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EXISTENCE OR NON-EXISTENCE OF THE PLANE WAVE SOLUTIONS OF EINSTEIN'S FIELD EQUATIONS FOR COUPLING OF VARIOUS FIELD

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ABSTRACT:

In the present paper we have studied $\left[\sqrt{t_1^2+t_2^2}\,/\,z\right]$ type plane wave solutions of Einstein's field equations in general theory of relativity in case where the zero-mass scalar field coupled with gravitational and electromagnetic field and established the existence of these two types of plane wave solutions in V4. Furthermore, we have considered the case of massive scalar field and shown that the non-existence of these two types of plane wave solutions in general relativity.

Keywords :- Plane wave solutions, Einstein's field equations, Massive scalar field, Gravitational and electromagnetic field, Einstein's generalities and Non-Flat solutions.

INTRODUCTION :

H. Takeno [7] -(1961) has obtained the non-flat plane wave solutions g_{ij} of the field equations $R_{ij}=0$ and established the existence of $(z-t)$—type and (t/z)—type plane waves for purely gravitational case in four-dimensional empty region of space-times.

On the lines of Takeno, Kadhao and Thengane [6]-(2002) have also obtained the plane wave solutions g_{ij} of field equations $R_{ij}=0$ in purely gravitational case by reformulating Takeno's definition of plane wave. Investing the line element,

$$ds^2=-Ady^2-\left(\frac{t_1^2+t_2^2}{zt_2}\right)^2 Bdz^2+\left(\frac{t_1}{t_2}\right)^2 2Bdt_1^2+2Bdt_2^2 \tag{1}$$

for $\left[\sqrt{t_1^2+t_2^2}\,/\,z\right]$ - type plane wave.

We have obtained the solutions g_{ij} in the form of P with the relations of non-vanishing components of Ricci tensor as

$$\begin{aligned} P=\left(\frac{z^4}{t_1^2+t_2^2}\right)R_{22}=\left(\frac{-z^3}{t_1}\right)R_{23}=\left(\frac{-z^3}{t_1}\right)R_{24} \\ =\frac{z^2(t_1^2+t_2^2)}{t_1t_2}R_{34}=\frac{z^2(t_1^2+t_2^2)}{t_1^2}\,R_{33} \\ =\frac{z^2(t_1^2+t_2^2)}{t_2^2}R_{44} \\ =\frac{\bar{\bar{A}}}{2A}-\frac{\bar{A}^2}{4A^2}-\frac{\bar{A}\bar{B}}{2AB}+\frac{\bar{\bar{B}}}{2B}-\frac{3\bar{B}^2}{4B^2} \end{aligned} \tag{2}$$

for $\left[\sqrt{t_1^2+t_2^2}\,/\,z\right]$ - type plane wave.

In the paper we investigate whether this type of plane wave solutions exist in the case where zero mass scalar field coupled with gravitational field and the zero mass scalar field coupled with gravitational and electromagnetic field. Furthermore, we consider the coupling of massive scalar field with gravitational field and the massive scalar field with gravitational field and electromagnetic field also in V_4 to investigate the existence of this type of plane wave solutions of Einstein's field equations

$$R_{ij}=(-8\pi)\left[T_{ij}-\tfrac{1}{2}g_{ij}T\right]\quad (i,j=1,2,3,4) \tag{3}$$

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where R_{ij} is the Ricci tensor of the space-time,
T_{ij} is the energy momentum tensor,
g_{ij} is the fundamental tensor of the space-time,
and $T = T_i^i = g_{ij}T_{ij}$.

$\left[\sqrt{t_1^2 + t_2^2} \,/\, z\right]$ - type plane wave solutions.

1. Zero mass scalar field coupled with gravitational field

The energy momentum tensor of zero mass scalar field is given by

$$T_{ij} = \frac{1}{4\pi}\left[V_i V_j - \frac{1}{2} g_{ij} V_k V^k\right], \quad (k = 1,2,3,4) \qquad (4)$$

where V is scalar function of Z and $V_j = \frac{\partial V}{\partial x^j}, (x^j = y, z, t_1, t_2)$ (5)

Thus

$$V_1 = 0, V_2 = \left(\frac{-\sqrt{t_1^2+t_2^2}}{t_1}\right)\bar{V}, V_3 = \left(\frac{t_1}{z\sqrt{t_1^2+t_2^2}}\right)\bar{V}, V_4 = \left(\frac{t_2}{z\sqrt{t_1^2+t_2^2}}\right)\bar{V} \qquad (6)$$

where (-) bar denotes partial derivative with respect to Z.

