

The life and work of David W. Lewis (1944–2021)

THOMAS UNGER

1. LIFE

David William Lewis was born in Douglas, Isle of Man, on 21st February 1944. He attended Douglas High School where he developed an interest in physics and astronomy, and ultimately mathematics. He went on to the University of Liverpool, and after completing his BSc degree in 1965 commenced doctoral studies in topology under the guidance of Terry Wall. When his PhD funding ended in 1968, he found employment in UCD where he was appointed as an assistant lecturer at the Mathematics Department. He continued working on his doctoral thesis, shifting from topology to algebra and specifically to the area of quadratic and hermitian forms.



FIGURE 1.1. David and Anne Lewis on the occasion of David's retirement conference in 2009. Photograph: M. Mackey.

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David was awarded the PhD degree in 1979 by the National University of Ireland. He was promoted to senior lecturer in 1987 and awarded a DSc degree in 1992. In 1997 he was promoted to associate professor and then in 2006 to full professor. From 1999 to 2002 he was head of department. Colleagues have commented on his fairness and effectiveness in this onerous role.

From the late 1980s David was involved in an Erasmus exchange with the University of Ghent. This was an offshoot of a collaboration with Jan Van Geel, and resulted later on in two of David's four PhD students. From 1997 until 2006 he was the local coordinator of two successive European research training networks for PhD students and post-docs (both called *Algebraic K-Theory, Linear Algebraic Groups and Related Structures*), expertly managed by Ulf Rehmann at the University of Bielefeld. In this context a major conference and a smaller workshop were co-organized by David in UCD in 1999 and 2004, respectively.

For more on David's early years growing up in Douglas, his university career, and his working life at UCD, see the interview [50] by Gary McGuire.

After his retirement in 2009 David remained research active as emeritus professor for a good number of years. In late 2013 he was diagnosed with Parkinson's disease. This led to a gradual decline in his health. David passed away peacefully on 20th August 2021, his wife Anne having pre-deceased him by three years. David and Anne are survived by their three sons Alan, Stephen and Gareth and their families. Their only daughter Joanne had passed away at a young age.

David will be fondly remembered for his fine qualities as a mathematician and the pleasure of collaborating with him, and for his friendship, kindness, thoughtfulness, sense of humour, humility and dedication to his family.

2. WORK

David published more than 60 papers (including a number of surveys and expository papers), one volume of conference proceedings [2] and a book on matrix theory [41]. He also maintained a website about mathematicians from the Isle of Man and the Manx diaspora, cf. [44].

David's PhD thesis contained a number of significant results as well as the germs of ideas that were fleshed out in later papers. His early publications match up with the chapters in his thesis [28] almost one-to-one, cf. [25], [26], [27], [29], [30], [31].

David made numerous contributions to the algebraic theory of quadratic forms and related areas, such as central simple algebras with involution. Below I will describe some of those results with the aim of allowing the reader to form a reasonable impression of David's research interests and the impact of his work. My selection of topics is by no means exhaustive. I will also indicate some noteworthy extensions and generalizations by other researchers of David's work. For the benefit of the readers of the *Bulletin* I have kept the style expository.

2.1. Some background material. Consider a pair (R, σ) where R is a unital ring, not necessarily commutative, and $\sigma : R \rightarrow R$ is an involution, i.e., an anti-automorphism of order 2. With (R, σ) we can associate the Witt group $W^\varepsilon(R, \sigma)$ of isometry classes of nonsingular ε -hermitian forms $\varphi : M \times M \rightarrow R$, where metabolic forms are identified with zero. In this notation ε is a central element in R such that $\sigma(\varepsilon)\varepsilon = 1$, M is a finitely generated projective right R -module, ε -hermitian means that φ is bi-additive and satisfies $\varphi(x\alpha, y\beta) = \sigma(\alpha)\varphi(x, y)\beta$ and $\varphi(y, x) = \varepsilon\sigma(\varphi(x, y))$ for all $x, y \in M$ and all $\alpha, \beta \in R$, nonsingular means that the R -linear map $M \rightarrow M^*$, $x \mapsto [y \mapsto \varphi(x, y)]$ to the dual module (considered as a right R -module via $f\alpha := \sigma(\alpha)f$ for all $f \in M^*$ and all $\alpha \in R$) is an isomorphism, and metabolic means that M contains a direct summand

that coincides with its orthogonal module with respect to φ . The group operation is induced by the orthogonal sum $\varphi_1 \perp \varphi_2$, defined on $M_1 \oplus M_2$ by

$$\varphi_1 \perp \varphi_2(x_1 + x_2, y_1 + y_2) := \varphi_1(x_1, y_1) + \varphi_2(x_2, y_2)$$

for all $x_i, y_i \in M_i$, $i = 1, 2$. Often one only considers (or needs to consider) hermitian and skew-hermitian forms, which correspond to the cases $\varepsilon = 1$ and $\varepsilon = -1$, respectively.

