ORIGINAL PAPER

PADOVAN HYBRID QUATERNIONS AND SOME PROPERTIES

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Manuscript received: 05.11.2021; Accepted paper: 12.02.2022;
Published online: 30.03.2022.

Abstract. *We define Padovan hybrid quaternions by using Padovan hybrid numbers and Padovan quaternion. We give the basic operation properties of Padovan hybrid quaternion numbers. We give some properties and identities such as the Binet formula, sum formula, the matrix representation, characteristic equation, norm, characteristic and generating function for these quaternions.*

Keywords: *Padovan numbers; Padovan hybrid numbers; Padovan hybrid quaternions; Binet formula; generating function.*

1. INTRODUCTION

Number sequences have attracted the attention of many scientists for many years, as they find application in nature and many sciences. The Fibonacci numbers are the best known of them. Many generalizations of number sequences were described and studied [1-9]. One of these number sequences is also the Padovan number sequence. The Padovan sequence is named after Richard Padovan who attributed its discovery to Dutch architect Hans van der Laan in his 1994 essay Dom [10]. Later, Padovan numbers were studied by many researchers and their relationship with other number sequences was examined [11-16].

Padovan number sequence is defined by the recurrence relation below

$$P_n = P_{n-2} + P_{n-3}, n \geq 3$$

with $P_0 = P_1 = P_2 = 1$.

The characteristic equation obtained from this recurrence relation is

$$m^3 - m - 1 = 0 \text{ [15]}.$$

The roots of the characteristic equation are r_1, r_2 and r_3. Thus, the Binet formula of Padovan numbers is defined

$$P_n = \alpha {r_1}^n + \beta {r_2}^n + \gamma {r_3}^n.$$

Quaternions were first described by Irish mathematician William Rowan Hamilton in 1843 and applied to mechanics in three-dimensional space. Quaternion, in algebra, is a generalization of two-dimensional complex numbers to three dimensions. Quaternions and

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their rules were invented by Irish mathematician Sir William Rowan Hamilton in 1843. He devised them as a way of describing three-dimensional problems in mechanics.

Quaternions are an important number system used in different areas such as number theory, computer science, quantum physics, and analysis [17-21]. The product rule for quaternion units is shown in Table 1.

Table 1. Multiplication of quaternion unit

.	i	j	k
i	-1	k	$-j$
j	$-k$	-1	i
k	j	$-i$	-1

Quaternions have generally the following form

$$h = a + ib + jc + kd$$

where, i, j, k are basis quaternions and a, b, c, d are real numbers.

Thus, Padovan and Pell-Padovan quaternions are presented by Taşçı in [16]. The *n*th Padovan quaternion is defined by

$$QP_n = P_n + iP_{n+1} + jP_{n+2} + kP_{n+3}$$

where P_n is *n*th Padovan numbers.

In [16], the relation between Padovan quaternion terms is

$$QP_{n+4} = QP_{n+1} + QP_{n+2}.$$

The Binet formula of Padovan hybrid numbers is defined as follows

$$QP_n = A{r_1}^n \underline{r_1} + B{r_2}^n \underline{r_2} + C{r_3}^n \underline{r_3}$$

where $\underline{r_1} = 1 + ir_1 + j{r_1}^2 + k{r_1}^3$, $\underline{r_2} = 1 + ir_2 + j{r_2}^2 + k{r_2}^3$ and $\underline{r_3} = 1 + ir_3 + j{r_3}^2 + k{r_3}^3$.

A new set of numbers was introduced by Özdemir in [24]. Then, it was called a hybrid number. A hybrid number $\hbar$ has form as follows

$$\hbar = a + \mathbb{i}b + \varepsilon c + hd$$

where $\mathbb{i}, \varepsilon, h$ are hybrid bases and a, b, c, d are real numbers. This number system can be accepted as a generalization of the complex $i^2 = -1$, hyperbolic $h^2 = 1$ and dual number $\varepsilon^2 = 0$ systems. A hybrid number is a number created with any combination of the complex, hyperbolic and dual numbers satisfying the relation $ih = -hi = i + \varepsilon$. Because these numbers are a composition of dual, complex and hyperbolic numbers, it would be better to call them hybrid numbers instead of the generalized complex numbers.