From the line element (1) we have

$$g_{22} = -\left(\frac{t_1^2+t_2^2}{zt_2}\right)B, \quad g_{33} = 2\left(\frac{t_1}{t_2}\right)^2 B, \quad g_{44} = 2B, \qquad (7)$$

$$g^{22} = -\left(\frac{zt_2}{t_1^2+t_2^2}\right)\frac{1}{B}, \quad g^{33} = 2\left(\frac{t_2}{t_1}\right)^2 \frac{1}{2B}, \quad g^{44} = \frac{1}{2B}, \qquad (8)$$

$$\Rightarrow V_k V^k = 0 \qquad (9)$$

The equation (4) becomes

$$T_{ij} = \frac{1}{4\pi}[V_i V_j] \qquad (10)$$

From equation (10) we have

$$T = T_i^i = g^{ij}T_{ij} = 0 \qquad (11)$$

Then using (10) and (11), Einstein's field equation (3) becomes

$$R_{ij} = -2[V_i V_j] \qquad (12)$$

which further gives

$$R_{22} = -2\left(\frac{t_1^2 + t_2^2}{t_2^2}\right)\bar{V}^2,$$

$$R_{33} = -2\left(\frac{t_1^2}{t_1^2 + t_2^2}\right)\frac{\bar{V}^2}{z^2}, \quad R_{44} = \left(\frac{-2t_2^2}{t_1^2 + t_2^2}\right)\frac{\bar{V}^2}{z^2}$$

$$R_{23} = -2\left(\frac{-t_1}{z^3}\right)\bar{V}^2, \quad R_{24} = -2\left(\frac{-t_1}{z^3}\right)\bar{V}^2, \quad R_{34} = -2\left(\frac{t_1 t_2}{t_1^2+t_2^2}\right)\frac{\bar{V}^2}{z^2}, \qquad (13)$$

But from the line element (1) the non-vanishing components of Ricci tensor are related as

$$\left(\frac{z^4}{t_1^2+t_2^2}\right)R_{22} = z^2\left(\frac{t_1^2+t_2^2}{t_1^2}\right)R_{33} = z^2\left(\frac{t_1^2+t_2^2}{t_2^2}\right)R_{44} = \left(\frac{-z^3}{t_1}\right)R_{23} = \left(\frac{-z^3}{t_2}\right)R_{24} = z^2\left(\frac{t_1^2+t_2^2}{t_1 t_2}\right)R_{34}. \qquad (14)$$

It is observed that the equation (14) is compatible with equation (2) which obtained in case of purely gravitational field.

Hence, we have, $\left[\sqrt{t_1^2 + t_2^2} \,/\, z\right]$ - type plane wave solution exist in the case where zero mass scalar field is coupled with gravitational field.

2. Zero mass scalar field coupled with gravitational and electromagnetic field.

In this case the energy momentum tensor is given by

$$T_{ij} = \frac{1}{4\pi}\left[V_i V_j - \frac{1}{2} g_{ij} V_k V^k\right] + E_{ij}, \qquad (15)$$

Where E_{ij} denotes electromagnetic energy momentum tensor.

But as in previous case $V_k V^k = 0$.

Therefore equation (15) becomes

$$T_{ij} = \frac{1}{4\pi}[V_i V_j] + E_{ij}, \qquad (16)$$

From equation (16) we have

$$T = T_i^i = g^{ij}T_{ij} = 0 \qquad (17)$$

Hence Einstein's field equation (3) becomes

$$R_{ij} = -2[V_i V_j + 4\pi E_{ij}] \qquad (18)$$

Then equation (18) yields

$$R_{22} = -2\left[\left(\frac{t_1^2+t_2^2}{z^4}\right)\bar{V}^2 + 4\pi E_{22}\right],$$

$$R_{23} = -2\left[\left(\frac{-t_1}{z^3}\right)\bar{V}^2 + 4\pi E_{23}\right],$$

$$R_{33} = -2\left[\left(\frac{t_1^2}{z^2(t_1^2+t_2^2)}\right)\bar{V}^2 + 4\pi E_{33}\right],$$

$$R_{24} = -2\left[\left(\frac{-t_1}{z^3}\right)\bar{V}^2 + 4\pi E_{24}\right]$$

$$R_{44} = -2\left[\left(\frac{t_2^2}{z^2(t_1^2+t_2^2)}\right)\bar{V}^2 + 4\pi E_{44}\right],$$

$$R_{34} = -2\left[\left(\frac{t_1 t_2}{z^2(t_1^2+t_2^2)}\right)\bar{V}^2 + 4\pi E_{34}\right] \qquad (19)$$

But from the line element (1) the non-vanishing components of Ricci tensor are related as

$$\left(\frac{z^4}{t_1^2+t_2^2}\right)R_{22} = z^2\left(\frac{t_1^2+t_2^2}{t_1^2}\right)R_{33} = z^2\left(\frac{t_1^2+t_2^2}{t_2^2}\right)R_{44} = \left(\frac{-z^3}{t_1}\right)R_{23} = \left(\frac{-z^3}{t_2}\right)R_{24} = z^2\left(\frac{t_1^2+t_2^2}{t_1 t_2}\right)R_{34} \qquad (20)$$

The equation (20) is compatible with (19). Therefore, $\left[\sqrt{t_1^2 + t_2^2} \,/\, z\right]$ - type plane wave

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solutions exist in the case where zero mass scalar field is coupled with gravitational and electromagnetic field.

