Ground Penetrating Radar
The first peer-reviewed scientific journal dedicated to GPR

Frequency domain deterministic-stochastic analysis of the transient current induced along a ground penetrating radar dipole antenna over a lossy half-space

Anna Šušnjara[1] (corresponding author), Dragan Poljak[1], Vicko Dorić[1],
Sébastien Lalléchère[2], Khalil el Khamlichi Drissi[2],
Pierre Bonnet[2] & Françoise Paladian[2]

[1] Department of Electronics and Computer Science, Faculty of Electrical Engineering, Mechanical Engineering and Naval Architecture (FESB), University of Split, Split, Croatia
ansusnja@fesb.hr, dpoljak@fesb.hr, vdoric@fesb.hr

[2] Université Clermont Auvergne, Centre national de la recherche scientifique (CNRS), SIGMA Clermont, Institut Pascal, Clermont-Ferrand, France
sebastien.lallechere@uca.fr, khalil.drissi@uca.fr, pierre.bonnet@ uca.fr, francoise.paladian@uca.fr

Abstract

This paper deals with the stochastic analysis of transient current induced along a ground penetrating radar (GPR) antenna. The antenna is modelled as a horizontal dipole and is placed over a lossy half-space. The electromagnetic formulation of the problem is based on the Pocklington's integro-differential equation in the frequency domain, which is solved by means of the Galerkin-Bubnov indirect boundary element method. The transient solution is obtained by using the inverse fast Fourier transform. The paper aims to investigate the variability of the current due to key uncertain parameters, such as the soil permittivity and conductivity, and the wire distance from the half-space. Stochastic assumptions are incorporated in the model by means of the stochastic collocation technique. Computational examples present the mean value of current distributed along the wire with the confidence margins. Sensitivity analysis is obtained, i.e., the uncertainty in the output is apportioned to different sources of uncertainty in the model input thus giving a better insight into model reliability.

Keywords: Ground Penetrating Radar (GPR); Electromagnetic modelling; Galerkin-Bubnov indirect boundary element method; Stochastic collocation technique; Antennas; Lossy half-space.

1. Introduction

Ground penetrating radar (GPR) is used in civil engineering, archaeology, and many other areas. GPR antennas are moved over the

https://doi.org/10.26376/GPR2018008

surface of the inspected soil or structure, while emitting and receiving electromagnetic (EM) waves. In order to extract accurate and useful information from the received EM field, it is important to have as much a priori information as possible [1]. Such information includes a good understanding of the electromagnetic properties of the involved media and used antennas [2]. However, the knowledge about these properties is inevitably stochastic in its nature.

Many researchers have studied the EM behaviour of GPR antennas, by using different techniques that can be classified in two main categories: frequency domain (FD) [3] and time domain (TD) [4]-[10] techniques. A stochastic analysis of the transient response of a GPR antenna has been presented in [11]-[13]. In [11] the unknown current along the wire above the lossy-half space is governed by the space-time Hallen integral equation. The deterministic solution is featured by GB-IBEM method. The stochastic response is obtained with respect to uncertain antenna position (height) and uncertain ground conductivity. The work done in [12] and [13] present the stochastic current response for the wire buried in the lossy ground which may be found useful not only in GPR purposes but in other areas, for example in the design of lighting protection for electrical settlements.

As a counterpoise to time domain analysis, the stochastic analysis of frequency domain response is presented in the present paper. Stochastic Collocation (SC) method is combined with a direct EM solver to assess the variability of the current induced on a GPR dipole antenna, due to the uncertain nature of the soil and antenna height. The dipole is assumed to be thin and is placed above a lossy half-space, with its axis parallel to the air-soil interface: such simple geometry is especially convenient for testing new computational approaches and methods. The formulation of the problem, implemented in our deterministic EM solver, is based on a FD solution of Pocklington's integro-differential equation, by means of Galerkin-Bubnov Indirect Boundary Element Method (GB-IBEM) [3]; the transient response is then obtained via inverse Fast Fourier's transform [14].

The paper is organized as follows. Section 2 outlines the employed FD integral equation approach and related numerical solution (Sub-section 2.1); the theoretical basis of the Stochastic Collocation method are also presented (Sub-section 2.2). Section 3 brings computational examples, while in Section 4 general conclusions are given.

https://doi.org/10.26376/GPR2018008

2. Electromagnetic modelling method

2.1 Deterministic frequency-domain analysis

The geometry of interest is a straight, thin and horizontal dipole placed at height *h* above a lossy half-space (see Figure 1). The wire is considered as a perfect conductor, with a voltage source applied to the gap in the centre of the antenna. The source has a Gaussian shaped waveform.