There have been some studies on hybrid numbers [22-28]. The product rule for hybrid units is shown in Table 2.

Table 2. Multiplication of hybrid unit

.	$\mathbb{i}$	ε	h
$\mathbb{i}$	-1	$1-h$	$\varepsilon+\mathbb{i}$
ε	$1+h$	0	$-\varepsilon$
h	$-\varepsilon-\mathbb{i}$	ε	1

Kızılateş defined the Fibonacci and Lucas hybrid numbers and examined some of their properties in [25]. Padovan hybrid numbers are introduced in [26] and the authors gave some results about them. The *nth* Padovan hybrid is defined by

$$HP_n = P_n + \mathbb{i}P_{n+1} + \varepsilon P_{n+2} + hP_{n+3} \tag{1}$$

where $\mathbb{i}, \varepsilon, h$ are unit hybrid and P_n is nth Padovan numbers. The following recurrence relation between Padovan quaternion terms is

$$HP_{n+4} = HP_{n+1} + HP_{n+2} \tag{2}$$

Binet formula of Padovan hybrid numbers are defined as follows

$$HP_n = Ar_1{}^n r_1{}^* + Br_2{}^n r_2{}^* + Cr_3{}^n r_3{}^*$$

where $r_1{}^* = 1 + \mathbb{i}r_1 + \varepsilon r_1{}^2 + hr_1{}^3$, $r_2{}^* = 1 + \mathbb{i}r_2 + \varepsilon r_2{}^2 + hr_2{}^3$ and $r_3{}^* = 1 + \mathbb{i}r_3 + \varepsilon r_3{}^2 + hr_3{}^3$.

Dağdeviren et al. introduced Horadam hybrid quaternions and studied Fibonacci and Lucas hybrid quaternions [11]. In this paper, we define Padovan hybrid quaternions and give some of their properties. We prove some theorems about Padovan hybrid quaternions. In addition, we find the Binet formula, generating functions, sum formula and matrix representation of Padovan Hybrid quaternions.

2. PADOVAN HYBRID QUATERNIONS

Definition 2.1. We denote the set of Padovan hybrid quaternions by HQP_n and define as follows

$$HQP_n = HP_n + iHP_{n+1} + jHP_{n+2} + kHP_{n+3}$$

where i, j and k are quaternion units and HP_n are nth Padovan hybrid numbers.

From (1), we know that

$$HP_n = P_n + \mathbb{i}P_{n+1} + \varepsilon\, P_{n+2} + h\, P_{n+3}$$

where $\mathbb{i}, \varepsilon$ and h are hybrid units and P_n are nth Pell numbers.

Therefore, the nth Padovan hybrid quaternion can also be written as follows

$$HQP_n = (P_n + \mathbb{i}P_{n+1} + \varepsilon\, P_{n+2} + h\, P_{n+3})$$

$$+(P_{n+1} + \mathbb{i}P_{n+2} + \varepsilon\, P_{n+3} + h\, P_{n+4})i$$
$$+(P_{n+2} + \mathbb{i}P_{n+3} + \varepsilon\, P_{n+4} + h\, P_{n+5})j$$

$$+(P_{n+3}+\mathbb{i}P_{n+4}+\varepsilon\, P_{n+5}+h\, P_{n+6})k$$

$$=QP_n+\mathbb{i}QP_{n+1}+\varepsilon QP_{n+2}+hQP_{n+3}$$

where QP_n are nth Padovan quaternion numbers.

Thus, we obtain

$$HQP_n=HP_n+iHP_{n+1}+jHP_{n+2}+kHP_{n+3}$$

$$=QP_n+\mathbb{i}QP_{n+1}+\varepsilon QP_{n+2}+hQP_{n+3}.$$

The following recurrence relation between Padovan hybrid quaternion terms is

$$HQP_n=HQP_{n-2}+HQP_{n-3}.$$

Accordingly, the characteristic equation is as follows

$$r^3-r-1=0.$$

The roots of the equation are

$$r_1\approx 1.3247,$$

$$r_2\approx -0.66236-0.56228i,$$

$$r_3\approx -0.66236+0.56228i.$$

Definition 2.3. Let HQP_n and HQP_m be any two Padovan hybrid quaternions. The addition and subtraction of the Padovan hybrid quaternions are defined by