Here are some examples (if $\varepsilon = 1$, we write W instead of W^1):

$$W(\mathbb{Z}, \text{id}) \cong W(\mathbb{R}, \text{id}) \cong W^{\pm 1}(\mathbb{C}, -) \cong W(\mathbb{H}, -) \cong \mathbb{Z},$$

$$W(\mathbb{C}, \text{id}) \cong W^{-1}(\mathbb{H}, -) \cong \mathbb{Z}/2\mathbb{Z},$$

$$W^{-1}(\mathbb{R}, \text{id}) \cong W^{\pm 1}(\mathbb{R} \times \mathbb{R}, \hat{\ }) = 0,$$

$$W(\mathbb{Q}_2, \text{id}) \cong \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$

where \mathbb{Z} , \mathbb{R} and \mathbb{C} are the integers, real numbers and complex numbers, as usual, \mathbb{H} is Hamilton's quaternion algebra, \mathbb{Q}_2 is the field of 2-adic numbers, $-$ denotes conjugation, $\hat{\ }$ denotes the exchange involution, and \cong denotes isomorphism.

If R is commutative, the tensor product of R -modules induces a multiplication that turns $W^\varepsilon(R, \sigma)$ into a ring. If 2 is invertible in R various simplifications can be made.

If $R = F$ is a field, $\varepsilon = 1$ and $\sigma = \text{id}_F$ we obtain the Witt ring $W(F) := W^1(F, \text{id}_F)$ of classes of symmetric bilinear forms on finite-dimensional F -vector spaces. If the characteristic of F is different from 2, any symmetric bilinear form $b : V \times V \rightarrow F$ on a finite-dimensional F -vector space V can be uniquely identified with a quadratic form q_b over F via $q_b(x) := b(x, x)$ and vice versa via $b_q(x, y) := \frac{1}{2}(q(x+y) - q(x) - q(y))$. Let us consider a quadratic form $q : V \rightarrow F$ where $\dim_F V = n$. After choosing an F -basis (e_1, \dots, e_n) of V we can represent q by the symmetric matrix $(b_q(e_i, e_j)) \in M_n(F)$. A different choice of basis yields a congruent matrix. If q_1 and q_2 are quadratic forms over F such that their associated matrices are congruent, then q_1 and q_2 are isometric, and we write $q_1 \simeq q_2$. It is a standard result that if the characteristic of F is different from 2, one can find a basis of V that is orthogonal with respect to b_q , i.e., such that the matrix of q is a diagonal matrix $\text{diag}(a_1, \dots, a_n)$. We then write $q \simeq \langle a_1, \dots, a_n \rangle$ and note that

$$\langle a_1, \dots, a_n \rangle = \langle a_1 \rangle \perp \dots \perp \langle a_n \rangle.$$

The quadratic form $\langle a_1, \dots, a_n \rangle$ is nonsingular if and only if $\det \text{diag}(a_1, \dots, a_n) \neq 0$ if and only if a_1, \dots, a_n are nonzero. Furthermore, $\langle a_1, \dots, a_n \rangle$ is isotropic over F if the quadratic polynomial $\sum_{i=1}^n a_i x_i^2$ has a nontrivial zero over F . Arbitrary permutations of the entries of $\langle a_1, \dots, a_n \rangle$, as well as multiplication of the entries by nonzero squares, give rise to isometric forms. For example, every a_i that is a nonzero square in F can be replaced by 1. In particular, we can view the nonzero entries a_i as elements of the square class group $F^\times / F^{\times 2}$.

Using diagonal notation, the sum and product in $W(F)$ are induced by

$$\langle a_1, \dots, a_n \rangle \perp \langle b_1, \dots, b_m \rangle = \langle a_1, \dots, a_n, b_1, \dots, b_m \rangle$$

and

$$\langle a_1, \dots, a_n \rangle \otimes \langle b_1, \dots, b_m \rangle = \langle a_1 b_1, a_1 b_2, \dots, a_i b_j, \dots, a_n b_m \rangle,$$

respectively. Hyperbolic forms are finite orthogonal sums of the hyperbolic plane $\langle 1, -1 \rangle$. They coincide with the metabolic forms in characteristic not 2, and so they are identified with the zero element of $W(F)$. The identity element of $W(F)$ is the class of the form $\langle 1 \rangle$. If q is a quadratic form over F , we denote its class in $W(F)$ by $[q]$. For example,

$$0 = [\langle 1, -1 \rangle] = [\langle 2, -2 \rangle], \quad 1 = [\langle 1 \rangle], \quad 2 = [\langle 1, 1 \rangle].$$