3. Massive scalar field coupled with gravitational field.

The energy momentum tensor of massive scalar field is given by

$$T_{ij} = \frac{1}{4\pi}\left[V_i V_j - \frac{1}{2} g_{ij}(V_k V^k - m^2 V^2)\right], \quad (k = 1,2,3,4.) \tag{21}$$

where V is scalar function of Z and $V_j = \frac{\partial V}{\partial x^j}, (\because x^i = y, z, t_1, t_2.)$

m is the mass associated with massive scalar field.

Thus

$$V_1 = 0, V_2 = \left(\frac{-\sqrt{t_1^2+t_2^2}}{z^2}\right)\bar{V}, V_3 = \left(\frac{t_1}{z\sqrt{t_1^2+t_2^2}}\right)\bar{V}, \quad V_4 = \left(\frac{t_2}{z\sqrt{t_1^2+t_2^2}}\right)\bar{V} \tag{22}$$

where (-) bar denotes partial derivative with respect to Z.

From the line element (1) we have

$$g_{22} = -\left(\frac{t_1^2+t_2^2}{zt_2}\right)B, \quad g_{33} = 2\left(\frac{t_1}{t_2}\right)^2 B, \quad g_{44} = 2B, \tag{23}$$

$$g^{22} = -\left(\frac{zt_2}{t_1^2+t_2^2}\right)\frac{1}{B}, \quad g^{33} = 2\left(\frac{t_2}{t_1}\right)^2\frac{1}{2B}, \quad g^{44} = \frac{1}{2B}, \tag{24}$$

$$\Rightarrow V_k V^k = 0 \tag{25}$$

Therefore equation (21) implies

$$T_{ij} = \frac{1}{4\pi}\left[V_i V_j - \frac{1}{2} g_{ij} m^2 V^2\right], \tag{26}$$

Equation (26) yields

$$T = T_i^i = g^{ij} T_{ij} = \frac{1}{2\pi} m^2 V^2 \tag{27}$$

Using (26) and (27), Einstein's field equation (3) becomes

$$R_{ij} = -2\left[V_i V_j - \frac{1}{2} g_{ij} m^2 V^2\right] \tag{28}$$

which further yields

$$R_{22} = -2\left[\left(\frac{t_1^2+t_2^2}{z^4}\right)\bar{V}^2 + \frac{1}{2}\left(\frac{t_1^2+t_2^2}{zt_2}\right) B m^2 V^2\right],$$

$$R_{33} = -2\left[\left(\frac{t_1^2}{t_1^2+t_2^2}\right)\frac{\bar{V}^2}{z^2} - \left(\frac{t_1}{t_2}\right)^2 B m^2 V^2\right],$$

$$R_{44} = -2\left[\left(\frac{t_2^2}{z^2(t_1^2+t_2^2)}\right)\bar{V}^2 - B m^2 V^2\right],$$

$$R_{23} = -2\left(\frac{-t_1}{z^3}\right)\bar{V}^2$$

$$R_{24} = -2\left(\frac{-t_2}{z^3}\right)\bar{V}^2,$$

$$R_{34} = -2\left(\frac{t_1 t_2}{z^3(t_1^2+t_2^2)}\right)\bar{V}^2. \tag{29}$$

But from the line element (1) the non-vanishing components of Ricci tensor are related as

$$\left(\frac{z^4}{t_1^2+t_2^2}\right) R_{22} = z^2\left(\frac{t_1^2+t_2^2}{t_1^2}\right) R_{33} = z^2\left(\frac{t_1^2+t_2^2}{t_2^2}\right) R_{44} = \left(\frac{-z^3}{t_1}\right) R_{23} = \left(\frac{-z^3}{t_2}\right) R_{24} = z^2\left(\frac{t_1^2+t_2^2}{t_1 t_2}\right) R_{34}. \tag{30}$$

It is to be noted that here the equation (29) is an incompatible with the equation (30) which is obtained in the case of purely gravitational field.

Therefore, $\left[\sqrt{t_1^2+t_2^2}\,/\,z\right]$ - type plane wave solution does not exist in the case where zero mass scalar field is coupled with gravitational and electromagnetic field.

Remarks: If $m^2 = 0$ then equation (29) is compatible with (30) and we have a result as in the case where the zero mass scalar field is coupled with the gravitational field.