$$HQP_n\pm HQP_m=(HP_n\pm HP_m)\pm i(HP_{n+1}\pm HP_{m+1})$$

$$\pm j(HP_{n+2}\pm HP_{m+2})\pm k(HP_{n+3}\pm HP_{m+3})$$

or

$$HQP_n\pm HQP_m=(QP_n\pm QP_m)\pm \mathbb{i}(QP_{n+1}\pm QP_{m+1})$$

$$\pm\varepsilon(QP_{n+2}\pm QP_{m+2})\pm h(QP_{n+3}\pm QP_{m+3}).$$

Definition 2.4. Let HQP_n and HQP_m be any two Padovan hybrid quaternions. Multiplication of the Padovan hybrid quaternions is defined by

$$HQP_nHQP_m=(HP_n+iHP_{n+1}+jHP_{n+2}+kHP_{n+3})(HP_m+iHP_{m+1}+jHP_{m+2}+kHP_{m+3})$$

$$=(HP_nHP_m-HP_{n+1}HP_{m+1}-HP_{n+2}HP_{m+2}-HP_{n+3}HP_{m+3})$$

$$+i(HP_nHP_{m+1}+HP_{n+1}HP_m+HP_{n+2}HP_{m+3}-HP_{n+2}HP_{m+2})$$

$$+j(HP_nHP_{m+2}-HP_{1_{n+1}}HP_{m+3}+HP_{1_{n+2}}HP_m+HP_{1_{n+3}}HP_{m+1})$$

$$+k(HP_nHP_{m+3}+HP_{n+1}HP_{m+2}-HP_{n+2}HP_{m+1}+HP_{n+3}HP_m)$$

Similarly, we get

$$HQP_nHQP_m = (QP_n + 𝕚QP_{n+1} + \varepsilon QP_{n+2} + hQP_{n+3})(QP_m + 𝕚QP_{m+1} + \varepsilon QP_{m+2} + hQP_{m+3})$$

$$= (QP_nQP_m - QP_{n+1}QP_{m+1} + QP_{n+3}QP_{m+3})$$

$$+𝕚(QP_nQP_{m+1} + QP_{n+1}QP_m + QP_{n+1}QP_{m+3} - QP_{n+3}QP_{m+1}$$

$$+ QP_{n+1}QP_{m+3}) + \varepsilon(QP_nQP_{m+1} + QP_{n+2}QP_m - QP_{n+2}QP_{m+3}$$

$$- QP_{n+3}QP_{m+1} + QP_{n+3}QP_{m+2}) + h(QP_nQP_{m+3} - QP_{n+1}QP_{m+2}$$

$$+ QP_{n+2}QP_{m+1} + QP_{n+3}QP_m).$$

We show the scalar part of Padovan hybrid quaternions by S_{HQP_n} and define it as

$$S_{HQP_n} = HP_n.$$

We denote the vector part of Padovan hybrid quaternions by S_{HQP_n} and define it as

$$V_{HQP_n} = iHP_{n+1} + jHP_{n+2} + kHP_{n+3}.$$

Thus, any Padovan hybrid quaternion HQP_n can be written as

$$HQP_n = S_{HQP_n} + V_{HQP_n}.$$

Theorem 2.5. *Binet formula of Padovan hybrid quaternions is as follows*

$$HQP_n = Ar_1{}^n r_1{}^* \underline{r_1} + Br_2{}^n r_2{}^* \underline{r_2} + Cr_3{}^n r_3{}^* \underline{r_3}$$

where

$$r_1{}^* = 1 + 𝕚r_1 + \varepsilon r_1{}^2 + hr_1{}^3,\ \underline{r_1} = 1 + ir_1 + jr_1{}^2 + kr_1{}^3,$$

$$r_2{}^* = 1 + 𝕚r_2 + \varepsilon r_2{}^2 + hr_2{}^3,\ \underline{r_2} = 1 + ir_2 + jr_2{}^2 + kr_2{}^3,$$

$$r_3{}^* = 1 + 𝕚r_3 + \varepsilon r_3{}^2 + hr_3{}^3,\ \underline{r_3} = 1 + ir_3 + jr_3{}^2 + kr_3{}^3.$$