For later use, let's look at a more elaborate example:

Example 2.1. Consider the field of rational numbers $F = \mathbb{Q}$ and let $q = [\langle 2, 3 \rangle] \in W(F)$. Then:

$$\begin{aligned} q^2 &= [\langle 2, 3 \rangle \otimes \langle 2, 3 \rangle] = [\langle 4, 6, 6, 9 \rangle] = [\langle 1, 6, 6, 1 \rangle], \\ 2^2 &= [\langle 1, 1 \rangle \otimes \langle 1, 1 \rangle] = [\langle 1, 1, 1, 1 \rangle], \\ q^2 - 2^2 &= [\langle 1, 6, 6, 1 \rangle \perp \langle -1, -1, -1, -1 \rangle] = [\langle 6, 6, -1, -1 \rangle], \\ q(q^2 - 2^2) &= [\langle 2, 3 \rangle \otimes \langle 6, 6, -1, -1 \rangle] = [\langle 12, 12, -2, -2, 18, 18, -3, -3 \rangle] \\ &= [\langle 3, 3, -2, -2, 2, 2, -3, -3 \rangle] = [\langle 2, -2 \rangle \perp \langle 2, -2 \rangle \perp \langle 3, -3 \rangle \perp \langle 3, -3 \rangle] = 0. \end{aligned}$$

The study of Witt rings of fields started with Witt's seminal paper from 1937 and received a major boost by Pfister in the 1960s. The resulting *algebraic theory of quadratic forms* has been a fruitful research area from its inception, with deep connections to many other areas in mathematics. The standard references are the books by Lam [23], Scharlau [56] and Knus [21]. The monograph [12] focusses on the modern geometric theory of quadratic forms. For more on the history of quadratic forms I refer the reader to [57].

Let K be a field of characteristic $\neq 2$. A finite-dimensional K -algebra A is called central simple over K if A has no non-trivial two-sided ideals and the centre of A is K . Let σ be an involution on A and let $F = \{a \in K \mid \sigma(a) = a\}$ be the fixed field of σ . The pair (A, σ) is called a *central simple F -algebra with involution* (in this terminology we emphasize F rather than K even though $Z(A) = K$). If $F = K$, then σ is said to be of the first kind. Otherwise σ is said to be of the second kind (or unitary), and we must have $[K : F] = 2$. In this case, it is customary to allow the possibility that K is not a quadratic extension field of F , but a double-field isomorphic to $F \times F$. When this happens, A is not simple, but a product of two simple F -algebras that are mapped to each other by σ . The motivation for allowing this possibility is that it can occur after scalar extension. For ease of exposition I will ignore this situation and assume that the centre of A is a field. The standard reference is *The Book of Involutions* [22], which also contains many notes with historical pointers.

Real square matrices with transposition $(M_n(\mathbb{R}), t)$ and complex square matrices with conjugate transposition $(M_n(\mathbb{C}), *)$ are easy examples of central simple \mathbb{R} -algebras with involution of the first, and second kind, respectively. Hamilton's quaternion algebra with quaternion conjugation $(\mathbb{H}, \bar{})$ is a central simple \mathbb{R} -algebra with involution of the first kind.

If (D, ϑ) is a central simple F -algebra with involution where D is a division algebra and if $h : V \times V \rightarrow D$ is a nonsingular ε -hermitian form over (D, ϑ) , where V is a finite-dimensional right D -vector space, then $(\text{End}_D(V), \text{ad}_h)$ is a central simple F -algebra with involution of the same kind as ϑ . Here ad_h , the adjoint involution of h , is defined by the property

$$h(x, f(y)) = h(\text{ad}_h(f)(x), y), \quad \forall x, y \in V, \quad \forall f \in \text{End}_D(V).$$

In fact, *all* central simple F -algebras with involution are of this form for some D , unique up to isomorphism, and some h , determined up to multiplication by a scalar in F^\times . For example, $(M_n(\mathbb{R}), t) \cong (\text{End}_{\mathbb{R}}(\mathbb{R}^n), \text{ad}_h)$, where $h = n \times \langle 1 \rangle := \langle 1, \dots, 1 \rangle$ (n copies of 1).