The current induced along the wire is governed by the Pocklington's integro-differential equation in the frequency domain [3]. Such equation is derived by enforcing the interface conditions for the tangential components of the electric field at the wire surface:

$$\vec{e}_x \cdot \left(\vec{E}^{exc} + \vec{E}^{sct}\right) = 0 \tag{1}$$

where $\vec{e}_x$ is the unit vector, the excitation field $\vec{E}^{exc}$ is composed by the incident field $\vec{E}^{inc}$ and the field reflected from the ground $\vec{E}^{ref}$:

$$\vec{E}^{exc} = \vec{E}^{inc} + \vec{E}^{ref} \tag{2}$$

The scattered electric field is given as:

$$\vec{E}^{sct} = -j\omega\vec{A} - \nabla\varphi \tag{3}$$

where $\vec{A}$ represents the magnetic vector potential, φ is the electric scalar potential and ω is angular frequency, $\omega=2\pi f$. According to the thin wire approximation, only the axial component of the magnetic vector potential exists, therefore Equation (3) becomes [3]:

$$E_x^{sct} = -j\omega A_x - \frac{\partial\varphi}{\partial x} \tag{4}$$

The axial component of the magnetic vector potential and the electric scalar potential are given by:

$$A_x = \frac{\mu}{4\pi}\int_{-L/2}^{L/2} I(x')g(x,x')dx' \tag{5}$$

and

$$\varphi(x) = -\frac{1}{j4\pi\omega\varepsilon_0}\int_{-L/2}^{L/2} \frac{\partial I(x')}{\partial x'} g(x,x')dx' \tag{6}$$

https://doi.org/10.26376/GPR2018008

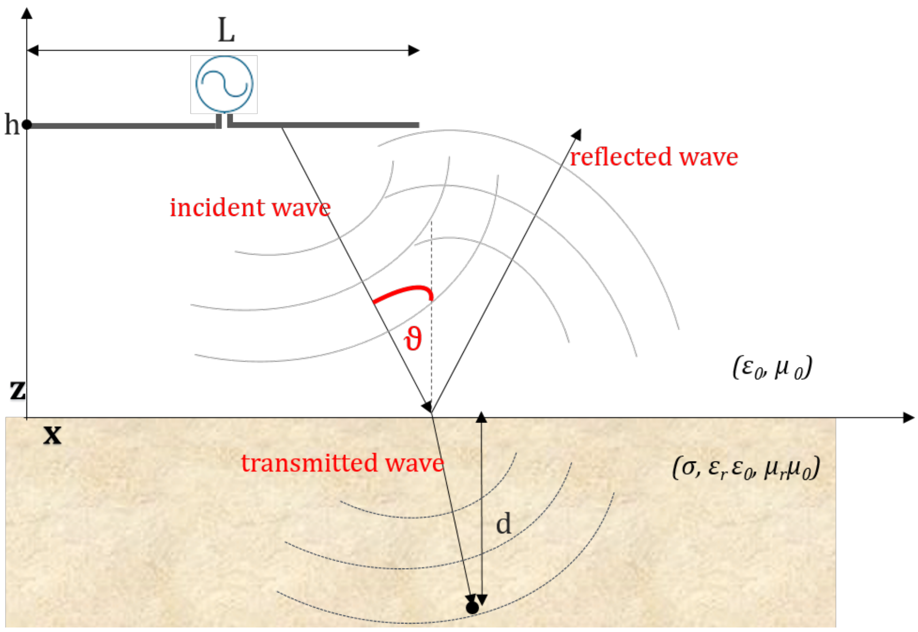


FIG. 1 – A GPR dipole antenna horizontally placed above a lossy half-space.

respectively. The induced current along the wire is represented by $I(x')$, while $g(x,x')$ denotes the total Green function [3]:

$$g(x,x') = g_0(x,x') - R_{TM} g_i(x,x') \tag{7}$$

where $g_0(x,x')$ is the free space Green function:

$$g_0(x,x') = \frac{e^{-jk_0R_0}}{R_0} \tag{8}$$

and $g_i(x,x')$ arises from the image theory:

$$g_i(x,x') = \frac{e^{-jk_0R_i}}{R_i} \tag{9}$$

The distance from the source point on the wire, or its image in the ground, to the observation point is denoted by R_0 and R_i, respectively. The influence of the lossy half-space is taken into account by means of the reflection coefficient R_{TM}, which for a transverse magnetic polarisation is given by:

$$R_{TM} = \frac{n\,\cos\vartheta - \sqrt{n - (\sin\vartheta)^2}}{n\,\cos\vartheta + \sqrt{n - (\sin\vartheta)^2}} \tag{10}$$

https://doi.org/10.26376/GPR2018008

where n stands for the refraction index and θ is the incident angle:

$$n = \sqrt{\varepsilon_r - j\sigma/(\omega\varepsilon_0)} \qquad \theta = \arctan\left[|x - x'|/(2h)\right] \tag{11}$$