Proof: By using Definition 2.1 and the Binet formula for the Padovan hybrid numbers, we get

$$HQP_n = HP_n + iHP_{n+1} + jHP_{n+2} + kH$$

$$= (Ar_1{}^n r_1{}^* + Br_2{}^n r_2{}^* + Cr_3{}^n r_3{}^*) + i(Ar_1{}^{n+1} r_1{}^* + Br_2{}^{n+1} r_2{}^* + Cr_3{}^{n+1} r_3{}^*)$$

$$+j(Ar_1{}^{n+2} r_1{}^* + Br_2{}^{n+2} r_2{}^* + Cr_3{}^{n+2} r_3{}^*) + k(Ar_1{}^n r_1{}^* + Br_2{}^n r_2{}^* + Cr_3{}^n r_3{}^*)$$

$$= Ar_1{}^n r_1{}^*(1 + ir_1 + jr_1{}^2 + kr_1{}^3) + Br_2{}^n r_2{}^*(1 + ir_2 + jr_2{}^2 + kr_2{}^3)$$

$$+Cr_3{}^n r_3{}^*(1 + ir_3 + jr_3{}^2 + kr_3{}^3)$$

Thus, we have

$$HQP_n = Ar_1{}^n r_1{}^* \underline{r_1} + Br_2{}^n r_2{}^* \underline{r_2} + Cr_3{}^n r_3{}^* \underline{r_3}.$$

Theorem 2.6. *The generating function of the Padovan hybrid quaternions is*

$$G(x) = \frac{HQP_0(1-x^2) + HQP_1 x}{(1-x^2-x^3)}.$$

Proof: We know that

$$G(x) = \sum_{n=0}^{\infty} HQP_n x^n = HQP_0 + HQP_1 x + HQP_2 x^2 + \cdots + HQP_n x^n + \cdots$$

Let us multiply this equation by $x^2,\ x^3$, respectively. So, the following equations are obtained.

$$G(x) = HQP_0 + HQP_1 x + HQP_2 x^2 + \cdots + HQP_n x^n + \cdots,$$

$$x^2 G(x) = x^2 HQP_0 + x^3 HQP_1 + x^4 HQP_2 + \cdots,$$

$$x^3 G(x) = x^3 HQP_0 + x^4 HQP_1 + x^5 HQP_2 + \cdots.$$

If we take some calculations then we get the following equation

$$G(x)(1-x^2-x^3) = HQP_0 + HQP_1 x - x^2 HQP_0$$

$$G(x) = \frac{HQP_0(1-x^2) + HQP_1 x}{(1-x^2-x^3)}.$$

Thus, the proof is completed.

Theorem 2.7. *The sum of the Padovan hybrid quaternions is as follows*

$$\sum_{k=1}^{n} HQP_n = HQP_{n+5} - HP_5 HP_0 - (iHP_1 + jHP_, + kHP_1).$$

Proof:

$$\sum_{k=1}^{n} HQP_n = HQP_1 + HQP_2 + HQP_3 + \cdots + HQP_n$$

$$= (HP_1 + iHP_2 + jHP_3 + kHP_4) + (HP_2 + iHP_3 + jHP_4 + kHP_5) + \cdots + (HP_n + iHP_{n+1} + jHP_{n+2} + kHP_{n+3})$$

$$= (HP_1 + HP_2 + HP_3 + \cdots + HP_n) + i(HP_2 + HP_3 + \cdots + HP_{n+1}) + j(HP_3 + \cdots + HP_{n+2}) + k(HP_4 + \cdots + HP_{n+3})$$

$$= (HP_{n+5} - HP_2 - HP_3) + i(HP_{n+6} - HP_2 - HP_3 - HP_1) + j(HP_{n+7} - 2HP_2 - HP_3 - HP_1) + k(HP_{n+8} - 2HP_2 - 2HP_3 - HP_1)$$

$$= HQP_{n+5} - HP_5 + i(-HP_1 - HP_5) + j(-HP_3 - HP_5) + k(-HP_1 - 2HP_5)$$

$$= HQP_{n+5} - HP_5(1 + i + j + 2k) - iHP_1 - jHP_3 - kHP_1$$

$$= HQP_{n+5} - HP_5 HP_0 - iHP_1 - jHP_3 - kHP_1.$$

Thus, the proof is obtained.