Because of this correspondence between involutions and ε -hermitian forms, central simple algebras with involution can be thought of as generalizations of quadratic forms. However, quadratic forms also show up in other ways. Important examples are obtained via the K -linear reduced trace map $\text{Trd}_A : A \rightarrow K$, which is defined as follows: let Ω be a splitting field of A , i.e., $A \otimes_K \Omega \cong M_n(\Omega)$ (such a field always exist, cf. [22, Theorem (1.1)]). Then for $a \in A$, $\text{Trd}_A(a)$ is the trace of the matrix of the image of a

under scalar extension to Ω . One can show that $\text{Trd}_A(a) \in K$ and is independent of the choice of Ω . The trace form of A is the symmetric bilinear form

$$T_A : A \times A \rightarrow K, (x, y) \mapsto \text{Trd}_A(xy).$$

The associated quadratic form $x \mapsto T_A(x, x) = \text{Trd}_A(x^2)$ is usually also denoted by T_A . The involution trace form of (A, σ) ,

$$T_{(A, \sigma)} : A \times A \rightarrow K, (x, y) \mapsto \text{Trd}_A(\sigma(x)y),$$

is symmetric bilinear over $F = K$ if σ is of the first kind and hermitian over $(K, \sigma|_K)$ if σ is of the second kind. The forms T_A and $T_{(A, \sigma)}$ are both nonsingular.

2.2. Exact sequences of Witt groups. In many situations, Witt groups cannot be computed explicitly but can be related to other Witt groups via exact sequences. Let F be a field of characteristic not 2 and let $F(\sqrt{a})$ be a quadratic field extension of F . Denote the map induced by $\sqrt{a} \mapsto -\sqrt{a}$ on the field $F(\sqrt{a})$ by $-$ as usual. Milnor and Husemoller showed that there is an exact sequence

$$0 \rightarrow W(F(\sqrt{a}), -) \xrightarrow{\pi} W(F) \xrightarrow{\rho} W(F(\sqrt{a})), \quad (2.1)$$

where π is induced by the trace $\text{Tr}_{F(\sqrt{a})/F}$ and ρ is induced by base change to $F(\sqrt{a})$, cf. [51, Appendix 2].

Let $b \in F$ be nonzero and let D denote the generalized quaternion algebra $(a, b)_F$, i.e., the F -algebra generated by symbols i and j that satisfy $i^2 = a$, $j^2 = b$ and $ij = -ji$. We assume that D is a division algebra and denote the map induced by $i \mapsto -i$, $j \mapsto -j$ by $-$ as well since we can identify $F(\sqrt{a})$ with a subfield of D in the obvious way. For example, if $F = \mathbb{R}$ and $a = b = -1$, then $D = \mathbb{H}$ and $F(\sqrt{a}) = \mathbb{C}$.

In the spirit of (2.1) David proved that the sequence

$$0 \rightarrow W(D, -) \rightarrow W(F(\sqrt{a}), -) \rightarrow W^{-1}(D, -) \rightarrow W(F(\sqrt{a})) \quad (2.2)$$

is exact in his 1979 paper [26]. (For ease of exposition I will not describe the maps that occur in this exact sequence and those in the remainder of this section. They are similar to the maps π and ρ in (2.1).) In [3, Appendix 2] this sequence was generalized by Parimala, Sridharan and Suresh to the exact sequence

$$W^\varepsilon(A, \sigma) \rightarrow W^\varepsilon(\tilde{A}, \sigma_1) \rightarrow W^{-\varepsilon}(A, \sigma) \rightarrow W^\varepsilon(\tilde{A}, \sigma_2), \quad (2.3)$$

where (A, σ) is a central simple F -algebra with involution, \tilde{A} is the centralizer of a skew-symmetric unit λ in A with the property that $F(\lambda)$ is a quadratic extension of F , $\sigma_1 = \sigma|_A$, and $\sigma_2 = \text{Int}(\mu^{-1}) \circ \sigma_1$ for a skew-symmetric unit μ in A that anti-commutes with λ , where Int denotes inner automorphism. (Note that not all central simple algebras with involution contain such elements λ and μ .) The “key exact sequence” (2.3) was used by Bayer-Fluckiger and Parimala in their proof of Serre’s Conjecture II for classical groups, cf. [4, 5].

In [31] David extended the sequences (2.1) and (2.2) further to the right, resulting in the exact sequences

$$0 \rightarrow W(F(\sqrt{a}), -) \rightarrow W(F) \rightarrow W(F(\sqrt{a})) \rightarrow W(F) \rightarrow W(F(\sqrt{a}), -) \rightarrow 0 \quad (2.4)$$

and

$$\begin{aligned} 0 \rightarrow W(D, -) \rightarrow W(F(\sqrt{a}), -) \rightarrow W^{-1}(D, -) \rightarrow W(F(\sqrt{a})) \\ \rightarrow W^{-1}(D, -) \rightarrow W(F(\sqrt{a}), -) \rightarrow W(D, -) \rightarrow 0. \end{aligned} \quad (2.5)$$

Anybody looking at the exact sequences (2.4) and (2.5) will surely not be able to resist folding them into exact polygons. In fact, it turns out that these sequences are special cases of an exact octagon involving Witt groups of Clifford algebras, as David showed in [34]: Let C denote the Clifford algebra $C(q)$ of a nonsingular quadratic form q over