Inserting Equations (4)-(7) in (2) leads to Pocklington's integro-differential equation for the unknown current induced along the wire axis [3]:

$$E_x^{exc} = \frac{j\omega\mu}{4\pi}\int_{-L/2}^{L/2} I(x')g(x,x')dx' + -\frac{1}{j4\pi\omega\varepsilon_0}\frac{\partial}{\partial x}\int_{-L/2}^{L/2}\frac{\partial I(x')}{\partial x'}g(x,x')dx' \tag{12}$$

The unknown current is calculated via GB-IBEM. More details can be found elsewhere, e.g. in [3]. In order to obtain the transient response, the solution is then transformed to the TD. First, the current is calculated for the specified frequency range, thus obtaining the transfer function of a system $H(f)$ in the FD. Then, $H(f)$ is multiplied by the spectrum of the Gaussian pulse given by [11]:

$$V(f) = V_0\sqrt{\pi}\, t_w\, e^{-(\pi f t_w)^2}\, e^{-j2\pi f t_0} \tag{13}$$

where V_0 is the amplitude, t_0 is the time delay of the Gaussian pulse and t_w is the half-width of the Gaussian pulse in the TD. Finally, the frequency response is transformed to the TD by using the inverse fast Fourier transform. For this purpose, the Matlab function *"ifft"* is used [14], [15].

2.2 Stochastic collocation technique

Stochastic studies, in general, are carried out in two steps [16]. The first step implies the definition of random input data of the given model; input random variables (RV) are statistically described by assigning the corresponding statistical distributions. The second step is to solve a stochastic numerical or analytical model to obtain the statistical description of the output value of interest.

The SC method has been widely used in recent stochastic EM computations: the main advantages are its non-intrusive nature and simplicity. A theoretical background of this approach can be found in [17], for the sake of completeness some basics are given in this section.

The fundamental principle of the SC method is the polynomial approximation of the considered output Y for N uncertain input

https://doi.org/10.26376/GPR2018008

parameters (N is defined as random dimension of the problem). The random input parameter Z is given as [17]:

$$Z = Z^0 + \hat{u} \tag{14}$$

where Z^0 is the initial value and $\hat{u}$ is a random variable with a statistical distribution. The output of interest Y is expanded over a stochastic space by using the Lagrangian basis functions:

$$Y(Z^0; t) = \sum_{\mathrm{i}=1}^{N_{SC}} Y_{\mathrm{i}}(Z^0) L_{\mathrm{i}}(t) \tag{15}$$

where $L_i(t)$ is the Lagrangian basis function given by:

$$L_{\mathrm{i}}(t) = \prod_{\mathrm{j}=1,\mathrm{j}\neq\mathrm{i}}^{N_{SC}} \frac{(t - t_{\mathrm{j}})}{(t_{\mathrm{j}} - t_{\mathrm{i}})} \tag{16}$$

and t_i is i-th collocation point, also called sigma point. The reasons for choosing the Lagrange polynomials as basis function come from its property $L_i(t_j) = \delta_{ij}$ leading to: $Y_i(Z^0) = Y(Z^0; t_i)$ [17]. Following the definition for the statistical moments, the expected value of the output Y of interest can be calculated as:

$$\langle Y(Z^0; t) \rangle = \sum_{\mathrm{i}=1}^{N_{SC}} Y_{\mathrm{i}}(Z^0)\, \mathrm{w}_{\mathrm{i}} \tag{17}$$

and the variance is:

$$\mathrm{Var}[Y(Z^0; t)] = \sigma_{\mathrm{E}}^2 = \sum_{\mathrm{i}=0}^{N_{SC}} \mathrm{w}_{\mathrm{i}} Y_{\mathrm{i}}^2 - \langle Y \rangle^2 \tag{18}$$

where w_i denotes the weight given by:

$$\mathrm{w}_{\mathrm{i}} = \int_{\mathrm{D}} L_{\mathrm{i}}(t)\, \mathrm{pdf}(t) \mathrm{d}t \tag{19}$$

for the input RV with the probability density function pdf(t). The order of the approximation depends on number of sigma points, N_{SC}. A higher N_{SC} implies a better approximation, but at the cost of computational effort. The dimensionality of the problem can be increased to the desired extent. In this paper higher dimensions are included via the tensor product rule. Therefore the number of required deterministic

https://doi.org/10.26376/GPR2018008

simulations is given as $(N_{SC})^N$. However, the SC technique suffers from the "curse of dimensionality;" hence, the computation for a very high number of input RVs by using the tensor product is not practical.

The SC method provides means for performing a Sensitivity Analysis (SA) of a given model. In this paper, the variance based sensitivity analysis is used [17]. This approach is part of a more general Global Sensitivity Analysis (GSA) method [18], which provides the means for estimating the influence of a collection of random inputs on the output of interest. In this paper, for a random dimension $N = 3$, the impact factor of each random input variable is obtained as:

$$\mathrm{I_i} = \mathrm{Var}(E;\ \mathrm{RV_i})/\mathrm{Var}(E) \tag{20}$$

where i = 1, 2, 3. This expression corresponds to the first order Sobol like indices [18]. The impact of the collection of input RVs is given as:

$$\mathrm{I'_{jk}} = \mathrm{Var}\big(E;\ \mathrm{RV_j}, \mathrm{RV_k}\big)/\mathrm{Var}(E) \tag{21}$$

where j =1, 2, 3 and k = 1, 2, 3, j≠k. The measure of mutual interaction of each combination of random input variables is defined through higher order indices. For example, second order indices are obtained by combining Equations (21) and (22). More details are found in [18].