Theorem 2.8. *The following equation is provided.*

$$HQP_{n-1} + HQP_{n-2} = HQP_{n+1}, n \geq 2$$

Proof: From Definition 2.1, we get the following

$$\begin{aligned} HQP_{n-1} + HQP_{n-2} &= HP_{n-1} + iHP_n + jHP_{n+1} + kHP_{n+2} \\ &\quad + HP_{n-2} + iHP_{n-1} + jHP_n + kHP_{n+1} \\ &= (HP_{n-1} + HP_{n-2}) + i(HP_n + HP_{n-1}) \\ &\quad + j(HP_{n+1} + HP_n) + k(HP_{n+2} + HP_{n+1}). \end{aligned}$$

From (1.2), we have

$$\begin{aligned} HQP_{n-1} + HQP_{n-2} &= HP_{n+1} + iHP_{n+2} + jHP_{n+3} + kHP_{n+4} \\ &= HQP_{n+1}. \end{aligned}$$

This completes the proof.

Theorem 2.9. *The following sums are provided for Padovan hybrid quaternions.*

$$\boldsymbol{i})\quad \sum_{k=1}^{n} HQP_{2k} = HQP_{2n+3} - HQP_3,$$

$$\boldsymbol{ii})\quad \sum_{k=1}^{n} HQP_{2k+1} = HQP_{2n+4} - HQP_2.$$

Proof: $\boldsymbol{i}$)

$$\sum_{k=1}^{n} HQP_{2k} = HQP_2 + HQP_4 + HQP_6 + \cdots + HQP_{2n}$$

by using Theorem 2.8, we have

$$\begin{aligned} \sum_{k=1}^{n} HQP_{2k} &= (HQP_5 - HQP_3) + (HQP_7 - HQP_5) \\ &\quad + (HQP_9 - HQP_7) + \cdots + (HQP_{2n+3} - HQP_{2n+1}) \\ &= HQP_{2n+3} - HQP_3. \end{aligned}$$

$\boldsymbol{ii}$)

$$\sum_{k=0}^{n} HQP_{2k+1} = HQP_1 + HQP_3 + HQP_5 + \cdots + HQP_{2n+1}$$

by using Theorem 2.8, we have

$$\sum_{k=0}^{n} HQP_{2k+1} = (HQP_4 - HQP_2) + (HQP_6 - HQP_4) + (HQP_8 - HQP_6)$$
$$+ \cdots + (HQP_{2n+4} - HQP_{2n+2})$$
$$= HQP_{2n+4} - HQP_2.$$

Lemma 2.10. *The following equation is provided.*

$$HQP_n + HQP_{n+1} + HQP_{n+2} = HQP_{n+5}.$$

Proof: The proof is easily done by using Theorem 2.8.

Theorem 2.11. For $n \geq 0$, the *following equations are provided.*

$\boldsymbol{i}$) $HQP_n - iHQP_{n+1} - jHQP_{n+2} - kHQP_{n+3} = HP_n + HP_{n+2} + HP_{n+4} + HP_{n+6}$,

$\boldsymbol{ii}$) $HQP_n - \mathbb{i}HQP_{n+1} - \varepsilon HQP_{n+2} - hHQP_{n+3} = QP_n + QP_{n+2} - 2QP_{n+3} - QP_{n+6}$.