F , and let $C' := C(q \perp \langle a \rangle)$ where $a \in F$ is nonzero. The algebra C carries the two natural involutions σ_1 and σ_{-1} , where $\sigma_{\pm 1}(x) = \pm x$ for all $x \in V$, where V is the finite-dimensional F -vector space on which q is defined. The following octagon is exact:

$$\begin{array}{ccccc}
 & & W(C, \sigma_{-1}) & \longrightarrow & W(C', \sigma_{-1}) \\
 & \nearrow & & & \searrow \\
 W(C', \sigma_1) & & & & W^{-1}(C, \sigma_1) \\
 \uparrow & & & & \downarrow \\
 W(C, \sigma_1) & & & & W^{-1}(C', \sigma_1) \\
 & \nwarrow & & & \swarrow \\
 & & W^{-1}(C', \sigma_{-1}) & \longleftarrow & W^{-1}(C, \sigma_{-1})
 \end{array}$$

In [33], David obtained equivariant versions (i.e., the forms are invariant under the action of a finite group) of (2.4) and (2.5), rolled up into exact octagons. These are related to work of Ranicki on L -groups, cf. [55].

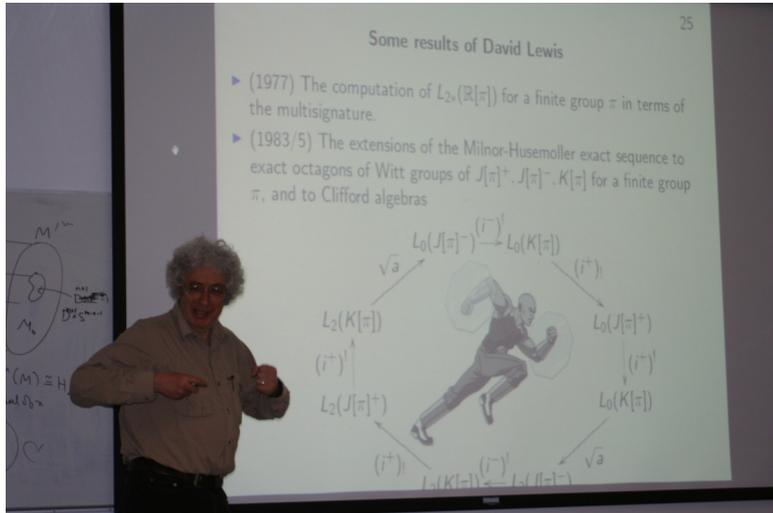


FIGURE 2.1. Andrew Ranicki (1948–2018) on the occasion of David’s retirement conference in 2009. Photograph: S. McGarraghy.

Inspired by (2.4), Grenier-Boley and Mahmoudi extended (2.3) further to the right and obtained the exact octagon

$$\begin{array}{ccccc}
 & & W^\varepsilon(A, \sigma) & \longrightarrow & W^\varepsilon(\tilde{A}, \sigma_1) \\
 & \nearrow & & & \searrow \\
 W^{-\varepsilon}(\tilde{A}, \sigma_2) & & & & W^{-\varepsilon}(A, \sigma) \\
 \uparrow & & & & \downarrow \\
 W^\varepsilon(A, \sigma) & & & & W^\varepsilon(\tilde{A}, \sigma_2) \\
 & \nwarrow & & & \swarrow \\
 & & W^\varepsilon(\tilde{A}, \sigma_1) & \longleftarrow & W^{-\varepsilon}(A, \sigma)
 \end{array}$$

which remains valid in the equivariant case. As an application they proved that if (A, σ) is a central simple algebra with involution of the first kind over F , then $W(F)$ is finite if

and only if both $W^\varepsilon(A, \sigma)$ and $W^{-\varepsilon}(A, \sigma)$ are finite, generalizing a similar observation of David's for quaternion algebras, cf. [31].

The exact sequences of Lewis, Grenier-Boley and Mahmoudi, and Parimala, Sridharan and Suresh were further generalized to Witt groups of Azumaya algebras with involution by First in his impressive 2022 paper [14], which also contains a wealth of information about Azumaya algebras with involution and several important applications.