3. Computational examples

A horizontal dipole antenna, with length $L = 1$ m and radius $a = 6.74$ mm, is placed above a lossy half-space. Both L and a are considered as deterministic parameters. Three parameters are modelled as random input variables, with uniform distributions: the antenna height (i.e., its distance from the air-soil interface) $h \sim \cup(12, 18)$ cm, the soil relative permittivity $\varepsilon_r \sim \cup(14, 18)$, and the soil conductivity $\sigma \sim \cup(0.1, 9.9)$ mS/m. The considered type of soil is an average one and the ranges of expected permittivity and conductivity values are taken from [1]. The output of interest is the transient current in the centre of the wire. The frequency range for the transfer function $H(f)$ is 10 Hz – 28.64 GHz. For the Gaussian pulse of Equation (13): $V_0 = 1\,V$, $t_0 = 1.43$ ns, and $t_w = 2/3$ ns.

In order to investigate to which extent the three random input variables impact the output of interest, all possible combinations of input RVs are taken into account, starting from three univariate cases

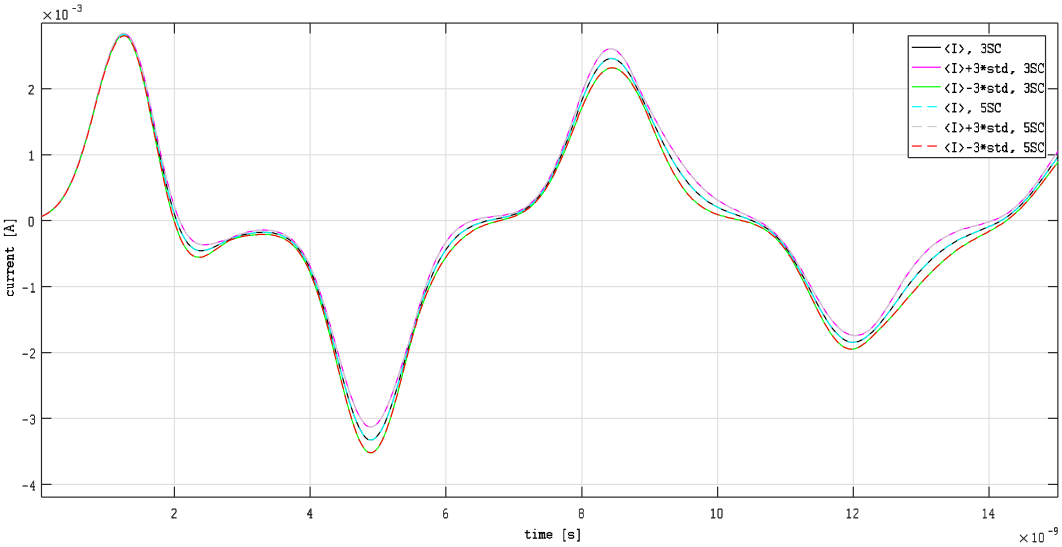
https://doi.org/10.26376/GPR2018008

where $N = 1$, then considering all bivariate cases with $N = 2$, and finally studying the multivariate case with random dimension $N = 3$. The variance-based approach is used to obtain the Sensitivity Analysis (SA) of the presented stochastic model. The experimental design (ED) is built for 3 and 5 SC points, thus implying $3^3 + 5^3 = 152$ different deterministic simulations. The complete stochastic analysis is done as a post processing of results obtained for the defined ED, by using equations similar to (17) and (18). The execution time of a single deterministic simulation is 17.78 minutes on ASUS PC with i5-5200 CPU and 2.20 GHz processor, which is impractical for traditional Monte Carlo simulations. In order to ensure good convergence with Monte Carlo approach, at least 10,000 – 100,000 simulations are necessary. On the other hand, according to the literature, the SC method with far less simulations is proven to have good convergence, especially for small number of input RVs, which is the case in this computational setup [11], [12], [16], [17].