Proof:

$\boldsymbol{i}$) $HQP_n - iHQP_{n+1} - jHQP_{n+2} - kHQP_{n+3} = (HP_n + iHP_{n+1} + jHP_{n+2} + kHP_{n+3})$

$-i(HP_{n+1} + iHP_{n+2} + jHP_{n+3} + kHP_{n+4}) - j(HP_{n+2} + iHP_{n+3} + jHP_{n+4} + kHP_{n+5})$

$-k(HP_{n+3} + iHP_{n+4} + jHP_{n+5} + kHP_{n+6})$

$= HP_n + HP_{n+2} + HP_{n+4} + HP_{n+6} + iHP_{n+1} + jHP_{n+2} + kHP_{n+3} - iHP_{n+1} - kHP_{n+3}$
$+ jHP_{n+4} - jHP_{n+2} + k\,HP_{n+3} - i\,HP_{n+5} - kHP_{n+3} - jHP_{n+4} + iHP_{n+5}$

$= HP_n + HP_{n+2} + HP_{n+4} + HP_{n+6}$.

$\boldsymbol{ii}$)$HQP_n - \mathbb{i}HQP_{n+1} - \varepsilon HQP_{n+2} - hHQP_{n+3} = (QP_n + \mathbb{i}QP_{n+1} + \varepsilon QP_{n+2} + hQP_{n+3})$

$-\mathbb{i}(QP_{n+1} + \mathbb{i}QP_{n+2} + \varepsilon QP_{n+3} + hQP_{n+4}) - \varepsilon(QP_{n+2} + \mathbb{i}QP_{n+3} + \varepsilon QP_{n+4} + hQP_{n+5})$

$-h(QP_{n+3} + \mathbb{i}QP_{n+4} + \varepsilon QP_{n+5} + hQP_{n+6})$

$= QP_n + \mathbb{i}QP_{n+1} + \varepsilon QP_{n+2} + hQP_{n+3} - \mathbb{i}QP_{n+1} + QP_{n+2} - (1-h)QP_{n+3} - (\varepsilon + \mathbb{i})QP_{n+4}$

$-\varepsilon QP_{n+2} - (1+h)QP_{n+3} + \varepsilon QP_{n+5} - hQP_{n+3} + (\varepsilon + \mathbb{i})QP_{n+4} - \varepsilon QP_{n+5} - QP_{n+6}$

$= QP_n + QP_{n+2} - 2QP_{n+3} - QP_{n+6}$.

So, the proof is obtained.

Definition 2.12. The conjugates of the Padovan hybrid quaternion can be defined

a) Quaternion conjugate: $\overline{HQP_n}$, $\overline{HQP_n} = \overline{QP_n} + \mathbb{i}\overline{QP_{n+1}} + \varepsilon\,\overline{QP_{n+2}} + h\overline{QP_{n+3}}$,

b) Hybrid Conjugate: $(HQP_n)^C$, $(HQP_n)^C = QP_n - \mathbb{i}QP_{n+1} - \varepsilon\,QP_{n+2} - hQ\,P_{n+3}$,

c) Total Conjugate : $(HQP_n)^{\mathrm{f}}$, $(HQP_n)^{\mathrm{f}} = \overline{QP_n} - \mathbb{i}\overline{QP_{n+1}} - \varepsilon\,\overline{QP_{n+2}} - h\overline{QP_{n+3}}$.

Definition 2.13. The norm of the Padovan hybrid quaternions is defined as follows

$$N(HQP_n) = {QP_n}^2 + {QP_{n+1}}^2 + {QP_{n+2}}^2 + {QP_{n+3}}^2$$

or

$$N(HQP_n) = {HP_n}^2 + {HP_{n+1}}^2 + {HP_{n+2}}^2 + {HP_{n+3}}^2$$

Lemma 2.14. *For $n \geq 0$, the following equations are provided.*

i) $(HQP_n)^C + HQP_n = 2QP_n$,

ii) $\overline{HQP_n} + HQP_n = 2HP_n$,

iii) $\overline{HQP_n} + (HQP_n)^{\dagger} = 2\overline{QP_n}$,

iv) $(HQP_n)^C + (HQP_n)^{\dagger} = 2(HP_n)^C$.