2.3. Annihilating polynomials. In Example 2.1 we saw that the element $q = [\langle 2, 3 \rangle] \in W(\mathbb{Q})$ is a root of the polynomial $p(x) = x(x^2 - 2^2)$, which is then said to be an annihilating polynomial of q . In his 1987 paper [38], David proved:

Theorem 2.2. *Let F be a field of characteristic not 2. Let φ be any quadratic form of dimension n over F , and let $q = [\varphi] \in W(F)$. Then $p_n(q) = 0$ in $W(F)$, where $p_n(x)$ is the monic integer polynomial defined by*

$$p_n(x) := \begin{cases} x(x^2 - 2^2)(x^2 - 4^2) \cdots (x^2 - n^2) & \text{if } n \text{ is even} \\ (x^2 - 1^2)(x^2 - 3^2) \cdots (x^2 - n^2) & \text{if } n \text{ is odd} \end{cases}.$$

In other words, the theorem says that Witt rings of fields of characteristic not 2 are integral rings. This had been known for a long time, but David's result provided the first examples of polynomials that annihilated particular classed of quadratic forms. There are several proofs of Theorem 2.2, but the slickest is due to Leung. It goes via induction on n , using the recurrence relation $p_n(x) = (x + n)p_{n-1}(x - 1)$, cf. [38, Comment 1].

As an application of Theorem 2.2 David obtained the standard structural properties of Witt rings, the main ones being: they have no odd torsion, no odd-dimensional zero divisors and no nontrivial idempotents; their prime ideals are determined by the orderings of the underlying field. Proofs of these facts are mostly omitted from [38], but can be found in David's 1989 paper [40]. For a description of subsequent work on annihilating polynomials by David and others up to 2000, I refer to David's survey [43]. David's doctoral students Seán McGarraghy and Stefan De Wannemacker (1971–2013) also worked on this topic, cf. [10], [11], [49], [48].

2.4. Levels of division algebras. A field F is called real if -1 cannot be written as a sum of squares of elements of F (real fields are sometimes called formally real fields, but this practice is slowly disappearing). Note that the characteristic of F must thus be zero. By the Artin-Schreier theorem (see for example [23, VIII, Theorem 1.10]), F is real if and only if F has at least one ordering.

If -1 can be written as a sum of squares in F , this begs the question how many squares are needed. The level $s(F)$ of a non-real field F is defined to be the smallest integer n such that -1 is a sum of n squares in F . The level of a real field is defined to be ∞ . In 1932 van der Waerden posed the following problem, cf. [58]:

Wenn in einem Körper die Zahl -1 Summe von 3 Quadraten ist, so auch von 2 Quadraten; wenn von 5, 6 oder 7, so auch von 4; wenn von 15 oder weniger, so auch von 8.

In other words, he asked if 1, 2, 4 and 8 are the only possible values of the level ≤ 15 . In 1934, H. Kneser proved that this is indeed the case, and that in addition all multiples of 16, except those of the form $2^{8h}g + 16h$, could occur, cf. [20, Satz 2]. The fields $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$ and $\mathbb{Q}(\sqrt{-7})$ have level 1, 2 and 4, respectively. Also, at the time, algebraic number fields were known to have level 1, 2, 4 or ∞ , but there were no known fields of finite level > 4 . It was not until 1963 that the complete solution was given by Pfister. He showed that the level of a field is either ∞ or a 2-power, and that every 2-power occurs as the level of some field, cf. [52].

The notion of level also makes sense for unitary rings that are not necessarily commutative, but Pfister's 2-power result no longer holds in general. For example, the ring $\mathbb{Z}/4\mathbb{Z}$ has level 3. A result of note is the topological proof of Dai, Lam and Peng of the fact that for any given positive integer n , the integral domain

$$\frac{\mathbb{R}[x_1, \dots, x_n]}{(1 + x_1^2 + \dots + x_n^2)}$$

has level n , cf. [9].

For noncommutative rings several generalizations of the level have also been investigated. For example, the sublevel $\underline{s}(R)$ of a ring R is the smallest positive integer n such that 0 can be written as a sum of $n + 1$ squares in R , and is ∞ if 0 cannot be written as a sum of squares.

In their 1985 paper [24] Leep, Shapiro and Wadsworth investigated sums of squares in central simple algebras. Recall that these are matrix algebras with entries in division algebras, finite-dimensional over their centre (assumed to be of characteristic not 2). They observed that by a result of Griffin and Krusemeyer (namely that if R is a commutative ring with $2 \in R^\times$ and $n \geq 2$, then every element of $M_n(R)$ is a sum of 3 squares [16]) it suffices to consider division algebras D . They showed that $\underline{s}(D) < \infty$ if and only if $s(D) < \infty$ if and only if each element of D is a sum of squares in D , cf. [24, Theorem D]. (Note that if D is actually a field, then this result is an easy exercise.) In their investigations the trace form T_D played a central role. David settled their Conjecture 3.6 and proved:

Theorem 2.3 ([35]). *0 is a nontrivial sum of squares in D if and only if the trace form T_D is weakly isotropic.*

David had a particular interest in levels of quaternion division algebras. In 1989 he proved:

Theorem 2.4 ([39, Propositions 2 and 3]).