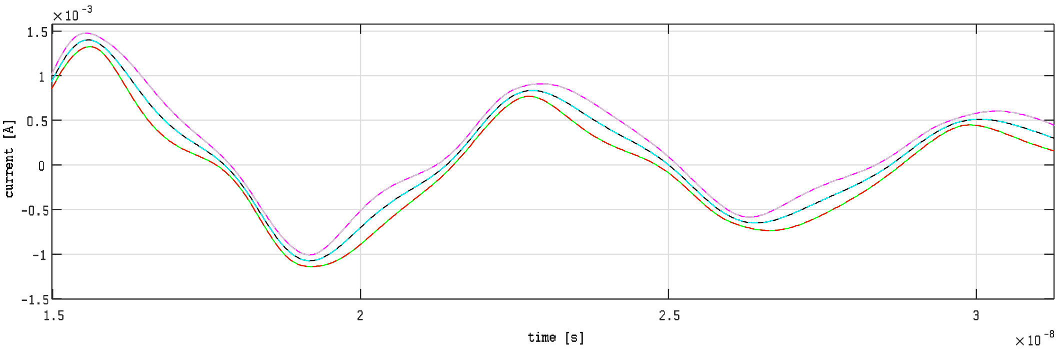
Figure 2(a) exhibits the mean trend of the transient current at the centre of the wire, when all three input variables are random (N=3). The results are presented for SC simulations with 3 and 5 sigma points. The crude estimate of confidence intervals is given as the mean ± 3 standard deviations. It is apparent that throughout the whole time interval the standard deviation of the current is not large. The confidence interval is almost negligible for the early time response: in the beginning of the simulation only the peaks exhibit a noticeable deviation from the mean trend. In the later time instances the deviation around the mean trend becomes larger and more or less uniform.

Figure 2(b) shows the subsequent time interval, from 15 ns to about 30 ns, for the same case study; from this plot, the convergence and accuracy of the stochastic approach can be appreciated. The results for 3 and 5 sigma points show a satisfactory agreement. This proves that, in order to access second order statistics (mean and standard deviation) for the current in the middle of the wire with three random input variables (ε_r, σ, and h) only 3 sigma points are required, i.e., 27 deterministic simulations. In the case of univariate scenario, when only σ is random, 5 sigma points were necessary; however, as it is going to be demonstrated later, the influence of this variable on the current is very small.

https://doi.org/10.26376/GPR2018008



(a)



(b)

FIG. 2 – Statistics of the current at the centre of wire, for the multivariate test case: ⟨I⟩ is the stochastic expected value and std is the standard deviation: **(a)** Focus on interval 0-1.5 s **(b)** Focus on the time interval 15-30 ns.

The impact factors of each random input, calculated by using Equation (20), are shown in Figure 3. Throughout the whole simulation interval the height of the antenna has the highest impact. The influence of the first random variable, ε_r, cannot be ignored in the time instants where the current reaches its local minimum and maximum values. The overall influence of soil conductivity is small.

The impact factor of the combinations of input variables obtained from Equation (21) is depicted in Figure 4. The domination of the antenna height is obvious. The interactions of RVs have negligible impact on the total variation of the current.

https://doi.org/10.26376/GPR2018008

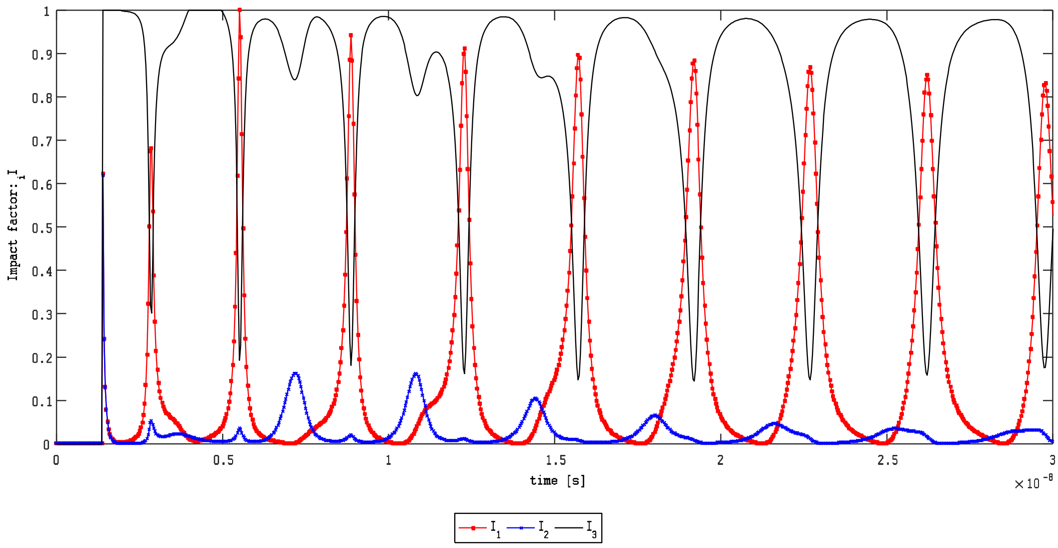


FIG. 3 – Impact factors of each input RV: $RV_1 = \varepsilon_r$, $RV_2 = \sigma$, $RV_3 = h$.