Proof: We will give proof of ii). Others are done similarly.

ii) $\overline{HQP_n} + HQP_n$

$$= QP_n - \mathbb{i}QP_{n+1} - \varepsilon\, \mathrm{Q}P_{n+2} - hQ\, P_{n+3} + QP_n + \mathbb{i}QP_{n+1} + +\varepsilon\, \mathrm{Q}P_{n+2} + hQ\, P_{n+3}$$

$$= \boldsymbol{QP_n}.$$

Definition 2.15. The character of Padovan hybrid quaternions is defined as follows

$$C(HQP_n) = {QP_n}^2 + (QP_{n+1} - QP_{n+2})^2 - {QP_{n+2}}^2 - {QP_{n+3}}^2$$

or

$$C(HQP_n) = -QP_{n+1}QP_{n+2} - QP_{n+2}QP_{n+1} - QP_nQP_{n+1} - QP_{n+1}QP_n.$$

Also, Padovan hybrid quaternion number can be represented in matrix form.

Theorem 2.16. *For $n \in \mathbb{N}$, an array of Padovan hybrid quaternion number is defined as*

$$\varphi HQP_n = \begin{bmatrix} QP_n + \mathrm{Q}P_{n+2} & 2QP_{n+1} - \mathrm{Q}P_{n+2} + QP_n \\ \mathrm{Q}P_{n+2} + QP_n & QP_n - \mathrm{Q}P_{n+2} \end{bmatrix}.$$

Proof: A matrix form of a hybrid number, denoted by $\varphi(a + \mathbb{i}b + \varepsilon c + hd)$, given as

$$\varphi(a + \mathbb{i}b + \varepsilon c + hd) = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \mathrm{c}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} + d\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \text{ [26]}$$

Now, let's find a matrix form of Padovan hybrid quaternion numbers.

$$\varphi HQP_n = QP_n\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + QP_{n+1}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \mathrm{Q}P_{n+2}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} + QP_{n+3}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} QP_n + \mathrm{Q}P_{n+2} & QP_{n+1} - \mathrm{Q}P_{n+2} + QP_{n+3} \\ \mathrm{Q}P_{n+2} - QP_{n+1} + QP_{n+3} & QP_n - \mathrm{Q}P_{n+2} \end{bmatrix}$$

$$= \begin{bmatrix} QP_n + \mathrm{Q}P_{n+2} & QP_{n+1} - \mathrm{Q}P_{n+2} + QP_{n+1} + QP_n \\ \mathrm{Q}P_{n+2} - QP_{n+1} + QP_{n+1} + QP_n & QP_n - \mathrm{Q}P_{n+2} \end{bmatrix}$$

$$= \begin{bmatrix} QP_n + QP_{n+2} & 2QP_{n+1} - QP_{n+2} + QP_n \\ QP_{n+2} + QP_n & QP_n - QP_{n+2} \end{bmatrix}.$$

Thus, the proof is obtained.
Now, we calculate determinant of φHQP_n.

$$\det(\varphi HQP_n) = \begin{vmatrix} QP_n + QP_{n+2} & QP_{n+1} - QP_{n+2} + QP_{n+3} \\ QP_{n+2} - QP_{n+1} + QP_{n+3} & QP_n - QP_{n+2} \end{vmatrix}$$

$$= 2|QP_{n+2}QP_n - QP_nQP_{n+2} - QP_{n+1}QP_n - QP_{n+1}QP_{n+2}|$$

$$= 2|QP_{n+2}QP_n - QP_nQP_{n+2} + C(HQP_n) + QP_{n+2}QP_{n+1} + QP_nQP_{n+1}|$$

$$= 2|C(HQP_n) + QP_{n+2}QP_{n+3} - QP_n(QP_{n+2} - QP_{n+1})|.$$

Theorem 2.17. *Let $n \geq 1$ be integer. We have*

$$\begin{pmatrix} HQP_{n+2} & HQP_{n+1} & HQP_n \\ HQP_{n+1} & HQP_n & HQP_{n-1} \\ HQP_n & HQP_{n-1} & HQP_{n-2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} HQP_2 & HQP_1 & HQP_0 \\ HQP_1 & HQP_0 & HQP_{-1} \\ HQP_0 & HQP_{-1} & HQP_{-2} \end{pmatrix}$$

where HQP_n is the nth Padovan hybrid quaternions.