- (1) *There exist quaternion division algebras of level $2^k + 1$ for all $k \geq 1$.*
- (2) *There exist quaternion division algebras of level 2^k for all $k \geq 0$.*

The proof consists of constructing explicit families of quaternion division algebras whose levels are these prescribed values. For example, consider the rational function field $K = \mathbb{R}(x, y, z)$, the Laurent series field $F = K((t))$, and let $a = x^2 + y^2 + z^2$. Then $D = (a, t)_F$ is a division algebra of level 3, cf. [39, Proposition 1].

In light of Theorem 2.4, it is natural to ask a) whether values other than 2^k and $2^k + 1$ can occur as the level of a quaternion division algebra, and b) if so, what these values are. In 2008, Hoffmann answered question a) in the affirmative. He proved that there are infinitely many quaternion division level values not of the form 2^k or $2^k + 1$, cf. [17]. Note that this result did not yield any explicit new values. Indeed, question b) still seems to be an open problem.

In [36] David wondered if for a quaternion division algebra D the level and sublevel are always related as follows: $s(D) = \underline{s}(D)$ or $s(D) = 1 + \underline{s}(D)$. Hoffmann showed that this is indeed the case, cf. [18]. In fact, Hoffmann came up with the idea of the proof at the retirement conference in honour of David in 2009, where he gave a survey talk on levels and sublevels of rings.

For more information on levels, I refer to David's 1987 survey [37] in issue 19 of this *Bulletin* and the updated version [42] from 2001, as well as Hoffmann's more recent survey [19].

2.5. Signatures of involutions. Let F be a real field and let P be an ordering of F . We can think of P as the set of nonnegative elements of F with respect to some total

order relation. For example, \mathbb{Q} and \mathbb{R} each have a unique ordering, \mathbb{C} has no orderings, $\mathbb{Q}(\sqrt{2})$ has two orderings (P_1 where $\sqrt{2}$ is positive, and P_2 where $-\sqrt{2}$ is positive), $\mathbb{R}(t)$ has infinitely many orderings, and $\mathbb{R}((t))$ has two orderings (one where t is positive and one where $-t$ is positive). The set of orderings of F is denoted X_F and called the space of orderings of F (as it is a topological space). Let $q : V \rightarrow F$ be a nonsingular quadratic form over F . The (Sylvester) signature of q at P is the integer

$$\operatorname{sgn}_P q := \#\{a_i \in P\} - \#\{a_i \in -P\}.$$

For example, if $F = \mathbb{Q}(\sqrt{2})$ and $q \simeq \langle 1, \sqrt{2} \rangle$, then

$$\operatorname{sgn}_{P_1} q = 2 - 0 = 2 \quad \text{and} \quad \operatorname{sgn}_{P_2} q = 1 - 1 = 0.$$

The total signature of q is the map

$$\operatorname{sgn} q : X_F \rightarrow \mathbb{Z}, \quad P \mapsto \operatorname{sgn}_P q.$$

The total signature yields a characterization of the torsion elements of the Witt ring: $[q] \in W_{\text{tors}}(F)$ if and only if $\operatorname{sgn}_P q = 0$ for all $P \in X_F$. If F is a nonreal field, then $W(F) = W_{\text{tors}}(F)$, i.e., every element of $W(F)$ is torsion. Furthermore, for either type of field the torsion order is 2-primary. In other words, $[q]$ is torsion if and only if there exists a positive integer ℓ such that $2^\ell \times q$ is a hyperbolic form (we also say that q is weakly hyperbolic). These fundamental results were established by Pfister in [53], and are referred to as *Pfister's local-global principle*. See also [23, VIII, §3].

Let (A, σ) be a central simple F -algebra with involution, let $K = Z(A)$ and let $P \in X_F$ (orderings are always considered on the fixed field F of σ). The involution σ is said to be positive at P if the involution trace form $T_{(A, \sigma)}$ is positive definite. This notion goes back to Weil [59]. A more fine-grained measure of positivity is given by the signature of σ at P ,

$$\operatorname{sgn}_P \sigma := \sqrt{\operatorname{sgn}_P T_{(A, \sigma)}},$$

introduced by David and Jean-Pierre Tignol for involutions of the first kind, cf. [46], and by Quéguiner for involutions of the second kind, cf. [54]. This definition extends the concept of signatures from quadratic forms to involutions. In the split case $(A, \sigma) = (\operatorname{End}_F(V), \operatorname{ad}_q)$ the signatures of the quadratic form q and its adjoint involution ad_q satisfy $\operatorname{sgn}_P \operatorname{ad}_q = |\operatorname{sgn}_P q|$. Such a relationship holds more generally for signatures of hermitian forms over central simple F -algebras with involution and their adjoint involutions. The details are too technical to be discussed here. The interested reader may consult [1].