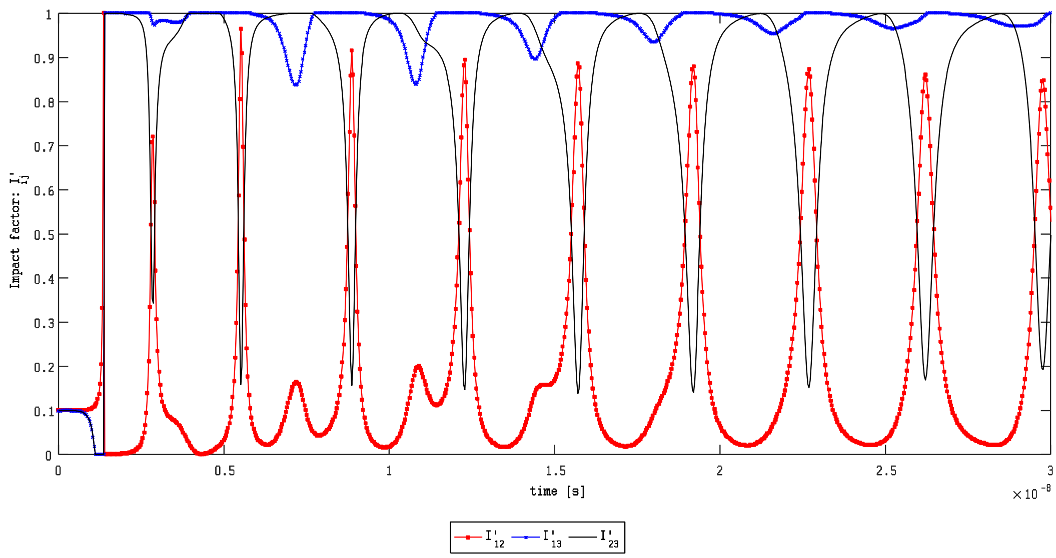


FIG. 4 – Impact factor of combinations of input random variables: $RV_{12} = [\varepsilon_r, \sigma]$, $RV_{13} = [\varepsilon_r, h]$, $RV_{23} = [\sigma, h]$.

4. CONCLUSIONS

This contribution presents a frequency domain stochastic-deterministic analysis of the current induced on a GPR dipole antenna placed above a lossy half-space. The current is governed by Pocklington's integro-

https://doi.org/10.26376/GPR2018008

differential equation in frequency domain, which is solved by means of Galerkin-Bubnov Indirect Boundary Element Method. The transient response is obtained by using inverse fast Fourier transform. The deterministic model is combined with Stochastic Collocation (SC) method in order to investigate to which extent random input parameters influence the induced current. Three input parameters are assumed to be uncertain: the permittivity and electrical conductivity of soil, and the wire distance from the soil, all following the prescribed uniform distributions. SC method showed a quick convergence in the calculation of stochastic moments, especially for the mean value. The obtained numerical results demonstrate that the variation of the wire height has the highest overall impact on the current distribution, for the considered average soil type.

The calculation of the current induced along the wire is the first step in analysing the electromagnetic behaviour of a GPR antenna and thus its scattered field. The stochastic approach in electromagnetic modelling is a natural and useful extension, arising from the inherent uncertainty of the parameters that belong to such environment. SC, in combination with existing deterministic models, is a powerful tool for the calculation of necessary statistics of the outputs of interest and allows a better understanding of the electromagnetic behaviour of a GPR system.

Acknowledgements

This work benefited from networking activities carried out within EU funded COST Action TU1208 “Civil Engineering Application of Ground Penetrating Radar” (www.gpradar.eu).

References

[1] H. M. Jol, Ground Penetrating Radar Theory and Applications, Amsterdam: Elsevier Science, 2009.

[2] C. Warren, N. Chiwaridzo, A. Giannopoulos, “Radiation Characteristics of a High-Frequency Antenna in Different Dielectric Environments,” Proc. 15th International Conference on Ground Penetrating Radar – GPR 2014, 30 June - 4 July 2014, Brussels, Belgium, pp. 767–772, doi: 10.1109/ICGPR.2014. 6970529.

[3] D. Poljak and V. Doric, “Transmitted field in the lossy ground from ground penetrating radar (GPR) dipole antenna,” Computational Methods and

https://doi.org/10.26376/GPR2018008

Experimental Measurements XVII 3, vol. 59, WIT Press, pp. 3-11, 2015, doi: 10.2495/CMEM150011.

[4] A. Giannopoulos, “Modelling ground penetrating radar by gprmax,” Construction and Building Materials, vol. 19, no. 10, pp. 755-762, December 2005, doi: 10.1016/j.conbuildmat.2005.06.007.

[5] L. Gurel and U. Oguz, “Three-dimensional fdtd modelling of a ground penetrating radar,” IEEE Transactions on Geoscience and Remote sensing, vol. 38, no. 4, pp. 1513–1521, 2000, doi: 10.1109/APS.2000.874882.

