriving (notably from Spain and Italy) by the 1970s, and several soon chose Al as research mentor. He took mentoring very seriously, and subjects pursued by students ranged far beyond his own research. His fifteen PhD students (having at present nearly over twice as many additional descendants) provide clear evidence of his success. In this way he also partially fulfilled some vicarious dreams, such as being a probabilist. Students and colleagues fondly recall his warmth, care, and special mannerisms (waving hands, telling jokes, teaching in bare feet, making amusing faces). But Al also carefully adapted his style to each student.

One comment from Carlo Morpurgo reflects a significant aspect of Al's immense value to his graduate students. "After a little while, [Al] laid out Plan A, which involved a sexy-sounding problem in spectral geometry. The problem was easy to state and with enough evidence to make one think that it was within reasonable reach. Alas, it turned out to be very hard, and to this day it is still unsolved....Al soon realized that I am not a guy that easily gives up, and he let me continue with that journey, supporting the small progress that I was making and offering a helping hand when I needed one. In 1992 I did not have enough for a thesis, and I was stuck on a difficult inequality. Al told me that he had a safer Plan B for me, but he still let me continue, until I was eventually able to prove the inequality and obtain satisfactory partial results toward the original problem. I regret not having thanked Al enough for giving me a hard PhD problem and for supporting my stubbornness,

Bar Harbor, Maine.
June 14, 1962.

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allowing me to go on with Plan A almost to the point of no return.”

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