The involution σ is said to be hyperbolic if there exists an element $e \in A$ such that $e^2 = e$ and $\sigma(e) = 1 - e$. Hyperbolic involutions were introduced in [6]. If $(A, \sigma) \cong (\operatorname{End}_D(V), \operatorname{ad}_h)$, then σ is hyperbolic if and only if h is hyperbolic. The involution σ is weakly hyperbolic if there exists a positive integer n such that the involution $* \otimes \sigma$ on $M_n(K) \otimes_K A \cong M_n(A)$ is hyperbolic, where $*$ denotes conjugate transposition. In 2003, David and I extended Pfister's local-global principle to central simple algebras with involution and to hermitian forms over such algebras, cf. [47]. See also [7].

2.6. Classification of involutions. Let F be a field of characteristic $\neq 2$, let V be a vector space of dimension n over F and let $q : V \rightarrow F$ be a nonsingular quadratic form on V . Assume that $q \simeq \langle a_1, \dots, a_n \rangle$. To q we can associate its dimension, $\dim(q) = n$, its determinant $d(q) = a_1 \cdots a_n \cdot F^{\times 2} \in F^\times / F^{\times 2}$, and its Hasse invariant $s(q) = \text{class of } \prod_{i < j} (a_i, a_j)_F \text{ in the Brauer group } \operatorname{Br}(F)$ (whose elements can be identified with the isomorphism classes of F -central division algebras, cf. [23, IV]) with the convention that $s(q) = 1$ if $n = 1$. If F is in addition a real field, then q has an associated total signature $\operatorname{sgn} q$, as we have seen above.

The dimension, determinant, Hasse invariant and total signature are called the “classical” invariants of quadratic form theory. They are isometry class invariants (and thus independent of the chosen diagonalization of q). In other words, if q_1 and q_2 are isometric, then they have the same classical invariants. The converse is false in general, but true under certain conditions on the third power of the *fundamental ideal* $I(F)$ of even-dimensional forms in the Witt ring $W(F)$. Indeed, in their seminal 1974 paper [13], Elman and Lam showed that if $I^3(F) = 0$, then quadratic forms are classified up to isometry by \dim , d and s , and if $I^3(F)$ is torsion-free, then quadratic forms are classified up to isometry by \dim , d , s and sgn .

In their 1998 paper [4] Bayer-Fluckiger and Parimala extended these classification results to isometry classes of hermitian forms over central simple algebras with involution for suitable generalizations of the classical invariants, under the assumption that $I^3(F(\sqrt{-1})) = 0$ (see [5]) and an additional assumption on F when the involution is unitary.

In 1999, David and Tignol [46] obtained similar classification results for conjugacy classes of involutions on a given central simple algebra, again for suitable generalizations of the classical invariants (including signatures of involutions as described in the previous section) and under certain assumptions on $I^3(F)$ (keeping [5] in mind since they use the results of [4]) and F . (Two involutions σ and σ' on a central simple algebra A are conjugate if and only if (A, σ) and (A, σ') are isomorphic as central simple algebras with involution.)

2.7. Sesquilinear Morita theory. Let R be a commutative ring, and let (A, σ) be an R -algebra with involution. Let P be a faithful finitely generated projective right A -module and let $h_0 : P \times P \rightarrow A$ be a nonsingular ε_0 -hermitian form over (A, σ) . Hermitian Morita theory asserts that the categories of (nonsingular) ε -hermitian forms over $(\text{End}_A(P), \text{ad}_{h_0})$ and of (nonsingular) $\varepsilon\varepsilon_0$ -hermitian forms over (A, σ) are equivalent. Furthermore, orthogonal sums and hyperbolic spaces are preserved under this correspondence, cf. [21, I, §9].

Restricting to central simple algebras with involution (over fields of characteristic not 2), the usefulness of hermitian Morita theory is immediately clear since questions about ε -hermitian forms can be reduced to forms over division algebras with involution, which is an advantage since in this situation—except in the case of skew-symmetric bilinear forms over fields—nonsingular forms are diagonalizable, cf. [32], and thus more amenable to computation.

The main result of [8] is a generalization of hermitian Morita theory on two levels: anti-automorphisms that are not assumed to be of order 2 and sesquilinear forms are considered instead of involutions and ε -hermitian forms, respectively. This 2013 paper, a collaboration with Anne Cortella, is David’s final one.

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Thomas Unger is associate professor in mathematics at UCD and a member of the editorial board of the *Bulletin*. David Lewis was his doctoral advisor.

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY COLLEGE DUBLIN, BELFIELD, DUBLIN 4, IRELAND

E-mail address: `thomas.unger@ucd.ie`