4. Massive scalar field coupled with gravitational & electromagnetic field.

In this case the energy momentum tensor is given by

$$T_{ij} = \frac{1}{4\pi}\left[V_i V_j - \frac{1}{2} g_{ij}(V_k V^k - m^2 V^2)\right] + E_{ij} \tag{31}$$

Where E_{ij} denotes electromagnetic energy momentum tensor.

But as in previous case $V_k V^k = 0$

Therefore, equations (31) becomes

$$T_{ij} = \frac{1}{4\pi}\left[V_i V_j + \frac{1}{2} g_{ij} m^2 V^2\right] \tag{32}$$

Also, equation (32) implies

$$T = T_i^i = g^{ij} T_{ij} = \frac{1}{2\pi} m^2 V^2 \tag{33}$$

Hence Einstein's field equation (3) becomes

$$R_{ij} = -2\left[V_i V_j - \frac{1}{2} g_{ij} m^2 V^2 + 4\pi E_{ij}\right] \tag{34}$$

Then from equation (33) we have,

$$R_{22} = -2\left[\left(\frac{t_1^2+t_2^2}{z^4}\right)\bar{V}^2 + \frac{1}{2}\left(\frac{t_1^2+t_2^2}{zt_2}\right)^2 B m^2 V^2 + 4\pi E_{22}\right],$$

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$$R_{33} = -2\left[\left(\frac{t_1^2}{t_1^2+t_2^2}\right)\frac{\bar{V}^2}{z^2} - \left(\frac{t_1}{t_2}\right)^2 Bm^2V^2 + 4\pi E_{33}\right],$$

$$R_{44} = -2\left[\left(\frac{t_2^2}{z^2(t_1^2+t_2^2)}\right)\bar{V}^2 - Bm^2V^2 + 4\pi E_{44}\right],$$

$$R_{23} = -2\left[\left(\frac{-t_1}{z^3}\right)\bar{V}^2 + 4\pi E_{23}\right],$$

$$R_{24} = -2\left[\left(\frac{-t_2}{z^3}\right)\bar{V}^2 + 4\pi E_{24}\right],$$

$$R_{34} = -2\left(\frac{t_1t_2}{z^3(t_1^2+t_2^2)}\right)\bar{V}^2 + 4\pi E_{34} \qquad (35)$$

But from the line element (1) the non-vanishing components of Ricci tensor are related as

$$\left(\frac{z^4}{t_1^2+t_2^2}\right)R_{22} = z^2\left(\frac{t_1^2+t_2^2}{t_1^2}\right)R_{33} = z^2\left(\frac{t_1^2+t_2^2}{t_2^2}\right)R_{44} = \left(\frac{-z^3}{t_1}\right)R_{23} = \left(\frac{-z^3}{t_2}\right)R_{24} = z^2\left(\frac{t_1^2+t_2^2}{t_1t_2}\right)R_{34}. \qquad (36)$$

The equation (35) is incompatible with equation (36) which is obtained in the case of purely gravitational field. Hence we have, $\left[\sqrt{t_1^2+t_2^2}\,/\,z\right]$-type plane wave solution of Einstein's field equation in general relativity doesn't exist in the case where zero massive scalar field is coupled with gravitational and electromagnetic field.

REMARK:

If $m^2 = 0$ then equation (35) is incompatible to (36) and we have a result as in the case where the zero-mass scalar field is coupled with the gravitational and electromagnetic field.

CONCLUSION:

The existence of $\left[\frac{\sqrt{t_1^2+t_2^2}}{z}\right]$ - type plane wave solutions in case where zero massive scalar field is coupled with gravitational field and zero mass scalar field is coupled with gravitational and electromagnetic field is characterized by (14),(20) respectively and the nonexistence of this plane wave solutions in the case where massive scalar filed is coupled with gravitational field and massive scalar filed is coupled with gravitational and electromagnetic field are shown by (30),(36) respectively.

REFERENCES:

The theory of the Relativity of motion -- Richard c. Tolman, Oxford Clarendon Press First edition 1934

Introduction to special Relativity --Wolfgang Rindler Oxford university Press.

Relativity --WGV ROSSER, EXETER UNIVERSITY, LONDON, BUTTERWORTHS 1964.

READABLE RELATIVITY, Bell, London Cambridge university Press, 1920.

Relativity, Einstein A., Methuen, London 1920

Kadhao S. R. and Thengane K. D. (2002), some plane wave solution of the field equations R_ij=0 in four-dimensional space-time, Einstein Foundation International, Volume 12, 2002, India.

Takeno H (1961), 'The Mathematical theory of plane gravitational wave in general relativity' Scientific report of the research institute for theoretical Physics, Hiroshima University, Hiroshima-Ken Japan. Published Date-01.01.1961, OSTI identifier 4751857 NSA-17-016858.

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