[6] T. Weiland, “A discretization model for the solution of Maxwell's equations for six-component fields,” Archiv fuer Elektronik und Uebertragungstechnik, vol. 31, p. 116–120, 1977.

[7] D. Poljak, S. Sesnic, D. Paric, and K. El Khamlichi Drissi, “Direct Time Domain Modeling of the Transient Field Transmitted in a Dielectric Half-Space for GPR Applications,” Proceedings of the International Conference on Electromagnetics in Advanced Applications (ICEAA), 7–11 September 2015, Torino, Italy, pp. 345–348, doi: 10.1109/ICEAA.2015.7297133.

[8] D. Poljak, S. Sesnic, A. Susnjara, D. Paric, K. El Khamlichi Drissi, S. Lallechere, “Direct time domain evaluation of the transient field transmitted into a lossy ground due to GPR antenna radiation,” Engineering Analysis with Boundary Elements, vol. 82, pp. 27–31, 2017, doi: 10.1016/ j.enganabound. 2017.05.007

[9] C. Warren, L. Pajewski, D. Poljak, A. Ventura, A. Giannopoulos, and S. Sesnic, “A comparison of Finite-Difference, Finite-Integration, and Integral-Equation methods in the Time Domain for modelling Ground Penetrating Radar antennas,” Proceedings of the 16th International Conference of Ground Penetrating Radar, 13–16 June 2016, Hong Kong, China, doi: 10.1109/ICGPR.2016. 7572676.

[10] C. Warren, S. Sesnic, A. Ventura, L. Pajewski, D. Poljak, and A. Giannopoulos, “Comparison of Time-Domain Finite-Difference, Finite-Integration, and Integral-Equation Methods for Dipole Radiation in Half-Space Environments,” Progress In Electromagnetics Research M, vol. 57, pp. 175–183, 2017, doi: 10.2528/PIERM17021602.

[11] S. Lalléchère, S. Antonijevic, K. El Khamlichi Drissi, and D. Poljak, “Optimized Numerical Models of Thin Wire above an Imperfect and Lossy Ground for GPR Statistics,” Proc. International Conference on Electromagnetics in Advanced Applications (ICEAA), 7–11 September 2015, Torino, Italy, pp. 907–910, doi: 10.1109/ICEAA.2015.7297246.

[12] S. Lalléchère, S. Sesnic, P. Bonnet, K. El Khamlichi Drissi, F. Paladian, D. Poljak, “Sensitivity analysis of the time transient currents induced along

https://doi.org/10.26376/GPR2018008

thin wires buried in lossy and uncertain environments," Proceedings of the 11th European Conference on Antennas and Propagation (EUCAP 2017), 19–24 March 2017, Paris, France doi: 10.23919/EuCAP.2017.7928291

[13] S. Sesnic, S. Lalléchère, D. Poljak, and K. El Khamlichi Drissi, "Stochastic collocation analysis of the transient current induced along the wire buried in a lossy medium," Computational Methods and Experimental Measurements XVII, Southampton, WIT Press, pp. 47–58, 2015, doi: 10.2495/CMEM150051.

[14] A. Susnjara, D. Poljak, S. Sesnic, and V. Doric, "Time Domain and Frequency Domain Integral Equation Method for the Analysis of Ground Penetrating (GPR) Antenna," Proceedings of the 24th International Conference on Software, Telecommunications and Computer Networks (SoftCOM 2016), 22–24 September 2016, Split, Croatia, pp. 1–4, doi: 10.1109/SOFTCOM.2016.7772184.

[15] Matlab Version 7.11.0.584 (R2010b), Massachusetts: The MathWorks Inc, 2010.

[16] S. Lalléchère, P. Bonnet, and F. Paladian, "Electrical stochastic modelling of cell for bio-electromagnetic compatibility applications," Annals of Telecommunications, vol. 69, pp. 295–308, 2014, doi: 10.1007/s12243-013-0364-9.

[17] H. Dodig, S. Lalléchère, P. Bonnet, and K. El Khamlichi Drissi, "Stochastic sensitivity of the electromagnetic distributions inside a human eye modeled with 3D hybrid BEM/FEM edge element method," Engineering Analysis with Boundary Elements, vol. 49, pp. 48–62, 2014, doi: 10.1016/j.enganabound.2014.04.005.

[18] A. Saltelli, M. Ratto, T. Andres, F. Campolongo, J. Cariboni, D. Gatelli, M. Saisana, and S. Tarantola, Global Sensitivity Analysis: The Primer, Ed. Wiley, January 2008.

[19] D. Poljak, Advanced Modeling in Computational Electromagnetic Compatibility. Hoboken, NJ: Wiley, 2007.

Appendix

Galerkin-Bubnov indirect boundary element solution of Pocklington integral equation

An operator form of Pocklington integro-differential equation (12) can be, for convenience, symbolically written as:

$$K\,I = E \tag{A1}$$

https://doi.org/10.26376/GPR2018008

where K is a linear operator and I is the unknown function to be found for a given excitation E.