Proof: We can perform the induction on n. For $n = 1$, we obtain

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} HQP_2 & HQP_1 & HQP_0 \\ HQP_1 & HQP_0 & HQP_{-1} \\ HQP_0 & HQP_{-1} & HQP_{-2} \end{pmatrix}$$

$$= \begin{pmatrix} HQP_1 + HQP_0 & HQP_0 + HQP_{-1} & HQP_{-1} + HQP_{-2} \\ HQP_2 & HQP_1 & HQP_0 \\ HQP_1 & HQP_0 & HQP_{-1} \end{pmatrix}.$$

From (2.1), we have

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} HQP_2 & HQP_1 & HQP_0 \\ HQP_1 & HQP_0 & HQP_{-1} \\ HQP_0 & HQP_{-1} & HQP_{-2} \end{pmatrix} = \begin{pmatrix} HQP_3 & HQP_2 & HQP_1 \\ HQP_2 & HQP_1 & HQP_{n-1} \\ HQP_1 & HQP_0 & HQP_{n-2} \end{pmatrix}.$$

Thus, for $n = 1$, the result is true.
We assume that it is true for n. Thus, we

$$\begin{pmatrix} HQP_{n+2} & HQP_{n+1} & HQP_n \\ HQP_{n+1} & HQP_n & HQP_{n-1} \\ HQP_n & HQP_{n-1} & HQP_{n-2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} HQP_2 & HQP_1 & HQP_0 \\ HQP_1 & HQP_0 & HQP_{-1} \\ HQP_0 & HQP_{-1} & HQP_{-2} \end{pmatrix}.$$

Now, let's show that it is true for $n + 1$.

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{n+1} \begin{pmatrix} HQP_2 & HQP_1 & HQP_0 \\ HQP_1 & HQP_0 & HQP_{-1} \\ HQP_0 & HQP_{-1} & HQP_{-2} \end{pmatrix}$$

$$=\begin{pmatrix}0&1&1\\1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}0&1&1\\1&0&0\\0&1&0\end{pmatrix}^n\begin{pmatrix}HQP_2&HQP_1&HQP_0\\HQP_1&HQP_0&HQP_{-1}\\HQP_0&HQP_{-1}&HQP_{-2}\end{pmatrix}$$

$$=\begin{pmatrix}0&1&1\\1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}HQP_{n+2}&HQP_{n+1}&HQP_n\\HQP_{n+1}&HQP_n&HQP_{n-1}\\HQP_n&HQP_{n-1}&HQP_{n-2}\end{pmatrix}$$

$$=\begin{pmatrix}HQP_{n+1}+HQP_n&HQP_n+HQP_{n-1}&HQP_{n-1}+HQP_{n-2}\\HQP_{n+2}&HQP_{n+1}&HQP_n\\HQP_{n+1}&HQP_n&HQP_{n-1}\end{pmatrix}.$$

From (2), we have

$$\begin{pmatrix}0&1&1\\1&0&0\\0&1&0\end{pmatrix}^{n+1}\begin{pmatrix}HQP_2&HQP_1&HQP_0\\HQP_1&HQP_0&HQP_{-1}\\HQP_0&HQP_{-1}&HQP_{-2}\end{pmatrix}=\begin{pmatrix}HQP_{n+3}&HQP_{n+2}&HQP_{n+1}\\HQP_{n+2}&HQP_{n+1}&HQP_n\\HQP_{n+1}&HQP_n&HQP_{n-1}\end{pmatrix}.$$

Thus, the proof is complete.

Theorem 2.17. *Let $n \geq 1$ be integer. Then, we have*

$$\begin{pmatrix}0&1&1\\1&0&0\\0&1&0\end{pmatrix}^n\begin{pmatrix}HQP_0\\HQP_1\\HQP_2\end{pmatrix}=\begin{pmatrix}HQP_n\\HQP_{n+1}\\HQP_{n+2}\end{pmatrix}.$$

Proof: The proof is seen by induction on n.

3. CONCLUSION

In this paper, we have introduced the Padovan hybrid quaternions. We give some important properties and identities such as Binet's formula, sum formula, the matrix representation, characteristic equation, norm, characteristic and generating function for these quaternions. We examine the matrix form of Padovan hybrid quaternions. We calculated the determinant of this matrix form. We have given some theorems about these matrix forms.

Acknowledgements: *We would like to thank the editor and referees for their valuable comments, and remarks that led to a great improvement of the paper.*

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