The unknown current is expanded into a finite sum of linearly independent basis functions $\{f_i\}$ with unknown complex coefficients α_i:

$$I \cong I_n \sum_{i=1}^{n} \alpha_i f_i \tag{A2}$$

Substituting (A2) into (A1) yields:

$$K\, I \cong I_n \sum_{i=1}^{n} \alpha_i K f_i = E_n = p_n(E) \tag{A3}$$

where $p_n(E)$ is called projection operator [19]. Now the residual R_n is formed as follows:

$$R_n = K\, I_n - E = p_n(E) - E \tag{A4}$$

In accordance to the definition of the scalar product of functions in Hilbert function space, the error R_n is weighted to zero with respect to certain weighting functions $\{W_j\}$, i.e.:

$$< R_n, W_j >= 0; \quad j = 1, 2, \dots n \tag{A5}$$

where the expression in brackets stands for a scalar product of functions given by:

$$< R_n, W_j >= \int_{\Omega} R_n W_j^{*} d\Omega \tag{A6}$$

where Ω denotes the actual calculation domain. As the operator K is linear, a system of linear equations is obtained by choosing $W_j = f_j$, which implies the Galerkin-Bubnov procedure. Thus, the following equation can be written:

$$\sum_{i=1}^{n} < K f_i, f_j >=< E, f_j > \quad j = 1, 2, \dots, n \tag{A7}$$

Equation (A7) is the strong Galerkin-Bubnov formulation of Pocklington integral equation (12). By using the integral equation kernel symmetry, by taking into account Dirichlet boundary conditions for the current at the free ends of the dipole, and after integration by parts, Equation (A7) becomes:

https://doi.org/10.26376/GPR2018008

$$\sum_{\mathrm{i}=1}^{n}\alpha_{\mathrm{i}}\frac{1}{j4\pi\varepsilon}\left\{\int_0^L\frac{df_{\mathrm{j}}(x)}{dx}\int_0^L\frac{df_{\mathrm{j}}(x)}{dx}g(x,x')dxdx' + k^2\int_0^L f_{\mathrm{j}}(x)\int_0^L f_{\mathrm{i}}(x)g(x,x')dxdx' + \int_0^L f_{\mathrm{j}}(x)Z_S(x)dx\right\} = \int_0^L E_x^{exc}(x)f_{\mathrm{j}}(x)dx\,,\quad \mathrm{j}=1,2,\dots n \tag{A8}$$

For the case of lossless conductors, Z_S=0. Equation (A8) represents the weak Galerkin-Bubnov formulation of the integral equation of (12). The resulting system of algebraic equations arising from the boundary element discretization of (A8) is given by [19]:

$$\sum_{\mathrm{j}=1}^{M}[Z]_{\mathrm{ji}}\{I\}_{\mathrm{i}} = \{V\}_{\mathrm{j}},\qquad \mathrm{j}=1,2,\dots M \tag{A9}$$

where $[Z]_{\mathrm{ji}}$ is the local matrix representing the interaction of the i-th source boundary element with the j-th observation boundary element:

$$[Z]_{\mathrm{ji}} = \frac{-1}{4j\pi\omega\varepsilon}\left(\int_{\Delta l_{\mathrm{j}}}\{D\}_{\mathrm{j}}\int_{\Delta l_{\mathrm{i}}}\{D'\}_{\mathrm{i}}^T g(x,x')dx\,dx' + k^2\int_{\Delta l_{\mathrm{j}}}\{f\}_{\mathrm{j}}\int_{\Delta l_{\mathrm{i}}}\{f\}_{\mathrm{i}}^T g(x,x')\,dxdx'\right) + \int_{\Delta l_{\mathrm{j}}} Z_L(x)\{f\}_{\mathrm{j}}\{f\}_{\mathrm{i}}^T dx \tag{A10}$$

The vector $\{I\}$ contains the unknown coefficients of the solution and represents the local voltage vector. The matrices $\{f\}$ and $\{f'\}$ contain the shape functions, $\{D\}$ and $\{D'\}$ contain their derivatives, M is the total number of line segments, and Δl_{i}, Δl_{j} are the widths of the i-th, j-th segments.

https://doi.org/10.26376/GPR2018008

The scientific paper that you have downloaded is included in Issue 2, Volume 1 (July 2018) of the journal *Ground Penetrating Radar* (ISSN 2533-3100; journal homepage: www.gpradar.eu/journal).

All *Ground Penetrating Radar* papers are processed and published in true open access, free to both Authors and Readers, thanks to the generous support of TU1208 GPR Association and to the voluntary efforts of the journal Editorial Board. The publication of Issue 2, Volume 1 is also supported by Adapis Georadar Teknik Ab (georadar.eu) and by IDS Georadar s.r.l. (idsgeoradar.com).

The present information sheet is obviously not part of the scientific paper.